

# Stabilization of second order evolution equations with unbounded feedback with time-dependent delay

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## Abstract

We consider abstract second order evolution equations with unbounded feedback with time-varying delay. Existence results are obtained under some realistic assumptions. We prove the exponential decay under some conditions by introducing an abstract Lyapunov functional. Our abstract framework is applied to the wave, to the beam and to the plate equations with boundary delays.

**Keywords** second order evolution equations, wave equations, time-varying delay, stabilization, Lyapunov functional.

## 1 Introduction

Time-delay often appears in many biological, electrical engineering systems and mechanical applications, and in many cases, delay is a source of instability [7]. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see e.g. [5, 16, 24, 17]). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

There are only a few works on Lyapunov-based technique for Partial Differential Equations (PDEs) with delay. Most of these works analyze the case of *constant delays*. Thus, stability conditions and exponential bounds were derived for some scalar heat and wave equations with constant delays and with Dirichlet boundary conditions without delay in [25, 26]. Stability and instability

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conditions for the wave equations with constant delay can be found in [17, 20]. The stability of linear parabolic systems with constant coefficients and internal constant delays has been studied in [8] in the frequency domain. Moreover we refer to [19] for the stability of second order evolution equation with constant delay in unbounded feedbacks.

Recently the stability of PDEs with *time-varying delays* was analyzed in [3, 6, 21, 22] via Lyapunov method. In the case of linear systems in an Hilbert space, the conditions of [3, 6, 22] assume that the operator acting on the delayed state is bounded (which means that this condition can not be applied to boundary delays for example). The stability of the 1-d heat and wave equations with boundary time-varying delays have been studied in [21] via Lyapunov functional.

The aim of this paper is to consider an abstract setting similar to [19] and as large as possible in order to contain a quite large class of problems with time-varying delay feedbacks (which contains in particular the results of [21] for the wave equation).

Before going on, let us present our abstract framework. Let  $H$  be a real Hilbert space with norm and inner product denoted respectively by  $\|\cdot\|_H$  and  $(\cdot, \cdot)_H$ . Let  $A : D(A) \rightarrow H$  be a self-adjoint positive operator with a compact inverse in  $H$ . Let  $V := D(A^{1/2})$  be the domain of  $A^{1/2}$ . Denote by  $D(A^{1/2})'$  the dual space of  $D(A^{1/2})$  obtained by means of the inner product in  $H$ .

Further, for  $i = 1, 2$ , let  $U_i$  be a real Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by  $\|\cdot\|_{U_i}$  and  $(\cdot, \cdot)_{U_i}$ , and let  $B_i \in \mathcal{L}(U_i, D(A^{1/2})')$ .

We consider the system described by

$$(1) \quad \begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 u_1(t) + B_2 u_2(t - \tau(t)) = 0, & t > 0, \\ \omega(0) = \omega_0, \dot{\omega}(0) = \omega_1, \\ u_2(t - \tau(0)) = f^0(t - \tau(0)), & 0 < t < \tau(0), \end{cases}$$

where  $t \in [0, \infty)$  represents the time,  $\tau(t) > 0$  is the time-varying delay,  $\omega : [0, \infty) \rightarrow H$  is the state of the system,  $\dot{\omega}$  is the time derivative of  $\omega$  and  $u_1 \in L^2([0, \infty), U_1)$ ,  $u_2 \in L^2([-\tau, \infty), U_2)$  are the input functions. The time-varying delay  $\tau(t)$  satisfies

$$(2) \quad \exists d < 1, \forall t > 0, \quad \dot{\tau}(t) \leq d < 1,$$

and

$$(3) \quad \exists M > 0, \forall t > 0, \quad 0 < \tau_0 \leq \tau(t) \leq M.$$

Moreover, we assume that

$$(4) \quad \forall T > 0, \tau \in W^{2, \infty}([0, T]).$$

Most of the linear equations modeling the vibrations of elastic structures with distributed control with delay can be written in the form (1), where  $\omega$  stands for the displacement field.

In many problems, coming in particular from elasticity, the inputs  $u_i$  are given in the feedback form  $u_i(t) = B_i^* \dot{\omega}(t)$ , which corresponds to collocated actuators and sensors. We obtain in this way the closed loop system

$$(5) \quad \begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 B_2^* \dot{\omega}(t - \tau(t)) = 0, & t > 0, \\ \omega(0) = \omega_0, \dot{\omega}(0) = \omega_1, \\ B_2^* \dot{\omega}(t - \tau(0)) = f^0(t - \tau(0)), & 0 < t < \tau(0). \end{cases}$$

The abstract second order evolution equations *without delay* or with *constant delay* of type (5) have been studied in [2] and [19] respectively. In these two papers, the exponential stability (or polynomial stability) is shown, under some conditions, via an observability inequality for solution of corresponding conservative system. In our case, for time-varying delay, this method can not be applied due to the loss of the time translation invariance. Hence we introduce new abstract Lyapunov functionals with exponential terms and an additional term, which take into account the dependence of the delay with respect to time. For the treatment of other problems with Lyapunov technique see [6, 18, 22].

Moreover, contrary to [17, 19], the existence results do not follow from standard semi-group theory because the spatial operator depends on time due to the time-varying delay. Therefore we use the variable norm technique of Kato [9, 10].

Hence the first natural question is the well-posedness of this system. In section 2 we will give a sufficient condition that guarantees that this system (5) is well-posed, where we closely follow the approach developed in [21] for the 1-d heat and wave equations. Secondly, we may ask if this system is dissipative. We show in section 3 that the condition

$$(6) \quad \exists 0 < \alpha < \sqrt{1-d}, \forall u \in V, \|B_2^* u\|_{U_2}^2 \leq \alpha \|B_1^* u\|_{U_1}^2$$

guarantees that the energy decays. Note further that if (6) is not satisfied, there exist cases where some instabilities may appear (see [17, 20, 27] for the wave equation with constant delay). Hence this assumption seems realistic.

In a third step, again under the condition (6), we prove the exponential decay of the system (5) by introducing an appropriate Lyapunov functional. Moreover we give the dependence of the decay rate with respect to the delay, in particular we show that if the delay increases the decay rate decreases. This is the content of section 4.

Finally we finish this paper by considering in section 5 different examples where our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new.

## 2 Well-posedness of the system

We aim to show that system (5) is well-posed. For that purpose, we use semi-group theory and an idea from [17]. Let us introduce the auxiliary variable

$z(\rho, t) = B_2^* \dot{\omega}(t - \tau(t)\rho)$  for  $\rho \in (0, 1)$  and  $t > 0$ . Note that  $z$  satisfies the following transport equation

$$\begin{cases} \tau(t) \frac{\partial z}{\partial t} + (1 - \dot{\tau}(t)\rho) \frac{\partial z}{\partial \rho} = 0, & 0 < \rho < 1, t > 0 \\ z(0, t) = B_2^* \dot{\omega}(t) \\ z(\rho, 0) = B_2^* \dot{\omega}(-\tau(0)\rho) = f^0(-\tau(0)\rho). \end{cases}$$

Therefore, the system (5) is equivalent to

$$(7) \quad \begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 z(1, t) = 0, & t > 0, \\ \tau(t) \frac{\partial z}{\partial t} + (1 - \dot{\tau}(t)\rho) \frac{\partial z}{\partial \rho} = 0, & t > 0, 0 < \rho < 1, \\ \omega(0) = \omega_0, \dot{\omega}(0) = \omega_1, z(\rho, 0) = f^0(-\tau(0)\rho), & 0 < \rho < 1, \\ z(0, t) = B_2^* \dot{\omega}(t), & t > 0. \end{cases}$$

If we introduce

$$U := (\omega, \dot{\omega}, z)^T,$$

then  $U$  satisfies

$$U' = (\dot{\omega}, \ddot{\omega}, \dot{z})^T = \left( \dot{\omega}, -A\omega(t) - B_1 B_1^* \dot{\omega}(t) - B_2 z(1, t), \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} \frac{\partial z}{\partial \rho} \right)^T.$$

Consequently the system (5) may be rewritten as the first order evolution equation

$$(8) \quad \begin{cases} U' = \mathcal{A}(t)U \\ U(0) = (\omega_0, \omega_1, f^0(-\tau(0)\cdot)), \end{cases}$$

where the time dependent operator  $\mathcal{A}(t)$  is defined by

$$\mathcal{A}(t) \begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} u \\ -A\omega - B_1 B_1^* u - B_2 z(1) \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} \frac{\partial z}{\partial \rho} \end{pmatrix},$$

with domain

$$(9) \quad D(\mathcal{A}(t)) := \{(\omega, u, z) \in V \times V \times H^1((0, 1), U_2); z(0) = B_2^* u, A\omega + B_1 B_1^* u + B_2 z(1) \in H\}.$$

We note that the domain of the operator  $\mathcal{A}(t)$  is independent of the time  $t$ , i.e.

$$(10) \quad \mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \forall t > 0.$$

Now, we introduce the Hilbert space

$$\mathcal{H} = V \times H \times L^2((0, 1), U_2)$$

equipped with the usual inner product

$$(11) \quad \left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle = \left( A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \tilde{\omega} \right)_H + (u, \tilde{u})_H + \int_0^1 (z(\rho), \tilde{z}(\rho))_{U_2} d\rho.$$

A general theory for equations of type (8) has been developed using semigroup theory [9, 10, 23]. The simplest way to prove existence and uniqueness results is to show that the triplet  $\{\mathcal{A}, \mathcal{H}, Y\}$ , with  $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$  for some fixed  $T > 0$  and  $Y = \mathcal{D}(\mathcal{A}(0))$ , forms a CD-system (or constant domain system, see [9, 10]). More precisely, the following theorem gives some existence and uniqueness results (for proof see Theorem 1.9 of [9] and also Theorem 2.13 of [10] or [1]).

**Theorem 2.1** [9] *Assume that*

- (i)  $Y = \mathcal{D}(\mathcal{A}(0))$  is a dense subset of  $\mathcal{H}$ ,
- (ii) (10) holds,
- (iii) for all  $t \in [0, T]$ ,  $\mathcal{A}(t)$  generates a strongly continuous semigroup on  $\mathcal{H}$  and the family  $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$  is stable with stability constants  $C$  and  $m$  independent of  $t$  (i.e. the semigroup  $(S_t(s))_{s \geq 0}$  generated by  $\mathcal{A}(t)$  satisfies  $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$ , for all  $u \in \mathcal{H}$  and  $s \geq 0$ ),
- (iv)  $\partial_t \mathcal{A}$  belongs to  $L_*^\infty([0, T], B(Y, \mathcal{H}))$ , the space of equivalent classes of essentially bounded, strongly measurable functions from  $[0, T]$  into the set  $B(Y, \mathcal{H})$  of bounded operators from  $Y$  into  $\mathcal{H}$ .

Then, problem (8) has a unique solution  $U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H})$  for any initial data in  $Y$ .

Our goal is then to check the above assumptions for system (8).

Let us suppose that

$$(12) \quad \exists 0 < \alpha \leq \sqrt{1-d}, \forall u \in V, \|B_2^*u\|_{U_2}^2 \leq \alpha \|B_1^*u\|_{U_1}^2,$$

where  $d$  is given by (2). Note that the choice of  $\alpha$  is possible since  $d < 1$  by (2).

The following lemma gives a sufficient condition to obtain (i):

**Lemma 2.2** *Assume that  $X = \{u \in V : B_1B_1^*u + B_2B_2^*u \in H\}$  is dense in  $H$ . Then*

$$(13) \quad \mathcal{D}(\mathcal{A}(0)) \text{ is dense in } \mathcal{H}.$$

**Proof.** Let  $(f, g, h)^\top \in \mathcal{H}$  be orthogonal to all elements of  $\mathcal{D}(\mathcal{A}(0))$ , namely

$$0 = \left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle = (\omega, f)_V + (u, g)_H + \int_0^1 (z(\rho), h(\rho))_{U_2} d\rho,$$

for all  $(\omega, u, z)^\top \in \mathcal{D}(\mathcal{A}(0))$ .

We first take  $\omega = 0$  and  $u = 0$  and  $z \in \mathcal{D}((0, 1), U_2)$ . As  $(0, 0, z)^\top \in \mathcal{D}(\mathcal{A}(0))$ , we get

$$\int_0^1 (z(\rho), h(\rho))_{U_2} d\rho = 0.$$

Since  $\mathcal{D}((0, 1), U_2)$  is dense in  $L^2((0, 1), U_2)$ , we deduce that  $h = 0$ .

In a second step, by taking  $\omega = 0$ ,  $z = B_2^*u$  and  $u \in X$ , we see that  $(0, u, B_2^*u)^T \in \mathcal{D}(\mathcal{A}(0))$  and therefore  $(u, g)_H = 0$ , for all  $u \in X$ . As  $X$  is dense in  $H$  by hypothesis, we deduce that  $g = 0$ .

The above orthogonality condition is then reduced to

$$0 = (\omega, f)_V, \forall (\omega, u, z)^T \in D(\mathcal{A}(0)).$$

By restricting ourselves to  $u = 0$  and  $z = 0$ , we obtain

$$(\omega, f)_V = 0, \forall (\omega, 0, 0)^T \in D(\mathcal{A}(0)).$$

But we easily check that  $(\omega, 0, 0)^T \in D(\mathcal{A}(0))$  if and only if  $\omega \in D(A)$ . Since  $D(A)$  is dense in  $V$  (equipped with the inner product  $\langle \cdot, \cdot \rangle_V$ ), we conclude that  $f = 0$ . ■

**Remark 2.3** As, by (12), the kernel  $\ker(B_1^*)$  of  $B_1^*$  is included in  $X$ , if  $\ker(B_1^*)$  is dense in  $H$ , then  $\mathcal{D}(\mathcal{A}(0))$  is dense in  $\mathcal{H}$ . ■

Now, we will show that the operator  $\mathcal{A}(t)$  generates a  $C_0$ -semigroup in  $\mathcal{H}$  and, by using the variable norm technique of Kato from [9], we will prove that system (8) (and then (5)) has a unique solution.

For that purpose, we introduce the following time-dependent inner product on  $\mathcal{H}$

$$\left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \left( A^{\frac{1}{2}}\omega, A^{\frac{1}{2}}\tilde{\omega} \right)_H + (u, \tilde{u})_H + q\tau(t) \int_0^1 (z(\rho), \tilde{z}(\rho))_{U_2} d\rho,$$

where  $q$  is a positive constant chosen such that

$$(14) \quad \frac{1}{\sqrt{1-d}} \leq q \leq \frac{2}{\alpha} - \frac{1}{\sqrt{1-d}}$$

with associated norm denoted by  $\|\cdot\|_t$ . This choice of  $q$  is possible since  $0 < \alpha \leq \sqrt{1-d}$  by (12). This new inner product is clearly equivalent to the usual inner product (11) on  $\mathcal{H}$ .

**Theorem 2.4** *Under the assumptions (2), (3), (4), (12) and (13), for an initial datum  $U_0 \in D(\mathcal{A}(t))$ , there exists a unique solution*

$$U \in C([0, +\infty), D(\mathcal{A}(t))) \cap C^1([0, +\infty), \mathcal{H})$$

to system (8).

**Proof.** We first notice that

$$(15) \quad \frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \forall t, s \in [0, T],$$

where  $\phi = (\omega, u, z)^\top$  and  $c$  is a positive constant. Indeed, for all  $s, t \in [0, T]$ , we have

$$\begin{aligned} \|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} &= \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right) \left(\|A^{\frac{1}{2}}\omega\|_H^2 + \|u\|_H^2\right) \\ &\quad + q \left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right) \int_0^1 \|z(\rho)\|_{U_2}^2 d\rho. \end{aligned}$$

We note that  $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$ . Moreover  $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$  for some  $c > 0$ . Indeed,

$$\tau(t) = \tau(s) + \dot{\tau}(a)(t-s), \quad \text{where } a \in (s, t),$$

and thus,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\dot{\tau}(a)|}{\tau(s)} |t-s|.$$

By (4),  $\dot{\tau}$  is bounded and therefore, there exists  $c > 0$  such that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0} |t-s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

by (3), which proves (15).

We now prove that  $\mathcal{A}(t)$  is dissipative up to a translation for a fixed  $t > 0$ . Take  $U = (\omega, u, z)^\top \in D(\mathcal{A}(t))$ . Then

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= \left\langle \left( \begin{array}{c} -A\omega - B_1 B_1^* u - B_2 z(1) \\ \frac{\dot{\tau}(t)\rho-1}{\tau(t)} \frac{\partial z}{\partial \rho} \end{array} \right), \left( \begin{array}{c} \omega \\ u \\ z \end{array} \right) \right\rangle_t \\ &= \left( A^{\frac{1}{2}}u, A^{\frac{1}{2}}\omega \right)_H - (A\omega + B_1 B_1^* u + B_2 z(1), u)_H \\ &\quad - q \int_0^1 \left( \frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} (1 - \dot{\tau}(t)\rho) d\rho. \end{aligned}$$

Since  $A\omega + B_1 B_1^* u + B_2 z(1) \in H$ , we obtain

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= \left( A^{\frac{1}{2}}u, A^{\frac{1}{2}}\omega \right)_H - \langle A\omega, u \rangle_{V', V} - \langle B_1 B_1^* u, u \rangle_{V', V} - \langle B_2 z(1), u \rangle_{V', V} \\ &\quad - q \int_0^1 \left( \frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} (1 - \dot{\tau}(t)\rho) d\rho \\ &= \langle A\omega, u \rangle_{V', V} - \langle A\omega, u \rangle_{V', V} - \|B_1^* u\|_{U_1}^2 - (z(1), B_2^* u)_{U_2} \\ &\quad - q \int_0^1 \left( \frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} (1 - \dot{\tau}(t)\rho) d\rho, \end{aligned}$$

by duality. By integrating by parts in  $\rho$ , we obtain

$$\begin{aligned} \int_0^1 \left( \frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} (1 - \dot{\tau}(t)\rho) d\rho &= \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} \left( \|z\|_{U_2}^2 \right) (1 - \dot{\tau}(t)\rho) d\rho \\ &= \frac{\dot{\tau}(t)}{2} \int_0^1 \|z\|_{U_2}^2 d\rho + \frac{1}{2} \|z(1)\|_{U_2}^2 (1 - \dot{\tau}(t)) \\ &\quad - \frac{1}{2} \|B_2^* u\|_{U_2}^2. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= -\|B_1^*u\|_{U_1}^2 - (z(1), B_2^*u)_{U_2} - \frac{q}{2}\|z(1)\|_{U_2}^2(1-\dot{\tau}(t)) + \frac{q}{2}\|B_2^*u\|_{U_2}^2 \\ &\quad - \frac{q\dot{\tau}(t)}{2} \int_0^1 \|z\|_{U_2}^2 d\rho. \end{aligned}$$

By Young's inequality and (12), we find

$$\langle \mathcal{A}(t)U, U \rangle_t \leq \left( \frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 \right) \|B_1^*u\|_{U_1}^2 + \left( \frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} \right) \|z(1)\|_{U_2}^2 + \kappa(t) \langle U, U \rangle_t,$$

where

$$(16) \quad \kappa(t) = \frac{(\dot{\tau}(t)^2 + 1)^{1/2}}{2\tau(t)}.$$

Observe that  $\frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 \leq 0$  and  $\frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} \leq 0$  since  $q$  satisfies (14).

This shows that

$$(17) \quad \langle \mathcal{A}(t)U, U \rangle_t - \kappa(t) \langle U, U \rangle_t \leq 0,$$

which means that the operator  $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$  is dissipative.

Moreover  $\dot{\kappa}(t) = \frac{\ddot{\tau}(t)\dot{\tau}(t)}{2\tau(t)(\dot{\tau}(t)^2+1)^{\frac{3}{2}}} - \frac{\dot{\tau}(t)(\dot{\tau}(t)^2+1)^{\frac{1}{2}}}{2\tau(t)^2}$  is bounded on  $[0, T]$  for all  $T > 0$  (by (3) and (4)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho-1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with  $\frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho-1)}{\tau(t)^2}$  bounded on  $[0, T]$  by (3) and (4). Thus

$$(18) \quad \frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0, T], B(D(\mathcal{A}(0)), \mathcal{H})),$$

the space of equivalence classes of essentially bounded, strongly measurable functions from  $[0, T]$  into  $B(D(\mathcal{A}(0)), \mathcal{H})$ .

Let us now prove that  $\lambda I - \mathcal{A}(t)$  is surjective for a fixed  $t > 0$  and any  $\lambda > 0$ .

Let  $(f, g, h)^T \in \mathcal{H}$ . We look for  $U = (\omega, u, z)^T \in D(\mathcal{A}(t))$  solution of

$$(\lambda I - \mathcal{A}(t)) \begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

or equivalently

$$(19) \quad \begin{cases} \lambda\omega - u = f \\ \lambda u + A\omega + B_1 B_1^* u + B_2 z(1) = g \\ \lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} \frac{\partial z}{\partial \rho} = h. \end{cases}$$

Suppose that we have found  $\omega$  with the appropriate regularity. Then, we have

$$u = -f + \lambda\omega \in V.$$

We can then determine  $z$ . Indeed  $z$  satisfies the differential equation

$$\lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} \frac{\partial z}{\partial \rho} = h$$

and the boundary condition  $z(0) = B_2^* u = -B_2^* f + \lambda B_2^* \omega$ . Therefore  $z$  is explicitly given by

$$z(\rho) = \lambda B_2^* \omega e^{-\lambda\tau(t)\rho} - B_2^* f e^{-\lambda\tau(t)\rho} + \tau(t) e^{-\lambda\tau(t)\rho} \int_0^\rho e^{\lambda\tau(t)\sigma} h(\sigma) d\sigma,$$

if  $\dot{\tau}(t) = 0$ , and

$$\begin{aligned} z(\rho) &= \lambda B_2^* \omega e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} - B_2^* f e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} \\ &\quad + \tau(t) e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} \int_0^\rho \frac{h(\sigma)}{1-\dot{\tau}(t)\sigma} e^{-\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma, \end{aligned}$$

otherwise. This means that once  $\omega$  is found with the appropriate properties, we can find  $z$  and  $u$ . In particular, we have, if  $\dot{\tau}(t) = 0$ ,

$$(20) \quad z(1) = \lambda B_2^* \omega e^{-\lambda\tau(t)} + z^0,$$

where  $z^0 = -B_2^* f e^{-\lambda\tau(t)} + \tau(t) e^{-\lambda\tau(t)} \int_0^1 e^{\lambda\tau(t)\sigma} h(\sigma) d\sigma$  is a fixed element of  $U_2$  depending only on  $f$  and  $h$ , and, otherwise

$$(21) \quad z(1) = \lambda B_2^* \omega e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} + z^0,$$

where  $z^0 = -B_2^* f e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} + \tau(t) e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} \int_0^1 \frac{h(\sigma)}{1-\dot{\tau}(t)\sigma} e^{-\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} d\sigma$  is a fixed element of  $U_2$  depending only on  $f$  and  $h$ .

It remains to find  $\omega$ . By (19),  $\omega$  must satisfy

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + B_2 z(1) = g + B_1 B_1^* f + \lambda f,$$

and thus by (20),

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau(t)} B_2 B_2^* \omega = g + B_1 B_1^* f + \lambda f - B_2 z^0 =: q,$$

where  $q \in V'$ , if  $\dot{\tau}(t) = 0$ , and by (21)

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} B_2 B_2^* \omega = g + B_1 B_1^* f + \lambda f - B_2 z^0 =: q,$$

where  $q \in V'$  otherwise. Assume  $\dot{\tau}(t) = 0$ . We take then the duality brackets  $\langle \cdot, \cdot \rangle_{V', V}$  with  $\phi \in V$  :

$$\left\langle \lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau(t)} B_2 B_2^* \omega, \phi \right\rangle_{V', V} = \langle q, \phi \rangle_{V', V}.$$

Moreover:

$$\begin{aligned} & \left\langle \lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau(t)} B_2 B_2^* \omega, \phi \right\rangle_{V', V} \\ &= \lambda^2 \langle \omega, \phi \rangle_{V', V} + \langle A\omega, \phi \rangle_{V', V} + \lambda (\langle B_1 B_1^* \omega, \phi \rangle_{V', V} + e^{-\lambda\tau(t)} \langle B_2 B_2^* \omega, \phi \rangle_{V', V}) \\ &= \lambda^2 (\omega, \phi)_H + \left( A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \phi \right)_H + \lambda ((B_1^* \omega, B_1^* \phi)_{U_1} + e^{-\lambda\tau(t)} (B_2^* \omega, B_2^* \phi)_{U_2}) \end{aligned}$$

because  $\omega \in V \subset H$ . Consequently, we arrive at the problem

$$(22) \quad \lambda^2 (\omega, \phi)_H + \left( A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \phi \right)_H + \lambda ((B_1^* \omega, B_1^* \phi)_{U_1} + e^{-\lambda\tau(t)} (B_2^* \omega, B_2^* \phi)_{U_2}) = \langle q, \phi \rangle_{V', V}, \forall \phi \in V.$$

The left hand side of (22) is continuous and coercive on  $V$ . Indeed, we have

$$\begin{aligned} & \left| \lambda^2 (\omega, \phi)_H + \left( A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \phi \right)_H + \lambda ((B_1^* \omega, B_1^* \phi)_{U_1} + e^{-\lambda\tau(t)} (B_2^* \omega, B_2^* \phi)_{U_2}) \right| \\ & \leq \lambda^2 \|\omega\|_H \|\phi\|_H + \left\| A^{\frac{1}{2}} \omega \right\|_H \left\| A^{\frac{1}{2}} \phi \right\|_H + \lambda (\|B_1^* \omega\|_{U_1} \|B_1^* \phi\|_{U_1} \\ & \quad + e^{-\lambda\tau(t)} \|B_2^* \omega\|_{U_2} \|B_2^* \phi\|_{U_2}) \\ & \leq C \lambda^2 \|\omega\|_V \|\phi\|_H + \left\| A^{\frac{1}{2}} \right\|_H^2 \|\omega\|_V \|\phi\|_V \\ & \quad + \lambda (\|B_1^*\|_{\mathcal{L}(V, U_1)}^2 \|\omega\|_V \|\phi\|_V + e^{-\lambda\tau(t)} \|B_2^*\|_{\mathcal{L}(V, U_2)}^2 \|\omega\|_V \|\phi\|_V) \\ & \leq C \|\omega\|_V \|\phi\|_V, \end{aligned}$$

and for  $\phi = \omega \in V$

$$\begin{aligned} & \lambda^2 \|\omega\|_H^2 + \left( A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \omega \right)_H + \lambda (\|B_1^* \omega\|_{U_1}^2 + e^{-\lambda\tau(t)} \|B_2^* \omega\|_{U_2}^2) \\ & \geq \left\| A^{\frac{1}{2}} \omega \right\|_H^2 \geq C \|\omega\|_V^2. \end{aligned}$$

Therefore, this problem (22) has a unique solution  $\omega \in V$  by Lax-Milgram's lemma. We can easily prove the same results in the case where  $\dot{\tau}(t) \neq 0$ . Moreover  $A\omega + B_1 B_1^* \omega + B_2 z(1) = g + \lambda f - \lambda^2 \omega \in H$ . In summary, we have found  $(\omega, u, z)^T \in D(\mathcal{A}(t))$  satisfying (19). Again as  $\kappa(t) > 0$ , this proves that

$$(23) \quad \lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective}$$

for some  $\lambda > 0$  and  $t > 0$ .

Then, (15), (17) and (23) imply that the family  $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$  is a stable family of generators in  $\mathcal{H}$  with stability constants independent of  $t$ , by Proposition 1.1 from [9]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are verified by (10), (13), (15), (17), (18) and (23), and thus, the problem

$$\begin{cases} \tilde{U}' = \tilde{\mathcal{A}}(t)\tilde{U} \\ \tilde{U}(0) = U_0 \end{cases}$$

has a unique solution  $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$  for  $U_0 \in D(\mathcal{A}(0))$ . The requested solution of (8) is then given by

$$U(t) = e^{\beta(t)} \tilde{U}(t)$$

with  $\beta(t) = \int_0^t \kappa(s) ds$ , because

$$\begin{aligned} U'(t) &= \kappa(t) e^{\beta(t)} \tilde{U}(t) + e^{\beta(t)} \tilde{U}'(t) \\ &= \kappa(t) e^{\beta(t)} \tilde{U}(t) + e^{\beta(t)} \tilde{\mathcal{A}}(t) \tilde{U}(t) \\ &= e^{\beta(t)} (\kappa(t) \tilde{U}(t) + \tilde{\mathcal{A}}(t) \tilde{U}(t)) \\ &= e^{\beta(t)} \mathcal{A}(t) \tilde{U}(t) = \mathcal{A}(t) e^{\beta(t)} \tilde{U}(t) \\ &= \mathcal{A}(t) U(t), \end{aligned}$$

which concludes the proof. ■

### 3 The decay of the energy

We now restrict the hypothesis (12) to obtain the decay of the energy. For that, we suppose that (6) holds, namely

$$\exists 0 < \alpha < \sqrt{1-d}, \forall u \in V, \|B_2^* u\|_{U_2}^2 \leq \alpha \|B_1^* u\|_{U_1}^2,$$

where  $d$  is the one from (2). Note that is possible since  $d < 1$  by (2).

Let us choose the following energy

$$(24) \quad E(t) := \frac{1}{2} \left( \|A^{\frac{1}{2}} \omega\|_H^2 + \|\dot{\omega}\|_H^2 + q\tau(t) \int_0^1 \|B_2^* \dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho \right),$$

where  $q$  is a positive constant satisfying

$$(25) \quad \frac{1}{\sqrt{1-d}} < q < \frac{2}{\alpha} - \frac{1}{\sqrt{1-d}},$$

that exists by (6). Note that this energy corresponds to the time-dependent inner product on  $\mathcal{H}$  defined before.

**Proposition 3.1** *If (2) and (6) hold, then for all  $(\omega_0, \omega_1, f^0(-\tau)) \in D(\mathcal{A}(t))$ , the energy of the corresponding regular solution of (5) is non-increasing and there exists a positive constant  $C$  depending only on  $\alpha, d$  and  $q$  such that*

$$(26) \quad E'(t) \leq -C \left( \|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau(t))\|_{U_2}^2 \right).$$

**Proof.** Deriving (24), we obtain

$$\begin{aligned} E'(t) &= \left( A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \dot{\omega} \right)_H + (\dot{\omega}, \ddot{\omega})_H + \frac{q\dot{\tau}(t)}{2} \int_0^1 \|B_2^* \dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho \\ &\quad + q\tau(t) \int_0^1 (B_2^* \dot{\omega}(t - \tau(t)\rho), B_2^* \ddot{\omega}(t - \tau(t)\rho))_{U_2} (1 - \dot{\tau}(t)\rho) d\rho. \end{aligned}$$

Since  $\ddot{\omega} = -(A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t - \tau(t))) \in H$ ,

$$\begin{aligned} E'(t) &= \langle A\omega, \dot{\omega} \rangle_{V', V} - \langle \dot{\omega}, A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t - \tau(t)) \rangle_{V, V'} \\ &\quad + \frac{q\dot{\tau}(t)}{2} \int_0^1 \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho \\ &\quad + q\tau(t) \int_0^1 (B_2^*\dot{\omega}(t - \tau(t)\rho), B_2^*\ddot{\omega}(t - \tau(t)\rho))_{U_2} (1 - \dot{\tau}(t)\rho) d\rho. \end{aligned}$$

Then

$$\begin{aligned} E'(t) &= \langle A\omega, \dot{\omega} \rangle_{V', V} - \langle \dot{\omega}, A\omega \rangle_{V, V'} - \langle \dot{\omega}, B_1B_1^*\dot{\omega} \rangle_{V, V'} - \langle \dot{\omega}, B_2B_2^*\dot{\omega}(t - \tau(t)) \rangle_{V, V'} \\ &\quad + \frac{q\dot{\tau}(t)}{2} \int_0^1 \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho \\ &\quad + q\tau(t) \int_0^1 (B_2^*\dot{\omega}(t - \tau(t)\rho), B_2^*\ddot{\omega}(t - \tau(t)\rho))_{U_2} (1 - \dot{\tau}(t)\rho) d\rho \\ &= -\|B_1^*\dot{\omega}\|_{U_1}^2 - (B_2^*\dot{\omega}, B_2^*\dot{\omega}(t - \tau(t)))_{U_2} + \frac{q\dot{\tau}(t)}{2} \int_0^1 \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho \\ &\quad + q\tau(t) \int_0^1 (B_2^*\dot{\omega}(t - \tau(t)\rho), B_2^*\ddot{\omega}(t - \tau(t)\rho))_{U_2} (1 - \dot{\tau}(t)\rho) d\rho. \end{aligned}$$

Moreover, recalling that  $z(\rho, t) = B_2^*\dot{\omega}(t - \tau(t)\rho)$  and thus  $z_\rho(\rho, t) = -\tau(t)B_2^*\ddot{\omega}(t - \tau(t)\rho)$ , we see that

$$\begin{aligned} &\int_0^1 (B_2^*\dot{\omega}(t - \tau(t)\rho), B_2^*\ddot{\omega}(t - \tau(t)\rho))_{U_2} (1 - \dot{\tau}(t)\rho) d\rho \\ &= -\frac{1}{\tau(t)} \int_0^1 \left( z(\rho, t), \frac{\partial z}{\partial \rho}(\rho, t) \right)_{U_2} (1 - \dot{\tau}(t)\rho) d\rho \\ &= -\frac{1}{2\tau(t)} \int_0^1 \frac{\partial}{\partial \rho} \left( \|z(\rho, t)\|_{U_2}^2 \right) (1 - \dot{\tau}(t)\rho) d\rho \\ &= -\frac{\dot{\tau}(t)}{2\tau(t)} \int_0^1 \|z(\rho, t)\|_{U_2}^2 d\rho - \frac{1 - \dot{\tau}(t)}{2\tau(t)} \|z(1, t)\|_{U_2}^2 + \frac{1}{2\tau(t)} \|z(0, t)\|_{U_2}^2 \\ &= -\frac{\dot{\tau}(t)}{2\tau(t)} \int_0^1 \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho - \frac{1 - \dot{\tau}(t)}{2\tau(t)} \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2 + \frac{1}{2\tau(t)} \|B_2^*\dot{\omega}(t)\|_{U_2}^2. \end{aligned}$$

Consequently,

$$E'(t) = -\|B_1^*\dot{\omega}\|_{U_1}^2 - (B_2^*\dot{\omega}, B_2^*\dot{\omega}(t - \tau(t)))_{U_2} - \frac{q(1 - \dot{\tau}(t))}{2} \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2 + \frac{q}{2} \|B_2^*\dot{\omega}(t)\|_{U_2}^2.$$

Young's inequality, (2) and (6) yield

$$E'(t) \leq \left( \frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 \right) \|B_1^*\dot{\omega}\|_{U_1}^2 + \left( \frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} \right) \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2.$$

Therefore, this estimate leads to

$$E'(t) \leq -C \left( \|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2 \right)$$

with

$$C = \min \left\{ \left( 1 - \frac{q\alpha}{2} - \frac{\alpha}{2\sqrt{1-d}} \right), \left( \frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2} \right) \right\}$$

which is positive according to the assumption (25). ■

**Remark 3.2** The choice to apply Young's inequality with a factor  $\sqrt{1-d}$  in the proof of the above proposition is made in order to give the stability result under the best assumption between  $\alpha$  and  $d$ . ■

**Remark 3.3** In the case where the delay is constant in time (and thus  $d = 0$ ), we recover some results from [19]. ■

**Remark 3.4** If (6) is not satisfied, there exist cases where instabilities may appear, see [17, 20, 27] for the wave equation with constant (in time) delay. Hence this condition appears to be quite realistic. ■

## 4 Exponential stability

In this section, we prove, under some assumptions, the exponential stability of (5) by using an appropriate abstract Lyapunov functional, defined by

$$(27) \quad \mathcal{E}(t) = E(t) + \gamma (\mathcal{E}_2(t) + (\mathcal{M}\omega(t), \dot{\omega}(t))_H),$$

where  $\gamma$  is a positive small constant that will be chosen later on,  $E$  is the standard energy defined by (24) with  $q$  verifying (25) and  $\mathcal{E}_2$  is defined by

$$(28) \quad \mathcal{E}_2(t) := q\tau(t) \int_0^1 e^{-2\delta\tau(t)\rho} \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho,$$

where  $\delta$  is a fixed positive real number. Moreover, the operator  $\mathcal{M} : V \rightarrow H$  satisfies the following assumptions

$$(29) \quad \exists C_0, C_1, C_2 > 0, \frac{d}{dt} (\mathcal{M}\omega(t), \dot{\omega}(t))_H \leq -C_0 E_0(t) + C_1 \|B_1^*\dot{\omega}(t)\|_{U_1}^2 + C_2 \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2,$$

where  $E_0$  is the natural energy for the problem without delay

$$E_0(t) := \frac{1}{2} \left( \left\| A^{\frac{1}{2}}\omega(t) \right\|_H^2 + \|\dot{\omega}(t)\|_H^2 \right),$$

and

$$(30) \quad \exists C > 0, \forall t > 0, \quad |(\mathcal{M}\omega(t), \dot{\omega}(t))_H| \leq C E_0(t).$$

First we note that the energies  $E$  and  $\mathcal{E}$  are equivalent, under (30).

**Lemma 4.1** *Assume (30). For  $\gamma$  small enough, there exists a positive constant  $C_3(\gamma)$  such that*

$$(31) \quad (1 - C\gamma)E(t) \leq \mathcal{E}(t) \leq C_3(\gamma)E(t), \text{ where } 1 - C\gamma > 0.$$

**Proof.** It is easy to see that

$$\mathcal{E}(t) \leq C_3(\gamma)E(t),$$

with  $C_3(\gamma) = \max(1 + \gamma C, 1 + 2\gamma)$  by (30), since  $e^{-2\delta\tau(t)\rho} \leq 1$ .

For the second inequality of (31), we note that, since  $\gamma\mathcal{E}_2(t) \geq 0$  and by (30),

$$\begin{aligned} \mathcal{E}(t) &\geq E(t) - C\gamma E_0(t) \\ &\geq (1 - C\gamma)E(t), \end{aligned}$$

and thus we obtain (31) with  $1 - C\gamma > 0$  for  $\gamma$  small enough ( $\gamma < 1/C$ ). ■

To prove the exponential decay of (5), we need the following lemma:

**Lemma 4.2** *Assume (2). Then*

$$(32) \quad \frac{d}{dt}\mathcal{E}_2(t) \leq -2\delta\mathcal{E}_2(t) + q \|B_2^*\dot{\omega}(t)\|_{U_2}^2.$$

**Proof.** Direct calculations show that

$$\frac{d}{dt}\mathcal{E}_2(t) = \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\tau(t) \int_0^1 (-2\delta\dot{\tau}(t)\rho) e^{-2\delta\tau(t)\rho} \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho + J,$$

where  $J$  is equal to

$$J := 2q\tau(t) \int_0^1 e^{-2\delta\tau(t)\rho} (B_2^*\dot{\omega}(t - \tau(t)\rho), B_2^*\dot{\omega}(t - \tau(t)\rho))_{U_2} (1 - \dot{\tau}(t)\rho) d\rho.$$

Recalling that  $z(\rho, t) = B_2^*\dot{\omega}(t - \tau(t)\rho)$  and then  $z_\rho(\rho, t) = -\tau(t)B_2^*\dot{\omega}(t - \tau(t)\rho)$ , we see that

$$J = -2q \int_0^1 e^{-2\delta\tau(t)\rho} \left( z(\rho, t), \frac{\partial z}{\partial \rho}(\rho, t) \right)_{U_2} (1 - \dot{\tau}(t)\rho) d\rho.$$

By integrating by parts in  $\rho$ , we obtain

$$\begin{aligned} J &= -J + 2q \int_0^1 e^{-2\delta\tau(t)\rho} \|z(\rho, t)\|_{U_2}^2 (-2\delta\tau(t)(1 - \dot{\tau}(t)\rho) - \dot{\tau}(t)) d\rho \\ &\quad - 2qe^{-2\delta\tau(t)} \|z(1, t)\|_{U_2}^2 (1 - \dot{\tau}(t)) + 2q \|z(0, t)\|_{U_2}^2, \end{aligned}$$

which yields

$$\begin{aligned} J &= q \int_0^1 e^{-2\delta\tau(t)\rho} \|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 (-2\delta\tau(t)(1 - \dot{\tau}(t)\rho) - \dot{\tau}(t)) d\rho \\ &\quad - qe^{-2\delta\tau(t)} \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2 (1 - \dot{\tau}(t)) + q \|B_2^*\dot{\omega}(t)\|_{U_2}^2. \end{aligned}$$

Consequently

$$\frac{d}{dt}\mathcal{E}_2(t) = -2\delta\mathcal{E}_2(t) - q(1 - \dot{\tau}(t))e^{-2\delta\tau(t)} \|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2 + q \|B_2^*\dot{\omega}(t)\|_{U_2}^2.$$

We thus get (32) by (2). ■

Now, we are able to state the main result of this paper:

**Theorem 4.3** *Assume that (2), (3), (6), (29) and (30) hold. Then there exist positive constants  $\nu$  and  $K$  such that*

$$E(t) \leq K e^{-\nu t} E(0), \quad \forall t > 0.$$

**Proof.** We have, by the definition (27) of  $\mathcal{E}$ ,

$$\frac{d}{dt} \mathcal{E}(t) = \frac{d}{dt} E(t) + \gamma \frac{d}{dt} \mathcal{E}_2(t) + \gamma \frac{d}{dt} (\mathcal{M}\omega(t), \dot{\omega}(t))_H.$$

By (26), (29) and (30),

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq \left( \frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 \right) \|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \left( \frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} \right) \|B_2^* \dot{\omega}(t - \tau(t))\|_{U_2}^2 \\ &\quad - 2\delta\gamma \mathcal{E}_2(t) + \gamma q \|B_2^* \dot{\omega}(t)\|_{U_2}^2 - \gamma C_0 E_0(t) + \gamma C_1 \|B_1^* \dot{\omega}(t)\|_{U_1}^2 \\ &\quad + \gamma C_2 \|B_2^* \dot{\omega}(t - \tau(t))\|_{U_2}^2. \end{aligned}$$

Using (6), we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq \left( \frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 + \gamma(q\alpha + C_1) \right) \|B_1^* \dot{\omega}(t)\|_{U_1}^2 \\ &\quad + \left( \frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} + \gamma C_2 \right) \|B_2^* \dot{\omega}(t - \tau(t))\|_{U_2}^2 - 2\delta\gamma \mathcal{E}_2(t) - \gamma C_0 E_0(t) \end{aligned}$$

We take now  $\gamma$  small enough, more precisely we take  $\gamma > 0$  such that

$$\gamma \leq \min \left( \frac{1 - \frac{\alpha}{2\sqrt{1-d}} - \frac{q\alpha}{2}}{q\alpha + C_1}, \frac{\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2}}{C_2} \right).$$

Note that  $(1 - \frac{\alpha}{2\sqrt{1-d}} - \frac{q\alpha}{2})/(q\alpha + C_1)$  and  $(\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2})/C_2$  are positive by the choice (25) of  $q$ . Then

$$\frac{d}{dt} \mathcal{E}(t) \leq -\gamma(2\delta\mathcal{E}_2(t) + C_0 E_0(t)).$$

As  $\tau(t) \leq M$  (by (3)), we have

$$\frac{d}{dt} \mathcal{E}(t) \leq -\gamma \left( C_0 E_0(t) + 2\delta e^{-2\delta M} q \tau(t) \int_0^1 \|B_2^* \dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2 d\rho \right),$$

and then, in view of definition of  $E$ , there exists a constant  $\gamma' > 0$  (depending on  $\gamma$  and  $\delta$ :  $\gamma' \leq \gamma \min(C_0, 4\delta e^{-2\delta M})$ ) such that

$$\frac{d}{dt} \mathcal{E}(t) \leq -\gamma' E(t).$$

By applying Lemma 4.1, we arrive at

$$\frac{d}{dt} \mathcal{E}(t) \leq -\frac{\gamma'}{C_3(\gamma)} \mathcal{E}(t).$$

Therefore

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\frac{\gamma'}{c_3(\gamma)}t}, \quad \forall t > 0,$$

and Lemma 4.1 allows to conclude the proof:

$$E(t) \leq \frac{1}{1-C\gamma}\mathcal{E}(t) \leq \frac{1}{1-C\gamma}\mathcal{E}(0)e^{-\frac{\gamma'}{c_3(\gamma)}t} \leq \frac{C_3(\gamma)}{1-C\gamma}E(0)e^{-\frac{\gamma'}{c_3(\gamma)}t}.$$

■

**Remark 4.4** In the proof of Theorem 4.3, we note that we can explicitly calculate the decay rate  $\nu$  of the energy, given by

$$\nu = \frac{\gamma}{C_3(\gamma)} \min(C_0, 4\delta e^{-2\delta M}),$$

with  $C_3(\gamma) = \max(1 + \gamma C, 1 + 2\gamma)$ ,

$$\gamma < \frac{1}{C}, \gamma \leq \frac{1 - \frac{\alpha}{2\sqrt{1-d}} - \frac{q\alpha}{2}}{q\alpha + C_1} \text{ and } \gamma \leq \frac{\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2}}{C_2}$$

(by Lemma 4.1 and Theorem 4.3), where  $C, C_0, C_1, C_2$  are given by (29) and (30),  $\alpha$  is defined by (6),  $q$  by (25) and  $\delta$  is a positive real number. Recalling that  $M$  is the upper bound of  $\tau$ , if the delay  $\tau$  becomes larger, the decay rate is slower. Moreover, we can choose  $\delta$  such that the decay of the energy is as quick as possible for given parameters. For that purpose, we note that the function  $\delta \rightarrow 4\delta e^{-2\delta M}$  admits a maximum at  $\delta = \frac{1}{2M}$  and that this maximum is  $\frac{2}{Me}$ . Thus the larger decay rate of the energy is given by

$$\nu_{max} = \frac{\gamma}{C_3(\gamma)} \min\left(C_0, \frac{2}{Me}\right).$$

■

## 5 Examples

We end up this paper by considering different examples for which our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new. In all examples, we assume that the delay function  $\tau$  satisfies the assumptions (2) to (4).

### 5.1 The wave equation

#### 5.1.1 The one dimensional wave equation

In this subsection, we show that our abstract framework apply to the 1-d wave equation:

$$(33) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x, t) - a \frac{\partial^2 u}{\partial x^2}(x, t) = 0, & 0 < x < \pi, t > 0, \\ u(0, t) = 0, & t > 0, \\ \frac{\partial u}{\partial x}(\pi, t) = -\alpha_1 \frac{\partial u}{\partial t}(\pi, t) - \alpha_2 \frac{\partial u}{\partial t}(\pi, t - \tau(t)), & t > 0, \\ u(x, 0) = u^0(x), \frac{\partial u}{\partial t}(x, 0) = u^1(x), & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\pi, t - \tau(0)) = f^0(t - \tau(0)), & 0 < t < \tau(0), \end{cases}$$

where  $\alpha_1, \alpha_2 > 0, a > 0$ . This system has been studied in [21], we also refer to [27] for a constant delay. First, we rewrite this system in the form (5). For that purpose, we introduce  $H = L^2(0, \pi)$  and the operator  $A : D(A) \rightarrow H$  defined by

$$A\varphi = -a \frac{d^2}{dx^2} \varphi$$

where  $D(A) = \{\varphi \in H^2(0, \pi); \varphi(0) = \frac{\partial \varphi}{\partial x}(\pi) = 0\}$ . The operator  $A$  is self-adjoint and positive with a compact inverse in  $H$ . We now define  $U = U_1 = U_2 = \mathbb{R}$  and the operators  $B_i : U \rightarrow D(A^{\frac{1}{2}})'$  given by

$$B_i k = \sqrt{\alpha_i} k \delta_\pi, i = 1, 2.$$

It is easy to verify that  $B_i^*(\varphi) = \sqrt{\alpha_i} \varphi(\pi)$  for  $\varphi \in D(A^{1/2})$  and thus  $B_i B_i^*(\varphi) = \alpha_i \varphi(\pi) \delta_\pi$  for  $\varphi \in D(A^{1/2})$  and  $i = 1, 2$ . Then the system (33) can be rewritten in the form (5). We notice that (12) is equivalent to

$$(34) \quad \exists 0 < \alpha \leq \sqrt{1-d}, \quad \alpha_2 \leq \alpha \alpha_1.$$

Taking  $\alpha = \alpha_2/\alpha_1$ , (34) is equivalent to

$$(35) \quad \alpha_2^2 \leq (1-d)\alpha_1^2,$$

which is the condition (10) from [21].

In Lemma 3.1 from [21], it is proved that  $D(\mathcal{A}(0))$  is dense in  $\mathcal{H}$ . Consequently, under the condition (35), by Theorem 2.4, this system is well-posed and by Proposition 3.1 the energy decays for  $\alpha_2^2 < (1-d)\alpha_1^2$ .

To prove the exponential stability of (33), we introduce the Lyapunov functional (27) with the operator  $\mathcal{M} : V \rightarrow H$  defined by

$$(36) \quad \mathcal{M}u = 2x \frac{\partial u}{\partial x}.$$

Then (29) holds with  $C_0 = 2, C_1 = \pi(1 + 2a\alpha_1^2)$  and  $C_2 = 2a\pi\alpha_2^2$  (see (48) from [21]) and (30) holds with  $C = 2\pi \max(1, 1/a)$ . Therefore, our abstract framework applies here and system (33) is exponentially stable under the previous hypotheses. We then recover the results from [21].

### 5.1.2 The multidimensional wave equation

In this subsection, we study the stability of the wave equation with boundary time varying delay. Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 1$ ) be an open bounded set with a boundary  $\Gamma$  of class  $C^2$ . We assume that  $\Gamma$  is divided into two parts  $\Gamma_D$  and  $\Gamma_N$ , i.e.  $\Gamma = \Gamma_D \cup \Gamma_N$ , with  $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$  and  $\Gamma_D \neq \emptyset$ . Moreover we assume that

$$\Gamma_N^2 \subseteq \Gamma_N^1 = \Gamma_N.$$

In this domain  $\Omega$ , we consider the initial boundary value problem

$$(37) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x, t) - \Delta u(x, t) = 0 & \text{in } \Omega \times (0, +\infty) \\ u(x, t) = 0 & \text{on } \Gamma_D \times (0, +\infty) \\ \frac{\partial u}{\partial \nu}(x, t) = -\alpha_1 \frac{\partial u}{\partial t}(x, t) \chi_{\Gamma_N^1} - \alpha_2 \frac{\partial u}{\partial t}(x, t - \tau(t)) \chi_{\Gamma_N^2} & \text{on } \Gamma_N \times (0, +\infty) \\ u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x) & \text{in } \Omega \\ \frac{\partial u}{\partial t}(x, t - \tau(0)) = f_0(x, t - \tau(0)) & \text{in } \Gamma_N^2 \times (0, \tau(0)), \end{cases}$$

where  $\nu(x)$  denotes the outer unit normal vector to the point  $x \in \Gamma$  and  $\partial u / \partial \nu$  is the normal derivative. Note that system (37) have been studied for instance in [4, 11, 12, 13, 14, 15] without delay and in [17] with a constant delay.

Let us denote by  $v \cdot w$  the Euclidean inner product between two vectors  $v, w \in \mathbb{R}^n$ . We assume that there exists  $x_0 \in \mathbb{R}^n$  such that denoting by  $m$  the standard multiplier

$$m(x) := x - x_0,$$

we have

$$(38) \quad m(x) \cdot \nu(x) \leq 0 \quad \text{on } \Gamma_D$$

and, for some positive constant  $\delta$ ,

$$(39) \quad m(x) \cdot \nu(x) \geq \delta > 0 \quad \text{on } \Gamma_N.$$

In the particular case where  $\Omega = O_1 \setminus O_2$ ,  $O_1$  and  $O_2$  being convex sets such that  $O_2 \subset O_1$ , the above assumptions (38), (39) hold with  $\Gamma_N = \partial O_1$  and  $\Gamma_D = \partial O_2$  for any  $x_0 \in O_2$ .

First, we rewrite this system in the form (5). For this purpose, we introduce  $H = L^2(\Omega)$  and the operator  $A : D(A) \rightarrow H$  defined by

$$A\varphi = -\Delta\varphi$$

where  $D(A) = \{\varphi \in H^2(\Omega) \cap V : \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma_N\}$ , where, as usual,

$$V = H_{\Gamma_D}^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}.$$

The operator  $A$  is self-adjoint and positive with a compact inverse in  $H$ . We now define  $U_1 = L^2(\Gamma_N^1)$ ,  $U_2 = L^2(\Gamma_N^2)$  and the operators  $B_i^* : V \rightarrow U_i$  as

$$(40) \quad B_i^* \varphi = \sqrt{\alpha_i} \varphi|_{\Gamma_N^i}, i = 1, 2,$$

where  $\varphi|_{\Gamma_N^i}$  is the trace operator for  $\varphi$ . The operator  $B_i : U_i \rightarrow V'$  is then defined by duality:

$$(41) \quad \langle B_i u, v \rangle_{V', V} = \int_{\Gamma_N^i} uv \, d\Gamma.$$

Thus the system (37) can be rewritten in the form (5). We notice that (12) is equivalent to (34) and then, as previously, to (35).

Note that the domain of the operator  $\mathcal{A}(t)$  defined in (9) is here

$$D(\mathcal{A}(t)) = \{(u, v, z)^T \in (E(\Delta, L^2(\Omega)) \cap V) \times V \times L^2(\Gamma_N^2; H^1(0, 1)) : \\ \frac{\partial u}{\partial \nu} = -\alpha_1 v \chi_{\Gamma_N^1} - \alpha_2 z(\cdot, 1) \chi_{\Gamma_N^2} \text{ on } \Gamma_N; v = z(\cdot, 0) \text{ on } \Gamma_N^2\},$$

where

$$E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}.$$

The hypothesis (13) holds thanks to Lemma 2.2 and Remark 2.3 because  $\mathcal{D}(\Omega) \subset \ker(B_1^*)$  and  $\mathcal{D}(\Omega)$  is dense in  $L^2(\Omega)$ .

Consequently, under the condition (35), this system is well-posed by Theorem 2.4 and the energy decays by Proposition 3.1 for  $\alpha_2^2 < (1-d)\alpha_1^2$ .

To prove the exponential stability of (37), we introduce the Lyapunov functional (27) with the operator  $\mathcal{M} : V \rightarrow H$  defined by

$$(42) \quad \mathcal{M}u = 2m \cdot \nabla u + (n-1)u.$$

Then we can easily prove that (30) holds by Poincaré's inequality. Moreover:

**Lemma 5.1** *Condition (29) holds.*

**Proof.** Let  $u \in H^2(\Omega)$ . Then the standard multiplier identity gives

$$(43) \quad \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} = - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx \\ + \int_{\Gamma_N} (m \cdot \nu)(u_t^2 - |\nabla u|^2) d\Gamma + \int_{\Gamma_N} [2m \cdot \nabla u + (n-1)u] \frac{\partial u}{\partial \nu} d\Gamma.$$

From (43) and Young's inequality, recalling that by (39)  $m \cdot \nu \geq \delta$  on  $\Gamma_N$ , we have

$$(44) \quad \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} \leq - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx \\ + \int_{\Gamma_N} (m \cdot \nu) u_t^2 d\Gamma - \delta \int_{\Gamma_N} |\nabla u|^2 d\Gamma + \frac{c}{\varepsilon} \int_{\Gamma_N} \left( \frac{\partial u}{\partial \nu} \right)^2 d\Gamma \\ + \varepsilon \int_{\Gamma_N} (|\nabla u|^2 + u^2) d\Gamma,$$

for some positive constants  $\varepsilon, c$ . Using the trace inequality and then Poincaré's Theorem, we have, for some  $c', c'' > 0$ ,

$$\int_{\Gamma_N} u^2 d\Gamma \leq c' \|u\|_{H^1(\Omega)}^2 \leq c'' \int_{\Omega} |\nabla u|^2 dx.$$

This estimate in (44) yields, for  $\varepsilon$  small enough ( $\varepsilon < \min(\delta, 1/(2c''))$ ),

$$(45) \quad \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} \leq -C_0 E_0(t) \\ + C \int_{\Gamma_N} u_t^2 d\Gamma + C \int_{\Gamma_N} \left( \frac{\partial u}{\partial \nu} \right)^2 d\Gamma,$$

for suitable positive constants  $C_0, C$ . Therefore, using the boundary condition (37) and Cauchy Schwarz's inequality in (45), we obtain (29). ■

Therefore, our abstract framework still applies and system (37) is exponentially stable under the above assumptions.

## 5.2 The beam equation

In this subsection, we show that our abstract framework can be applied to the 1-d beam equation:

$$(46) \quad \begin{cases} \frac{\partial^2 \omega}{\partial t^2}(x, t) + \frac{\partial^4 \omega}{\partial x^4}(x, t) = 0, & 0 < x < 1, t > 0, \\ \omega(0, t) = \frac{\partial \omega}{\partial x}(0, t) = 0, & t > 0, \\ \frac{\partial^2 \omega}{\partial x^2}(1, t) = 0, & t > 0, \\ \frac{\partial \omega}{\partial x^3}(1, t) = \alpha_1 \frac{\partial \omega}{\partial t}(1, t) + \alpha_2 \frac{\partial \omega}{\partial t}(1, t - \tau(t)), & t > 0, \\ \omega(x, 0) = \omega^0(x), \frac{\partial \omega}{\partial t}(x, 0) = \omega^1(x), & 0 < x < 1, \\ \frac{\partial \omega}{\partial t}(1, t - \tau(0)) = f^0(t - \tau(0)), & 0 < t < \tau(0), \end{cases}$$

where  $\alpha_1, \alpha_2 > 0$ . First, we rewrite this system in the form (5). For that purpose, we introduce  $H = L^2(0, 1)$  and the operator  $A : D(A) \rightarrow H$  defined by

$$A\varphi = \frac{d^4}{dx^4}\varphi$$

where  $D(A) = \{\varphi \in H^4(0, 1); \varphi(0) = \frac{\partial \varphi}{\partial x}(0) = \frac{\partial^2 \varphi}{\partial x^2}(1) = \frac{\partial^3 \varphi}{\partial x^3}(1) = 0\}$ , which is a self-adjoint and positive operator with a compact inverse in  $H$ . We now define  $U = U_1 = U_2 = \mathbb{R}$  and the operators  $B_i : U \rightarrow D(A^{\frac{1}{2}})'$  given by

$$B_i k = \sqrt{\alpha_i} k \delta_1, \quad i = 1, 2.$$

It is easy to verify that  $B_i^*(\varphi) = \sqrt{\alpha_i} \varphi(1)$  for  $\varphi \in D(A^{1/2})$  and thus  $B_i B_i^*(\varphi) = \alpha_i \varphi(1) \delta_1$  for  $\varphi \in D(A^{1/2})$  and  $i = 1, 2$ . Then the system (46) can be rewritten in the form (5). We notice that (12) is equivalent to (34) and by taking  $\alpha = \alpha_2/\alpha_1$ , (34) is equivalent to (35).

By Lemma 2.2 and Remark 2.3, (13) holds, because  $\mathcal{D}(0, 1) \subset \ker(B_1^*)$  and  $\mathcal{D}(0, 1)$  is dense in  $H$ . Hence, under the condition (35), this system is well-posed by Theorem 2.4 and the energy decays by Proposition 3.1 for  $\alpha_2^2 < (1-d)\alpha_1^2$ .

To prove the exponential stability of (46), we introduce the Lyapunov functional (27) with the operator  $\mathcal{M} : V \rightarrow H$  defined by (36).

The following lemma shows that (29) and (30) hold.

**Lemma 5.2** *The conditions (29) and (30) hold.*

**Proof.** Condition (30) follows directly from Young's inequality:

$$\begin{aligned} |(\mathcal{M}\omega, \dot{\omega})_H| &= \left| 2 \int_0^1 x \frac{\partial \omega}{\partial x}(x, t) \frac{\partial \omega}{\partial t}(x, t) dx \right| \\ &\leq \int_0^1 \left( \left( \frac{\partial \omega}{\partial x}(x, t) \right)^2 + \left( \frac{\partial \omega}{\partial t}(x, t) \right)^2 \right) dx. \end{aligned}$$

For the other assertion, we note that

$$\frac{d}{dt}(\mathcal{M}\omega, \dot{\omega})_H = \int_0^1 \left( 2x \frac{\partial^2 \omega}{\partial x \partial t}(x, t) \frac{\partial \omega}{\partial t}(x, t) - 2x \frac{\partial \omega}{\partial x}(x, t) \frac{\partial^4 \omega}{\partial x^4}(x, t) \right) dx.$$

But, by integrating by parts, we obtain

$$2 \int_0^1 x \frac{\partial^2 \omega}{\partial x \partial t}(x, t) \frac{\partial \omega}{\partial t}(x, t) dx = - \int_0^1 \left( \frac{\partial \omega}{\partial t}(x, t) \right)^2 dx + \left( \frac{\partial \omega}{\partial t}(1, t) \right)^2.$$

Moreover, again integrating by parts yields

$$\begin{aligned} \int_0^1 x \frac{\partial \omega}{\partial x}(x, t) \frac{\partial^4 \omega}{\partial x^4}(x, t) dx &= - \int_0^1 \frac{\partial \omega}{\partial x}(x, t) \frac{\partial^3 \omega}{\partial x^3}(x, t) dx - \int_0^1 x \frac{\partial^2 \omega}{\partial x^2}(x, t) \frac{\partial^3 \omega}{\partial x^3}(x, t) dx \\ &\quad + \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^3 \omega}{\partial x^3}(1, t), \end{aligned}$$

with

$$\int_0^1 x \frac{\partial^2 \omega}{\partial x^2}(x, t) \frac{\partial^3 \omega}{\partial x^3}(x, t) dx = - \frac{1}{2} \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx + \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial x^2}(1, t) \right)^2,$$

and

$$\int_0^1 \frac{\partial \omega}{\partial x}(x, t) \frac{\partial^3 \omega}{\partial x^3}(x, t) dx = - \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx + \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^2 \omega}{\partial x^2}(1, t) - \frac{\partial \omega}{\partial x}(0, t) \frac{\partial^2 \omega}{\partial x^2}(0, t).$$

Consequently

$$\begin{aligned} \int_0^1 x \frac{\partial \omega}{\partial x}(x, t) \frac{\partial^4 \omega}{\partial x^4}(x, t) dx &= \frac{3}{2} \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx - \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^2 \omega}{\partial x^2}(1, t) \\ &\quad + \frac{\partial \omega}{\partial x}(0, t) \frac{\partial^2 \omega}{\partial x^2}(0, t) - \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial x^2}(1, t) \right)^2 + \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^3 \omega}{\partial x^3}(1, t). \end{aligned}$$

Therefore, the boundary conditions satisfied by  $\omega$  lead to

$$\begin{aligned} \frac{d}{dt}(\mathcal{M}\omega, \dot{\omega})_H &= - \int_0^1 \left( \frac{\partial \omega}{\partial t}(x, t) \right)^2 dx + \left( \frac{\partial \omega}{\partial t}(1, t) \right)^2 - 3 \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx \\ &\quad - 2 \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^3 \omega}{\partial x^3}(1, t). \end{aligned}$$

By Young's inequality, we have

$$\left| -2 \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^3 \omega}{\partial x^3}(1, t) \right| \leq \epsilon \left( \frac{\partial \omega}{\partial x}(1, t) \right)^2 + \frac{1}{\epsilon} \left( \frac{\partial^3 \omega}{\partial x^3}(1, t) \right)^2, \quad \forall \epsilon > 0.$$

Moreover by trace inequality and Poincaré's inequality, there exists a constant  $C > 0$  such that

$$\left( \frac{\partial \omega}{\partial x}(1, t) \right)^2 \leq C \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx.$$

Thus, by the dissipation condition at 1 of (46),

$$\begin{aligned} \left| -2 \frac{\partial \omega}{\partial x}(1, t) \frac{\partial^3 \omega}{\partial x^3}(1, t) \right| &\leq C\epsilon \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx + \frac{2\alpha_1^2}{\epsilon} \left( \frac{\partial \omega}{\partial t}(1, t) \right)^2 \\ &\quad + \frac{2\alpha_2^2}{\epsilon} \left( \frac{\partial \omega}{\partial t}(1, t - \tau(t)) \right)^2. \end{aligned}$$

Therefore it holds

$$\begin{aligned} \frac{d}{dt} (\mathcal{M}\omega, \dot{\omega})_H &\leq - \int_0^1 \left( \frac{\partial \omega}{\partial t}(x, t) \right)^2 dx - (3 - C\epsilon) \int_0^1 \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 dx \\ &\quad + \left( 1 + \frac{2\alpha_1^2}{\epsilon} \right) \left( \frac{\partial \omega}{\partial t}(1, t) \right)^2 + \frac{2\alpha_2^2}{\epsilon} \left( \frac{\partial \omega}{\partial t}(1, t - \tau(t)) \right)^2, \quad \forall \epsilon > 0 \end{aligned}$$

It suffices to take  $\epsilon \leq 2/C$ , to obtain

$$\begin{aligned} \frac{d}{dt} (\mathcal{M}\omega, \dot{\omega})_H &\leq - \int_0^1 \left( \left( \frac{\partial \omega}{\partial t}(x, t) \right)^2 + \left( \frac{\partial^2 \omega}{\partial x^2}(x, t) \right)^2 \right) dx + C_1 \left( \frac{\partial \omega}{\partial t}(1, t) \right)^2 \\ &\quad + C_2 \left( \frac{\partial \omega}{\partial t}(1, t - \tau(t)) \right)^2, \end{aligned}$$

with  $C_1, C_2 > 0$ , which corresponds to (29). ■

Therefore, by our abstract framework the system (46) is exponentially stable under the above assumptions.

### 5.3 The plate equation

In this subsection, we study the stability of the plate equation with boundary time-varying delay. Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 1$ ) be an open bounded set with a boundary  $\Gamma$  of class  $C^2$ . We assume that  $\Gamma$  is divided into two parts  $\Gamma_D$  and  $\Gamma_N$ , i.e.  $\Gamma = \Gamma_D \cup \Gamma_N$ , with  $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$  and  $\Gamma_D \neq \emptyset$ . Moreover we assume that

$$\Gamma_N^2 \subseteq \Gamma_N^1 = \Gamma_N.$$

In this domain  $\Omega$ , we consider the initial boundary value problem

$$(47) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x, t) + \Delta^2 u(x, t) = 0 & \text{in } \Omega \times (0, +\infty) \\ u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0 & \text{on } \Gamma_D \times (0, +\infty) \\ \Delta u(x, t) = 0 & \text{on } \Gamma_N \times (0, +\infty) \\ \frac{\partial \Delta u}{\partial \nu}(x, t) = \alpha_1 \frac{\partial u}{\partial t}(x, t) \chi_{\Gamma_N} + \alpha_2 \frac{\partial u}{\partial t}(x, t - \tau(t)) \chi_{\Gamma_N^2} & \text{on } \Gamma_N \times (0, +\infty) \\ u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x) & \text{in } \Omega \\ \frac{\partial u}{\partial t}(x, t - \tau(0)) = f_0(x, t - \tau(0)) & \text{in } \Gamma_N^2 \times (0, \tau(0)). \end{cases}$$

We assume that (38) holds with the standard multiplier  $m(x) := x - x_0$ , for some  $x_0 \in \mathbb{R}^n$ . Note that the hypothesis (39) is not necessary.

To rewrite this system in the form (5), we introduce  $H = L^2(\Omega)$  and the operator  $A : D(A) \rightarrow H$  given by

$$A\varphi = \Delta^2\varphi$$

where  $D(A) = \{\varphi \in H^4(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_D, \Delta u = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma_N\}$ . The operator  $A$  is self-adjoint and positive with a compact inverse in  $H$ . The operators  $B_1^*$  and  $B_2^*$  are here given by (40) and  $B_1, B_2$  by (41) with  $U_1 = L^2(\Gamma_N^1), U_2 = L^2(\Gamma_N^2)$ .

Thus the system (47) can be rewritten in the form (5). We notice that (12) is equivalent to (34) and then, as previously, to (35).

By Lemma 2.2 and Remark 2.3, we see that (13) holds because  $\mathcal{D}(\Omega) \subset \ker(B_1^*)$  and  $\mathcal{D}(\Omega)$  is dense in  $L^2(\Omega)$ . Therefore, under the hypothesis (35), this system is well-posed by Theorem 2.4 and the energy decays by Proposition 3.1 for  $\alpha_2^2 < (1-d)\alpha_1^2$ .

To prove the exponential stability of (47), we introduce the Lyapunov functional (27) with the operator  $\mathcal{M} : V \rightarrow H$  defined by (42). Then we can easily prove that (30) holds by Poincaré's theorem. Moreover:

**Lemma 5.3** *Condition (29) holds.*

**Proof.** Direct calculation gives

$$(48) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} (2m \cdot \nabla u + (n-1)u) u_t dx &= \int_{\Omega} 2m \cdot \nabla u_t u_t dx + (n-1) \int_{\Omega} u_t^2 dx \\ &\quad - \int_{\Omega} (2m \cdot \nabla u) \Delta^2 u dx - (n-1) \int_{\Omega} u \Delta^2 u dx. \end{aligned}$$

By Green's formula, we find

$$\int_{\Omega} 2m \cdot \nabla u_t u_t dx = -n \int_{\Omega} u_t^2 dx + \int_{\Gamma} (m \cdot \nu) u_t^2 d\Gamma.$$

Moreover again two applications of Green's formula lead to

$$\int_{\Omega} (2m \cdot \nabla u) \Delta^2 u dx = 2 \int_{\Omega} \Delta(m \cdot \nabla u) \Delta u dx - 2 \int_{\Gamma} \frac{\partial}{\partial \nu} (m \cdot \nabla u) \Delta u d\Gamma + 2 \int_{\Gamma} \frac{\partial \Delta u}{\partial \nu} (m \cdot \nabla u) d\Gamma,$$

with

$$\Delta(m \cdot \nabla u) \Delta u = 2(\Delta u)^2 + m \cdot \nabla(\Delta u) \Delta u = 2(\Delta u)^2 + \frac{1}{2} m \cdot \nabla((\Delta u)^2).$$

Then

$$\begin{aligned} \int_{\Omega} (2m \cdot \nabla u) \Delta^2 u dx &= 4 \int_{\Omega} (\Delta u)^2 dx + \int_{\Omega} m \cdot \nabla((\Delta u)^2) dx - 2 \int_{\Gamma} \frac{\partial}{\partial \nu} (m \cdot \nabla u) \Delta u d\Gamma \\ &\quad + 2 \int_{\Gamma} \frac{\partial \Delta u}{\partial \nu} (m \cdot \nabla u) d\Gamma \\ &= 4 \int_{\Omega} (\Delta u)^2 dx - n \int_{\Omega} (\Delta u)^2 dx + \int_{\Gamma} (m \cdot \nu) (\Delta u)^2 d\Gamma \\ &\quad - 2 \int_{\Gamma} \frac{\partial}{\partial \nu} (m \cdot \nabla u) \Delta u d\Gamma + 2 \int_{\Gamma} \frac{\partial \Delta u}{\partial \nu} (m \cdot \nabla u) d\Gamma, \end{aligned}$$

by Green's formula. For the last term of (48), we use again two times Green's formula,

$$\int_{\Omega} u \Delta^2 u dx = \int_{\Omega} (\Delta u)^2 dx - \int_{\Gamma} \frac{\partial u}{\partial \nu} \Delta u d\Gamma + \int_{\Gamma} \frac{\partial \Delta u}{\partial \nu} u d\Gamma.$$

Consequently, (48) becomes

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (2m \cdot \nabla u + (n-1)u) u_t dx &= - \int_{\Omega} (u_t^2 + 3(\Delta u)^2) dx + \int_{\Gamma} (m \cdot \nu) (u_t^2 - (\Delta u)^2) d\Gamma \\ &+ \int_{\Gamma} \left( 2 \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma + (n-1) \frac{\partial u}{\partial \nu} \right) \Delta u d\Gamma \\ &- \int_{\Gamma} \frac{\partial \Delta u}{\partial \nu} (2(m \cdot \nabla u) + (n-1)u) d\Gamma. \end{aligned}$$

As  $u = \partial u / \partial \nu = 0$  on  $\Gamma_D$ ,  $\nabla u = 0$  on  $\Gamma_D$  and

$$\frac{\partial}{\partial \nu} (m \cdot \nabla u) = m \cdot \nu \frac{\partial^2 u}{\partial \nu^2} = (m \cdot \nu) \Delta u \quad \text{on } \Gamma_D.$$

Therefore the boundary conditions of (47) implies

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (2m \cdot \nabla u + (n-1)u) u_t dx &= - \int_{\Omega} (u_t^2 + 3(\Delta u)^2) dx - \int_{\Gamma_D} (m \cdot \nu) (\Delta u)^2 d\Gamma \\ &+ \int_{\Gamma_N} (m \cdot \nu) u_t^2 d\Gamma + 2 \int_{\Gamma_D} (m \cdot \nu) (\Delta u)^2 d\Gamma \\ &- \int_{\Gamma_N} \frac{\partial \Delta u}{\partial \nu} (2(m \cdot \nabla u) + (n-1)u) d\Gamma. \end{aligned}$$

By (38), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (2m \cdot \nabla u + (n-1)u) u_t dx &\leq - \int_{\Omega} (u_t^2 + 3(\Delta u)^2) dx + \int_{\Gamma_N} (m \cdot \nu) u_t^2 d\Gamma \\ &- \int_{\Gamma_N} \frac{\partial \Delta u}{\partial \nu} (2(m \cdot \nabla u) + (n-1)u) d\Gamma. \end{aligned}$$

From Young's inequality, we deduce that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (2m \cdot \nabla u + (n-1)u) u_t dx &\leq - \int_{\Omega} (u_t^2 + 3(\Delta u)^2) dx + c \int_{\Gamma_N} u_t^2 d\Gamma \\ &+ \frac{C}{\epsilon} \int_{\Gamma_N} \left( \frac{\partial \Delta u}{\partial \nu} \right)^2 d\Gamma + \epsilon \int_{\Gamma_N} ((\nabla u)^2 + u^2) d\Gamma, \end{aligned}$$

with  $C, c > 0$ . We conclude the proof of this lemma by using a trace inequality, Poincaré's inequality and the boundary condition of (47). ■

In conclusion, our abstract framework applies again and system (47) is exponentially stable under the previous hypotheses.

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