Elements of stochastic calculus via regularisation

A la mémoire de Paul André Meyer

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Summary. This paper first summarizes the foundations of stochastic calculus via regularization and constructs through this procedure Itô and Stratonovich integrals. In the second part, a survey and new results are presented in relation with finite quadratic variation processes, Dirichlet and weak Dirichlet processes.

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1 Introduction

Stochastic integration via regularization is a technique of integration developed in a series of papers by the authors starting from [46], continued in [47, 48, 49, 50, 45] and later carried out by other authors, among them [51, 12, 13, 55, 54, 56, 58, 17, 16, 18, 19, 24]. Among some recent applications to finance, we refer for instance to [32, 4].

This approach constitutes a counterpart of a discretization approach initiated by Föllmer ([20]) and continued by many authors, see for instance [2, 22, 15, 14, 11, 23].

The two theories run parallel and, at the axiomatic level, almost all the results we obtained via regularization can essentially be translated in the language of discretization.

The advantage of using regularization lies in the fact that this approach is natural and relatively simple, and easily connects to other approaches. We now list some typical features of stochastic calculus via regularization.

- Two fundamental notions are the quadratic variation of a process, see Definition 2 and the forward integral, see Definition 1. Calculus via regularization is first of all a calculus related to finite quadratic variation processes, see section 4. Itô integrals with respect to continuous semimartingales can be defined through forward integrals, see Section 3; this makes classical stochastic calculus appear as a particular instance of calculus via regularization. Let the integrator be a classical Brownian motion W and the integrand a measurable adapted process H such that $\int_0^T H_t^2 dt < \infty$ a.s., where a.s. means almost surely. We will show in section 3.5 that the forward integral $\int_0^\cdot H d^- W$ coincides with the Itô integral $\int_0^\cdot H dW$. On the other hand, the discretization approach constitues a sort of Riemann-Stieltjes type integral and only allows integration of processes that are not too irregular, see Remark 14.
- Calculus via regularization constitutes a bridge between non causal and causal calculus operating through substitution formulae, see subsection 3.6. A precise link between forward integration and the theory of enlargement of filtrations may be given, see [47]. Our integrals can be connected to the well-known Skorohod type integrals, see again [47].
- With the help of symmetric integrals a calculus with respect to processes with a variation higher than 2 may be developed. For instance fractional Brownian motion is the prototype of such processes.
- This stochastic calculus constitutes somehow a barrier separating the pure pathwise calculus in the sense of T. Lyons and coauthors, see e.g. [36, 35, 31, 28], and any stochastic calculus taking into account an underlying probability, see Section 6.

This paper will essentially focuse on the first item.

The paper is organized as follows. First, in Section 2, we recall the basic definitions and properties of forward, backward, symmetric integrals and covariations. Justifying the related definitions and properties needs no particular effort. A significant example is the Young integral, see [57]. In Section 3 we redefine Itô integrals in the spirit of integrals via regularization and we prove some typical properties. We essentially define Itô integrals as forward integrals in a subclass and we then extend this definitionthrough functional analysis methods. Section 4 is devoted to finite quadratic variation processes.

In particular we establish C^1-stability properties and an Itô formula of C^2-type. Section 5 provides some survey material with new results related to the class of weak Dirichlet processes introduced by [12] with later developments discussed by [24, 7]. Considerations about Itô formulae under C^1-conditions are discussed as well.

2 Stochastic integration via regularization

2.1 Definitions and fundamental properties

In this paper T will be a fixed positive real number. By convention, any real continuous function f defined either on $[0,T]$ or $\mathbb{R}_+$ will be prolongated (with the same name) to the real line, setting

$$f(t) = \begin{cases} f(0) \text{ if } t \le 0 \\ f(T) \text{ if } t > T. \end{cases} \tag{1}$$

Let $(X_t)_{t\ge 0}$ be a continuous process and $(Y_t)_{t\ge 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}_+)$, i.e. for any $a > 0$, $\int_0^a |Y_t| dt < \infty$ a.s.

Our generalized stochastic integrals and covariations will be defined through a regularization procedure. More precisely, let $I^-(\varepsilon, Y, dX)$ (resp. $I^+(\varepsilon, Y, dX)$, $I^0(\varepsilon, Y, dX)$ and $C(\varepsilon, Y, X)$) be the ε-forward integral (resp. ε-backward integral, ε-symmetric integral and ε-covariation):

$$I^-(\varepsilon, Y, dX)(t) = \int_0^t Y(s) \frac{X(s+\varepsilon) - X(s)}{\varepsilon} ds; \quad t \ge 0, \tag{2}$$

$$I^+(\varepsilon, Y, dX)(t) = \int_0^t Y(s) \frac{X(s) - X(s-\varepsilon}{\varepsilon} ds; \quad t \ge 0, \tag{3}$$

$$I^0(\varepsilon, Y, dX)(t) = \int_0^t Y(s) \frac{X(s+\varepsilon) - X(s-\varepsilon)}{2\varepsilon} ds; \quad t \ge 0, \tag{4}$$

$$C(\varepsilon, X, Y)(t) = \int_0^t \frac{\big(X(s+\varepsilon) - X(s)\big)\big(Y(s+\varepsilon) - Y(s)\big)}{\varepsilon} ds; \quad t \ge 0. \tag{5}$$

Observe that these four processes are continuous.

Definition 1. *1) A family of processes $(H_t^{(\varepsilon)})_{t\in[0,T]}$ is said to converge to $(H_t)_{t\in[0,T]}$ in the* **ucp sense**, *if $\sup_{0\le t\le T} |H_t^{(\varepsilon)} - H_t|$ goes to 0 in probability, as $\varepsilon \to 0$.*

2) Provided the corresponding limits exist in the ucp sense, we define the following integrals and covariations by the following formulae

a) **Forward integral:** $\int_0^t Y d^- X = \lim_{\varepsilon\to 0^+} I^-(\varepsilon, Y, dX)(t).$

b) **Backward integral:** $\int_0^t Y d^+ X = \lim_{\varepsilon \to 0^+} I^+(\varepsilon, Y, dX)(t)$.

c) **Symmetric integral:** $\int_0^t Y d^\circ X = \lim_{\varepsilon \to 0+} I^\circ(\varepsilon, Y, dX)(t)$.

d) **Covariation:** $[X, Y]_t = \lim_{\varepsilon \to 0^+} C(\varepsilon, X, Y)(t)$. *When* $X = Y$ *we often put* $[X] = [X, X]$.

Remark 1. Let X, X', Y, Y' be four processes with X, X' continuous and Y, Y' having paths in $L^1_{loc}(\mathbb{R}_+)$. $\star$ will stand for one of the three symbols $-$, $+$ or $\circ$.

1. $(X, Y) \mapsto \int_0^\cdot Y d^\star X$ and $(X, Y) \mapsto [X, Y]$ are bilinear operations.
2. The covariation of continuous processes is a symmetric operation.
3. When it exists, $[X]$ is an increasing process.
4. If τ is a random time, $[X^\tau, X^\tau]_t = [X, X]_{t \wedge \tau}$ and
$$\int_0^t Y 1_{[0,\tau]} d^\star X = \int_0^t Y d^\star X^\tau = \int_0^t Y^\tau d^\star X^\tau = \int_0^{t \wedge \tau} Y d^\star X,$$
where X^τ is the process X stopped at time τ, defined by $X^\tau_t = X_{t \wedge \tau}$.
5. If ξ and η are two fixed r.v., $\int_0^\cdot (\xi Y_s) d^\star (\eta X_s) = \xi \eta \int_0^\cdot Y_s d^\star X_s$.
6. Integrals via regularization also have the following localization property. Suppose that $X_t = X'_t, Y_t = Y'_t, \forall t \in [0, T]$ on some subset Ω_0 of Ω. Then
$$1_{\Omega_0} \int_0^t Y_s d^\star X_s = 1_{\Omega_0} \int_0^t Y'_s d^\star X'_s, \quad t \in [0, T].$$
7. If Y is an elementary process of the type $Y_t = \sum_{i=1}^N A_i 1_{I_i}$, where A_i are random variables and (I_i) a family of real intervals with end-points $a_i < b_i$, then
$$\int_0^t Y_s d^\star X_s = \sum_{i=1}^N A_i (X_{b_i \wedge t} - X_{a_i \wedge t}).$$

Definition 2. *1) If* $[X]$ *exists,* X *is said to be a* **finite quadratic variation process** *and* $[X]$ *is called the* **quadratic variation** *of* X.
2) If $[X] = 0$, X *is called a* **zero quadratic variation process**.
3) A vector $(X^1, \ldots, X^n)$ *of continuous processes is said to have all its* **mutual covariations** *if* $[X^i, X^j]$ *exists for all* $1 \le i, j \le n$.
We will also use the terminology **bracket** *instead of covariation.*

Remark 2. 1) If $(X^1, \ldots, X^n)$ has all its mutual covariations, then
$$[X^i + X^j, X^i + X^j] = [X^i, X^i] + 2[X^i, X^j] + [X^j, X^j]. \tag{6}$$
From the previous equality, it follows that $[X^i, X^j]$ is the difference of two increasing processes, having therefore bounded variation; consequently the bracket is a classical integrator in the Lebesgue-Stieltjes sense.

2) Relation (6) holds as soon as three brackets among the four exist. More generally, by convention, an identity of the type $I_1 + \cdots + I_n = 0$ has the following meaning: if $n-1$ terms among the I_j exist, the remaining one also makes sense and the identity holds true.
3) We will see later, in Remark 23, that there exist processes X and Y such that $[X,Y]$ exists but does not have finite variation; in particular (X,Y) does not have all its mutual brackets.

The properties below follow elementarily from the definition of integrals via regularization.

Proposition 1. *Let $X = (X_t)_{t\geq 0}$ be a continuous process and $Y = (Y_t)_{t\geq 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}_+)$. Then*

1) $[X,Y]_t = \int_0^t Y d^+X - \int_0^t Y d^-X$.
2) $\int_0^t Y d^\circ X = \frac{1}{2}\left(\int_0^t Y d^+X + \int_0^t Y d^-X\right)$.
3) **Time reversal**. *Set* $\hat{X}_t = X_{T-t}, t \in [0,T]$. *Then*
   1. $\int_0^t Y d^\pm X = -\int_{T-t}^T \hat{Y} d^\mp \hat{X}, \quad 0 \leq t \leq T$;
   2. $\int_0^t Y d^\circ X = -\int_{T-t}^T \hat{Y} d^\circ \hat{X}, \quad 0 \leq t \leq T$;
   3. $[\hat{X},\hat{Y}]_t = [X,Y]_T - [X,Y]_{T-t}, \quad 0 \leq t \leq T$.
4) **Integration by parts**. *If Y is continuous,*

$$\begin{aligned} X_tY_t &= X_0Y_0 + \int_0^t X d^-Y + \int_0^t Y d^+X \\ &= X_0Y_0 + \int_0^t X d^-Y + \int_0^t Y d^-X + [X,Y]_t. \end{aligned}$$

5) **Kunita-Watanabe inequality**. *If X and Y are finite quadratic variation processes,*

$$\left|[X,Y]\right| \leq \left\{[X]\ [Y]\right\}^{1/2}.$$

6) If X is a finite quadratic variation process and Y is a zero quadratic variation process then (X,Y) has all its mutual brackets and $[X,Y]=0$.
7) Let X be a bounded variation process and Y be a process with locally bounded paths, and at most countably many discontinuities. Then
   a) $\int_0^t Y d^+X = \int_0^t Y d^-X = \int_0^t Y dX$, *where* $\int_0^t Y dX$ *is a Lebesgue-Stieltjes integral.*
   b) $[X,Y]=0$. *In particular a bounded variation and continuous process is a zero quadratic variation process.*

8) Let X be an absolutely continuous process and Y be a process with locally bounded paths. Then

$$\int_0^t Y d^+ X = \int_0^t Y d^- X = \int_0^t Y X' ds.$$

Remark 3. If Y has uncountably many discontinuities, 7) may fail. Take for instance $Y = 1_{\text{supp}\, dV}$, where V is an increasing continuous function such that $V'(t) = 0$ a.e. (almost everywhere) with respect to Lebesgue measure. Then $Y = 0$ Lebesgue a.e., and $Y = 1, dV$ a.e. Consequently

$$\int_0^t Y dV = V(t) - V(0), \quad I^-(\varepsilon, Y, dV)(t) = 0 \quad \int_0^t Y d^- V = 0.$$

Remark 4. Point 2) of Proposition 1 states that the symmetric integral is the average of the forward and backward integrals.

Proof of Proposition 1. Points 1), 2), 3), 4) follow immediately from the definition. For illustration, we only prove 3); operating a change of variable $u = T - s$, we obtain

$$\int_0^t Y_s \frac{X_s - X_{s-\varepsilon}}{\varepsilon} ds = -\int_{T-t}^T \hat{Y}_u \frac{\hat{X}_{u+\varepsilon} - \hat{X}_u}{\varepsilon} du, \quad 0 \le t \le T.$$

Since X is continuous, one can take the limit of both members and the result follows.

5) follows by Cauchy-Schwarz inequality which says that

$$\frac{1}{\varepsilon}\left|\int_0^t (X_{s+\varepsilon} - X_s)\,(Y_{s+\varepsilon} - Y_s) ds\right|$$
$$\le \left\{\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon} - X_s)^2 ds \; \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s)^2 ds\right\}^{\frac{1}{2}}.$$

6) is a consequence of 5).

7) Using Fubini, one has

$$\frac{1}{\varepsilon}\int_0^t Y_s (X_{s+\varepsilon} - X_s) ds = \frac{1}{\varepsilon}\int_0^t ds\, Y_s \int_s^{s+\varepsilon} dX_u$$
$$= \int_0^{t+\varepsilon} dX_u \, \frac{1}{\varepsilon}\int_{u-\varepsilon}^{u\wedge t} Y_s ds.$$

Since the jumps of Y are at most countable, $\frac{1}{\varepsilon}\int_{u-\varepsilon}^{u} Y_s ds \to Y_u, d|X|$ a.e. where $|X|$ denotes the total variation of X. Since $t \to Y_t$ is locally bounded, Lebesgue's convergence theorem implies that $\int_0^t Y d^- X = \int_0^t Y dX$.

The fact that $\int_0^t Y d^+ X = \int_0^t Y dX$ follows similarly.

b) is a consequence of point 1).

8) can be reached using similar elementary integration properties.

2.2 Young integral in a simplified framework

We will consider the integral defined by Young ([57]) in 1936, and implemented in the stochastic framework by Bertoin, see [3]. Here we will restrict ourselves to the case when integrand and integrator are Hölder continuous processes. As a result, that integral will be shown to coincide with the forward integral, but also with backward and symmetric ones.

Definition 3. *1. Let C^α be the set of Hölder continuous functions defined on $[0,T]$, with index $\alpha > 0$. Recall that $f : [0,T] \mapsto \mathbb{R}$ belongs to C^α if*

$$N_\alpha(f) := \sup_{0 \le s,t \le T} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < \infty.$$

2. If $X, Y : [0,T] \mapsto \mathbb{R}$ are two functions of class C^1, the Young integral of Y with respect to X on $[a,b] \subset [0,T]$ is defined as :

$$\int_a^b Y d^{(y)} X := \int_a^b Y(t) X'(t) dt, \quad 0 \le a \le b \le T.$$

To extend the Young integral to Hölder functions we need some estimate of $\int_0^T Y d^{(y)} X$ in terms of the Hölder norms of X and Y. More precisely, let X and Y be as in Definition 3 above; then in [15], it is proved:

$$\Big| \int_a^T (Y - Y(a)) d^{(y)} X \Big| \le C_\rho T^{1+\rho} N_\alpha(X) N_\beta(Y), \quad 0 \le a \le T, \tag{7}$$

where $\alpha, \beta > 0$, $\alpha + \beta > 1$, $\rho \in]0, \alpha+\beta-1[$, and C_ρ is a universal constant.

Proposition 2. *1. The map $(X,Y) \in C^1([0,T]) \times C^1([0,T]) \mapsto \int_0^\cdot Y d^{(y)} X$ with values in C^α, extends to a continuous bilinear map from $C^\alpha \times C^\beta$ to $\mathcal{C}^\alpha$. The value of this extension at point $(X,Y) \in \mathcal{C}^\alpha \times C^\beta$ will still be denoted by $\int_0^\cdot Y d^{(y)} X$ and called the* **Young integral** *of Y with respect to X.*
2. Inequality (7) is still valid for any $X \in C^\alpha$ and $Y \in C^\beta$.

Proof. 1. Let X, Y be of class $C^1([0,T])$ and

$$F(t) = \int_0^t Y d^{(y)} X = \int_0^t Y(s) X'(s) ds, \quad t \in [0,T].$$

For any $a, b \in [0,T]$, $a < b$, we have

$$F(b) - F(a) = \int_a^b \big(Y(t) - Y(a)\big) d^{(y)} X + Y(a)\big(X(b) - X(a)\big).$$

Then (7) implies

$$|F(b)-F(a)| \le C_\rho(b-a)^{1+\rho} N_\alpha(X) N_\beta(Y) + \sup_{0\le t\le T} |Y(t)|\, N_\alpha(X)(b-a)^\alpha; \quad (8)$$

consequently $F \in C^\alpha$.

Then the map $(X,Y) \in C^1([0,T]) \times C^1([0,T]) \mapsto \int_0^\cdot Y d^{(y)} X$, which is bilinear, extends to a continuous bilinear map from $C^\alpha \times \mathcal{C}^\beta$ to C^α.

2. is a consequence of point 1.

Before discussing the relation between Young integrals and integrals via regularization, here is useful technical result.

Lemma 1. *Let $0 < \gamma' < \gamma \le 1, \varepsilon > 0$. With $Z \in C^\gamma$ we associate*

$$Z_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \big(Z(u+\varepsilon) - Z(u)\big)du, \ t \in [0,T].$$

Then Z_ε converges to Z in $C^{\gamma'}$, as $\varepsilon \to 0$.

Proof. For any $0 \le t \le T$,

$$Z_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \big(Z(u+\varepsilon) - Z(u)\big)du = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} Z(u)du - \frac{1}{\varepsilon}\int_0^\varepsilon Z(u)du.$$

Setting $\Delta_\varepsilon(t) = Z_\varepsilon(t) - Z(t)$, we get

$$\begin{aligned}\Delta_\varepsilon(t) - \Delta_\varepsilon(s) &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} Z(u)du - Z(t) - \frac{1}{\varepsilon}\int_s^{s+\varepsilon} Z(u)du + Z(s)\\ &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \big(Z(u)-Z(t)\big)du - \frac{1}{\varepsilon}\int_s^{s+\varepsilon}\big(Z(u)-Z(s)\big)du,\end{aligned}$$

where $0 \le s \le t \le T$.

a) Suppose $0 \le s < s+\varepsilon < t$. The above inequality implies

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \le \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \big|Z(u)-Z(t)\big|du + \frac{1}{\varepsilon}\int_s^{s+\varepsilon}\big|Z(u)-Z(s)\big|du.$$

Since $Z \in C^\gamma$, then

$$\begin{aligned}|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| &\le \frac{N_\gamma(Z)}{\varepsilon}\Big(\int_t^{t+\varepsilon}(u-t)^\gamma du + \int_s^{s+\varepsilon}(u-s)^\gamma du\Big)\\ &\le \frac{2N_\gamma(Z)}{\gamma+1}\varepsilon^\gamma.\end{aligned}$$

But $\varepsilon < t-s$, consequently

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \le \frac{2N_\gamma(Z)}{\gamma+1}\varepsilon^{\gamma-\gamma'}|t-s|^{\gamma'}. \quad (9)$$

b) We now investigate the case $0 \le s < t < s+\varepsilon$. The difference $\Delta_\varepsilon(t) - \Delta_\varepsilon(s)$ may be decomposed as follows :

$$\Delta_\varepsilon(t) - \Delta_\varepsilon(s) = \frac{1}{\varepsilon}\int_{s+\varepsilon}^{t+\varepsilon} \big(Z(u) - Z(s+\varepsilon)\big)du - \frac{1}{\varepsilon}\int_s^t \big(Z(u) - Z(s)\big)du$$

$$+\frac{t-s}{\varepsilon}\big(Z(s+\varepsilon) - Z(s)\big) + Z(s) - Z(t).$$

Proceeding as in the previous step and using the inequality $0 < t-s < \varepsilon$, we obtain

$$\begin{aligned} |\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| &\le N_\gamma(Z)\Big(\frac{2}{\gamma+1}\frac{(t-s)^{\gamma+1}}{\varepsilon} + \frac{t-s}{\varepsilon^{1-\gamma}} + (t-s)^\gamma\Big) \\ &\le 2N_\gamma(Z)\frac{\gamma+2}{\gamma+1}\varepsilon^{\gamma-\gamma'}|t-s|^{\gamma'}. \end{aligned}$$

At this point, the above inequality and (9) directly imply that $N_{\gamma'}(Z_\varepsilon - Z) \le C\varepsilon^{\gamma-\gamma'}$ and the claim is finally established.

In the sequel of this section X and Y will denote stochastic processes.

Remark 5. If X and Y have a.s. Hölder continuous paths respectively of order α and β with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta > 1$. Then one can easily prove that $[X, Y] = 0$.

Proposition 3. *Let X, Y be two real processes indexed by $[0,T]$ whose paths are respectively a.s. in C^α and C^β, with $\alpha > 0, \beta > 0$ and $\alpha+\beta > 1$. Then the three integrals $\int_0^\cdot Y d^+X$, $\int_0^\cdot Y d^-X$ and $\int_0^\cdot Y d^\circ X$ exist and coincide with the Young integral $\int_0^\cdot Y d^{(y)}X$.*

Proof. We establish that the forward integral coincides with the Young integral. The equality concerning the two other integrals is a consequence of Proposition 1 1., 2. and Remark 5.

By additivity we can suppose, without lost generality, that $Y(0) = 0$.

Set

$$\Delta_\varepsilon(t) := \int_0^t Y d^{(y)}X - \int_0^t Y dX_\varepsilon,$$

where

$$X_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \big(X(u+\varepsilon) - X(u)\big)du,\ t \in [0,T].$$

Since $t \mapsto X_\varepsilon(t)$ is of class $C^1([0,T])$, then $\int_0^t Y dX_\varepsilon$ is equal to the Young integral $\int_0^t Y d^{(y)}X_\varepsilon$ and therefore

$$\Delta_\varepsilon(t) = \int_0^t Y d^{(y)}\big(X - X_\varepsilon\big).$$

Let α' be such that : $0 < \alpha' < \alpha$ and $\alpha' + \beta > 1$. Applying inequality (7) we obtain

$$\sup_{0 \le t \le T} |\Delta_\varepsilon(t)| \le C_\rho T^{1+\rho} N_{\alpha'}(X - X_\varepsilon) N_\beta(Y), \quad \rho \in]0, \alpha' + \beta - 1[.$$

Lemma 1 with $Z = X$ and $\gamma = \alpha$ directly implies that $\Delta_\varepsilon(t)$ goes to 0, uniformly a.s. on $[0, T]$, as $\varepsilon \to 0$, concluding the proof of the Proposition.

3 Itô integrals and related topics

The section presents the construction of Itô integrals with respect to continuous local martingales; it is based on McKean's idea (see section 2.1 of [37]), which fits the spirit of calculus via regularization.

3.1 Some reminders on martingales theory

In this subsection, we recall basic notions related to martingale theory, essentially without proofs, except when they help the reader. For detailed complements, see [30], chap. 1., in particular for definition of adapted and progressively measurable processes.

Let $(\mathcal{F}_t)_{t \ge 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions, see Definition 2.25, chap. 1 in [30].

An adapted process (M_t) of integrable random variables, i.e. verifying $E(|M_t|) < \infty, \ \forall t \ge 0$ is:

- an $(\mathcal{F}_t)$-martingale if $E(M_t | \mathcal{F}_s) = M_s, \quad \forall t \ge s$;
- a $(\mathcal{F}_t)$- submartingale if $E(M_t | \mathcal{F}_s) \ge M_s, \quad \forall t \ge s$

In this paper, all submartingales (and therefore all martingales) will be supposed to be continuous.

Remark 6. It follows from the definition that if $(M_t)_{t \ge 0}$ is a martingale, then $E(M_t) = E(M_0), \ \forall t \ge 0$. If $(M_t)_{t \ge 0}$ is a supermartingale (resp. submartingale) then $t \longrightarrow E(M_t)$ is decreasing (resp. increasing).

Definition 4. *A process X is said to be* **square integrable** *if $E(X_t^2) < \infty$ for each $t \ge 0$.*

When we speak of a martingale without specifying the σ-fields, we refer to the *canonical* filtration generated by the process and satisfying the usual conditions.

Definition 5. *1. A (continuous) process $(X_t)_{t\geq 0}$, is called a $(\mathcal{F}_t)$-***local martingale** *(resp. $(\mathcal{F}_t)$-***local submartingale***) if there exists an increasing sequence (τ_n) of stopping times such that $X^{\tau_n} 1_{\tau_n > 0}$ is an $(\mathcal{F}_t)$-martingale (resp. submartingale) and $\lim_{n\to\infty} \tau_n = +\infty$ a.s.*

Remark 7.
- An $(\mathcal{F}_t)$-martingale is an $(\mathcal{F}_t)$-local martingale. A bounded $(\mathcal{F}_t)$-local martingale is an $(\mathcal{F}_t)$-martingale.
- The set of $(\mathcal{F}_t)$-local martingales is a linear space.
- If M is an $(\mathcal{F}_t)$-local martingale and τ a stopping time, then M^τ is again an $(\mathcal{F}_t)$-local martingale.
- If M_0 is bounded, in the definition of a local martingale one can choose a localizing sequence (τ_n) such that each M^{τ_n} is bounded.
- A convex function of an $(\mathcal{F}_t)$-local submartingale is an $(\mathcal{F}_t)$-local submartingale.

Definition 6. *A process S is called a (continuous) $(\mathcal{F}_t)$-***semimartingale** *if it is the sum of an $(\mathcal{F}_t)$-local martingale and an $(\mathcal{F}_t)$-adapted continuous bounded variation process.*

A basic decomposition in stochastic analysis is the following.

Theorem 1. *(***Doob decomposition of a submartingale***)*

Let X be a $(\mathcal{F}_t)$-local submartingale. Then, there is an $(\mathcal{F}_t)$-local martingale M and an adapted, continuous, and finite variation process V (such that $V_0 = 0$) with $X = M + V$. The decomposition is unique.

Definition 7. *Let M be an $(\mathcal{F}_t)$-local martingale. We denote by $< M >$ the bounded variation process featuring in the Doob decomposition of the local submartingale M^2. In particular $M^2 - < M >$ is an $(\mathcal{F}_t)$-local martingale.*

In Corollary 2, we will prove that $< M >$ coincides with $[M, M]$, so that the skew bracket $< M >$ does not depend on the underlying filtration.

The following result will be needed in section 3.2.

Lemma 2. *Let $(M^n_{t\in[0,T]})$ be a sequence of $(\mathcal{F}_t)$ local martingales such that $M^n_0 = 0$ and $< M^n >_T$ converges to 0 in probability as $n \to \infty$. Then $M^n \to 0$ ucp, when $n \to \infty$.*

Proof. It suffices to apply to $N = M^n$ the following inequality stated in [30], Problem 5.25 Chap. 1, which holds for any $(\mathcal{F}_t)$-local martingale (N_t) such that $N_0 = 0$:

$$P\big(\sup_{0\leq u\leq t} |N_u| \geq \lambda\big) \leq P\big(< N >_t \geq \delta\big) + \frac{1}{\lambda^2} E\big[\delta \wedge < N >_t \big], \qquad (10)$$

for any $t \geq 0,\ \lambda, \delta > 0$.

An immediate consequence of the previous lemma is the following.

Corollary 1. *Let M be an $(\mathcal{F}_t)$-local martingale vanishing at zero, with $< M > = 0$. Then M is identically zero.*

3.2 The Itô integral

Let M be an $(\mathcal{F}_t)$-local martingale. We construct here the Itô integral with respect to M using stochastic calculus via regularization. We will proceed in two steps. First we define the Itô integral $\int_0^\cdot HdM$ for a smooth integrand process H as the forward integral $\int_0^\cdot Hd^-M$. Second, we extend $H \mapsto \int_0^\cdot HdM$ via functional analytical arguments. We remark that the classical theory of Itô integrals first defines the integral of simple step processes H, see Remark 9, for details.

Observe first that the forward integral of a continuous process H of bounded variation is well defined because Proposition 1 4), 7) imply that

$$\int_0^t Hd^-M = H_tM_t - H_0M_0 - \int_0^t Md^+H = H_tM_t - H_0M_0 - \int_0^t M_s dH_s. \quad (11)$$

Call $\mathcal{C}$ the vector algebra of adapted processes whose paths are of class C^0. This linear space, equipped with the metrizable topology which governs the ucp convergence, is an F-space. For the definition and properties of F-spaces, see [10], chapter 2.1. Remark that the set $\mathcal{M}_{\text{loc}}$ of continuous $(\mathcal{F}_t)$-local martingales is a closed linear subspace of $\mathcal{C}$, see for instance [24].

Denote by $\mathcal{C}^{BV}$ the $\mathcal{C}$ subspace of processes whose paths are a.s. continuous with bounded variation. The next observation is crucial.

Lemma 3. *If H is an adapted process in $\mathcal{C}^{BV}$ then $\left(\int_0^\cdot Hd^-M\right)$ is an $(\mathcal{F}_t)$-local martingale whose quadratic variation is given by*

$$< \int_0^\cdot Hd^-M >_t = < \int_0^\cdot H_s^2 d < M >_s .$$

Proof. We only sketch the proof. We restrict ourselves to prove that if M is a local martingale then $Y = \displaystyle\int_0^\cdot Hd^-M$ is a local martingale.

By localization, we can suppose that H, its total variation $\|H\|$ and M are bounded processes.

Let $0 \le s < t$. Since $H_t = H_0 + \int_s^t dH_u$, (11) implies

$$Y_t = H_sM_t - H_0M_0 - \int_0^s M_u dH_u + \int_s^t (M_t - M_u)dH_u. \quad (12)$$

Let (π_n) be a sequence of subdivisions of $[s,t]$, such that the mesh of (π_n) goes to zero when $n \to +\infty$. Since M is continuous, M and $\|H\|$ are bounded,

$$\Delta_n := \sum_{\pi_n} (M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i}),$$

goes to $\int_s^t (M_t - M_u)dH_u$ a.s. and in L^1. Consequently,

$$E\left(\int_s^t (M_t - M_u) dH_u\right) = \lim_{n\to\infty} E(\Delta_n|\mathcal{F}_s)$$

and

$$E(\Delta_n|\mathcal{F}_s) = \sum_{\pi_n} E\left((M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i})|\mathcal{F}_s\right).$$

But one has

$$\begin{aligned} &E\left((M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i})|\mathcal{F}_s\right) \\ &\qquad = E\left(E((M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i})|\mathcal{F}_{u_{i+1}})|\mathcal{F}_s\right) \qquad (13) \\ &\qquad = E\left((H_{u_{i+1}} - H_{u_i})E(M_t - M_{u_{i+1}}|\mathcal{F}_{u_{i+1}})|\mathcal{F}_s\right) \\ &\qquad = 0, \qquad (14) \end{aligned}$$

since H is adapted and M is a martingale.

Finally, taking the conditional expectation with respect to $\mathcal{F}_s$ in (12) yields

$$E\big[Y_t|\mathcal{F}_s\big] = H_s M_s - H_0 M_0 - \int_0^s M_u dH_u = Y_s.$$

Similar arguments show that $Y^2 - \displaystyle\int_0^\cdot H^2 d < M >$ is a martingale.

The previous lemma allows to extend the map $H \mapsto \int_0^t H d^- M$. Let $\mathcal{L}^2(d < M >)$ denote the set of progressively measurable processes such that

$$\int_0^T H^2 d < M >< \infty \text{ a.s.} \qquad (15)$$

$\mathcal{L}^2(d < M >)$ is an F-space with respect to the metrizable topology d_2 defined as follows: (H^n) converges to H when $n \to \infty$ if $\int_0^T (H_s^n - H_s)^2 d < M >_s \to 0$ in probability, when $n \to \infty$.

Remark 8. $\mathcal{C}^{BV}$ is dense in $\mathcal{L}^2(d < M >)$. Indeed, according to [30], lemma 2.7 section 3.2, simple processes are dense into $\mathcal{L}^2(d < M >)$. On the other hand, a simple process of the form $H_t = \xi 1_{]a,b]}$, ξ being $\mathcal{F}_a$ measurable, can be expressed as a limit of $H_t^n = \xi\phi^n$ where ϕ^n are continuous functions with bounded variation.

Let $\Lambda : \mathcal{C}^{BV} \to \mathcal{M}_{\text{loc}}$ be the map defined by $\Lambda H = \int_0^\cdot H d^- M$.

Lemma 4. *If $\mathcal{C}^{BV}$ (resp. $\mathcal{M}_{\text{loc}}$) is equipped with d_2 (resp. the ucp topology) then Λ is continuous.*

Proof. Let H^k be a sequence of processes in $\mathcal{C}^{BV}$, converging to 0 for d_2 when $k \to \infty$. Set $N^k = \int_0^\cdot H^k d^- M$. Lemma 3 implies that $< N^k >_T$ converges to 0 in probability. Finally Lemma 2 concludes the proof.

We can now easily define the Itô integral. Since $\mathcal{C}^{BV}$ is dense in $\mathcal{L}^2(d<M>)$ for d_2, Lemma 4 and standard functional analysis arguments imply that Λ uniquely and continuously extends to $\mathcal{L}^2(d<M>)$.

Definition 8. *If H belongs to $\mathcal{L}^2(d<M>)$, we put $\int_0^\cdot HdM := \Lambda H$ and we call this the* **Itô integral of H with respect to M**.

Proposition 4. *If H belongs to $\mathcal{L}^2(d<M>)$, then $(\int_0^\cdot HdM)$ is an $(\mathcal{F}_t)$-local martingale with bracket*

$$< \int_0^\cdot HdM >= \int_0^\cdot H^2 d<M> . \tag{16}$$

Proof. Let $H \in \mathcal{L}^2(d<M>)$. From Definition 8, $(\int_0^\cdot HdM)$ is an $(\mathcal{F}_t)$-local martingale. It remains to prove (16).

Since H belongs to $\mathcal{L}^2(d<M>)$, then there exists a sequence (H_n) of elements in $\mathcal{C}^{BV}$, such that $H_n \to H$ in $\mathcal{L}^2(d<M>)$.

Introduce $N_n = \displaystyle\int_0^\cdot H_n dM$ and $N'_n = N_n^2 - <N_n>$. According to lemma 4, $<N_n>= \displaystyle\int_0^\cdot H_n^2 d<M>$; now $N_n \to N$, ucp, $n \to \infty$ and $<N_n>$ goes to $\displaystyle\int_0^\cdot H^2 d<M>$ in the ucp sense, as $n \to \infty$. Therefore N'_n converges with respect to the ucp topology, to the local martingale $N^2 - \displaystyle\int_0^\cdot H^2 d<M>$. This actually proves (16).

Remark 9. 1. Recall that whenever $H \in \mathcal{C}^{BV}$

$$\int_0^\cdot HdM = \int_0^\cdot Hd^-M.$$

This property will be generalized in Propositions 6 and 2.

2. We emphasize that Itô stochastic integration based on adapted simple step processes and the previous construction, finally lead to the same object. If H is of the type $Y 1_{]a,b]}$ where Y is an $\mathcal{F}_a$ measurable random variable, it is easy to show that $\int_0^t HdM = Y(M_{t\wedge b} - M_{t\wedge a})$. Since the class of elementary processes obtained by linear combination of previous processes is dense in $\mathcal{L}^2(d<M>)$ and the map Λ is continuous, then $\int_0^\cdot HdM$ equals the classical Itô integral.

In Proposition 5 below we state the chain rule property.

Proposition 5. *Let $(M_t, t \ge 0)$ be an $(\mathcal{F}_t)$-local martingale, $(H_t, t \ge 0)$ be in $\mathcal{L}^2(d<M>)$, $N := \displaystyle\int_0^\cdot H_s dM_s$ and $(K_t, t \ge 0)$ be a $(\mathcal{F}_t)$-progressively measurable process such that $\displaystyle\int_0^T (H_s K_s)^2 d<M>_s < \infty$ a.s. Then*

$$\int_0^t K_s dN_s = \int_0^t H_s K_s dM_s, \quad 0 \le t \le T. \tag{17}$$

Proof. Since the map $\Lambda : H \in \mathcal{L}^2(d < M >) \mapsto \int_0^\cdot HdM$ is continuous, it suffices to prove (17) for H and K continuous and with bounded variation.

For simplicity we suppose $M_0 = H_0 = K_0 = 0$.

One has

$$\int_0^t KdN = \int_0^t (N_t - N_u) dK_u,$$

and

$$\begin{aligned} N_t - N_u &= \int_0^t (M_t - M_v) dH_v - \int_0^u (M_u - M_v) dH_v \\ &= (M_t - M_u) H_u + \int_u^t (M_t - M_v) dH_v, \end{aligned}$$

where $0 \le u \le t$.

Using Fubini's theorem one gets

$$\begin{aligned} \int_0^t KdN = \int_0^t (M_t - M_u)(H_u dK_u + K_u dH_u) \\ = \int_0^t (M_t - M_u) d(HK)_u = \int_0^t HKdM. \end{aligned}$$

3.3 Connections with calculus via regularizations

The next Proposition will show that, under suitable conditions, the Itô integral is a forward integral.

Proposition 6. *Let X be an $(\mathcal{F}_t)$-local martingale and suppose that (H_t) is progressively measurable and locally bounded.*

1. *If H has a left limit at each point then* $\int_0^\cdot H_s d^- X_s = \int_0^\cdot H_{s-} dX_s$.
2. *If $H_t = H_{t-}$, $d < X >_t$ a.e. (in particular if H is càdlàg), then* $\int_0^\cdot H_s d^- X_s = \int_0^\cdot H_s dX_s$.

Proof.

Since $s \mapsto \int_{s-\varepsilon}^s H_u du$ is continuous with bounded variation,

$$\begin{aligned} \int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big) dX_s = \int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big) d^- X_s \\ = X_t \Big(\frac{1}{\varepsilon}\int_{t-\varepsilon}^t H_u du\Big) - H_0 X_0 - \frac{1}{\varepsilon}\int_0^t (H_s - H_{s-\varepsilon}) X_s ds. \end{aligned}$$

The second integral in the right-hand side can be modified as follows

$$-\int_0^t (H_s - H_{s-\varepsilon})X_s ds = \int_0^t H_s(X_{s+\varepsilon} - X_s)ds - \int_{t-\varepsilon}^t H_s X_{s+\varepsilon} ds + H_0 \int_0^\varepsilon X_s ds.$$

Consequently

$$\int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big) dX_s = \frac{1}{\varepsilon}\int_0^t H_s(X_{s+\varepsilon} - X_s)ds + R_\varepsilon(t), \tag{18}$$

where

$$\begin{aligned} R_\varepsilon(t) &= X_t\Big(\frac{1}{\varepsilon}\int_{t-\varepsilon}^t H_s ds\Big) - \frac{1}{\varepsilon}\int_{t-\varepsilon}^t H_s X_{s+\varepsilon} ds + H_0\left(\frac{1}{\varepsilon}\int_0^\varepsilon X_s ds - X_0\right) \\ &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^t H_s(X_t - X_{s+\varepsilon})ds + H_0\left(\frac{1}{\varepsilon}\int_0^\varepsilon X_s ds - X_0\right) \end{aligned} \tag{19}$$

converges to zero ucp.

Under assumption 1, Lebesgue's dominated convergence theorem implies that $\frac{1}{\varepsilon}\int_{\cdot-\varepsilon}^{\cdot} H_s ds$ converges to H_- according to $\mathcal{L}^2(d < M >)$, so the left-hand side of equality (18) converges to the Itô integral $\int_0^\cdot H_{s-} dX_s$. This forces the right-hand side to converge to $\int_0^\cdot H_s d^- X_s$.

The proof of 2 is similar, remarking that $H_s = H_{s-}$, for $d < M >_s$ a.e.

When the integrator is a Brownian motion W, we will see in Theorem 2 below that the forward integral coincides with the Itô integral for any integrand in $\mathcal{L}^2(d < W >)$. This is no longer true when the integrator is a general semimartingale. The following example provides a martingale (M_t) and a deterministic integrand h such that the Itô integral $\int_0^t hdM$ and the forward integral $\int_0^t hd^- M$ exist, but are different.

Example 1. Let $\psi : [0, \infty[\mapsto \mathbb{R}$ verify $\psi(0) = 0$, ψ is continuous, increasing, and $\psi'(t) = 0$ a.e. (with respect to the Lebesgue measure). Let (M_t) be the process: $M_t = W_{\psi(t)},\ t \geq 0$, and h be the indicator function of the support of the positive measure $d\psi$. Since $W_t^2 - t$ is a martingale, $< W >_t = t$. Clearly (M_t) is a martingale and $< M >_t = \psi(t),\ t \geq 0$. Observe that $h = 0$ a.e. with respect to Lebesgue measure. Then $\int_0^\cdot h(s)\frac{M(s+\varepsilon) - M(s)}{\varepsilon} ds = 0$ and so $\int_0^\cdot hd^- M = 0$.

On the other hand, $h=1,\ d\psi$ a.e., implies $\int_0^t h dM = M_t,\ t \geq 0$.

Remark 10. A significant result of classical stochastic calculus is the Bichteler-Dellacherie theorem, see [43] Th. 22, Section III.7. In the regularization approach, an analogous property occurs: if the forward integral exists for a rich class of adapted integrands, then the integrator is forced to be a semimartingale. More precisely we recall the significant statement of [47], Proposition 1.2.

Let $(X_t, t \geq 0)$ be an $(\mathcal{F}_t)$-adapted and continuous process such that for any càdlàg, bounded and adapted process (H_t), the forward integral $\int_0^\cdot H d^- X$ exists. Then (X_t) is an $(\mathcal{F}_t)$-semimartingale.

From Proposition 6 we deduce the relation between skew and square bracket.

Corollary 2. *Let M be an $(\mathcal{F}_t)$-local martingale. Then $< M >= [M]$ and*

$$M_t^2 = M_0^2 + 2\int_0^t M d^- M + < M >_t . \tag{20}$$

Proof. The proof of (20) is very simple and is based on the following identity

$$(M_{s+\varepsilon} - M_s)^2 = M_{s+\varepsilon}^2 - M_s^2 - 2M_s(M_{s+\varepsilon} - M_s).$$

Integrating on $[0,t]$ leads to

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (M_{s+\varepsilon} - M_s)^2 ds &= \frac{1}{\varepsilon}\int_0^t M_{s+\varepsilon}^2 ds - \frac{1}{\varepsilon}\int_0^t M_s^2 ds - \frac{2}{\varepsilon}\int_0^t M_s(M_{s+\varepsilon} - M_s) ds \\ &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} M_s^2 ds - \frac{1}{\varepsilon}\int_0^\varepsilon M_s^2 ds - \frac{2}{\varepsilon}\int_0^t M_s(M_{s+\varepsilon} - M_s) ds.\end{aligned}$$

Therefore, taking the limit when $\varepsilon \to 0$, one obtains

$$[M]_t = M_t^2 - M_0^2 - 2\int_0^t M_s d^- M_s.$$

Since $t \mapsto M_t$ is continuous, the forward integral $\int_0^\cdot M d^- M$ coincides with the corresponding Itô integral. Consequently $M_t^2 - M_0^2 - [M]_t$ is a local martingale. This proves both $[M] =< M >$ and (20).

Corollary 3. *Let M, M' be two $(\mathcal{F}_t)$-local martingales. Then (M, M') has all its mutual covariations.*

Proof. Since M, M' and $M + M'$ are continuous local martingales, Corollary 2 directly implies that they have finite quadratic variation. The bilinearity property of the covariation directly implies that $[M, M']$ exists and equals

$$\frac{1}{2}([M + M'] - [M] - [M']).$$

Proposition 7. *Let M and M' be two $(\mathcal{F}_t)$-local martingales, H and H' be two progressively measurable processes such that*

$$\int_0^{\cdot} H^2 d < M >< \infty, \quad \int_0^{\cdot} H^2 d < M' >< \infty.$$

Then

$$\Big[\int_0^{\cdot} H dM, \int_0^{\cdot} H' dM'\Big]_t = \int_0^t H H' d[M, M']_t.$$

The next proposition provides a simple example of two processes (M_t) and (Y_t) such that $[M, Y]$ exists even though the vector (M, Y) has no mutual covariation.

Proposition 8. *Let (M_t) be an continuous $(\mathcal{F}_t)$-local martingale, (Y_t) a càdlàg and an $(\mathcal{F}_t)$-adapted process. If M and Y are independent then $[M, Y] = 0$.*

Proof. Let $\mathcal{Y}$ be the σ-field generated by (Y_t), and denote by $(\tilde{\mathcal{M}}_t)$ the smallest filtration satisfying the usual conditions and containing $(\mathcal{F}_t)$ and $\mathcal{Y}$, i.e., $\sigma(M_s, s \le t) \vee \mathcal{Y} \subset \tilde{\mathcal{M}}_t, \forall t \ge 0$. It is not difficult to show that (M_t) is also an $(\tilde{\mathcal{M}}_t)$-martingale.

Thanks to Proposition 1 1., it is sufficient to prove that

$$\int_0^t Y d^- M = \int_0^t Y d^+ M. \tag{21}$$

Proposition 6 implies that the left-hand side coincides with the $(\mathcal{M}_t)$-Itô integral $\int_0^t Y dM$.

Without restricting generality we suppose $M_0 = 0$. We proceed as in the proof of Proposition 6. Since a.s. $s \mapsto \displaystyle\int_s^{s+\varepsilon} Y_u du$ is continuous with bounded variation,

$$\int_0^t \Big(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} Y_u du\Big) d^- M_s = M_t \Big(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} Y_u du\Big) - \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s) M_s ds.$$

As the processes Y and M are independent, the forward integral in the left-hand side above is actually an Itô integral. Therefore, taking the limit when $\varepsilon \to 0$ and using Proposition 6, one gets

$$\int_0^t Y dM = \int_0^t Y d^- M = Y_t M_t - \int_0^t M d^- Y.$$

According to point 4) of Proposition 1, the right-hand side is equal to $\displaystyle\int_0^t Y d^+ M$; this proves (21).

3.4 The semimartingale case

We begin this section with a technical lemma which implies that the decomposition of a semimartingale is unique.

Lemma 5. *Let $(M_t, t \geq 0)$ be a $(\mathcal{F}_t)$-local martingale with bounded variation. Then (M_t) is constant.*

Proof. Since M has bounded variation, then Proposition 1, 7) implies that $[M] = 0$. Consequently Corollaries 1 and 2 imply that $M_t = M_0, \ t \geq 0$.

It is now easy to define stochastic integration with respect to continuous semimartingales.

Definition 9. *Let $(X_t, t \geq 0)$ be an $(\mathcal{F}_t)$-semimartingale with canonical decomposition $X = M+V$, where M (resp. V) is a continuous $(\mathcal{F}_t)$-local martingale (resp. bounded variation, continuous and $(\mathcal{F}_t)$-adapted process) vanishing at 0. Let $(H_t, t \geq 0)$ be an $(\mathcal{F}_t)$-progressively measurable process, satisfying*

$$\int_0^T H_s^2 d[M,M]_s < \infty, \quad \text{and} \quad \int_0^T |H_s| d\|V\|_s < \infty, \tag{22}$$

where $\|V\|_t$ is the total variation of V over $[0,t]$.

We set

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dV_s, \quad 0 \leq t \leq T.$$

Remark 11. 1. In the previous definition, the integral with respect to M (resp. V) is an Itô-type (resp. Stieltjes-type) integral.
2. It is clear that $\int_0^\cdot H_s dX_s$ is again a continuous $(\mathcal{F}_t)$-semimartingale, with martingale part $\int_0^\cdot H_s dM_s$ and bounded variation component $\int_0^\cdot H_s dV_s$.

Once we have introduced stochastic integrals with respect to continuous semimartingales, it is easy to define Stratonovich integrals.

Definition 10. *Let $(X_t, t \geq 0)$ be an $(\mathcal{F}_t)$-semimartingale and $(Y_t, t \geq 0)$ an $(\mathcal{F}_t)$-progressively measurable process. The* **Stratonovich** *integral of Y with respect to X is defined as follows*

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s dX_s + \frac{1}{2}[Y,X]_t; \quad t \geq 0, \tag{23}$$

if $[Y,X]$ and $\int_0^\cdot Y_s dX_s$ exist.

Remark 12. 1. Recall that conditions of type (22) ensure existence of the stochastic integral with respect to X.

2. If (X_t) and (Y_t) are $(\mathcal{F}_t)$-semimartingales, then $\int_0^{\cdot} Y_s \circ dX_s$ exists and is called the **Fisk-Stratonovich** integral.
3. Suppose that (X_t) is an $(\mathcal{F}_t)$-semimartingale and (Y_t) is a left continuous and $(\mathcal{F}_t)$-adapted process such that $[Y, X]$ exists. We already have observed (see Proposition 6) that $\displaystyle\int_0^{\cdot} Y_s dX_s$ coincides with $\displaystyle\int_0^{\cdot} Y_s d^- X_s$. Proposition 1 1) and 2) imply that the Stratonovich integral $\displaystyle\int_0^{\cdot} Y_s \circ dX_s$ is equal to the symmetric integral $\displaystyle\int_0^{\cdot} Y_s d^{\circ} X_s$.

At this point we can easily identify the covariation of two semimartingales.

Proposition 9. *Let $S^i = M^i + V^i$ be two $(\mathcal{F}_t)$-semimartingales, $i = 1, 2$, where M^i are local martingales and V^i bounded variation processes. One has $[S^1, S^2] = [M^1, M^2]$.*

Proof. The result follows directly from Corollary 3, Proposition 1 7), and the bilinearity of the covariation.

Corollary 4. *Let S^1, S^2 be two $(\mathcal{F}_t)$- semimartingales such that their martingale parts are independent. Then $[S^1, S^2] = 0$.*

Proof. It follows from Proposition 8.

The statement of Proposition 6 can be adapted to semimartingale integrators as follows.

Proposition 10. *Let X be an $(\mathcal{F}_t)$-semimartingale and suppose that (H_t) is adapted, with left limits at each point. Then $\displaystyle\int_0^{\cdot} H_s d^- X_s = \int_0^{\cdot} H_{s-} dX_s$. If H is càdlàg then $\displaystyle\int_0^{\cdot} H d^- X = \int_0^{\cdot} H dX$.*

Remark 13. 1. Forward integrals generalize not only classical Itô integrals but also the integral obtained from the theory of enlargements of filtrations, see e.g. [29]. Let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be two filtrations fulfilling the usual conditions with $\mathcal{F}_t \subset \mathcal{G}_t$ for all t. Let X be a $(\mathcal{G}_t)$-semimartingale which is $(\mathcal{F}_t)$-adapted. By Stricker's theorem, X is also an $(\mathcal{F}_t)$-semimartingale. Let H be a càdlàg bounded $(\mathcal{F}_t)$-adapted process. According to Proposition 10, the $(\mathcal{F}_t)$-Itô integral $\int_0^{\cdot} H dX$ equals the $(\mathcal{G}_t)$-Itô integral and it coincides with the forward integral $\int_0^{\cdot} H d^- X$.
2. The result stated above is false when H has no left limits at each point. Using a tricky example in [42], it is possible to exhibit a filtration $(\mathcal{G}_t)$, a $(\mathcal{G}_t)$-semimartingale $(X_t)_{t \geq 0}$ with natural filtration $\mathcal{F}_t^X$, a bounded and $(\mathcal{F}_t^X)$-progressively measurable process H, such that $\int_0^{\cdot} H d^- X$ equals the $(\mathcal{F}_t^X)$-Itô integral but differs from the $(\mathcal{G}_t)$-Itô integral. More precisely one has:

a) X is a 3-dimensional Bessel process with decomposition

$$X_t = W_t + \int_0^t \frac{1}{X_s} ds, \tag{24}$$

where W is an $(\mathcal{F}_t^X)$-Brownian motion,
b) X is a $(\mathcal{G}_t)$-semimartingale with decomposition $M + V$ where M is the local martingale part,
c) $H_t(\omega) = 1$ for $dt \otimes dP$-almost all $(t, \omega) \in [0, T] \times \Omega$,
d) $\beta_t = \int_0^t HdX$ is a $(\mathcal{G}_t)$-Brownian motion.

Property (d) implies that $I^-(\varepsilon, H, dX) = I^-(\varepsilon, 1, dX)$ so that $\int_0^t Hd^-X = X_t$. The $(\mathcal{F}_t^X)$-Itô integral $\int_0^t HdX$ equals $\int_0^t HdW + \int_0^t \frac{H_s}{X_s} ds$; Theorem 2 below and Proposition 1 8) imply that this integral coincides with $\int_0^t Hd^-X$. Since a Bessel process cannot be equal to a Brownian motion, the $(\mathcal{G}_t)$-Itô integral $\int_0^t HdX$ differs from the $(\mathcal{F}_t^X)$-Itô integral $\int_0^t HdX$. Indeed, the pathology comes from the integration with respect to the bounded variation process. In fact, according to ii), $[X]_t = [W]_t = t$; therefore M is a $(\mathcal{G}_t)$-Brownian motion. Theorem 2 below says that $\int_0^\cdot Hd^-M = \int_0^\cdot HdM$; the additivity of forward integrals and Itô integrals imply that $\int_0^\cdot Hd^-V \neq \int_0^\cdot HdV$. Consequently it can be deduced from Proposition 1 7) a) that the discontinuities of H are not a.s. countable. It can even be shown that the discontinuities of H are not negligible with respect to dV.

3.5 The Brownian case

In this section we will investigate the link between forward and Itô integration with respect to a Brownian motion. In this section (W_t) will denote a $(\mathcal{F}_t)$-Brownian motion.

The main result of this subsection is the following.

Theorem 2. *Let $(H_t, t \geq 0)$ be an $(\mathcal{F}_t)$-progressively measurable process satisfying $\int_0^T H_s^2 ds < \infty$ a.s. Then the Itô integral $\int_0^\cdot H_s dW_s$ coincides with the forward integral $\int_0^\cdot H_s d^-W_s$.*

Remark 14. 1. We would like to illustrate the advantage of using regularization instead of discretization ([20]) through the following example.
Let g be the indicator function of $\mathbb{Q} \cap \mathbb{R}_+$.
Let $\Pi = \{t_0 = 0, t_1, \cdots, t_N = T\}$ be a subdivision of $[0, T]$ and

$$I(\Pi, g, dW)_t := \sum_i g(t_i)\big(W(t_{i+1} \wedge t) - W(t_i \wedge t)\big); \quad 0 \leq t \leq T.$$

We remark that

$$I(\Pi, g, dW)_t = \begin{cases} 0 & \text{if } \Pi \subset \mathbb{R} \setminus \mathbb{Q} \\ W_t & \text{if } \Pi \subset \mathbb{Q}. \end{cases}$$

Therefore there is no canonical definition of $\int_0^t gdW$ through discretization. This is not surprising since g is not a.e. continuous and so is not Riemann integrable. On the contrary, integration via regularization seems drastically more adapted to define $\int_0^t gd^-W$, for any $g \in L^2([0,T])$, since this integral coincides with the classical Itô-Wiener integral.

2. In order to overcome this problem, McShane pointed out an alternative approximation scheme, see [38] chap. 2 and 3. McShane's stochastic integration makes use of the so-called *belated* partition; the integral is then even more general than Itô's one, and it includes in particular the function g above.

Proof. (of Theorem 2) 1) First, suppose in addition that H is a continuous process. Replacing X by W in (18) one gets

$$\int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big) dW_s = \frac{1}{\varepsilon}\int_0^t H_s(W_{s+\varepsilon} - W_s)ds + R_\varepsilon(t), \tag{25}$$

where the remainder term $R_\varepsilon(t)$ is given by (19).

Recall the maximal inequality ([52], chap. I.1): there exists a constant C such that for any $\phi \in L^2([0,T])$,

$$\int_0^T \Big(\sup_{0<\eta<1}\Big\{\frac{1}{\eta}\int_{(v-\eta)_+}^v \phi_v dv\Big\}\Big)^2 du \le C\int_0^T \phi_v^2 dv. \tag{26}$$

2) We claim that (25) may be extended to any progressively measurable process (H_t) satisfying $\int_0^\cdot H_s^2 ds < \infty$.

Set $H_t^n = n\int_{t-1/n}^t H_u du$ for $t \ge 0$. It is clear that as $n \to \infty$

- for a.e. t, H_t^n converges to H_t,
- (H_t^n) converges to (H_t) in $\mathcal{L}^2(d < W >)$ (i.e. $\int_0^\cdot (H_s^n - H_s)^2 ds$ goes to 0 in the ucp sense).

Since

$$< \int_0^\cdot \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big) dW_s >_t = \int_0^\cdot \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big)^2 ds,$$

(26) and Lemma 2 imply that (25) and (19) are valid.

3) Letting $\varepsilon \to 0$ in (25) and using once more (26), Lemma 2 allows to conclude the proof of Theorem 2.

3.6 Substitution formulae

We conclude Section 3 by observing that discretization makes it possible to integrate non adapted integrands in a context which is covered neither by Skorohod integration theory nor by enlargement of filtrations. A class of examples is the following.

Let $(X(t,x), t \geq 0, x \in \mathbb{R}^d)$ and $(Y(t,x),\ t \geq 0,\ x \in \mathbb{R}^d)$ be two families of continuous $(\mathcal{F}_t)$ semimartingales depending on a parameter x and $(H(t,x),\ t \geq 0,\ x \in \mathbb{R}^d)$ an $(\mathcal{F}_t)$ progressively measurable processes depending on x. Let Z be a $\mathcal{F}_T$-measurable r.v., taking its values in $\mathbb{R}^d$.

Under some minimal conditions of Garsia-Rodemich-Rumsey type, see for instance [49, 50], one has

$$\int_0^t H(s,Z)\, d^- X(s,Z) = \int_0^t H(s,x)\, dX(s,x)\Big|_{x=Z},$$

$$[X(\cdot,Z), Y(\cdot,Z)] = [X(\cdot,x), Y(\cdot,x)]\Big|_{x=Z}.$$

The first result is useful to prove existence results for SDEs driven by semimartingales, with anticipating initial conditions.

It is significant to remark that these substitution formulae give rise to anticipating calculus in a setting which is not covered by Malliavin non-causal calculus since our integrators may be general semimartingales, while Skorohod integrals apply essentially to Gaussian integrators or eventually to Poisson type processes. Note that the usual causal Itô calculus does not apply here since $(X(s,Z))_s$ is not a semimartingale (take for instance a r.v. Z which generates $\mathcal{F}_T$.)

4 Calculus for finite quadratic variation processes

4.1 Stability of the covariation

A basic tool of calculus via regularization is the stability of finite quadratic variation processes under C^1 transformations.

Proposition 11. *Let (X^1, X^2) be a vector of processes having all its mutual covariations and $f, g \in C^1(\mathbb{R})$. Then $[f(X^1), g(X^2)]$ exists and is given by*

$$[f(X^1), g(X^2)]_t = \int_0^t f'(X^1_s) g'(X^2_s) d[X^1, X^2]_s$$

Proof. By polarization and bilinearity, it suffices to consider the case when $X = X^1 = X^2$ and $f = g$. Using Taylor's formula, one can write

$$f(X_{s+\varepsilon}) - f(X_s) = f'(X_s)(X_{s+\varepsilon} - X_s) + R(s,\varepsilon)(X_{s+\varepsilon} - X_s), \quad s \geq 0,\ \varepsilon > 0,$$

where $R(s,\varepsilon)$ denotes a process which converges in the ucp sense to 0 when $\varepsilon \to 0$. Since f' is unifomly continuous on compacts,

$$(f(X_{s+\varepsilon}) - f(X_s))^2 = f'(X_s)^2 (X_{s+\varepsilon} - X_s)^2 + R(s,\varepsilon)(X_{s+\varepsilon} - X_s)^2.$$

Integrating from 0 to t yields

$$\frac{1}{\varepsilon}\int_0^t (f(X_{s+\varepsilon}) - f(X_s))^2 ds = I_1(t,\varepsilon) + I_2(t,\varepsilon)$$

where

$$I_1(t,\varepsilon) = \int_0^t f'(X_s)^2 \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} ds,$$
$$I_2(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t R(s,\varepsilon)(X_{s+\varepsilon} - X_s)^2 ds.$$

Clearly one has

$$\sup_{t\le T} |I_2(t,\varepsilon)| \le \sup_{s\le T} |R(s,\varepsilon)| \frac{1}{\varepsilon}\int_0^T (X_{s+\varepsilon} - X_s)^2 ds.$$

Since $[X]$ exists, $I_2(\cdot,\varepsilon) \xrightarrow{\text{ucp}} 0$. The result will follow if we establish

$$\frac{1}{\varepsilon}\int_0^\cdot Y_s d\mu_\varepsilon(s) \xrightarrow{\text{ucp}} \int_0^\cdot Y_s d[X,X]_s \tag{27}$$

where $\mu_\varepsilon(t) = \int_0^t \frac{ds}{\varepsilon}(X_{s+\varepsilon} - X_s)^2$ and Y is a continuous process. It is not difficult to verify that a.s., $\mu_\varepsilon(dt)$ converges to $d[X,Y]$, when $\varepsilon \to 0$; this finally implies (27).

4.2 Itô formulae for finite quadratic variation processes

Even though all Itô formulae that we will consider can be stated in the multi-dimensional case, see for instance [49], we will only deal here with dimension 1. Let $X = (X_t)_{t\ge 0}$ be a continuous process.

Proposition 12. *Suppose that $[X,X]$ exists and let $f \in C^2(\mathbb{R})$. Then*

$$\int_0^\cdot f'(X) d^- X \quad \text{and} \quad \int_0^\cdot f'(X) d^+ X \quad \text{exist.} \tag{28}$$

Moreover

a) $f(X_t) = f(X_0) + \int_0^t f'(X) d^\mp X \pm \frac{1}{2}\int_0^t f''(X_s) d[X,X]_s$,

b) $f(X_t) = f(X_0) + \displaystyle\int_0^t f'(X) d^\mp X \pm \frac{1}{2}[f'(X), X]_t$, *c)* $f(X_t) = f(X_0) + \int_0^t f'(X) d^\circ X.$

Proof. c) follows from b) summing up + and −.

b) follows from a), since Proposition 11 implies that

$$[f'(X), X]_t = \int_0^t f''(X) d[X, X].$$

The proof of a) and (28) is similar to that of Proposition 11, but with a second-order Taylor expansion.

The next lemma emphasizes that the existence of a quadratic variation is closely connected with the existence of some related forward and backward integrals.

Lemma 6. *Let* X *be a continuous process. Then* $[X, X]$ *exists* $\Longleftrightarrow \int_0^\cdot X d^- X$ *exists* $\Longleftrightarrow \int_0^\cdot X d^+ X$ *exists.*

Proof. Start with the identity

$$(X_{s+\varepsilon} - X_s)^2 = X_{s+\varepsilon}^2 - X_s^2 - 2X_s(X_{s+\varepsilon} - X_s) \tag{29}$$

and observe that, when $\varepsilon \to 0$,

$$\frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon}^2 - X_s^2) ds \to X_t^2 - X_0^2.$$

Integrating (29) from 0 to t and dividing by ε easily gives the equivalence between the first two assertions.

The equivalence between the first and third ones is similar, replacing ε with $-\varepsilon$ in (29).

Lemma 6 admits the following generalization.

Corollary 5. *Let* X *be a continuous process. The following properties are equivalent*

a) $[X, X]$ *exists;*

b) $\forall g \in C^1, \quad \int_0^\cdot g(X) d^- X$ *exists;*

c) $\forall g \in C^1, \quad \int_0^\cdot g(X) d^+ X$ *exists.*

Proof. The Itô formula stated in Proposition 12 1) implies a) $\Rightarrow$ b). b) $\Rightarrow$ a) follows setting $g(x) = x$ and using Lemma 6.

b) $\Leftrightarrow$ c) because of Proposition 1 1) which states that

$$\int_0^{\cdot} g(X)d^+X = \int_0^{\cdot} g(X)d^-X + [g(X), X],$$

and Proposition 11 saying that $[g(X), X]$ exists.

When X is a semimartingale, the Itô formula seen above becomes the following.

Proposition 13. *Let $(S_t)_{t\geq 0}$ be a continuous $(\mathcal{F}_t)$-semimartingale and f a function in $C^2(\mathbb{R})$. One has the following.*

1.
$$f(S_t) = f(S_0) + \int_0^t f'(S_u)dS_u + \frac{1}{2}\int_0^t f''(S_u)d[S,S]_u.$$

2. *Let (S_t^0) be another continuous $(\mathcal{F}_t)$-semimartingale. The following integration by parts holds:*
$$S_tS_t^0 = S_0S_0^0 + \int_0^t S_u dS_u^0 + \int_0^t S_u^0 dS_u + [S, S^0]_t.$$

Proof. We recall that Itô and forward integrals coincide, see Proposition 6; therefore point 1 is a consequence of Proposition 12.

2 stems from the integration by parts formula in Proposition 1 4).

4.3 Lévy area

In Corollary 5, we have seen that $\int_0^t g(X)d^-X$ exists when X is a one-dimensional finite quadratic variation process and $g \in C^1(\mathbb{R})$.

If $X = (X^1, X^2)$ is two-dimensional and has all its mutual covariations, consider $g \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. We naturally define, if it exists,

$$\int_0^t g(X) \cdot d^-X = \lim_{\varepsilon \to 0^+} I^-(\varepsilon, g(X) \cdot dX)(t),$$

where

$$I^-(\varepsilon, g(X) \cdot dX)(t) = \int_0^t g(X)(s) \cdot \frac{X(s+\varepsilon) - X(s)}{\varepsilon} ds; \quad 0 \leq t \leq T, \quad (30)$$

and $\cdot$ denotes the scalar product in $\mathbb{R}^2$.

With a 2-dimensional Itô formula of the same type as in Proposition 12, it is possible to show that $\int_0^t g(X) \cdot d^-X$ exists if $g = \nabla u$, where u is a potential of class C^2. If g is a general $C^1(\mathbb{R}^2)$ function, one cannot expect in general that $\int_0^t g(X) \cdot d^-X$ exists.

T. Lyons' rough paths approach, see for instance [36, 35, 31, 28, 8] has considered in detail the problem of the existence of integrals of the type $\int_0^t g(X) \cdot dX$. In this theory, the concept of Lévy area plays a significant role. Translating this in the present context one would say that the essential assumption is that $X = (X^1, X^2)$ has a Lévy area type process. This section will only make some basic observations on that topic from the perspective of stochastic calculus via regularization.

Given two classical semimartingales S^1, S^2, the classical notion of Lévy area is defined by

$$L(S^1, S^2)_t = \int_0^t S^1 dS^2 - \int_0^t S^2 dS^1,$$

where both integrals are of Itô type.

Definition 11. *Given two continuous processes X and Y, we put*

$$L(X,Y)_t = \lim_{\varepsilon \to 0^+} \int_0^t \frac{X_s Y_{s+\varepsilon} - X_{s+\varepsilon} Y_s}{\varepsilon} ds.$$

where the limit is understood in the ucp sense. $L(X,Y)$ is called the **Lévy area** *of the processes X and Y.*

Remark 15. The following properties are easy to establish.

1. $L(X,X) \equiv 0$.
2. The Lévy area is an antisymmetric operation, i.e.

$$L(X,Y) = -L(Y,X).$$

Using the approximation of symmetric integral we can easily prove the following.

Proposition 14. *$\int_0^\cdot X d^\circ Y$ exists if and only if $L(X,Y)$ exists. Moreover*

$$2\int_0^t X d^\circ Y = X_t Y_t - X_0 Y_0 + L(X,Y)_t$$

Recalling the convention that an equality among three objects implies that at least two among the three are defined, we have the following.

Proposition 15. *1.* $L(X,Y)_t = \int_0^t X d^\circ Y - \int_0^t Y d^\circ X.$

2. $L(X,Y)_t = \int_0^t X d^- Y - \int_0^t Y d^- X.$

Proof. 1. From Proposition 14 applied to X, Y and Y, X, and by antisymmetry of Lévy areas we have

$$2\int_0^t X d^\circ Y = X_t Y_t - X_0 Y_0 + L(X,Y)_t,$$
$$2\int_0^t Y d^\circ X = X_t Y_t - X_0 Y_0 - L(X,Y)_t.$$

Taking the difference gives 1.

2. follows from the definition of forward integrals.

Remark 16. If $[X, Y]$ exists, point 2 of Proposition 15 is a consequence of point 1 and of Proposition 1 1, 2.

For a real-valued process $(X_t)_{t\geq 0}$, Lemma 6 says that

$$[X, X] \quad \text{exists} \Leftrightarrow \int_0^\cdot X d^- X \quad \text{exists.}$$

Given a vector of processes $\underline{X} = (X^1, X^2)$ we may ask wether the following statement is true:

(X^1, X^2) has all its mutual brackets if and only if

$$\int_0^\cdot X^i d^- X^j \quad \text{exists,}$$

for $i, j = 1, 2$. In fact the answer is negative if the two-dimensional process X does not have a Lévy area.

Remark 17. Suppose that (X^1, X^2) has all its mutual covariations. Let $*$ stand for $\circ$, or $-$, or $+$. The following are equivalent.

1. The Lévy area $L(X^1, X^2)$ exists.
2. $\int_0^\cdot X^i d^* X^j$ exists for any $i, j = 1, 2$.

By Lemma 6, we first observe that $\int X^i d^\circ X^i$ exists since X^i is a finite quadratic variation process. In point 2, equivalence between the three cases $\circ, -$ and $+$ is obvious using Proposition 1 1 2. Equivalence between the existence of $\int_0^\cdot X^1 d^\circ X^2$ and $L(X^1, X^2)$ was already established in Proposition 14.

5 Weak Dirichlet processes

5.1 Generalities

Weak Dirichlet processes constitute a natural generalization of Dirichlet processes, which in turn naturally extend semimartingales. Dirichlet processes have been considered by many authors, see for instance [21, 2].

Let $(\mathcal{F}_t)_{t\geq 0}$ be a fixed filtration fulfilling the usual conditions. In the present section 5, (W_t) will denote a classical $(\mathcal{F}_t)$-Brownian motion. For simplicity, we shall stick to the framework of continuous processes.

Definition 12. *1. An $(\mathcal{F}_t)$-***Dirichlet process** *is the sum of an $(\mathcal{F}_t)$-local martingale M and a zero quadratic variation process A.*

*2. An $(\mathcal{F}_t)$-***weak Dirichlet process** *is the sum of an $(\mathcal{F}_t)$-local martingale M and a process A such that $[A, N] = 0$ for every continuous $(\mathcal{F}_t)$- local martingale N.*

In both cases, we will suppose $A_0 = 0$ a.s.

Remark 18. 1. The process (A_t) in the latter decomposition is $(\mathcal{F}_t)$-adapted.
2. Any $(\mathcal{F}_t)$-semimartingale is an $(\mathcal{F}_t)$-Dirichlet process.

The statement of the following proposition is essentially contained in [13].

Proposition 16. *1. Any $(\mathcal{F}_t)$-Dirichlet process is an $(\mathcal{F}_t)$-weak Dirichlet process.*

2. The decomposition $M + A$ is unique.

Proof. Point 1 follows from Proposition 1 6).

Concerning point 2, let X be a weak Dirichlet process with decompositions $X = M^1 + A^1 = M^2 + A^2$. Then $0 = M + A$ where $M = M^1 - M^2$, $A = A^1 - A^2$. We evaluate the covariation of both members against M to obtain

$$0 = [M] + [M, A^1] - [M, A^2] = [M].$$

Since $M_0 = A_0 = 0$ and M is a local martingale, Corollary 1 gives $M = 0$.

The class of semimartingales with respect to a given filtration is known to be stable with respect to C^2 transformations, as Proposition 13 implies. Proposition 11 says that finite quadratic variation processes are stable under C^1 transformations.

It is possible to show that the class of weak Dirichlet processes with finite quadratic variation (as well as Dirichlet processes) is stable with respect to the same type of transformations. We start with a result which is a slight improvement (in the continuous case) of a result obtained by [7].

Proposition 17. *Let X be a finite quadratic variation process which is $(\mathcal{F}_t)$-weak Dirichlet, and $f \in C^1(\mathbb{R})$. Then $f(X)$ is also weak Dirichlet.*

Proof. Let $X = M + A$ be the corresponding decomposition. We express $f(X_t) = M^f + A^f$ where

$$M_t^f = f(X_0) + \int_0^t f'(X) dM, \quad A_t^f = f(X_t) - M_t^f.$$

Let N be a local martingale. We have to show that $[f(X) - M^f, N] = 0$.

By additivity of the covariation, and the definition of weak Dirichlet process, $[X, N] = [M, N]$ so that Proposition 11 implies $[f(X), N]_t = \int_0^t f'(X_s) d[M, N]_s$.

On the other hand, Proposition 7 gives

$$[M^f, N]_t = \int_0^t f'(X_s) d[M, N]_s,$$

and the result follows.

Remark 19. 1. If X is an $(\mathcal{F}_t)$- Dirichlet process, it can be proved similarly that $f(X)$ is an $(\mathcal{F}_t)$- Dirichlet process; see [2] and [51] for details.
2. The class of Lyons-Zheng processes introduced in [51] consitutes a natural generalization of reversible semimartingales, see Definition 13. The authors proved that this class is also stable through C^1 transformation.
3. Suppose that $(\mathcal{F}_t)$ is the canonical filtration associated with a Brownian motion W. Then a continuous $(\mathcal{F}_t)$-adapted process D is weak Dirichlet if and only if D is it is the sum of an $(\mathcal{F}_t)$-local martingale and a process A such that $[A, W] = 0$. See [9], Corollary 3.10.

We also report a Girsanov type theorem established by [7] at least in a discretization framework.

Proposition 18. *Let $X = (X_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-weak Dirichlet process, and Q a probability equivalent to P on $\mathcal{F}_T$. Then $X = (X_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-weak Dirichlet process with respect to Q.*

Proof. We set $D_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$; D is a positive local martingale.

Let L be the local martingale such that $D_t = \exp(L_t - \frac{1}{2}[L]_t)$. Let $X = M + A$ be the corresponding decomposition. It is well-known that $\tilde{M} = M - [M, L]$ is a local martingale under Q. So, X is a Q-weak Dirichlet process.

As mentioned earlier, Dirichlet processes are stable with respect to C^1 transformations. In applications, in particular to control theory, one often needs to know the nature of process $(u(t, D_t))$ where $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$ and D is a Dirichlet process. The following result was established in [24].

Proposition 19. *Let (S_t) be a continuous $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation; let $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$. Then $(u(t, S_t))$ is a $(\mathcal{F}_t)$-weak Dirichlet process.*

Remark 20. There is no reason for $(u(t, S_t))$ to have a finite quadratic variation since the dependence of u on the first argument t may be very rough. A fortiori $(u(t, S_t))$ will not be Dirichlet. Consider for instance u only depending on time, deterministic, with infinite quadratic variation.

Examples of Dirichlet processes (respectively weak Dirichlet processes) arise directly from classical Brownian motion W.

Example 2. Let f be of class $C^0(\mathbb{R})$, $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$.

1. If f is C^1, then $X = f(W)$ is a $(\mathcal{F}_t)$-Dirichlet process.
2. $u(t, W_t)$ is an $(\mathcal{F}_t)$-weak Dirichlet process, but not Dirichlet in general.

3. $f(W)$ is not always a Dirichlet process, not even of finite quadratic variation as shown by Proposition 20.

The Example and Remark above easily show that the class of $(\mathcal{F}_t)$-Dirichlet processes strictly includes the class of $(\mathcal{F}_t)$-semimartingales.

More sophisticated examples of weak Dirichlet processes may be found in the class of the so called *Volterra* type processes, se e.g. [12, 13]

Example 3. Let $(N_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-local martingale, $G : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ a continuous random field such that $G(t, \cdot)$ is $(\mathcal{F}_s)$-adapted for each t. Set

$$X_t = \int_0^t G(t,s) dN_s.$$

Then (X_t) is an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $M+A$, where $M_t = \int_0^t G(s,s) dN_s$.

Suppose that $[G(\cdot, s_1); G(\cdot, s_2)]$ exists for any s_1, s_2. With some additional technical assumption, one can show that A is a finite quadratic variation process with

$$[A]_t = 2 \int_0^t \left(\int_0^{s_2} [G(\cdot, s_1); G(\cdot, s_2)] \circ dM_{s_1} \right) \circ dM_{s_2};$$

this iterated Stratonovich integral can be expressed as the sum $C_1(t) + C_2(t)$ where

$$C_1(t) = \int_0^t [G(\cdot, s); G(\cdot, s)] d[M]_s,$$
$$C_2(t) = 2 \int_0^t \left(\int_0^{s_2} [G(\cdot, s_1); G(\cdot, s_2)] dM_{s_1} \right) dM_{s_2}.$$

Example 4. Take for N a Brownian motion W and $G(t,s) = B_{(t-s)\vee 0}$ where B is a Brownian motion independent of W. Then $[A] = \int_0^t (t-s) ds = \frac{t^2}{2}$.

One significant motivation for considering Dirichlet (respectively weak Dirichlet) processes comes from the study of generalized diffusion processes, typically solutions of stochastic differential equations with distributional drift.

Such processes were investigated using stochastic calculus via regularization by [18, 19]. We try to express here just a guiding idea. The following particular case of such equations is motivated by random media modelization:

$$dX_t = dW_t + b'(X_t) dt, \quad X_0 = x_0 \tag{31}$$

where b is a continuous function. Typically, b could be the realization of a continuous process, independent of W, stopped outside a finite interval.

We shall not recall the precise meaning of the solution of (31). In [18, 19] a rigorous sense is given to a solution (in the distribution laws) and existence and uniqueness are established for any initial conditions.

Here we shall just attempt to convince the reader that the solution is a Dirichlet process. For this we define the real function h of class C^1 by

$$h(x) = \int_0^x e^{-b(y)} dy.$$

We set $\sigma_0 = h' \circ h^{-1}$. We consider the unique solution in law of the equation

$$dY_t = \sigma_0(Y_t) dW_t, Y_0 = h(x_0)$$

which exists because of classical Stroock-Varadhan arguments ([53]); so Y is clearly a semimartingale, thus a Dirichlet process. The process $X = h^{-1}(Y)$ is a Dirichlet process since h^{-1} is of class C^1. If b were of class C^1, (31) would be an ordinary stochastic differential equation, and it could be shown that X is the unique solution of that equation. In the present case X will still be the solution of (31), considered as a generalized stochastic differential equation.

We now consider the case when the drift is time inhomogeneous as follows:

$$dX_t = dW_t + \partial_x b(t, X_t) dt, X_0 = x_0 \tag{32}$$

where $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function of class C^1 in time. Then it is possible to find a $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ such that the *solution* (X_t) of (32) can be expressed as $(k(t, Y_t))$ for some semimartingale Y; so X will be an $(\mathcal{F}_t)$-weak Dirichlet process. For this and more general situations, see [44].

5.2 Itô formula under weak smoothness assumptions

In this section, we formulate and prove an Itô formula of C^1 type. As for the C^2 type Itô formula, the next Theorem is stated in the one-dimensional framework only in spite of its validity in the multidimensional case.

Let $(S_t)_{t \geq 0}$ be a semimartingale and $f \in C^2$. We recall the classical Itô formula, as a particular case of Proposition 13: :

$$f(S_t) = f(S_0) + \int_0^t f'(S_s) dS_s + \frac{1}{2} \int_0^t f''(S_s) d[S, S]_s.$$

Using Proposition 6 and Definition 10 (Stratonovich integrals), we obtain

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S_s) dS_s + \frac{1}{2} [f'(S), S]_t \\ &= f(S_0) + \int_0^t f'(S) \circ dS. \end{aligned} \tag{33}$$

We observe that in formulae (33), only the first derivative of f appears. Besides, we know that $f(S)$ is a Dirichlet process if $f \in C^1(\mathbb{R})$.
At this point we may ask if formulae (33) remains valid when f is in $C^1(\mathbb{R})$ only; a partial answer will be given in Theorem 3 below.

Definition 13. *Let (S_t) be a continuous semimartingale; set $\hat{S}_t = S_{T-t}$ for $t \in [0,T]$. S is called a* **reversible semimartingale** *if $(\hat{S}_t)_{t\in[0,T]}$ is again a semimartingale.*

Theorem 3. *([45]) Let S be a reversible semimartingale indexed by $[0,T]$ and $f \in C^1(\mathbb{R})$. Then one has*

$$f(S_t) = f(S_0) + \int_0^t f'(S)dS + R_t = f(S_0) + \int_0^t f'(S) \circ dS$$

where $R = \frac{1}{2}[f'(S), S]$.

Remark 21. After the pioneering work of [5], which expressed the remainder term (R_t) with the help of generalized integral with respect to local time, two papers appeared: [22] in the case of Brownian motion and [22] and [45] for multidimensional reversible semimartingales. Later, an incredible amount of contributions on that topic have been published. We cannot give the precise content of each paper; a non-exhaustive list is [1, 14, 15, 23, 24, 39, 40]. Among the C^1-type Itô formulae in the framework of generalized Stratonovich integral with respect to Lyons-Zheng processes, it is also important to quote [33, 34, 51].

Example 5. i) Classical $(\mathcal{F}_t)$-Brownian motion W is a reversible semimartingale, see for instance [22, 41, 19]. More precisely $\hat{W}_t = W_T + \beta_t + \int_0^t \frac{\hat{W}_s}{T-s} ds$, where β is a $(\mathcal{G}_t)$-Brownian motion and $(\mathcal{G}_t)$ is the natural filtration associated with $\hat{W}_t$.
ii) Let (X_t) be the solution of the stochastic differential equation

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,$$

with $\sigma, b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ Lipschitz with at most linear growth, $\sigma \geq c > 0$. Then (X_t) is a reversible semimartingale; see for instance [19]. Moreover if $f \in W^{1,2}_{\text{loc}}$, it is proved in [19] that $(f(X_t))$ is an $(\mathcal{F}_t)$-Dirichlet process.

Proof. (of Theorem 3). We use in an essential way the Banach-Steinhaus theorem for F-spaces; see for instance [10] chap. 2.1.

Define two maps $T^{\pm}_{\varepsilon}$ from the F-space $C^0(\mathbb{R})$ to the F-space $\mathcal{C}([0,T])$, which consists of all continuous processes indexed by $[0,T]$, by

$$T^-_{\varepsilon} g = \int_0^{\cdot} g(S_s) \frac{S_{s+\varepsilon} - S_s}{\varepsilon} ds,$$

$$T_\varepsilon^+ g = \int_0^\cdot g(S_s) \frac{S_s - S_{s-\varepsilon}}{\varepsilon} ds.$$

These operators are linear and continuous. Moreover, for each $g \in C^0$ we have

$$\lim_{\varepsilon \to 0} T_\varepsilon^- g = \int_0^\cdot g(S) dS,$$

because of Proposition 6 which says that $\int_0^t g(S) dS$ is also an Itô integral.

Since $\hat{S}$ is a semimartingale, for the same reasons as above,

$$\int_{T-t}^T g(\hat{S}) d^- \hat{S} \tag{34}$$

also exists and equals an Itô integral.

Using Proposition 1 3), it follows that $\int_0^\cdot g(S) d^+ S$ also exists.

Therefore the Banach-Steinhaus theorem implies that

$$g \mapsto \int_0^\cdot g(S) d^- S, \quad g \mapsto \int_0^\cdot g(S) d^+ S,$$

are continuous maps from $C^0(\mathbb{R})$ to $\mathcal{C}([0,T])$; by additivity, so are also

$$g \mapsto [g(S), S], \quad g \mapsto \int_0^\cdot g(S) d^\circ S.$$

Let $f \in C^1(\mathbb{R})$, $(\rho_\varepsilon)_{\varepsilon > 0}$ be a family of mollifiers converging to the Dirac measure at zero. We set $f_\varepsilon = f \star \rho_\varepsilon$ where $\star$ denotes convolution. Since f_ε is of class C^2, by the "smooth" Itô formula stated at Proposition 13 and by Proposition 1 1) and 2), we have

$$f_\varepsilon(S_t) = f_\varepsilon(S_0) + \int_0^t f'_\varepsilon(S) dS + \frac{1}{2}[f'_\varepsilon(S), S],$$
$$f_\varepsilon(S_t) = f_\varepsilon(S_0) + \int_0^t f'_\varepsilon(S) d^\circ S.$$

Since f'_ε goes to f' in $C^0(\mathbb{R})$, we can take the limit term by term and

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S) dS + \frac{1}{2}[f'(S), S], \\ f(S_t) &= f(S_0) + \int_0^t f'(S) d^\circ S. \end{aligned} \tag{35}$$

Remark 12 says that the latter symmetric integral is in fact a Stratonovich integral.

Corollary 6. *If* $(S_t)_{t\in[0,T]}$ *is a reversible semimartingale and* $g \in C^0(\mathbb{R})$*, then* $[g(S), S]$ *exists and has zero quadratic variation.*

Proof. Let $g \in C^0(\mathbb{R})$ and let $S = M + V$ be the decomposition of S as a sum of a local martingale M and a finite variation process V, such that $V_0 = 0$. Let $f \in C^1(\mathbb{R})$ such that $f' = g$. We know that $f(S)$ is a Dirichlet process with local martingale part

$$M_t^f = f(S_0) + \int_0^t g(S)dM.$$

Let A^f be its zero quadratic variation component. Using Thereom 3, we have

$$A_t^f = \int_0^t g(S)dV + \frac{1}{2}[g(S), S].$$

$\int_0^\cdot g(S)dV$ has finite variation, therefore it has zero quadratic variation; since so does also A^f, the result follows immediately.

Proposition 20. *Let* $g \in C^0(\mathbb{R})$ *such that* $g(W)$ *is a finite quadratic variation process. Then* g *has bounded variation on compacts.*

Proof. Suppose that $g(W)$ is of finite quadratic variation. We already know that W is a reversible semimartingale. By Corollary 6, $[W, g(W)]$ exists and it is a zero quadratic variation process. Since $[W]$ exists, we deduce that $(g(W), W)$ has all its mutual covariations. In particular $[g(W), W]$ has bounded variation because of Remark 2. Let f be such that $f' = g$; Theorem 3 implies that $f(W)$ is a semimartingale. A celebrated result of Çinlar, Jacod, Protter and Sharpe [6] asserts that $f(W)$ is a $(\mathcal{F}_t)$-semimartingale if and only if f is a difference of two convex functions; this finally allows to conclude that g has bounded variation on compacts.

Remark 22. Given two processes X and Y, the covariations $[X]$ and $[X, Y]$ may exist even if Y is not of finite quadratic variation. In particular (X, Y) may not have all its mutual covariations. For instance, if X has bounded variation, and Y is any continuous process, then $[X, Y] = 0$, see Proposition 1 7 b). A less trivial exemple is provided by $X = W$, $Y = g(W)$ where g is continuous but not of bounded variation, see Proposition 20.

Remark 23. ([22]). When S is a Brownian motion, Theorem 3 and Corollary 6 are in fact respectively valid for $f \in W^{1,2}_{\text{loc}}(\mathbb{R})$ and $g \in L^2_{\text{loc}}(\mathbb{R})$.

6 Final remarks

We conclude this paper with some considerations about calculus related to processes having no quadratic variation. On this, the reader can consult [13, 27, 26]. In [13] one defines a notion of $n-$ covariation $[X^1, \dots, X^n]$ of n processes $X^1, \dots, X^n$ and the n-variation of a process X.

We recall some basic significant results related to those papers.

1. For a process X having a 3-variation, it is possible to write an Itô formula of the type
$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^\circ X_s - \frac{1}{12}\int_0^t f^{(3)}(X_s)d[X,X,X]_s.$$
Moreover one-dimensional stochastic differential equations driven by a strong 3-variation were considered in [13].
2. Let $B = B^H$ be a fractional Brownian motion with Hurst index $H > \frac{1}{6}$ and f a function of class C^6. It is shown in [27, 26] that
$$f(B_t) = f(B_0) + \int_0^t f'(B)d^\circ B.$$
3. Using more sophisticated integrals via regularization, other types of Itô formulae can be written for any H in $]0,1[$; see [26].
4. In [25], it is shown that stochastic calculus via regularization is *almost pathwise*. Suppose for instance that X is a semimartingale or a fractional Brownian motion, with Hurst index $H > \frac{1}{2}$; then its quadratic variation $[X]$ is a limit of $C(\varepsilon, X, X)$ not only ucp as in (5), but also *uniformly a.s.* Similarly, if X is semimartingale and Y is a suitable integrand, the Itô integral $\int_0^\cdot Y dX$ is approximated by $I^-(\varepsilon, Y, dX)$ not only ucp as in (2), but also uniformly a.s.

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