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Energy decay estimates for the damped plate equation with a local degenerated dissipation

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Dedicated to the memory of J.L. Lions

Abstract

We consider the Euler–Bernoulli plate equation in a bounded open set Ω of $\mathbb{R}^2$ with a degenerated local damping term. This dissipation is effective in a subset ω of Ω and the damping coefficient may vanish in some subset of dimension one of ω. We show that the usual observability inequality for the undamped problem implies polynomial decay estimates for the damped problem. Our method can be applied for other PDE's such as the wave equation or the Schrödinger equation.

Keywords: Interior damping; Singular coefficient; Observability inequality

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1. Introduction and statement of the main result

Let Ω be a bounded open subset of $\mathbb{R}^2$ with boundary Γ and let ω be an arbitrary non-empty subset of Ω. Let $a = a(x) \in L^\infty(\Omega)$ be such that $a(x) \geqslant 0$ for almost all $x \in \Omega$ and suppose that

$$\int_\omega \frac{\mathrm{d}x}{a^p} < \infty, \tag{1}$$

for some $p > 0$.

We consider the following initial and boundary value problem modelling the damped vibrations of a simply supported plate:

$$\begin{aligned} &y'' + \Delta^2 y + a(x)y' = 0 \quad \text{in } \Omega \times (0,\infty),\\ &y = \Delta y = 0 \quad \text{on } \Gamma \times (0,\infty),\\ &y(0) = y^0 \quad \text{in } \Omega,\\ &y'(0) = y^1 \quad \text{in } \Omega. \end{aligned} \tag{2}$$

The function a will be called "the localizing coefficient" since it determines where the damping is effective in Ω.

The difficulty of the problem we consider consists in the fact that we don't assume that the localizing coefficient satisfies a condition of the form $a(x) \geqslant a_0 > 0$ in ω, as in most of the previous literature (cf. [16] and references therein).

Localizing coefficients satisfying only (1) were considered, in the case of the wave equation, in [12,14,11],

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by applying multipliers techniques. As far as we know the present paper is the first one which tackles the plate equation with a local degenerated dissipation. Our method essentially relies on a classical observability inequality (see [9]) for the undamped problem and it can be applied for other partial differential equations (see [3] for applications to the wave and to the Schrödinger equations). Although the problem we consider is linear, the technique we use in order to get polynomial decay estimates has common points with the methods developed in [7] and in [12] for the semilinear wave equation with linear or nonlinear damping.

We notice that, after formally taking the L^2 scalar product of the first equation in (2) by y', simple calculations show that for every $t \geqslant 0$ we have

$$E'(t) = -\int_\Omega a(x)|y'(x,t)|^2\,\mathrm{d}x \leqslant 0,$$

where

$$E(t) = \frac{1}{2}\int_\Omega (|y'(x,t)|^2 + |\Delta y(x,t)|^2)\,\mathrm{d}x, \quad \forall t \geqslant 0, \tag{3}$$

is the energy of the plate.

The above relation implies that for any $T > 0$ we have

$$E(0) - E(T) = \int_{\Omega\times(0,T)} a(x)|y'(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4}$$

In order to formulate our results we consider the Hilbert space

$$H = [H^2(\Omega)\cap H_0^1(\Omega)] \times L^2(\Omega),$$

where by $H^s(\Omega)$, $s \in \mathbb{R}$ we denote the usual Sobolev spaces. We endowe this space with the norm

$$\left\|\begin{pmatrix} y \\ z\end{pmatrix}\right\|_H^2 = \int_\Omega (|\Delta y|^2 + |z|^2)\,\mathrm{d}x, \quad \forall \begin{pmatrix} y \\ z\end{pmatrix} \in H.$$

We also consider the linear operator

$$A : D(A) \to H$$

defined by

$$A\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} v \\ -\Delta^2 u - av\end{pmatrix}, \quad \forall \begin{pmatrix} u \\ v\end{pmatrix} \in D(A), \tag{5}$$

where

$$D(A) = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \;\middle|\; \begin{array}{c} \phi \in H^4(\Omega) \text{ and} \\ \phi = \Delta\phi = 0 \text{ on } \Gamma \\ \psi \in H_0^1(\Omega)\cap H^2(\Omega) \end{array} \right\} \tag{6}$$

Moreover we need the domain of the powers of the operator A defined recursively by

$$D(A^m) = \{U \in D(A^{m-1}) | AU \in D(A^{m-1})\}, \quad \forall m \geqslant 2.$$

The operator $A^m : D(A^m) \to H$ is as usually defined by $A^m U = A(A^{m-1}U)$, for all $U \in D(A^m)$. It is well-known $D(A^m)$ is a Hilbert space when equipped with the graph norm of A^m. Some useful properties of the operator A defined above are summarized in the result below. Since these properties are either well-known or very easy to prove (see, e.g., [4]), this result is stated without proof.

Proposition 1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth boundary (say C^∞) or let Ω be a rectangle. Moreover suppose that $m \geqslant 1$. Then the following assertions hold true:*

(1) *The operator A defined by (5) is the generator of a strongly continuous semigroup of contractions in H.*

(2) *If $a \in C^{2m-2}(\bar\Omega)$ then we have the imbedding*

$$D(A^m) \subset H^{2m+2}(\Omega) \times H^{2m}(\Omega), \tag{7}$$

and the graph norm on $D(A^m)$ is equivalent to the restriction to $D(A^m)$ of the usual norm in $H^{2m+2}(\Omega)\times H^{2m}(\Omega)$.

(3) *The operator A generates a semigroup of linear contractions in $D(A^m)$.*

As a consequence of the result above we obtain the following existence and regularity property.

Corollary 2. *Suppose that Ω and a satisfy the assumptions in Proposition 1 and that $\binom{y^0}{y^1} \in D(A^m)$. Then the initial and boundary value problem (2) admits a unique solution y such that*

$$\begin{pmatrix} y \\ y' \end{pmatrix} \in \bigcap_{j=0}^{m} C^{m-j}([0,\infty), D(A^j)).$$

Moreover, for all $j \in \{0,1,\ldots,m\}$ we have

$$\left\|\begin{pmatrix} y(t) \\ y'(t)\end{pmatrix}\right\|_{D(A^j)} \leqslant \left\|\begin{pmatrix} y^0 \\ y^1\end{pmatrix}\right\|_{D(A^j)}, \quad \forall t \geqslant 0.$$

Let us now consider the undamped problem corresponding to (2), i.e., the initial and boundary value problem

$$\begin{cases} \phi'' + \Delta^2\phi = 0, \\ \phi|_{\partial\Omega} = \Delta\phi|_{\partial\Omega} = 0, \\ \phi(0) = \phi^0, \ \phi'(0) = \phi^1. \end{cases} \tag{8}$$

The main result of this paper is:

Theorem 3. *Let* $m \geqslant 1$, *let* $\Omega \subset \mathbb{R}^2$ *be an open bounded set with smooth boundary or let* Ω *be a rectangle. Suppose that* ω *is an open subset of* Ω *and that the positive function* $a \in L^\infty(\bar{\Omega})$ *satisfies condition* (1). *Moreover, suppose that there exist a constant* $C > 0$ *and a time* $T > 0$ *such that every solution of* (8) *satisfies the inequality*

$$\int_0^T \int_\omega |\phi'|^2 \geqslant C \int_\Omega (|\phi^1|^2 + |\Delta\phi^0|^2)\, dx,$$

$$\forall \begin{pmatrix} \phi^0 \\ \phi^1 \end{pmatrix} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega). \tag{9}$$

Then there exists a constant $L_m > 0$ (*depending on* $\|a^{-1}\|_{L^p(\omega)}$ *and on* $\|a\|_{L^\infty(\Omega)}$) *such that the solutions of* (2) *satisfy the estimate*:

$$E(t) \leqslant \frac{L_m}{(1+t)^{2pm}} \left\| \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \right\|_{D(A^m)}, \quad \forall t \geqslant 0, \tag{10}$$

for all $\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \in D(A^m)$.

Remark 4. In the particular case of Ω being a rectangle Jaffard showed in [6] that (9) holds for all open set $\omega \subset \Omega$. Taking in consideration the fact that assumption (1) implies that $a(x) > 0$ for x in an open subset $\omega_1 \subset \Omega$ it follows that, in this case, the result in Theorem 3 is not sharp, since the solutions of (2) decay exponentially in the energy space.

Remark 5. Suppose that $a \geqslant a_0 > 0$ in ω_1, where ω_1 is an open subset of ω. According to a result in [10] the solutions of (2) decay exponentially if and only if the same property holds for the system obtained by replacing the plate equation by the Schrödinger equation (and by keeping the same damping region). On the other hand, if Ω is a disk, the results in [1,5,8] imply that the solutions of the damped Schrödinger equation decay exponentially if and only if ω_1 satisfies the "geometric optics condition". Thus, if Ω is a disk, then the solutions of (2) decay exponentially if and only if $a \geqslant a_0 > 0$ in ω_1 and ω_1 satisfies the geometric optics condition. If we suppose that Ω is a disk one can easily construct open sets ω_1 and ω_2 not satisfying the geometric optics condition such that $\omega = \omega_1 \cup \omega_2$ satisfies the geometric optics condition. Thus if Ω is a disk, a satisfies the assumption in Theorem 3 but a is not bounded by below on ω, then the solutions of (2) have no exponential decay. However, in the above situation, the conclusion of Theorem 3 still holds.

The plan of this paper is as follows: in Section 2 we give some preliminary lemmas that will be useful for the proof of Theorem 3. Section 3 is devoted to the proof of the main result.

2. Some preliminaries

In this section we first give several results, which will be useful in the proof of the main Theorem 3.

We first recall a version of the Gagliardo–Nirenberg inequality which is a particular case of Lemma 1 in [12]. As usual, we denote by $H^m(\Omega)$ the Sobolev space formed by the functions in $L^2(\Omega)$ which have distributional derivatives, up to the order m, in $L^2(\Omega)$.

Lemma 4. *Suppose that* $m \geqslant 1$ *and that* $\Omega \subset \mathbb{R}^2$ *is an open set with smooth boundary or that* Ω *is a rectangle. Then there exists a constant* $C > 0$, *depending only on* Ω, *such that*

$$\|\vartheta\|_{L^\infty(\Omega)} \leqslant C \|\vartheta\|_{H^{2m}(\Omega)}^{\theta} \|\vartheta\|_{L^2(\Omega)}^{1-\theta}, \quad \forall \vartheta \in H^{2m},$$

where $\theta = 1/2m$.

We also need the following technical lemma, which can be proved by using Lemma 3.3 in [7] (see also Ammari and Tucsnak [2]).

Lemma 5. *Let* $(\xi_k)_{k\in N}$ *be a sequence of positive real numbers satisfying*

$$\xi_{k+1} \leqslant \xi_k - C\xi_{k+1}^{2+\alpha}, \quad \forall k \geqslant 0,$$

where $C > 0$ and $\alpha > -1$ are constants. Then there exists a positive constant M such that

$$\xi_k \leqslant \frac{M}{(1+k)^{1/1+\alpha}}, \qquad \forall k \geqslant 0. \tag{11}$$

We next recall several classical results in control theory which give, in particular, the interpretation of (9) in the terms of system theory.

Throughout this section, U, H and Y are Hilbert spaces, $A : D(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ on H, $B \in L(U,H)$ is an input operator and $C \in L(H,Y)$ is an observation operator.

Definition 6. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on H and let $C \in L(H,Y)$. The pair (A,C) is exactly observable in time $T > 0$ if there exists a constant $m > 0$ such that

$$\int_0^T \|C\mathbb{T}_t x\|_U^2 \, \mathrm{d}t \geqslant m\|x\|_H^2 \qquad \forall x \in H.$$

The invariance of observability under output feedback is known for bounded input and observation operators (and beyond). More precisely (see, for instance, [15, Remark 6.5]) the following result holds.

Proposition 7. *Suppose that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on H, $C \in L(H,Y)$, $B \in L(U,H)$ and that $K \in L(Y,U)$. Then the pair (A,C) is exactly observable in time T if and only if the pair $(A+BKC, C)$ is exactly observable in time T.*

By applying Proposition 7 to our specific context we obtain the following result.

Proposition 8. *Suppose that every solution of (8) satisfies the inequality (9). Then there exists a constant $m > 0$ such that*

$$m(\|(\Delta y)(0)\|_{L^2(\Omega)}^2 + \|y'(0)\|_{L^2(\Omega)}^2) \leqslant \int_0^T \int_\omega |y'|^2 + \int_0^T \int_\Omega a|y'|^2. \tag{12}$$

for all $y \in C(0,T;H^2(\Omega)) \cap C^1(0,T;L^2(\Omega))$ satisfying (2).

Proof. Define

$$H = [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega),$$

and consider the operator $A : D(A) \to H$ defined by (5) and (6). Moreover, define the input space $U = L^2(\Omega)$ and the input operator $B : U \to H$ by

$$Bu = \begin{pmatrix} 0 \\ \sqrt{a}u \end{pmatrix} \qquad \forall u \in U.$$

The output space Y is defined by

$$Y = L^2(\Omega) \times L^2(\omega),$$

whereas the observation operator C is given by

$$C\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \sqrt{a}\psi \\ \psi|_\omega \end{pmatrix} \qquad \forall \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in H.$$

Finally we define the feedback operator $K \in L(Y,U)$ by

$$K\begin{pmatrix} y \\ z \end{pmatrix} = y \qquad \forall \begin{pmatrix} y \\ z \end{pmatrix} \in Y.$$

A simple calculation shows that

$$BKC\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ a\psi \end{pmatrix} \qquad \forall \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in H.$$

The above relation combined to (5) implies that,

$$(A+BKC)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ -\Delta^2\phi \end{pmatrix} \qquad \forall \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in D(A),$$

The above relation and (9) imply the exact observability in time T for the couple $(A+BKC, C)$. This implies, by Proposition 7, that the couple (A,C) is exactly observable in time T which is exactly the conclusion (12). □

3. Proof of the main result

Proof of Theorem 3. We first tackle the case $m = 1$.

Let y be the solution of (2). By the Hölder inequality, we obtain that

$$\int_{\Omega\times(0,T)} a|y'|^2 \leqslant \left(\int_{\Omega\times(0,T)} a\right)^{1/p+1} \times \left(\int_{\Omega\times(0,T)} a|y'|^{2+2/p}\right)^{p/p+1}, \tag{13}$$

and that

$$\int_{\omega\times(0,T)} |y'|^2 \leqslant \left(\int_{\omega\times(0,T)} a^{-p}\right)^{1/p+1} \times \left(\int_{\Omega\times(0,T)} |y'|^{2+2/p}\right)^{p/p+1}, \tag{14}$$

where $p > 0$ is such that (1) holds true. Relation (13), (14) combined to Proposition 8 imply that there exists a constant C, depending on $\|a^{-1}\|_{L^p(\omega)}$ and on $\|a\|_{L^\infty(\Omega)}$, such that

$$\|y^0\|^2_{H^2(\Omega)} + \|y^1\|^2_{L^2(\Omega)} \leqslant C\left(\int_{\Omega\times(0,T)} a|y'|^{2+2/p}\right)^{p/p+1}. \tag{15}$$

Since $\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \in D(A)$, by Corollary 2, we have that

$$y' \in C([0,\infty); H^2(\Omega)).$$

By applying the Gagliardo–Nirenberg inequality given in Lemma 4 (for $m = 1$) we get that

$$\|y'(s)\|_{L^\infty(\Omega)} \leqslant C\|y'(s)\|^{1/2}_{H^2(\Omega)}\|y'(s)\|^{1/2}_{L^2(\Omega)}, \quad \forall s \in [0,T], \tag{16}$$

where $C > 0$ is a constant depending only on Ω.

We next use again the notation

$$E(t) = \tfrac{1}{2}(\|y(t)\|^2_{H^2(\Omega)} + \|y'(t)\|^2_{L^2(\Omega)}),$$

and we follow the pattern of the proof in [12, p. 13]. We notice that (15) combined to Proposition 8 implies that there exists a constant $C > 0$ such that, for all

$$\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \in D(A)$$

we have

$$E(0) \leqslant C \sup_{s\in[0,T]} (\|y'(s)\|^{1/p+1}_{H^2}\|y'(s)\|^{1/p+1}_{L^2}) \times \left(\int_{\Omega\times(0,T)} a|y'|^2\right)^{p/p+1}. \tag{17}$$

Relation (17) implies, by using (4), that

$$E(0) \leqslant C \sup_{s\in[0,T]} (\|y'(s)\|^{1/p+1}_{H^2}\|y'(s)\|^{1/p+1}_{L^2}) \times (E(0) - E(T))^{p/p+1}. \tag{18}$$

If we denote

$$F(t) = \left\| \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \right\|^2_{D(A)}, \quad \forall t \geqslant 0,$$

then from Corollary 2 there exists a constant $C > 0$ such that

$$\|y'(s)\|^2_{H^2(\Omega)} \leqslant CF(0), \quad \text{and}$$

$$\|y'(s)\|^2_{L^2(\Omega)} \leqslant CE(0) \ \forall s \geqslant 0.$$

The two above inequalities combined to (18) imply that

$$[E(0)]^{(2p+1/2(p+1))} \leqslant C[F(0)]^{1/2(p+1)}(E(0) - E(T))^{p/p+1}. \tag{19}$$

Since the function $t \to E(t)$ is non-increasing, relation (19) implies, after a simple calculation, that there exists a constant $C_1 > 0$ such that

$$E(T) \leqslant E(0) - C_1 \frac{[E(T)]^{1+1/2p}}{[F(0)]^{1/2p}}. \tag{20}$$

We follow now the method used in Russell [13].

Estimate (20) remains valid in successive intervals $[kT, (k+1)T]$, $k = 0, 1, \ldots$. Moreover, by Corollary 2, we have $F(t) \leqslant F(0)$, for all $t \geqslant 0$. It follows that there exists a constant $C_2 > 0$ such that

$$E((k+1)T) \leqslant E(kT) - C_1 \frac{[E((k+1)T)]^{1+1/2p}}{[F(0)]^{1/2p}}, \quad \forall k \geqslant 0. \tag{21}$$

If we define

$$\xi_k = \frac{E(kT)}{F(0)}, \quad k = 0, 1, \ldots$$

then (21) can be rewritten as

$$\xi_{k+1} \leqslant \xi_k - C_2 \xi_{k+1}^{1+1/2p}, \qquad \forall k \geqslant 0.$$

By applying Lemma 5 with $\alpha = (1/2p) - 1$ we obtain the existence of a constant $M > 0$ such that

$$E(kT) \leqslant \frac{M}{(1+k)^{2p}} \left\| \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \right\|^2_{D(A)}, \qquad \forall k \geqslant 0.$$

Using again the fact that E is non-increasing, the above relation implies that there exists a constant $L_1 > 0$, depending on $\|a^{-1}\|_{L^p(\omega)}$ and on $\|a\|_{L^\infty(\Omega)}$, such that

$$E(t) \leqslant \frac{L_1}{(1+t)^{2p}} \left\| \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \right\|^2_{D(A)}, \qquad \forall t > 0.$$

We have thus obtained our main result in the case $m = 1$.

For $m \geqslant 2$ the result can be obtained by induction over m. More precisely, suppose that $n \geqslant 2$ is such that the assertion in Theorem 3 holds for $m \in \{1, \ldots, n-1\}$ and let

$$\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \in D(A^n).$$

Then, by using the semigroup property, we have that

$$E(2t) \leqslant \frac{L_{n-1}}{(1+t)^{2(n-1)p}} \left\| \begin{pmatrix} y(t) \\ y'(t), \end{pmatrix} \right\|^2_{D(A^{n-1})}, \qquad \forall t > 0. \tag{22}$$

for some constant $L_{n-1} > 0$. Moreover, there exists a constant $C > 0$ such that

$$\left\| \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \right\|^2_{D(A^{n-1})} \leqslant \left\| A^{n-1} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \right\|^2_H, \qquad \forall t \geqslant 0. \tag{23}$$

If we denote

$$\begin{pmatrix} z(t) \\ z'(t) \end{pmatrix} = A^{n-1} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \tag{24}$$

then $z(t)$ is the solution of (2) with initial data

$$A^{n-1} \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \in D(A).$$

Since we already proved the assertion of the theorem for $m = 1$ it follows that

$$\left\| \begin{pmatrix} z(t) \\ z'(t) \end{pmatrix} \right\|^2_H \leqslant \frac{L_1}{(1+t)^{2p}} \left\| \begin{pmatrix} z(0) \\ z'(0) \end{pmatrix} \right\|^2_{D(A)}. \tag{25}$$

On the other hand from (24) it follows that

$$\left\| \begin{pmatrix} z(0) \\ z'(0) \end{pmatrix} \right\|_{D(A)} \leqslant C \left\| \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \right\|_{D(A^n)}, \tag{26}$$

for some constant $C > 0$. Relations (23), (25) and (26) imply that the conclusion of the theorem also holds for $m = n$. We have thus ended our proof. □

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