Exponential stability of the wave equation with boundary time-varying delay

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Abstract

We consider the wave equation with a time - varying delay term in the boundary condition in a bounded and smooth domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions, we prove exponential stability of the solution. These results are obtained by introducing suitable energies and suitable Lyapounov functionals. Such analysis is also extended to a nonlinear version of the model.

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1 Introduction

We are interested in the effect of a time–varying delay in boundary stabilization of the wave equation in domains of $\mathbb{R}^n$. Delay effects arise in many pratical problems and it is well known that they can induce some unstabilities, see [5, 6, 7, 25, 30].

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_D and Γ_N, i.e. $\Gamma = \Gamma_D \cup \Gamma_N$, with $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$.

In this domain Ω, we consider the initial boundary value problem

$$u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_D \times (0,+\infty) \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau(t)) \quad \text{on} \quad \Gamma_N \times (0,+\infty) \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{1.4}$$

$$u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) \quad \text{in } \Gamma_N \times (0,\tau(0)), \tag{1.5}$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\dfrac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau(t) > 0$ is the time-varying delay, μ_1 and μ_2 are positive real numbers and the initial datum (u_0, u_1, f_0) belongs to a suitable space.

On the function τ we assume that there exists a positive constant $\overline{\tau}$ such that

$$0 \leq \tau(t) \leq \overline{\tau}, \quad \forall\, t > 0. \tag{1.6}$$

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Moreover, we assume

$$\tau'(t) < 1 \quad \forall\, t > 0, \tag{1.7}$$

and

$$\tau \in W^{2,\infty}([0,T]), \quad \forall\, T > 0. \tag{1.8}$$

We are interested in giving an exponential stability result for such a problem.
Let us denote by $\langle v, w\rangle$ or, equivalently, by $v{\cdot}w$ the euclidean inner product between two vectors $v, w \in \mathbb{R}^n$.

We assume that there exists $x_0 \in \mathbb{R}^n$ such that denoting by m the standard multiplier

$$m(x) := x - x_0,$$

we have

$$m(x)\cdot\nu(x) \le 0 \quad \text{on} \quad \Gamma_D \tag{1.9}$$

and, for some positive constant δ,

$$m(x)\cdot\nu(x) \ge \delta \quad \text{on} \quad \Gamma_N. \tag{1.10}$$

It is well–known that if $\mu_2 = 0$, that is in absence of delay, the energy of problem $(1.1) - (1.5)$ is exponentially decaying to zero. See for instance Chen [3], Lagnese [16, 17], Lasiecka and Triggiani [18], Komornik and Zuazua [15], Komornik [13, 14]. On the contrary, if $\mu_1 = 0$, that is if we have only the delay part in the boundary condition on Γ_N, system $(1.1) - (1.5)$ becomes unstable. See, for instance Datko, Lagnese and Polis [7].

The above problem, with both $\mu_1, \mu_2 > 0$ and a constant delay τ, has been studied in one space dimension by Xu, Yung and Li [30] and on networks by Nicaise and Valein [26] and in higher space dimension by Nicaise and Pignotti [25]. Assuming that

$$\mu_2 < \mu_1 \tag{1.11}$$

in [25], a stabilization result in general space dimension is given, by using a suitable observability estimate. This is done by applying inequalities obtained from Carleman estimates for the wave equation by Lasiecka, Triggiani and Yao in [19] and by using compactness-uniqueness arguments.

The case of time–varying delay has been studied by Nicaise, Valein and Fridman [27] in one space dimension. In [27] an exponential stability result is given, under the condition

$$\mu_2 < \sqrt{1-d}\,\mu_1 \tag{1.12}$$

where d is a constant such that

$$\tau'(t) \le d < 1, \quad \forall\, t > 0. \tag{1.13}$$

Here, we extend this result to general space dimension. Moreover, we remove the hypothesis

$$\tau(t) \ge \tau_0 > 0, \quad \forall\, t > 0, \tag{1.14}$$

assumed in [27], that is the delay may degenerate.

We will study also a nonlinear version of the above model. Consider the system

$$u_{tt}(x,t) - \Delta u(x,t) = 0 \ \text{ in } \Omega\times(0,+\infty) \tag{1.15}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_D\times(0,+\infty) \tag{1.16}$$

$$\frac{\partial u}{\partial\nu}(x,t) = -\beta_1(u_t(x,t)) - \beta_2(u_t(x,t-\tau(t))) \quad \text{on} \quad \Gamma_N\times(0,+\infty) \tag{1.17}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{1.18}$$

$$u_t(x,t-\tau(0)) = g_0(x,t-\tau(0)) \quad \text{in } \Gamma_N\times(0,\tau(0)), \tag{1.19}$$

where $\beta_j : \mathbb{R}\to\mathbb{R}, j = 1,2,$ satisfy suitable growth assumptions. In particular we assume

$$|\beta_j(s)| \le c_j|s|, \quad \forall s\in\mathbb{R}, \quad j = 1,2, \tag{1.20}$$

for some positive constants c_1, c_2 and

$$\beta_2(s) \cdot s \geq 0, \quad \forall s \in \mathbb{R}. \tag{1.21}$$

Moreover we assume

$$\exists\, \gamma_1 > 0,\, \forall x,\, y \in \mathbb{R},\, (\beta_1(x) - \beta_1(y))(x - y) \geq \gamma_1 (x - y)^2, \tag{1.22}$$

and

$$\exists\, \gamma_2 > 0,\, \forall x,\, y \in \mathbb{R},\, |\beta_2(x) - \beta_2(y)| \leq \gamma_2 |x - y|. \tag{1.23}$$

Note that (1.20) and (1.23) imply $c_2 \leq \gamma_2$ and from (1.20) and (1.22) we deduce

$$\beta_1(s) \cdot s \geq \gamma_1 s^2, \quad \forall s \in \mathbb{R}. \tag{1.24}$$

Under a suitable relation between the above coefficients we can give a well–posedness result and an exponential stability estimate for problem (1.15) – (1.19). To prove the well–posedness of the nonlinear model we need to assume (1.14). In our opinion, this is only a technical assumption but at the moment we are not able to remove it.

The paper is organized as follows. Well–posedness of the problems is analysed in section 2 using semigroup theory. In subsection 2.1 we study the well-posedness of problem (1.1) – (1.5), while in subsection 2.2 we concentrate on problem (1.15) – (1.19). In section 3 and section 4 we prove the exponential stability of the linear and nonlinear problems respectively.

2 Well-posedness of the problems

Using semigroup theory we can give the well–posedness of problem (1.1) – (1.5) and problem (1.15) – (1.19).

2.1 Linear problem

Let us set

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Gamma_N,\ \rho \in (0, 1),\ t > 0. \tag{2.1}$$

Then, problem (1.1) – (1.5) is equivalent to

$$\begin{aligned}
& u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) && (2.2)\\
& \tau(t) z_t(x, \rho, t) + (1 - \tau'(t)\rho) z_\rho(x, \rho, t) = 0 \quad \text{in} \quad \Gamma_N \times (0,1) \times (0, +\infty) && (2.3)\\
& u(x,t) = 0 \quad \text{on} \quad \Gamma_D \times (0, +\infty) && (2.4)\\
& \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 z(x, 1, t) \quad \text{on} \quad \Gamma_N \times (0, +\infty) && (2.5)\\
& z(x, 0, t) = u_t(x,t) \quad \text{on} \quad \Gamma_N \times (0, \infty) && (2.6)\\
& u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega && (2.7)\\
& z(x, \rho, 0) = f_0(x, -\rho\tau(0)) \quad \text{in} \quad \Gamma_N \times (0, 1). && (2.8)
\end{aligned}$$

To prove the well-posedness of (2.2) – (2.8) we have to distinguish two cases. First, we assume also (1.14), i.e. we assume

$$0 < \tau_0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t > 0. \tag{2.9}$$

In the second case we assume only (1.6).

2.1.1 First case

Assume for the moment that (2.9) holds.

If we denote by

$$U := (u, u_t, z)^T,$$

then

$$U' = (u_t, u_{tt}, z_t)^T = \left(u_t, \Delta u, \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho\right)^T.$$

Therefore, problem (2.2) – (2.8) can be rewritten as

$$\left\{ \begin{array}{l} U' = \mathcal{A}(t)U \\ U(0) = (u_0, u_1, f_0(\cdot, -\cdot\, \tau(0)))^T \end{array} \right. \tag{2.10}$$

where the operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}(t)) := \Big\{ & (u, v, z)^T \in \big(E(\Delta, L^2(\Omega)) \cap V\big) \times V \times L^2(\Gamma_N; H^1(0,1)) \ : \\ & \frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1) \text{ on } \Gamma_N; \ v = z(\cdot, 0) \text{ on } \Gamma_N \Big\}, \end{aligned} \tag{2.11}$$

where, as usual,

$$V = H^1_{\Gamma_D}(\Omega) = \{ \ u \in H^1(\Omega) \ : \ u = 0 \quad \text{on} \ \ \Gamma_D \ \},$$

and

$$E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}.$$

Notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \quad \forall t > 0. \tag{2.12}$$

Recall that for a function $u \in E(\Delta, L^2(\Omega))$, then $\partial u / \partial \nu$ belongs to $H^{-1/2}(\Gamma_N)$ and the next Green formula is valid (see section 1.5 of [9])

$$\int_\Omega \nabla u \nabla w dx = -\int_\Omega \Delta u w dx + \langle \frac{\partial u}{\partial \nu}; w \rangle_{\Gamma_N}, \forall w \in H^1_{\Gamma_D}(\Omega), \tag{2.13}$$

where $\langle \cdot; \cdot \rangle_{\Gamma_N}$ means the duality pairing between $H^{-1/2}(\Gamma_N)$ and $H^{1/2}(\Gamma_N)$.

Note further that for $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$, $\dfrac{\partial u}{\partial \nu}$ belongs to $L^2(\Gamma_N)$, since $z(\cdot, 1)$ is in $L^2(\Gamma_N)$. Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := V \times L^2(\Omega) \times L^2(\Gamma_N \times (0,1)) \tag{2.14}$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx + \int_{\Gamma_N} \int_0^1 z(x, \rho)\tilde{z}(x, \rho) d\rho d\Gamma. \tag{2.15}$$

A general theory for equations of type (2.10) has been developed using semigroup theory [11, 12, 28]. The simplest way to prove existence and uniqueness results is to show that the triplet $\{\mathcal{A}, \mathcal{H}, Y\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ for some fixed $T > 0$ and $Y = \mathcal{D}(\mathcal{A}(0))$, forms a CD-system (or constant domain system, see [11, 12]). More precisely, the following theorem give some existence and uniqueness results and is proved in Theorem 1.9 of [11] (see also Theorem 2.13 of [12] or [1])

Theorem 2.1 *Assume that*

(i) $Y = \mathcal{D}(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$,

(ii) (2.12) *holds,*

(iii) for all $t \in [0,\, T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,\, T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e. the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, *for all* $u \in \mathcal{H}$ *and* $s \geq 0$*),*

(iv) $\partial_t \mathcal{A}$ *belongs to* $L^\infty_*([0,\, T],\, B(Y,\, \mathcal{H}))$, *the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,\, T]$ *into the set* $B(Y,\, \mathcal{H})$ *of bounded operators from* Y *into* $\mathcal{H}$.

Then, problem (2.10) *has a unique solution* $U \in C([0,\, T], Y) \cap C^1([0,\, T], \mathcal{H})$ *for any initial datum in* Y.

Our goal is then to check the above assumptions for problem (2.10).

Lemma 2.2 $D(\mathcal{A}(0))$ *is dense in* $\mathcal{H}$.

Proof. The proof is the same as the one of Lemma 2.1 of [27], we give it for the sake of completeness. Let $(f,\, g,\, h)^\top \in \mathcal{H}$ be orthogonal to all elements of $D(\mathcal{A}(0))$, namely

$$0 = \left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega \{\nabla u(x)\nabla f(x) + v(x)g(x)\}dx + \int_{\Gamma_N}\int_0^1 z(x,\rho)h(x,\rho)d\rho d\Gamma,$$

for all $(u,\, v,\, z)^\top \in D(\mathcal{A}(0))$.

We first take $u = 0$ and $v = 0$ and $z \in \mathcal{D}(\Gamma_N \times (0,\, 1))$. As $(0,\, 0,\, z)^\top \in D(\mathcal{A}(0))$, we get

$$\int_{\Gamma_N}\int_0^1 z(x,\rho)h(x,\rho)d\rho d\Gamma = 0.$$

Since $\mathcal{D}(\Gamma_N \times (0,\, 1))$ is dense in $L^2(\Gamma_N \times (0,\, 1))$, we deduce that $h = 0$.

In the same manner, by taking $u = 0$, $z = 0$ and $v \in \mathcal{D}(\Omega)$ we see that $g = 0$.

The above orthogonality condition is then reduced to

$$0 = \int_\Omega \nabla u \nabla f dx,\ \forall (u,\, v,\, z)^\top \in D(\mathcal{A}(0)).$$

By restricting ourselves to $v = 0$ and $z = 0$, we obtain

$$\int_\Omega \nabla u(x)\nabla f(x)dx = 0,\ \forall (u,\, 0,\, 0)^\top \in D(\mathcal{A}(0)).$$

But we easily check that $(u,\, 0,\, 0)^\top \in D(\mathcal{A}(0))$ if and only if $u \in D(\Delta) = \{v \in E(\Delta, L^2(\Omega)) \cap V : \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N\}$, the domain of the Laplace operator with mixed boundary conditions. Since it is well known that $D(\Delta)$ is dense in V (equipped with the inner product $< .,\, . >_V$), we conclude that $f = 0$. ■

Assuming

$$\mu_2 \leq \sqrt{1-d}\,\mu_1, \tag{2.16}$$

we will show that $\mathcal{A}(t)$ generates a C_0 semigroup on $\mathcal{H}$ and using the variable norm technique of Kato from [11] and Theorem 2.1, that problem (2.10) has a unique solution.

Let ξ be a positive real number such that

$$\frac{\mu_2}{\sqrt{1-d}} \leq \xi \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{2.17}$$

Note that, from (2.16), such a constant ξ exists.

Let us define on the Hilbert space $\mathcal{H}$ the following time-dependent inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t := \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx + \xi\tau(t) \int_{\Gamma_N} \int_0^1 z(x,\rho)\tilde{z}(x,\rho) d\rho d\Gamma. \tag{2.18}$$

Using this time-dependent inner product and Theorem 2.1 we obtain the following existence and uniqueness result:

Theorem 2.3 *For any initial datum* $U_0 \in \mathcal{D}(\mathcal{A}(0))$ *there exists a unique solution*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{H})$$

of system (2.10).

Proof. We first notice that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \le e^{\frac{c}{2\tau_0}|t-s|}, \ \forall t,\ s \in [0,\ T], \tag{2.19}$$

where $\phi = (u,\ v,\ z)^\top$ and c is a positive constant. Indeed, for all $s,\ t \in [0,\ T]$, we have

$$\begin{aligned} \|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} &= \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right) \int_\Omega (|\nabla u(x)|^2 + v^2) dx \\ &+ \xi \left(\tau(t) - \tau(s) e^{\frac{c}{\tau_0}|t-s|}\right) \int_{\Gamma_N} \int_0^1 z(x,\rho)^2 d\rho d\Gamma. \end{aligned}$$

We notice that $1 - e^{\frac{c}{\tau_0}|t-s|} \le 0$. Moreover $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \le 0$ for some $c > 0$. Indeed,

$$\tau(t) = \tau(s) + \tau'(a)(t-s), \quad \text{where } a \in (s,\ t),$$

and thus,

$$\frac{\tau(t)}{\tau(s)} \le 1 + \frac{|\tau'(a)|}{\tau(s)} |t-s|.$$

By (1.8), τ' is bounded and therefore,

$$\frac{\tau(t)}{\tau(s)} \le 1 + \frac{c}{\tau_0} |t-s| \le e^{\frac{c}{\tau_0}|t-s|},$$

by (2.9), which proves (2.19).

Now we calculate $\langle \mathcal{A}(t)U, U \rangle_t$ for a fixed t. Take $U = (u,v,z)^T \in \mathcal{D}(\mathcal{A}(t))$. Then,

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= \left\langle \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_t \\ &= \int_\Omega \{\nabla v(x) \nabla u(x) + v(x)\Delta u(x)\} dx - \xi \int_{\Gamma_N} \int_0^1 (1 - \tau'(t)\rho) z_\rho(x,\rho) z(x,\rho) d\rho d\Gamma. \end{aligned}$$

So, by Green's formula,

$$\langle \mathcal{A}(t)U, U \rangle_t = \int_{\Gamma_N} \frac{\partial u}{\partial \nu}(x) v(x) d\Gamma - \xi \int_{\Gamma_N} \int_0^1 (1 - \tau'(t)\rho) z_\rho(x,\rho) z(x,\rho) d\rho d\Gamma. \tag{2.20}$$

Integrating by parts in ρ, we get

$$\begin{aligned} \int_{\Gamma_N} \int_0^1 z_\rho(x,\rho) z(x,\rho)(1 - \tau'(t)\rho) d\rho d\Gamma &= \int_{\Gamma_N} \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} z^2(x,\rho)(1 - \tau'(t)\rho) d\rho d\Gamma \\ &= \frac{\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 z^2(x,\rho) d\rho d\Gamma + \frac{1}{2} \int_{\Gamma_N} \{z^2(x,1)(1 - \tau'(t)) - z^2(x,0)\} d\Gamma. \end{aligned} \tag{2.21}$$

Therefore, from (2.20) and (2.21),

$$\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t &= \int_{\Gamma_N} \frac{\partial u}{\partial \nu}(x)v(x)d\Gamma - \frac{\xi}{2}\int_{\Gamma_N} \{z^2(x,1)(1-\tau'(t)) - z^2(x,0)\}d\Gamma \\
&\quad -\frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma \\
&= -\int_{\Gamma_N} (\mu_1 v(x) + \mu_2 z(x,1))v(x)d\Gamma - \frac{\xi}{2}\int_{\Gamma_N} \{z^2(x,1)(1-\tau'(t)) - z^2(x,0)\}d\Gamma \\
&\quad -\frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma \\
&= -\mu_1\int_{\Gamma_N} v^2(x)d\Gamma - \mu_2\int_{\Gamma_N} z(x,1)v(x)d\Gamma - \frac{\xi}{2}\int_{\Gamma_N} z^2(x,1)(1-\tau'(t))d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} v^2(x)d\Gamma \\
&\quad -\frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma,
\end{aligned}$$

from which follows, using Cauchy-Schwarz's inequality and (1.13),

$$\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t \le \left(-\mu_1 + \frac{\mu_2}{2\sqrt{1-d}} + \frac{\xi}{2}\right)\int_{\Gamma_N} v^2(x)d\Gamma \\
+\left(\frac{\mu_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2}\right)\int_{\Gamma_N} z^2(x,1)d\Gamma + \kappa(t)\langle U, U\rangle_t,
\end{aligned} \tag{2.22}$$

where

$$\kappa(t) = \frac{(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)}. \tag{2.23}$$

Now, observe that from (2.17),

$$-\mu_1 + \frac{\mu_2}{2\sqrt{1-d}} + \frac{\xi}{2} \le 0, \quad \frac{\mu_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2} \le 0.$$

Then,

$$\langle \mathcal{A}(t)U,\, U\rangle_t - \kappa(t)\langle U,\, U\rangle_t \le 0, \tag{2.24}$$

which means that the operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Moreover $\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2+1)^{\frac{1}{2}}} - \frac{\tau'(t)(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)^2}$ is bounded on $[0,\, T]$ for all $T > 0$ (by (1.8) and (2.9)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with $\frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ bounded on $[0,\, T]$ by (1.8) and (2.9). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0,\, T],\, B(D(\mathcal{A}(0)),\, \mathcal{H})), \tag{2.25}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,\, T]$ into $B(D(\mathcal{A}(0)),\, \mathcal{H})$.

Now, we will show that $\lambda I - \mathcal{A}(t)$ is surjective for fixed $t > 0$ and $\lambda > 0$. Given $(f, g, h)^T \in \mathcal{H}$, we seek $U = (u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t))\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

that is verifying

$$\left\{\begin{array}{l} \lambda u - v = f \\ \lambda v - \Delta u = g \\ \lambda z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = h. \end{array}\right. \tag{2.26}$$

Suppose that we have found u with the appropriated regularity. Then,

$$v := \lambda u - f \in V \tag{2.27}$$

and we can determine z. Indeed, by (2.11),

$$z(x,0) = v(x), \quad \text{for} \ \ x \in \Gamma_N, \tag{2.28}$$

and, from (2.26),

$$\lambda z(x,\rho) + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho(x,\rho) = h(x,\rho), \quad \text{for} \quad x \in \Gamma_N, \ \rho \in (0,1). \tag{2.29}$$

Then, by (2.28) and (2.29), we obtain

$$z(x,\rho) = v(x) e^{-\lambda\rho\tau(t)} + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^\rho h(x,\sigma) e^{\lambda\sigma\tau(t)} d\sigma,$$

if $\tau'(t) = 0$, and

$$z(x,\rho) = v(x) e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} + e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \int_0^\rho \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma,$$

otherwise. So, from (2.27),

$$\begin{aligned} z(x,\rho) = {} & \lambda u(x) e^{-\lambda\rho\tau(t)} - f(x) e^{-\lambda\rho\tau(t)} \\ & + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^\rho h(x,\sigma) e^{\lambda\sigma\tau(t)} d\sigma, \quad \text{on } \Gamma_N \times (0,1), \end{aligned} \tag{2.30}$$

if $\tau'(t) = 0$, and

$$\begin{aligned} z(x,\rho) = {} & \lambda u(x) e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} - f(x) e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \\ & + e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \int_0^\rho \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma, \text{ on } \Gamma_N \times (0,1) \end{aligned} \tag{2.31}$$

otherwise.

In particular, if $\tau'(t) = 0$

$$z(x,1) = \lambda u(x) e^{-\lambda\tau(t)} + z_0(x), \quad x \in \Gamma_N, \tag{2.32}$$

with $z_0 \in L^2(\Gamma_N)$ defined by

$$z_0(x) = -f(x) e^{-\lambda\tau(t)} + \tau(t) e^{-\lambda\tau(t)} \int_0^1 h(x,\sigma) e^{\lambda\sigma\tau(t)} d\sigma, \quad x \in \Gamma_N, \tag{2.33}$$

and, if $\tau'(t) \neq 0$

$$z(x,1) = \lambda u(x) e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0(x), \quad x \in \Gamma_N, \tag{2.34}$$

with $z_0 \in L^2(\Gamma_N)$ defined by

$$\begin{aligned} z_0(x) = {} & -f(x) e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} \\ & + e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} \int_0^1 \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma, \quad x \in \Gamma_N. \end{aligned} \tag{2.35}$$

It remains to find u. By (2.27) and (2.26), the function u satisfies

$$\lambda(\lambda u - f) - \Delta u = g,$$

that is

$$\lambda^2 u - \Delta u = g + \lambda f. \tag{2.36}$$

Problem (2.36) can be reformulated as

$$\int_\Omega (\lambda^2 u - \Delta u) w dx = \int_\Omega (g + \lambda f) w dx, \quad \forall w \in H^1_{\Gamma_D}(\Omega). \tag{2.37}$$

Integrating by parts,

$$\begin{aligned}\int_\Omega (\lambda^2 u - \Delta u) w dx &= \int_\Omega (\lambda^2 uw + \nabla u \nabla w) dx - \int_{\Gamma_N} \frac{\partial u}{\partial \nu} w d\Gamma \\ &= \int_\Omega (\lambda^2 uw + \nabla u \nabla w) dx + \int_{\Gamma_N} (\mu_1 v + \mu_2 z(x,1)) w d\Gamma.\end{aligned}$$

If $\tau'(t) = 0$, by (2.27) and (2.32), we have

$$\int_\Omega (\lambda^2 u - \Delta u) w dx = \int_\Omega (\lambda^2 uw + \nabla u \nabla w) dx + \int_{\Gamma_N} \{\mu_1(\lambda u - f) w + \mu_2(\lambda u e^{-\lambda\tau(t)} + z_0) w\} d\Gamma,$$

and if $\tau'(t) \neq 0$, by (2.27) and (2.34),

$$\int_\Omega (\lambda^2 u - \Delta u) w dx = \int_\Omega (\lambda^2 uw + \nabla u \nabla w) dx + \int_{\Gamma_N} \{\mu_1(\lambda u - f) w + \mu_2(\lambda u e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0) w\} d\Gamma.$$

Therefore, (2.37) can be rewritten as

$$\begin{aligned}&\int_\Omega (\lambda^2 uw + \nabla u \nabla w) dx + \int_{\Gamma_N} (\mu_1 + \mu_2 e^{-\lambda\tau(t)}) \lambda uw d\Gamma \\ &\quad = \int_\Omega (g + \lambda f) w dx + \mu_1 \int_{\Gamma_N} f w d\Gamma - \mu_2 \int_{\Gamma_N} z_0 w d\Gamma, \quad \forall w \in H^1_{\Gamma_D}(\Omega),\end{aligned} \tag{2.38}$$

if $\tau'(t) = 0$, and

$$\begin{aligned}&\int_\Omega (\lambda^2 uw + \nabla u \nabla w) dx + \int_{\Gamma_N} (\mu_1 + \mu_2 e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))}) \lambda uw d\Gamma \\ &\quad = \int_\Omega (g + \lambda f) w dx + \mu_1 \int_{\Gamma_N} f w d\Gamma - \mu_2 \int_{\Gamma_N} z_0 w d\Gamma, \quad \forall w \in H^1_{\Gamma_D}(\Omega),\end{aligned} \tag{2.39}$$

otherwise. As the left-hand side of (2.38) or (2.39) is coercive on $H^1_{\Gamma_D}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $u \in H^1_{\Gamma_D}(\Omega)$ of (2.38) or (2.39).

If we consider $w \in \mathcal{D}(\Omega)$ in (2.38) or (2.39), u solves in $\mathcal{D}'(\Omega)$

$$\lambda^2 u - \Delta u = g + \lambda f, \tag{2.40}$$

and thus $u \in E(\Delta, L^2(\Omega))$.

Using Green's formula (2.13) in (2.38) and using (2.40), we obtain, if $\tau'(t) = 0$

$$\int_{\Gamma_N} (\mu_1 + \mu_2 e^{-\lambda\tau(t)}) \lambda uw d\Gamma + \langle \frac{\partial u}{\partial \nu}; w \rangle_{\Gamma_N} = \mu_1 \int_{\Gamma_N} f w d\Gamma - \mu_2 \int_{\Gamma_N} z_0 w d\Gamma,$$

from which follows

$$\frac{\partial u}{\partial \nu} + (\mu_1 + \mu_2 e^{-\lambda\tau(t)}) \lambda u = \mu_1 f - \mu_2 z_0 \quad \text{on } \Gamma_N. \tag{2.41}$$

Therefore, from (2.41),

$$\frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1) \quad \text{on} \;\; \Gamma_N,$$

where we have used (2.27) and (2.32).

We find the same result if $\tau'(t) \neq 0$.

So, we have found $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$ which verifies (2.26), and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective} \tag{2.42}$$

for any $\lambda > 0$ and $t > 0$.

Then, (2.19), (2.24) and (2.42) imply that the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t, by Proposition 1.1 from [11]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are verified by (2.12), (2.19), (2.24), (2.25), (2.42) and Lemma 2.2, and thus, the problem

$$\left\{ \begin{array}{l} \tilde{U}' = \tilde{\mathcal{A}}(t)\tilde{U} \\ \tilde{U}(0) = U_0 \end{array} \right.$$

has a unique solution $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$ for $U_0 \in D(\mathcal{A}(0))$. The requested solution of (2.10) is then given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t) = \int_0^t \kappa(s)ds$, because

$$\begin{array}{rcl} U'(t) & = & \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}'(t) \\ & = & \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ & = & e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ & = & e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\beta(t)}\tilde{U}(t) \\ & = & \mathcal{A}(t)U(t), \end{array}$$

which concludes the proof. ■

2.1.2 The general case

In this subsection (1.6) only holds, so τ may be also degenerate, i.e. $\tau(t) = 0$ for some times t. Taking

$$\tau_\epsilon(t) = \tau(t) + \epsilon, \quad \forall\; 0 < \epsilon < \epsilon_0$$

then

$$0 < \epsilon \leq \tau_\epsilon(t) \leq \bar{\tau} + \epsilon, \tag{2.43}$$

i.e. τ_ϵ satisfies (2.9). Therefore, by Theorem 2.3, there exists a unique solution

$$U_\epsilon = (u^\epsilon, v^\epsilon, z^\epsilon)^T \in C([0, +\infty), \mathcal{D}(\mathcal{A}_\epsilon(t))) \cap C^1([0, \infty), \mathcal{H})$$

for $U_{\epsilon,0} \in \mathcal{D}(\mathcal{A}_\epsilon(0))$, of problem

$$\left\{ \begin{array}{l} U_\epsilon' = \mathcal{A}_\epsilon(t)U_\epsilon \\ U_\epsilon(0) = (u_0, u_1, f_0(\cdot, -\cdot\, \tau_\epsilon(0)))^T = U_{\epsilon,0}, \end{array} \right. \tag{2.44}$$

where the operator $\mathcal{A}_\epsilon(t)$ is defined by

$$\mathcal{A}_\epsilon(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau_\epsilon'(t)\rho - 1}{\tau_\epsilon(t)} z_\rho \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t) + \epsilon} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}_\epsilon(t)) = \mathcal{D}(\mathcal{A}(t)).$$

The aim is then to take the limit of $(u_\epsilon)_{0<\epsilon<\epsilon_0}$ when ϵ tends to 0.

To pass at the limit, we need to have more regularity on the solution and, for that purpose, we use Theorem 2.13 of [11] (see also Theorem 3.2.3 of [1]).

We now fix $0 < \epsilon < \epsilon_0$. We consider the family of Hilbert spaces

$$X = X_0 = \mathcal{H}, \quad X_1 = \left(V \cap H^{3/2}(\Omega)\right) \times V \times L^2(\Gamma_N; H^1(0,1)),$$

$$X_2 = \left(V \cap H^{3/2}(\Omega)\right) \times \left(V \cap H^{3/2}(\Omega)\right) \times L^2(\Gamma_N; H^2(0,1)),$$

with the usual norms

$$\|\cdot\|_0 = \|\cdot\|_{\mathcal{H}}, \quad \|\cdot\|_1 = \|\cdot\|_{X_1}, \quad \|\cdot\|_2 = \|\cdot\|_{X_2}.$$

We can easily check that

$$X_2 \hookrightarrow X_1 \hookrightarrow X_0 = X$$

and

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2.$$

Let $Y = \mathcal{D}(\mathcal{A}_\epsilon(t))$. Y is a dense subset of $X = X_0 = \mathcal{H}$ and a subset of X_1. Indeed, by a result of Lions and Magenes [22], if $u \in H^1_{\Gamma_D}(\Omega)$, $\Delta u \in L^2(\Omega)$ and $\partial u/\partial \nu \in L^2(\Gamma_N)$, then $u \in H^{3/2}(\Omega)$. Consequently

$$\mathcal{D}(\mathcal{A}_\epsilon(t)) \cap X_1 = \mathcal{D}(\mathcal{A}_\epsilon(t)) = Y, \quad \forall t \in [0,T].$$

The family of operators $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(t) : t \in [0,\, T]\}$ is a stable family of generators in $X = \mathcal{H}$ with stability constants independent of t (see the previous subsection).

We have that

$$\frac{d}{dt}\mathcal{A}_\epsilon(t) \in L^\infty_*([0,\, T],\, B(D(\mathcal{A}_\epsilon(0)),\, X)) \cap L^\infty_*([0,\, T],\, B(D(\mathcal{A}_\epsilon(0)) \cap X_2,\, X_1)),$$

$$\frac{d^2}{dt^2}\mathcal{A}_\epsilon(t) \in L^\infty_*([0,\, T],\, B(D(\mathcal{A}_\epsilon(0)) \cap X_2,\, X)),$$

because

$$\frac{d}{dt}\mathcal{A}_\epsilon(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\tau''(t)(\tau(t)+\epsilon)\rho - \tau'(t)(\tau'(t)\rho-1)}{(\tau(t)+\epsilon)^2} z_\rho \end{pmatrix},$$

and

$$\frac{d^2}{dt^2}\mathcal{A}_\epsilon(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{[\tau'''(\tau+\epsilon)\rho + \tau''\tau'\rho - \tau''(\tau'\rho-1) - \tau'(\tau''\rho)](\tau+\epsilon)^2 + 2\tau'(\tau+\epsilon)[\tau''(\tau+\epsilon)\rho - \tau'(\tau'\rho-1)]}{(\tau+\epsilon)^4} z_\rho \end{pmatrix},$$

and by (1.6) and (1.8).

Finally, again with a result of [22] and as $\frac{\tau'(t)\rho-1}{\tau(t)+\epsilon}$ is bounded on $[0,T]$ by (1.6) and (1.8), if $\phi \in \mathcal{D}(\mathcal{A}_\epsilon(t))$ and $\mathcal{A}_\epsilon(t)\phi \in X$, then $\phi \in X_1$ with

$$\|\phi\|_1 \leq \nu(\|\mathcal{A}_\epsilon(t)\phi\|_0 + \|\phi\|_0),$$

and, if $\phi \in \mathcal{D}(\mathcal{A}_\epsilon(t))$ and $\mathcal{A}_\epsilon(t)\phi \in X_1$, then $\phi \in X_2$ with

$$\|\phi\|_2 \leq \nu(\|\mathcal{A}_\epsilon(t)\phi\|_1 + \|\phi\|_0).$$

Introduce now the space $D^2(0)$ defined by

$$D^2(0) = \{\phi \in \mathcal{D}(\mathcal{A}(0)) \cap X_2 \,:\, -\mathcal{A}(0)\phi \in \mathcal{D}(\mathcal{A}(0))\}.$$

Therefore, by the result of [11] (see also [1]), for all initial data $U_{\epsilon,0} \in D^2(0)$, there exists a unique solution $U_\epsilon \in C^1([0,T],\mathcal{H}) \cap C([0,T],\mathcal{D}(\mathcal{A}_\epsilon(0)))$ of (2.44) which satisfies, moreover,

$$\frac{d^2}{dt^2}U_\epsilon \in C([0,T],\mathcal{H}).$$

We then have more regularity of the solution with more regular initial data. Therefore, we can give a sense to the derivative of the stronger energy $\tilde{E}_\epsilon$ defined as follows:

$$\tilde{E}_\epsilon(t) = \frac{1}{2}\int_\Omega \Big((\Delta u_\epsilon)^2 + (\nabla u_{\epsilon,t})^2 \Big)\, dx + \frac{q\tau_\epsilon(t)}{2}\int_{\Gamma_N}\int_0^1 u^2_{\epsilon,tt}(x,t-\tau_\epsilon(t)\rho)d\rho d\Gamma, \tag{2.45}$$

for $(u_0,u_1,f_0(\cdot,-\cdot\,\tau_\epsilon(0)))^T \in D^2(0)$, where q is a suitable positive constant. Then the derivative of $\tilde{E}_\epsilon$ gives

$$\begin{aligned}\tilde{E}'_\epsilon(t) &= \int_\Omega (u_{\epsilon,ttt}u_{\epsilon,tt} + \nabla u_{\epsilon,t}\nabla u_{\epsilon,tt})\, dx + \frac{q\tau'_\epsilon(t)}{2}\int_{\Gamma_N}\int_0^1 u^2_{\epsilon,tt}(x,t-\tau_\epsilon(t)\rho)d\rho d\Gamma \\ &+q\tau_\epsilon(t)\int_{\Gamma_N}\int_0^1 u_{\epsilon,tt}(x,t-\tau_\epsilon(t)\rho)u_{\epsilon,ttt}(x,t-\tau_\epsilon(t)\rho)(1-\tau'_\epsilon(t)\rho)d\rho d\Gamma.\end{aligned}$$

By Green's formula and integrating by parts in ρ, we obtain

$$\tilde{E}'_\epsilon(t) = \int_{\Gamma_N}\frac{\partial u_{\epsilon,t}}{\partial\nu}u_{\epsilon,tt}d\Gamma - \frac{q}{2}\int_{\Gamma_N} u^2_{\epsilon,tt}(x,t-\tau_\epsilon(t))(1-\tau'_\epsilon(t))d\Gamma + \frac{q}{2}\int_{\Gamma_N} u^2_{\epsilon,tt}(x,t)d\Gamma.$$

Since u_ϵ satisfies (2.44),

$$\frac{\partial u_{\epsilon,t}}{\partial\nu} = -\mu_1 u_{\epsilon,tt}(t) - \mu_2 u_{\epsilon,tt}(t-\tau_\epsilon(t))(1-\tau'_\epsilon(t)),$$

and we obtain

$$\begin{aligned}\tilde{E}'_\epsilon(t) = \Big(\frac{q}{2}-\mu_1\Big)\int_{\Gamma_N} u^2_{\epsilon,tt}(t)d\Gamma - \mu_2(1-\tau'_\epsilon(t))\int_{\Gamma_N} u_{\epsilon,tt}(x,t-\tau_\epsilon(t))u_{\epsilon,tt}(x,t)d\Gamma \\ -\frac{q}{2}\int_{\Gamma_N} u^2_{\epsilon,tt}(x,t-\tau_\epsilon(t))(1-\tau'_\epsilon(t))d\Gamma.\end{aligned}$$

By Cauchy-Schwarz's inequality, we get, for $\alpha > 0$,

$$\begin{aligned}\tilde{E}'_\epsilon(t) &\le \Big(\frac{q}{2}-\mu_1+\frac{\alpha\mu_2(1-\tau'_\epsilon(t))}{2}\Big)\int_{\Gamma_N} u^2_{\epsilon,tt}(t)d\Gamma \\ &+\Big(\frac{\mu_2(1-\tau'_\epsilon(t))}{2\alpha} - \frac{q(1-\tau'_\epsilon(t))}{2}\Big)\int_{\Gamma_N} u^2_{\epsilon,tt}(x,t-\tau_\epsilon(t))d\Gamma.\end{aligned}$$

Let $\tau'_{min} = \min\limits_{t\in[0,T]} \tau'(t)$ and assume that

$$(1-d)(1-\tau'_{min}) \le 2. \tag{2.46}$$

By (2.16) and (2.46), we have

$$\mu_2 \le \frac{\sqrt{2}}{\sqrt{1-\tau'_{min}}}\mu_1,$$

which implies

$$\frac{\mu_2}{2\alpha} \le 2\mu_1 + \mu_2\alpha(\tau'_{min}-1),$$

with $\alpha = \frac{1}{\sqrt{2(1-\tau'_{min})}}$. Consequently we can choose $q > 0$ such that

$$\frac{\mu_2}{2\alpha} \le q \le 2\mu_1 + \mu_2\alpha(\tau'_{min}-1),$$

and thus

$$\tilde{E}'_\epsilon(t) \le 0.$$

Under the assumption (2.46), we have

$$\tilde{E}_\epsilon(t) \le \tilde{E}_\epsilon(0), \quad \forall t > 0,$$

i.e., for all $0 < \epsilon < \epsilon_0$ and $t > 0$,

$$\begin{aligned} &\int_\Omega \Big((u_{\epsilon,tt})^2 + (\nabla u_{\epsilon,t})^2 \Big) dx + q\tau_\epsilon(t) \int_{\Gamma_N} \int_0^1 u^2_{\epsilon,tt}(x, t - \tau_\epsilon(t)\rho) d\rho d\Gamma \\ &\le \int_\Omega \Big((\Delta u_0)^2 + (\nabla u_1)^2 \Big) dx + q(\tau(0) + \epsilon) \int_{\Gamma_N} \int_0^1 f^2_{0,t}(x, -(\tau(0) + \epsilon)\rho) d\rho d\Gamma. \end{aligned} \tag{2.47}$$

Therefore, assuming that $(f_{0,t}(x, -(\tau(0)+\epsilon)\rho)_{0<\epsilon<\epsilon_0}$ is bounded on $L^2(\Gamma_N \times (0,1))$, the sequence $(u_\epsilon)_\epsilon$ is bounded on $H^1((0,T);V) \cap H^2((0,T);L^2(\Omega))$, and thus, there exists $u \in H^1((0,T);V) \cap H^2((0,T);L^2(\Omega))$ such that, up to a subsequence,

$$u_\epsilon \rightharpoonup u \quad \text{in } H^1((0,T);V) \cap H^2((0,T);L^2(\Omega)).$$

The limit u then satisfies (1.1) in $\mathcal{D}'(\Omega \times (0,T))$ and (1.2), (1.5). Moreover u satisfies (1.3) since $u_{\epsilon,t}\,|_{\Gamma_N} \to u_t\,|_{\Gamma_N}$ in $L^2((0,T) \times \Gamma_N)$ and by using Lebesgue's convergence theorem. In the same manner, we find that u verifies (1.5), since, by change of variable and by (2.47) we have

$$\int_{\Gamma_N} \int_{t-(\tau(t)+\epsilon)}^t u^2_{\epsilon,tt}(x,t) dt d\Gamma \le C, \quad \forall t \in [0,T],$$

and thus

$$\int_{\Gamma_N} \int_{-\tau(0)}^0 u^2_{\epsilon,tt}(x,t) dt d\Gamma \le C, \quad \forall t \in [0,T].$$

In conclusion we have proved the next existence result.

Theorem 2.4 *Assume* (2.46) *and let* $(f_{0,t}(x, -(\tau(0)+\epsilon)\rho)_{0<\epsilon<\epsilon_0}$ *be bounded on* $L^2(\Gamma_N \times (0,1))$. *Then, for all initial data* $U_0 \in D^2(0)$, *there exists a unique solution* $u \in H^1((0,T);V) \cap H^2((0,T);L^2(\Omega))$ *of* (2.44).

2.2 Nonlinear problem

Here we restrict ourselves to the case where (2.9) holds.
As previously, if we set $z(x,\rho,t)$ as in (2.1), problem (1.15) – (1.19) is equivalent to

$$u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{2.48}$$

$$\tau(t) z_t(x,\rho,t) + (1 - \tau'(t)\rho) z_\rho(x,\rho,t) = 0 \quad \text{in} \quad \Gamma_N \times (0,1) \times (0,+\infty) \tag{2.49}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_D \times (0,+\infty) \tag{2.50}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -\beta_1(u_t(x,t)) - \beta_2(z(x,1,t)) \quad \text{on} \quad \Gamma_N \times (0,+\infty) \tag{2.51}$$

$$z(x,0,t) = u_t(x,t) \quad \text{on} \quad \Gamma_N \times (0,\infty) \tag{2.52}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{2.53}$$

$$z(x,\rho,0) = g_0(x, -\rho\tau(0)) \quad \text{in} \quad \Gamma_N \times (0,1). \tag{2.54}$$

Then problem (2.48) – (2.54) can be rewritten as

$$\left\{ \begin{array}{l} U' = \mathcal{A}(t) U \\ U(0) = (u_0, u_1, g_0(\cdot, -\cdot\,\tau(0)))^T \end{array} \right. \tag{2.55}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho-1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}(t)) := \Big\{ \ & (u,v,z)^T \in \big(E(\Delta, L^2(\Omega)) \cap V\big) \times V \times L^2(\Gamma_N; H^1(0,1)) \ : \\ & \frac{\partial u}{\partial \nu} = -\beta_1(v) - \beta_2(z(\cdot,1)) \text{ on } \Gamma_N; \ v = z(\cdot,0) \text{ on } \Gamma_N \ \Big\}. \end{aligned} \tag{2.56}$$

Notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e. (2.12) holds. Note further that for $(u,v,z)^T \in \mathcal{D}(\mathcal{A}(t))$, $\partial u/\partial \nu$ belongs to $L^2(\Gamma_N)$, by (1.20) and since $z(\cdot,1)$ is in $L^2(\Gamma_N)$.

We observe that the operator $\mathcal{A}(t)$ defined before is nonlinear (due to the domain (2.56) of the operator $\mathcal{A}(t)$) and therefore the technique developed in Section 2 can not be applied here. For nonlinear operators $\mathcal{A}(t)$ similar results exist (see [4, 8, 10, 20]) but for maximal operators $\mathcal{A}(t)$ with one inner product independent of t. For our system we need a variant of such results for maximal monotone operators $\mathcal{A}(t)$ for a time-dependent inner product depending "smoothly" on t.

We have the following result from [10] (see also [24]):

Theorem 2.5 *Let X be a real separable Hilbert space. For a fixed $T > 0$ and any time $t \in [0,T]$ we assume that there exists an inner product $\langle .,. \rangle_t$ on X depending "smoothly" on t in the following sense: there exists $c > 0$ such that*

$$\frac{\|u\|_t}{\|u\|_s} \leq e^{c|t-s|}, \quad \forall u \in X, \forall t, s \in [0, T]. \tag{2.57}$$

Assume furthermore that:

(i) for all $t \in [0,T]$, $A(t)$ is a maximal monotone operator for the inner product $\langle .,. \rangle_t$;

(ii) the domain $\mathcal{D}(A(t))$ of $A(t)$ is independent of t, for all $t \in [0,T]$;

(iii) there exists a positive constant K such that

$$\|A(t)u - A(s)u\|_0 \leq K\,|t-s|\,(1 + \|u\|_0 + \|A(s)u\|_0), \forall u \in \mathcal{D}(A(t)), \forall s, t \in [0,T], \tag{2.58}$$

where here $\|.\|_0 = \|.\|_{t=0}$. Then for all $v \in \mathcal{D}(A(t))$ the evolution equation

$$\begin{cases} u' + A(t)u = 0 \quad \text{for } 0 \leq t \leq T \\ u(0) = v \end{cases} \tag{2.59}$$

has a unique solution $u \in C([0,T]; X)$ such that $u(t)$ belongs to $\mathcal{D}(A(t))$ for all $t \in [0,T]$, its strong derivative $u'(t) = -A(t)u(t)$ exists and is continuous except at a countable numbers of values t.

Therefore to prove the existence and uniqueness of the solution of (2.55), we define an inner product depending "smoothly" on t.

For that, we assume that

$$\gamma_2 \leq \gamma_1 \sqrt{1-d} \tag{2.60}$$

holds, where γ_1, γ_2 is defined by (1.22) and (1.23).

Let ξ be a positive real number such that

$$\frac{\gamma_2}{\sqrt{1-d}} \leq \xi \leq 2\gamma_1 - \frac{\gamma_2}{\sqrt{1-d}}. \tag{2.61}$$

Note that, from (2.60), such a constant ξ exists.

Let us define on the Hilbert space $\mathcal{H}$ the following time-dependent inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t := \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx + \xi\tau(t) \int_{\Gamma_N} \int_0^1 z(x,\rho)\tilde{z}(x,\rho) d\rho d\Gamma, \tag{2.62}$$

where ξ is defined by (2.61).

Note that if β_1 and β_2 are linear, i.e. $\beta_i(s) = \mu_i s$ with $\mu_i > 0$, the assumptions (2.16) and (2.60), and (2.17) and (2.61) are exactly the same.

The aim of this section is then to prove the following theorem:

Theorem 2.6 *Assume* (1.13), (1.20), (1.21), (1.22), (1.23), (2.9), (2.60) *hold. Moreover assume that β_2 is nondecreasing. For any initial datum $U_0 \in \mathcal{D}(\mathcal{A}(0))$, then there exists a unique solution*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}(t))) \cap C^1([0, +\infty), \mathcal{H})$$

of problem (2.55).

To prove Theorem 2.6, we thus check that (2.57) holds and that

$$\tilde{\mathcal{A}}_-(t) = -\tilde{\mathcal{A}}(t) = -\mathcal{A}(t) + \kappa(t)I \tag{2.63}$$

satisfies the assumptions (i) to (iii) of Theorem 2.5, where κ is defined by (2.23).

The proof of (2.57) is the same that in Theorem 2.3, so we omit it.

We clearly have (ii) for $\tilde{\mathcal{A}}_-(t)$ since $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\tilde{\mathcal{A}}_-(t))$.

Therefore, it remains to show (i) and (iii), which is the aim of the three following lemmas.

Lemma 2.7 *Assume* (1.22), (1.23), (2.60) *and* (2.61) *hold. Then $\tilde{\mathcal{A}}_-(t)$ is a monotone operator in $\mathcal{H}$ for the inner product $\langle ., . \rangle_t$ for any fixed $t \geq 0$, i.e.:*

$$\left\langle \tilde{\mathcal{A}}_-(t)\phi_1 - \tilde{\mathcal{A}}_-(t)\phi_2, \phi_1 - \phi_2 \right\rangle_t \geq 0, \quad \forall \phi_1,\, \phi_2 \in \mathcal{D}(\tilde{\mathcal{A}}_-(t)). \tag{2.64}$$

Proof. First, from the definition of $\mathcal{A}(t)$, for $\phi_i = (u_i, v_i, z_i)^T \in \mathcal{D}(\mathcal{A}(t))$,

$$\begin{aligned}\langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t = &\int_\Omega \{(\nabla v_1 - \nabla v_2)(\nabla u_1 - \nabla u_2) + (\Delta u_1 - \Delta u_2)(v_1 - v_2)\}dx \\ &+\xi \int_{\Gamma_N} \int_0^1 \left(\frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho}\right)(z_1 - z_2)(\tau'(t)\rho - 1)d\rho d\Gamma.\end{aligned}$$

So, by Green's formula,

$$\begin{aligned}\langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t = &\int_{\Gamma_N} (v_1 - v_2)\frac{\partial}{\partial \nu}(u_1 - u_2)d\Gamma \\ &+\xi \int_{\Gamma_N} \int_0^1 \left(\frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho}\right)(z_1 - z_2)(\tau'(t)\rho - 1)d\rho d\Gamma.\end{aligned}$$

Integrating by parts in ρ, we get

$$\begin{aligned}&\int_{\Gamma_N} \int_0^1 \left(\frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho}\right)(z_1 - z_2)(\tau'(t)\rho - 1)d\rho d\Gamma = \frac{1}{2}\int_{\Gamma_N} \int_0^1 (\tau'(t)\rho - 1)\frac{\partial}{\partial \rho}(z_1 - z_2)^2 d\rho d\Gamma \\ &= -\frac{\tau'(t)}{2}\int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma + \frac{\tau'(t) - 1}{2}\int_{\Gamma_N} (z_1(x,1) - z_2(x,1))^2 d\Gamma \\ &\quad +\frac{1}{2}\int_{\Gamma_N} (z_1(x,0) - z_2(x,0))^2 d\Gamma.\end{aligned}$$

Therefore

$$\begin{aligned}\langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t = &\int_{\Gamma_N} (v_1 - v_2)\frac{\partial}{\partial \nu}(u_1 - u_2)d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma \\ &-\frac{\xi(1 - \tau'(t))}{2}\int_{\Gamma_N} (z_1(x,1) - z_2(x,1))^2 d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} (z_1(x,0) - z_2(x,0))^2 d\Gamma.\end{aligned}$$

As $\phi_i \in \mathcal{D}(\mathcal{A}(t))$ for $i = 1, 2$, we obtain

$$\begin{aligned}\langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2\rangle_t &= -\int_{\Gamma_N} (z_1(x,0) - z_2(x,0))(\beta_1(z_1(x,0)) - \beta_1(z_2(x,0)))d\Gamma \\ &- \int_{\Gamma_N} (z_1(x,0) - z_2(x,0))(\beta_2(z_1(x,1)) - \beta_2(z_2(x,1)))d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 (z_1 - z_2)^2 d\rho d\Gamma \\ &- \frac{\xi(1-\tau'(t))}{2}\int_{\Gamma_N} (z_1(x,1) - z_2(x,1))^2 d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} (z_1(x,0) - z_2(x,0))^2 d\Gamma.\end{aligned}$$

From (1.22), (1.23) and Cauchy-Schwarz's inequality

$$\begin{aligned}\langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2\rangle_t \leq \left(\frac{\xi}{2} - \gamma_1 + \frac{\gamma_2}{2\sqrt{1-d}}\right)\int_{\Gamma_N} (z_1(x,0) - z_2(x,0))^2 d\Gamma \\ + \left(\frac{\gamma_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2}\right)\int_{\Gamma_N} (z_1(x,1) - z_2(x,1))^2 d\Gamma \\ - \frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 (z_1 - z_2)^2 d\rho d\Gamma.\end{aligned}$$

By (2.61) and the definition (2.23) of κ, we get

$$\langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2\rangle_t \leq \kappa(t)\xi\tau(t)\int_{\Gamma_N}\int_0^1 (z_1 - z_2)^2 d\rho d\Gamma \leq \kappa(t)\langle \phi_1 - \phi_2, \phi_1 - \phi_2\rangle_t.$$

By the definition (2.63) of $\tilde{\mathcal{A}}_-(t)$, we obtain (2.64). ■

Lemma 2.8 *Assume that* (1.8), (1.20), (1.21), (1.22) *and* (2.9) *hold. Moreover assume that* β_2 *is nondecreasing. Then* $\tilde{\mathcal{A}}_-(t)$ *is a maximal operator in* $\mathcal{H}$*, i.e. for all* $(f,g,h)^T \in \mathcal{H}$*, there exists* $(u,v,z)^T \in \mathcal{D}(\tilde{\mathcal{A}}_-(t))$ *such that*

$$(I + \tilde{\mathcal{A}}_-(t))(u,v,z)^T = (f,g,h)^T. \tag{2.65}$$

Proof. Given $(f,g,h)^T \in \mathcal{H}$, we seek $U = (u,v,z)^T \in \mathcal{D}(\tilde{\mathcal{A}}_-(t))$ solution of

$$\begin{cases} (1+\kappa(t))u - v = f \\ (1+\kappa(t))v - \Delta u = g \\ (1+\kappa(t))z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = h. \end{cases} \tag{2.66}$$

In the beginning of this proof we follow the proof of Theorem 2.3. Suppose that we have found u with the appropriated regularity. Then v is given by

$$v := (1+\kappa(t))u - f \in V, \tag{2.67}$$

and z by

$$\begin{aligned} z(x,\rho) = (1+\kappa(t))u(x)e^{-(1+\kappa(t))\rho\tau(t)} - f(x)e^{-(1+\kappa(t))\rho\tau(t)} \\ + \tau(t)e^{-(1+\kappa(t))\rho\tau(t)}\int_0^\rho h(x,\sigma)e^{(1+\kappa(t))\sigma\tau(t)}d\sigma \quad \text{on } \Gamma_N \times (0,1), \end{aligned} \tag{2.68}$$

if $\tau'(t) = 0$, and

$$\begin{aligned} z(x,\rho) = (1+\kappa(t))u(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} - f(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} \\ + e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)}\int_0^\rho \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma \quad \text{on } \Gamma_N \times (0,1) \end{aligned} \tag{2.69}$$

otherwise.

In particular, if $\tau'(t) = 0$

$$z(x,1) = (1+\kappa(t))u(x)e^{-(1+\kappa(t))\tau(t)} + z_0(x), \quad x \in \Gamma_N, \tag{2.70}$$

with $z_0 \in L^2(\Gamma_N)$ defined by

$$z_0(x) = -f(x)e^{-(1+\kappa(t))\tau(t)} + \tau(t)e^{-(1+\kappa(t))\tau(t)} \int_0^1 h(x,\sigma)e^{(1+\kappa(t))\sigma\tau(t)}d\sigma, \quad x \in \Gamma_N, \tag{2.71}$$

and, if $\tau'(t) \neq 0$

$$z(x,1) = (1+\kappa(t))u(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} + z_0(x), \quad x \in \Gamma_N, \tag{2.72}$$

with $z_0 \in L^2(\Gamma_N)$ defined by

$$\begin{aligned} z_0(x) = {} & -f(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} \\ & + e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))} \int_0^1 \frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma, \quad x \in \Gamma_N. \end{aligned} \tag{2.73}$$

By (2.66), as in the proof of Theorem 2.3, the function u satisfies

$$(1+\kappa(t))^2u - \Delta u = g + (1+\kappa(t))f, \tag{2.74}$$

which can be reformulated as

$$\int_\Omega ((1+\kappa(t))^2u - \Delta u)wdx = \int_\Omega (g + (1+\kappa(t))f)wdx, \quad \forall w \in H^1_{\Gamma_D}(\Omega). \tag{2.75}$$

Integrating by parts and since $(u,v,z)^T \in \mathcal{D}(\mathcal{A}(t))$, we obtain

$$\int_\Omega ((1+\kappa(t))^2uw + \nabla u\nabla w)dx + \int_{\Gamma_N} (\beta_1(v) + \beta_2(z(x,1)))wd\Gamma = \int_\Omega (g + (1+\kappa(t))f)wdx,$$

for all $w \in H^1_{\Gamma_D}(\Omega)$.

Assume that $\tau'(t) = 0$. Using (2.67) and (2.70), we get

$$\gamma(u,w) = F(w), \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{2.76}$$

where the form γ (linear on w but not on u) is defined by

$$\begin{aligned} \gamma(u,w) = {} & \int_\Omega ((1+\kappa(t))^2uw + \nabla u\nabla w)dx \\ & + \int_{\Gamma_N} \left(\beta_1((1+\kappa(t))u - f) + \beta_2((1+\kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0)\right) wd\Gamma, \end{aligned}$$

and the linear form F is defined by

$$F(w) = \int_\Omega (g + (1+\kappa(t))f)wdx.$$

Introducing the (nonlinear) mapping

$$\mathcal{B} : V \to V' : u \to \mathcal{B}u,$$

where $\mathcal{B}u(w) = \gamma(u,w)$, we see that (2.76) is equivalent to

$$\mathcal{B}u = F,$$

since F clearly belongs to V'. This means that the solvability of (2.76) is equivalent to the surjectivity of $\mathcal{B}$. This surjectivity is obtained using Corollary II.2.2 of [29], which states that $\mathcal{B}$ is surjective if $\mathcal{B}$ is monotone, hemicontinuous, bounded and coercive. Let us then check these properties.

We first prove that $\mathcal{B}$ is monotone, i.e.

$$[\mathcal{B}u - \mathcal{B}v](u - v) \geq 0, \quad \forall u, v \in V. \tag{2.77}$$

In view of the definition of $\mathcal{B}$,

$$\begin{aligned}
[\mathcal{B}u - \mathcal{B}v](u - v) &= \int_\Omega \left((1 + \kappa(t))^2 (u - v)^2 + |\nabla(u - v)|^2\right) dx \\
&+ \int_{\Gamma_N} \left(\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)\right)(u - v) d\Gamma \\
&+ \int_{\Gamma_N} \left(\beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) - \beta_2((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0)\right)(u - v) d\Gamma.
\end{aligned}$$

By (1.22),

$$\begin{aligned}
&\int_{\Gamma_N} \left(\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)\right)(u - v) d\Gamma \\
&= \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \left(\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)\right) \\
&\qquad . \left(((1 + \kappa(t))u - f) - ((1 + \kappa(t))v - f)\right) d\Gamma \geq 0,
\end{aligned}$$

and as β_2 is nondecreasing,

$$\begin{aligned}
&\int_{\Gamma_N} \left(\beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) - \beta_2((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0)\right)(u - v) d\Gamma \\
&= \frac{e^{(1+\kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \left(\beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) - \beta_2((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0)\right) \\
&\qquad . \left(((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) - ((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0)\right) d\Gamma \geq 0.
\end{aligned}$$

This two estimates clearly imply (2.77).

The boundedness of $\mathcal{B}$ follows from the properties (1.20) satisfied by β_1 and β_2, the fact that $1+\kappa(t)$ is bounded by (2.9) and (1.8), Cauchy-Schwarz's inequality and a trace theorem (reminding that f and z_0 are fixed).

The hemicontinuity of $\mathcal{B}$ means that the function $s \to \mathcal{B}(u+sw)(w)$ is continuous for each u, $w \in V$. As

$$\begin{aligned}
\mathcal{B}(u + sw)(w) &= \int_\Omega ((1 + \kappa(t))^2 (u + sw)w + \nabla(u + sw)\nabla w) dx \\
&+ \int_{\Gamma_N} (\beta_1((1 + \kappa(t))(u + sw) - f) + \beta_2((1 + \kappa(t))(u + sw)e^{-(1+\kappa(t))\tau(t)} + z_0))w d\Gamma,
\end{aligned}$$

this follows from the continuity of β_1 and β_2.

It remains to check the coerciveness of $\mathcal{B}$, i.e.

$$\frac{\mathcal{B}u(u)}{\|u\|_V} \to \infty \quad \text{if } \|u\|_V \to +\infty. \tag{2.78}$$

From the definition of $\mathcal{B}$, we have, since $1 + \kappa(t) > 1$,

$$\mathcal{B}u(u) \geq \|u\|_V^2 + \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f)u d\Gamma + \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0)u d\Gamma.$$

We deduce, by (1.24) and (1.21)

$$\begin{aligned}
\mathcal{B}u(u) &\geq \|u\|_V^2 + \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f) f d\Gamma \\
&- \frac{e^{(1+\kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) z_0 d\Gamma.
\end{aligned}$$

By Cauchy-Schwarz's inequality, (2.9), (1.20), a trace theorem and the fact that $1+\kappa(t)$ is bounded, we obtain that there exists $C>0$ such that

$$\left|\frac{1}{1+\kappa(t)}\int_{\Gamma_N}\beta_1((1+\kappa(t))u-f)fd\Gamma\right|\le C\left(\|u\|_{H^1(\Omega)}+\|f\|_{H^1(\Omega)}\right)\|f\|_{H^1(\Omega)}$$

and

$$\begin{gathered}\left|\frac{e^{(1+\kappa(t))\tau(t)}}{1+\kappa(t)}\int_{\Gamma_N}\beta_2((1+\kappa(t))ue^{-(1+\kappa(t))\tau(t)}+z_0)z_0d\Gamma\right|\\ \le C\left(\|u\|_{H^1(\Omega)}+\|z_0\|_{H^1(\Omega)}\right)\|z_0\|_{H^1(\Omega)}.\end{gathered}$$

Then

$$\mathcal{B}u(u)\ge\|u\|_V^2-C\left(\left(\|u\|_{H^1(\Omega)}+\|f\|_{H^1(\Omega)}\right)\|f\|_{H^1(\Omega)}+\left(\|u\|_{H^1(\Omega)}+\|z_0\|_{H^1(\Omega)}\right)\|z_0\|_{H^1(\Omega)}\right),$$

which implies (2.78).

Therefore, by Corollary II.2.2 of [29], there exists $u\in V$ solution of (2.76). If $\tau'(t)\neq 0$, we obtain the same result by similar arguments. This function u satisfies (2.74) by choosing test function in $\mathcal{D}(\Omega)$ and then satisfies

$$\frac{\partial u}{\partial\nu}=-\beta_1(v)-\beta_2(z(.,1))\quad\text{on }\Gamma_N$$

by Green's formula.

So we have found $(u,v,z)^T\in\mathcal{D}(\tilde{\mathcal{A}}_-(t))$ which verifies (2.65). ■

It remains to show (iii) of Theorem 2.5 to finish the proof of Theorem 2.6. This is the aim of the following lemma.

Lemma 2.9 *Assume that* (1.8), (1.13) *and* (2.9) *hold. Then* (2.58) *holds.*

Proof. Let $\phi=(u,v,z)^T\in\mathcal{D}(\tilde{\mathcal{A}}_-(t))$. By definition (2.63) of $\tilde{\mathcal{A}}_-(t)$, we have

$$\begin{aligned}\left\|\tilde{\mathcal{A}}_-(t)\phi-\tilde{\mathcal{A}}_-(s)\phi\right\|_0&=\|(\kappa(t)-\kappa(s))\phi-(\mathcal{A}(t)\phi-\mathcal{A}(s))\phi\|_0\\&\le|\kappa(t)-\kappa(s)|\,\|\phi\|_0+\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0.\end{aligned}\tag{2.79}$$

As

$$\kappa'(t)=\frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2+1)^{\frac12}}-\frac{\tau'(t)(\tau'(t)^2+1)^{\frac12}}{2\tau(t)^2}$$

is bounded on [0, T] for all $T>0$ (by (2.9) and (1.8)), by the mean value theorem there exists $K>0$ such that

$$|\kappa(t)-\kappa(s)|\,\|\phi\|_0\le K\,|t-s|\,\|\phi\|_0.\tag{2.80}$$

Moreover

$$\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0^2=\xi\tau(0)\int_{\Gamma_N}\int_0^1\left(\frac{\tau'(t)\rho-1}{\tau(t)}-\frac{\tau'(s)\rho-1}{\tau(s)}\right)^2z_\rho^2d\rho d\Gamma.$$

In addition

$$\left(\frac{\tau'(t)\rho-1}{\tau(t)}\right)'=\frac{\tau''(t)\tau(t)\rho-\tau'(t)^2\rho+\tau'(t)}{\tau(t)^2}$$

is bounded on $[0,T]$ for all $T>0$ by (2.9) and (1.8). By the mean value theorem, we then obtain that there exists $K>0$ such that

$$\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0^2\le K^2\,|t-s|^2\,\xi\tau(0)\int_{\Gamma_N}\int_0^1z_\rho^2(x,\rho)d\rho d\Gamma.$$

Moreover by (2.9) and (1.13)

$$\left(\frac{\tau(t)}{\tau'(t)\rho-1}\right)^2 \leq \frac{\bar{\tau}^2}{(1-d)^2}.$$

Therefore

$$\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0^2 \leq \left(\frac{K\bar{\tau}}{1-d}\right)^2 |t-s|^2\, \xi\tau(0)\int_{\Gamma_N}\int_0^1 \left(\frac{\tau'(t)\rho-1}{\tau(t)} z_\rho(x,\rho)\right)^2 d\rho d\Gamma.$$

This leads to

$$\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0^2 \leq \left(\frac{K\bar{\tau}}{1-d}\right)^2 |t-s|^2 \|\mathcal{A}(t)\phi\|_0^2,$$

and thus to

$$\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0 \leq \left(\frac{K\bar{\tau}}{1-d}\right) |t-s| \|\mathcal{A}(t)\phi\|_0.$$

By definition (2.63) of $\tilde{\mathcal{A}_-}(t)$, we have

$$\|\mathcal{A}(t)\phi\|_0 \leq \left\|\tilde{\mathcal{A}_-}(t)\phi\right\|_0 + \kappa(t)\|\phi\|_0,$$

with $\kappa(t)$ bounded on $[0,T]$. Consequently there exists $C>0$ such that

$$\|\mathcal{A}(t)\phi-\mathcal{A}(s)\phi\|_0 \leq C|t-s|\left(\left\|\tilde{\mathcal{A}_-}(t)\phi\right\|_0 + \|\phi\|_0\right). \tag{2.81}$$

Therefore (2.79), (2.80) and (2.81) imply (2.58). ■

Proof of Theorem 2.6. The assumptions of Theorem 2.5 were verified for $\tilde{\mathcal{A}_-}(t)$ and the inner product $\langle .,.\rangle_t$ defined by (2.62). Consequently the evolution equation

$$\left\{\begin{array}{l}\tilde{U}'+\tilde{\mathcal{A}_-}(t)\tilde{U}=0\\ \tilde{U}(0)=U_0\in\mathcal{D}(\mathcal{A}(t)),\end{array}\right. \tag{2.82}$$

has a unique solution $\tilde{U}\in C([0,T];\mathcal{H})$ such that $\tilde{U}(t)$ belongs to $\mathcal{D}(\tilde{\mathcal{A}_-}(t))$ for all $t\in[0,T]$, its strong derivative $\tilde{U}'(t)=-\tilde{\mathcal{A}_-}(t)\tilde{U}(t)$ exists and is continuous except at a countable numbers of values t.

The requested solution of (2.55) is then given by

$$U(t)=e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t)=\int_0^t\kappa(s)ds$. ■

3 Stability result for the linear problem

In this section, we will give an exponential stability result for problem (1.1) – (1.5) under the assumption (1.12). We define the energy of system (1.1) – (1.5) as

$$E(t):=\frac{1}{2}\int_\Omega\{u_t^2+|\nabla u|^2\}dx+\frac{\xi}{2}\tau(t)\int_0^1\int_{\Gamma_N}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma, \tag{3.1}$$

where ξ is a positive constant such that

$$2\mu_1-\frac{\mu_2}{\sqrt{1-d}}-\xi>0, \quad \text{and} \quad \xi-\frac{\mu_2}{\sqrt{1-d}}>0. \tag{3.2}$$

Note that from (1.12) such a constant ξ exists. We have the following identity.

Proposition 3.1 *For any regular solution of problem* (1.1) − (1.5) *we have*

$$
\begin{aligned}
E'(t) = -\mu_1 \int_{\Gamma_N} u_t^2(x,t)d\Gamma - \int_{\Gamma_N} \mu_2 u_t(x,t)u_t(x,t-\tau(t))d\Gamma \\
-\frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t-\tau(t))(1-\tau'(t))d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma.
\end{aligned}
\tag{3.3}
$$

Proof. Differentiating (3.1) we obtain

$$
\begin{aligned}
E'(t) = \int_{\Omega}\{u_t u_{tt} + \nabla u \nabla u_t\}dx + \frac{\xi}{2}\tau'(t)\int_0^1\int_{\Gamma_N} u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \\
+\xi\tau(t)\int_{\Gamma_N}\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma,
\end{aligned}
$$

and then, applying Green's formula,

$$
\begin{aligned}
E'(t) = \int_{\Gamma_N} u_t \frac{\partial u}{\partial \nu} d\Gamma + \frac{\xi}{2}\tau'(t)\int_0^1\int_{\Gamma_N} u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma \\
+\xi\tau(t)\int_{\Gamma_N}\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma.
\end{aligned}
\tag{3.4}
$$

Now, observe that, if $\tau(t) \neq 0$,

$$
u_t(x,t-\tau(t)\rho) = -\tau^{-1}(t)u_\rho(x,t-\tau(t)\rho),
$$

and

$$
u_{tt}(x,t-\tau(t)\rho) = \tau^{-2}(t)u_{\rho\rho}(x,t-\tau(t)\rho).
$$

Therefore,

$$
\begin{aligned}
&\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho \\
&= -\tau^{-3}(t)\int_0^1 u_\rho(x,t-\tau(t)\rho)u_{\rho\rho}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho \\
&= -\tau^{-3}(t)[u_\rho^2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)]_0^1 \\
&\quad +\tau^{-3}(t)\int_0^1 u_\rho(x,t-\tau(t)\rho)u_{\rho\rho}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho \\
&\quad -\tau'(t)\tau^{-3}(t)\int_0^1 u_\rho(x,t-\tau(t)\rho)u_\rho(x,t-\tau(t)\rho)d\rho.
\end{aligned}
\tag{3.5}
$$

Then, from (3.5),

$$
\begin{aligned}
&\int_0^1 u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho \\
&= -\frac{1}{2}\tau'(t)\tau^{-3}(t)\int_0^1 u_\rho^2(t-\tau(t)\rho)d\rho \\
&\quad -\frac{\tau^{-1}(t)}{2}u_t^2(x,t-\tau(t))(1-\tau'(t)) + \frac{\tau^{-1}(t)}{2}u_t^2(x,t) \\
&= -\frac{1}{2}\tau'(t)\tau^{-1}(t)\int_0^1 u_t^2(x,t-\tau(t)\rho)d\rho \\
&\quad -\frac{\tau^{-1}(t)}{2}u_t^2(x,t-\tau(t))(1-\tau'(t)) + \frac{\tau^{-1}(t)}{2}u_t^2(x,t).
\end{aligned}
\tag{3.6}
$$

Using (3.4), (3.6) and the boundary condition (1.3) on Γ_N, we have

$$\begin{aligned} E'(t) = -\mu_1 \int_{\Gamma_N} u_t^2(x,t)d\Gamma - \mu_2 \int_{\Gamma_N} u_t(x,t)u_t(x,t-\tau(t))d\Gamma \\ +\frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma - \frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t-\tau(t))(1-\tau'(t))d\Gamma. \end{aligned} \tag{3.7}$$

So, for any time t such that $\tau(t) \neq 0$, the identity (3.3) is proved.

Now, let t be such that $\tau(t) = 0$. Then, from (3.4) we have

$$E'(t) = -(\mu_1 + \mu_2)\int_{\Gamma_N} u_t^2(x,t)d\Gamma + \frac{\xi}{2}\tau'(t)\int_{\Gamma_N} u_t^2(x,t)d\Gamma. \tag{3.8}$$

Therefore, identity (3.3) is proved for all times $t > 0$. ∎

Proposition 3.2 *For any regular solution of problem* (1.1) – (1.5) *the energy decays and there exists a positive constant C such that*

$$E'(t) \le -C\int_{\Gamma_N} \{u_t^2(x,t) + u_t^2(x,t-\tau(t))\}d\Gamma. \tag{3.9}$$

Proof. In the case of $\tau(t) \neq 0$, from (3.7), applying Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} E'(t) \le -\mu_1 \int_{\Gamma_N} u_t^2(x,t)d\Gamma + \frac{1}{\sqrt{1-d}}\frac{\mu_2}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma + \sqrt{1-d}\,\frac{\mu_2}{2}\int_{\Gamma_N} u_t^2(x,t-\tau(t))d\Gamma \\ -\frac{\xi}{2}(1-\tau'(t))\int_{\Gamma_N} u_t^2(x,t-\tau(t))d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma, \end{aligned}$$

from which easily follows (3.9) recalling (3.2). In the case of $\tau(t) = 0$, from (3.8) easily follows (3.9) observing that by (3.2)

$$\xi < 2\mu_1 < \frac{2(\mu_1+\mu_2)}{d}. \qquad ∎$$

Remark 3.3 The choice to apply Cauchy-Schwarz's inequality with a factor $\sqrt{1-d}$ in the proof of the above proposition is made in order to give the stability result under the best assumption between μ_1 and μ_2.

Now, let us introduce the Lyapounov functional

$$\hat{E}(t) = E(t) + \gamma\left\{\int_\Omega [2m\cdot\nabla u + (n-1)u]u_t dx + \mathcal{E}(t)\right\}, \tag{3.10}$$

where γ is a positive small constant that we will choose later on and $\mathcal{E}(t)$ is defined by

$$\mathcal{E}(t) := \xi\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma. \tag{3.11}$$

Note that, from Poincaré's Theorem, the functional $\hat{E}$ is equivalent to the energy E, that is there exist two positive constant d_1, d_2 such that

$$d_1\hat{E}(t) \le E(t) \le d_2\hat{E}(t), \quad \forall\, t \ge 0. \tag{3.12}$$

Moreover, we denote by $E_S(\cdot)$ the standard energy for wave equation without delay, that is

$$E_S(t) := \frac{1}{2}\int_\Omega (u_t^2(x,t) + |\nabla u(x,t)|^2)dx. \tag{3.13}$$

The following estimate holds true.

Lemma 3.4 *For any regular solution of problem* (1.1) − (1.5),

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u+(n-1)u]u_t dx\right\}\\&\qquad\leq -C_0E_S(t)+C\left\{\int_{\Gamma_N}[u_t^2(x,t)+u_t^2(x,t-\tau(t))]d\Gamma\right\},\end{aligned}\tag{3.14}$$

for suitable positive constants C_0, C.

Proof. The standard multiplier identity gives

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u+(n-1)u]u_t dx\right\}&=-\int_\Omega\{u_t^2+|\nabla u|^2\}dx\\+\int_{\Gamma_N}(m\cdot\nu)(u_t^2-|\nabla u|^2)d\Gamma&+\int_{\Gamma_N}[2m\cdot\nabla u+(n-1)u]\frac{\partial u}{\partial\nu}d\Gamma.\end{aligned}\tag{3.15}$$

From (3.15) and Young's inequality, recalling that by (1.10) $m\cdot\nu\geq\delta$ on Γ_N, we have

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u+(n-1)u]u_t dx\right\}&\leq-\int_\Omega\{u_t^2+|\nabla u|^2\}dx\\+\int_{\Gamma_N}(m\cdot\nu)u_t^2d\Gamma-\delta\int_{\Gamma_N}|\nabla u|^2d\Gamma&+\frac{c}{\varepsilon}\int_{\Gamma_N}\left(\frac{\partial u}{\partial\nu}\right)^2d\Gamma\\+\varepsilon\int_{\Gamma_N}(|\nabla u|^2+u^2)d\Gamma,&\end{aligned}\tag{3.16}$$

for some positive constants ε, c. From (3.16), using the trace's inequality and Poincaré's Theorem, for ε small enough we deduce

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u+(n-1)u]u_t dx\right\}&\leq -C_0E_S(t)\\+C\int_{\Gamma_N}u_t^2d\Gamma+C\int_{\Gamma_N}\left(\frac{\partial u}{\partial\nu}\right)^2d\Gamma,&\end{aligned}\tag{3.17}$$

for suitable positive constants C_0, C. Therefore, using the boundary condition (1.3) and Cauchy-Schwarz's inequality in (3.17), we obtain (3.14) ∎.

We can also estimate the component $\mathcal{E}(\cdot)$ in the Lyapounov functional (3.10).

Lemma 3.5 *For any regular solution of problem* (1.1) − (1.5),

$$\frac{d}{dt}\mathcal{E}(t)\leq-2\mathcal{E}(t)+\xi\int_{\Gamma_N}u_t^2d\Gamma\,.\tag{3.18}$$

Proof. Differentiating (3.11) we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)&=\xi\tau'(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\&-2\xi\tau'(t)\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}\rho u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\&+2\xi\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}u_t(x,t-\tau(t)\rho)u_{tt}(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma.\end{aligned}\tag{3.19}$$

Now, let us suppose $\tau(t) \neq 0$ and integrate by parts the last term in (3.19). We obtain

$$\begin{aligned}
&\int_0^1 e^{-2\tau(t)\rho} u_t(x, t-\tau(t)\rho) u_{tt}(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho \\
&\quad = -\tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} u_\rho(x, t-\tau(t)\rho) u_{\rho\rho}(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho \\
&\quad = \tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} u_\rho(x, t-\tau(t)\rho) u_{\rho\rho}(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho \\
&\quad -\tau'(t)\tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} u_\rho^2(x, t-\tau(t)\rho) d\rho \\
&\quad -2\tau^{-2}(t) \int_0^1 e^{-2\tau(t)\rho} u_\rho^2(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho \\
&\quad -\tau^{-3}(t) \left[e^{-2\tau(t)\rho} u_\rho^2(x, t-\tau(t)\rho)(1-\tau'(t)\rho) \right]_0^1
\end{aligned} \tag{3.20}$$

and then

$$\begin{aligned}
&\int_0^1 e^{-2\tau(t)\rho} u_t(x, t-\tau(t)\rho) u_{tt}(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho \\
&\quad = -\frac{1}{2}\tau'(t)\tau^{-1}(t) \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t-\tau(t)\rho) d\rho \\
&\quad - \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho \\
&\quad - \frac{\tau^{-1}(t)}{2} e^{-2\tau(t)} u_t^2(x, t-\tau(t))(1-\tau'(t)) + \frac{\tau^{-1}(t)}{2} u_t^2(x,t).
\end{aligned} \tag{3.21}$$

Now, substituting identity (3.21) in (3.19), we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \xi\tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t-\tau(t)\rho) d\rho d\Gamma \\
&-2\xi\tau'(t)\tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} \rho u_t^2(x, t-\tau(t)\rho) d\rho d\Gamma \\
&-\xi\tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t-\tau(t)\rho) d\rho d\Gamma \\
&-2\xi\tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t-\tau(t)\rho)(1-\tau'(t)\rho) d\rho d\Gamma \\
&-\xi e^{-2\tau(t)} \int_{\Gamma_N} u_t^2(x, t-\tau(t))(1-\tau'(t)) d\Gamma + \xi \int_{\Gamma_N} u_t^2(x,t) d\Gamma,
\end{aligned} \tag{3.22}$$

and so

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= -2\xi\tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t-\tau(t)\rho) d\rho d\Gamma \\
&-\xi e^{-2\tau(t)} \int_{\Gamma_N} u_t^2(x, t-\tau(t))(1-\tau'(t)) d\Gamma + \xi \int_{\Gamma_N} u_t^2(x,t) d\Gamma
\end{aligned} \tag{3.23}$$

from which immediately follows estimate (3.18) for t such that $\tau(t) \neq 0$. In the case of $\tau(t) = 0$, note that from (3.19) we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= \xi\tau'(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\ &\le \xi d\int_{\Gamma_N}\int_0^1 u_t^2(x,t)d\rho d\Gamma = \xi d\int_{\Gamma_N} u_t^2(x,t)d\Gamma\\ &= \xi d\int_{\Gamma_N} u_t^2(x,t)d\Gamma - 2\mathcal{E}(t).\end{aligned} \tag{3.24}$$

Then, even in this case, we obtain (3.18). ■

Now, we can deduce the exponential stability estimate for problem (1.1) – (1.5).

Theorem 3.6 *Assume* (1.12). *There exist positive constants* C_1, C_2 *such that for any solution of problem* (1.1) – (1.5),

$$E(t) \le C_1 E(0)e^{-C_2 t}, \quad \forall t \ge 0. \tag{3.25}$$

Proof. From Proposition 3.2, Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}\frac{d}{dt}\hat{E}(t) \le -C&\left\{\int_{\Gamma_N}[u_t^2(x,t)+u_t^2(x,t-\tau(t))]d\Gamma\right\}\\ &+\gamma\left(-C_0E_S(t)+\tilde{C}\int_{\Gamma_N}[u_t^2(x,t)+u_t^2(x,t-\tau(t))]d\Gamma - 2\mathcal{E}(t)\right).\end{aligned} \tag{3.26}$$

Then, for γ sufficiently small, we can estimate

$$\frac{d}{dt}\hat{E}(t) \le -\gamma C_0 E_S(t) - 2\gamma\mathcal{E}(t). \tag{3.27}$$

Now, observe that by assumption (1.6) on $\tau(t)$, we can deduce

$$\begin{aligned}\mathcal{E}(t) &\ge \xi\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\overline{\tau}}u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma\\ &\ge \frac{c\xi\tau(t)}{2}\int_{\Gamma_N}\int_0^1 u_t^2(x,t-\tau(t)\rho)d\rho d\Gamma,\end{aligned} \tag{3.28}$$

for some positive constant c.

Therefore, from (3.27) and (3.28),

$$\frac{d}{dt}\hat{E}(t) \le -\gamma C_0E_S(t) - 2\gamma\mathcal{E}(t) \le -cE(t) \le -C\hat{E}(t), \tag{3.29}$$

for suitable positive constants c, C, where we used also the first inequality in (3.12). This clearly implies

$$\hat{E}(t) \le e^{-Ct}\hat{E}(0),$$

and so, using (3.12),

$$E(t) \le C_1 e^{-C_2 t}E(0),$$

for suitable constants $C_1, C_2 > 0$. ■

4 Nonlinear stability result

In this section we consider the problem $(1.15)-(1.19)$ with β_1, β_2 satisfying (1.20), (1.24). Moreover we assume

$$\gamma_1 > \frac{c_2}{\sqrt{1-d}}, \tag{4.1}$$

where γ_1, c_2 are the constants in (1.20) and (1.24) (which comes from (1.20) and (1.22)) and d is as in (1.13).

We define the energy associated to the problem as in (3.1) with the constant ξ such that

$$2\gamma_1 - \frac{c_2}{\sqrt{1-d}} - \xi > 0 \quad \text{and} \quad \xi - \frac{c_2}{\sqrt{1-d}} > 0. \tag{4.2}$$

Note that, from assumption (4.1), such a constant ξ exists.

Notice that (2.60) implies (4.1). Moreover the existence of ξ verifying (2.61) guarantees that ξ verifies (4.2), since $c_2 \le \gamma_2$.

The following identity holds true.

Proposition 4.1 *For any regular solution of problem* $(1.15)-(1.18)$ *we have*

$$\begin{aligned} E'(t) = &-\int_{\Gamma_N} u_t(x,t)\beta_1(u_t(x,t))d\Gamma - \int_{\Gamma_N} u_t(x,t)\beta_2(u_t(x,t-\tau(t)))d\Gamma \\ &-\frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t-\tau(t))(1-\tau'(t))d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma. \end{aligned} \tag{4.3}$$

Proof. The proof is analogous to the one of Proposition 3.1, so we omit the details. ∎

Proposition 4.2 *For any regular solution of problem* $(1.15)-(1.18)$ *the energy decays and there exists a positive constant* C *such that*

$$E'(t) \le -C\int_{\Gamma_N} \{u_t^2(x,t) + u_t^2(x,t-\tau(t))\}d\Gamma. \tag{4.4}$$

Proof. In the case of $\tau(t) \neq 0$, from (4.3), we obtain, by (1.20) and (1.24),

$$\begin{aligned} E'(t) \le &-\gamma_1\int_{\Gamma_N} u_t^2(x,t)d\Gamma + \int_{\Gamma_N} c_2|u_t(x,t)||u_t(x,t-\tau(t))|d\Gamma \\ &-\frac{\xi}{2}(1-\tau'(t))\int_{\Gamma_N} u_t^2(x,t-\tau(t))d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma. \end{aligned}$$

Then, applying Cauchy-Schwarz's inequality, we have

$$\begin{aligned} E'(t) \le &-\gamma_1\int_{\Gamma_N} u_t^2(x,t)d\Gamma + \frac{\xi}{2}\int_{\Gamma_N} u_t^2(x,t)d\Gamma \\ &-\frac{\xi}{2}(1-\tau'(t))\int_{\Gamma_N} u_t^2(x,t-\tau(t))d\Gamma + \frac{c_2}{2\sqrt{1-d}}\int_{\Gamma_N} u_t^2(x,t)d\Gamma \\ &+\sqrt{1-d}\,\frac{c_2}{2}\int_{\Gamma_N} u_t^2(x,t-\tau(t))d\Gamma. \end{aligned} \tag{4.5}$$

From (4.5) estimate (4.4) easily follows recalling that ξ satisfies (4.2).

If t is such that $\tau(t)=0$, then from (4.3) we deduce

$$E'(t) = -\int_{\Gamma_N} u_t(t)\beta_1(u_t(t))d\Gamma - \int_{\Gamma_N} u_t(t)\beta_2(u_t(t))d\Gamma \\ +\frac{\xi}{2}\tau'(t)\int_{\Gamma_N} u_t^2(t)d\Gamma.$$

Then, from (1.24) and (1.21)

$$E'(t) \le -(\gamma_1 - \frac{\xi}{2}d)\int_{\Gamma_N} u_t^2(t)d\Gamma,$$

and this clearly gives (4.4) observing that by (4.2)

$$\gamma_1 > \frac{\xi}{2} + \frac{c_2}{2\sqrt{1-d}} > \frac{\xi}{2} > \frac{\xi}{2}d. \qquad \blacksquare$$

Now, let $\hat{E}$ be the Lyapounov functional introduced in (3.10) with a small enough positive constant γ and let $\mathcal{E}$ be defined as in (3.11).

Even in this case lemma 3.4 holds true. Indeed inequality (3.17) is obtained without using the boudary condition (1.3) on Γ_N. From (3.17) we easily deduce estimate (3.14) for suitable positive constants C_0, C, using the boundary condition (1.17) and the assumptions (1.20) on the functions β_1, β_2.

We can estimate also the component $\mathcal{E}(\cdot)$ in the Lyapounov functional (3.10) as in the previous case, and so analogously to the linear case, we can deduce an exponential stability estimate for problem (1.15) – (1.19).

Theorem 4.3 *Assume* (4.1). *There exist positive constants* D_1, D_2 *such that for any solution of problem* (1.15) – (1.19),

$$E(t) \le D_1 E(0) e^{-D_2 t}, \quad \forall t \ge 0. \tag{4.6}$$

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