Skorohod calculus for Gaussian processes via rough paths methods

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Private message

Happy Birthday Süleyman!

Sketch of the talk

1. Introduction
   - Brief rough paths review
   - Rough paths and fractional Brownian motion
   - Aim of the talk

2. Skorohod type formula
   - Stratonovich type formula
   - Skorohod type formula

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Rough path assumptions

Regularity of X: $X \in \mathcal{C}^{\gamma}(\mathbb{R}^d)$ with $\gamma > 0$.

Set: $\mathbf{X}^{\mathbf{1}}_{st} \equiv X_t - X_s$.

Iterated integrals: X allows to define

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \ldots, i_n) = \int_{s \le u_1 < \cdots < u_n \le t} dX_{u_1}(i_1)\, dX_{u_2}(i_2) \cdots dX_{u_n}(i_n),$$

for $0 \le s < t \le T$, $n \le \lfloor 1/\gamma \rfloor$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$.

Regularity of the iterated integrals: $\mathbf{X}^{\mathbf{n}} \in \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n})$, where

$$\mathcal{N}[g; \mathcal{C}_2^{\kappa}] \equiv \sup_{0 \le s < t \le T} \frac{|g_{st}|}{|t - s|^{\kappa}}$$

Main rough paths result

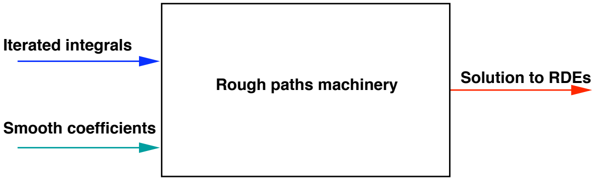
Theorem (loose formulation): Under the assumption of the previous slide, plus regularity assumptions on σ, one can

1. Obtain change of variables formula of Itô's type
2. Solve equations of the form $dY_t = \sigma(Y_t)dX_t, \qquad Y_0 = a$

Moreover, the application

$$\begin{aligned} F : \mathbb{R}^n \times \mathcal{C}_2^{\gamma}(\mathbb{R}^d) \times \cdots \times \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n}) &\longrightarrow \mathcal{C}^{\gamma}(\mathbb{R}^m) \\ (a, \mathbf{X^1}, \ldots, \mathbf{X^n}) &\mapsto Y \end{aligned}$$

is a continuous map (Lyons-Qian, Friz-Victoir, Gubinelli).



Meaning of the second order integral

Definition

The double integral associated to X is an element $\{\mathbf{X}^{\mathbf{2}}_{st}(i_1, i_2);\ s \leq t,\ 1 \leq i_1, i_2 \leq d\}$ satisfying:

(i) The regularity condition $\mathbf{X}^{\mathbf{2}} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$.

(ii) The multiplicative property

$$\mathbf{X}^{\mathbf{2}}_{st}(i_1, i_2) - \mathbf{X}^{\mathbf{2}}_{su}(i_1, i_2) - \mathbf{X}^{\mathbf{2}}_{ut}(i_1, i_2)$$
$$= \mathbf{X}^{\mathbf{1}}_{su}(i_1)\, \mathbf{X}^{\mathbf{1}}_{ut}(i_2) = [X_u(i_1) - X_s(i_1)]\ [X_t(i_2) - X_u(i_2)]\,.$$

(iii) The geometric relation

$$\mathbf{X}^{\mathbf{2}}_{st}(i_1, i_2) + \mathbf{X}^{\mathbf{2}}_{st}(i_2, i_1) = \mathbf{X}^{\mathbf{1}}_{st}(i_1)\, \mathbf{X}^{\mathbf{1}}_{st}(i_2).$$

Meaning of iterated integrals (2)

Remark:

The above relations are satisfied when X is smooth, with

$$\mathbf{X}^{\mathbf{2}}_{st}(i,j) = \int_s^t \delta X_{su}(i)\, dX_u(j).$$

Definition of n^{th} order iterated integrals:

Can also be given by a set of algebraic and analytic relations

Geometric and weakly geometric rough paths

Remark:

- The stack $\{\mathbf{X^n};\, n \le \lfloor 1/\gamma \rfloor\}$ as defined above is called a weakly geometric rough path above X
  $\hookrightarrow$ allows a reasonable differential calculus
- When there exists a family X^ε such that
  - X^ε is smooth
  - $\mathbf{X}^{\mathbf{n},\varepsilon}$ is the n^{th} Riemann integral of X^ε
  - $\mathbf{X^n} = \lim_{\varepsilon\to 0} \mathbf{X}^{\mathbf{n},\varepsilon}$

  then one has a so-called geometric rough path above X
  $\hookrightarrow$ easier physical interpretation

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Canonical example: fractional Brownian motion

- $B = (B(1), \ldots, B(d))$
- $B(i)$ centered Gaussian process, independence of coordinates
- Variance of the increments:

$$E[|B_t(i) - B_s(i)|^2] = |t - s|^{2H}$$

- $H^- \equiv$ Hölder-continuity exponent of B
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$, most natural generalization of BM

Motivations: Engineering, Finance, Biophysics

Rough paths and fBm

Situation 1: $H > 1/4$

$\hookrightarrow$ 3 possible geometric rough paths constructions for B.

- Malliavin calculus tools, with Volterra representation
- Regularization of the fBm path (Coutin-Qian, Friz-Victoir)
- Analytic approximation (Unterberger)

Situation 2: $d = 1$

$\hookrightarrow$ Then one can take $\mathbf{B}^{\mathbf{n}}_{st} = \frac{(B_t - B_s)^n}{n!}$

Situation 3: $H \leq 1/4$, $d > 1$

The constructions by approximation diverge

Existence result by dyadic approximation (Lyons-Victoir)

Recent advances (Unterberger, Nualart-T)

for weakly geometric rough path construction

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Aim of the talk

For the case $H \leq 1/4$, $d > 1$:

- Consider a rather general multidimensional Gaussian process X
- Assume that X gives raise to a weakly geometric rough path
- Then show a Skorohod type change of variable for X

Conclusion: Weakly geometric $\Longrightarrow$ Skorohod change of variable

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Assumptions

Consider:

- $\gamma > 0$ with $\lfloor 1/\gamma \rfloor = N$
- $X \in \mathcal{C}_1^{\gamma}$, admitting a rough path of order N
- $f \in C_b^{N+1}(\mathbb{R}^d; \mathbb{R})$
- $0 \leq s < t \leq T$
- An arbitrary partition $\Pi_{st} = \{s = t_0, \ldots, t_n = t\}$ of $[s, t]$

Stratonovich type formula

Theorem

Under the previous conditions:

(1) *The following limit exists:*

$$\mathcal{J}_{st}\left(\nabla f(X)\, dX\right) :=$$

$$\lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \left[\partial_i f(X_{t_q})\, \mathbf{X}^{\mathbf{1}}_{t_q,t_{q+1}}(i) + \sum_{k=2}^{N} \partial^k_{i_k \dots i_1} f(X_{t_q})\, \mathbf{X}^{\mathbf{k}}_{t_q,t_{q+1}}(i_k, \dots, i_1) \right]$$

(2) *We have*

$$f(X_t) - f(X_s) = \mathcal{J}_{st}\left(\nabla f(X)\, dX\right) = \int_s^t \langle \nabla f(X_u),\, dX_u \rangle_{\mathbb{R}^d} \qquad (1)$$

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Additional assumptions

Suppose:

- $\gamma > 0$ with $\lfloor 1/\gamma \rfloor = N$
- $X \in \mathcal{C}_1^{\gamma}$, admitting a rough path of order N
- $f \in C_b^{N+1}(\mathbb{R}^d; \mathbb{R})$
- X is a centered Gaussian process with independent coordinates
- If $R_t := E[|X_t(1)|^2]$, then $t \mapsto R_t$ has finite variations

Skorohod type formula

Theorem

Under the previous conditions:

(1) *The following limit exists:*

$$\mathcal{J}_{st}\left(\nabla f(X) \diamond dX\right) :=$$

$$\lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \left[\partial_i f(X_{t_q}) \diamond \mathbf{X}^{\mathbf{1}}_{t_q,t_{q+1}}(i) + \sum_{k=2}^{N} \partial^k_{i_k \ldots i_1} f(X_{t_q}) \diamond \mathbf{X}^{\mathbf{k}}_{t_q,t_{q+1}}(i_k,\ldots,i_1) \right]$$

(2) *We have*

$$f(X_t) - f(X_s) = \mathcal{J}_{st}\left(\nabla f(X) \diamond dX\right) + \frac{1}{2}\int_s^t \Delta f(x_u)\, dR_u.$$

(3) *If $X \equiv fBm$ with $H \in (0,1)$, then $\mathcal{J}_{st}\left(\nabla f(X) \diamond dX\right)$ can be interpreted in the dual sense of Malliavin calculus*

Strategy of the proof

Brief summary:

- Start from the fact that the Stratonovich sum is convergent:

$$\sum_{q=0}^{n-1}\left[\partial_i f(X_{t_q})\,\mathbf{X}^{\mathbf{1}}_{t_q,t_{q+1}}(i)+\sum_{k=2}^{N}\partial^k_{i_k\ldots i_1}f(X_{t_q})\,\mathbf{X}^{\mathbf{k}}_{t_q,t_{q+1}}(i_k,\ldots,i_1)\right]$$

- Compute the corrections between $\partial^k_{i_k\ldots i_1}f(X_{t_q})\,\mathbf{X}^{\mathbf{k}}_{t_q,t_{q+1}}(i_k,\ldots,i_1)$ and $\partial^k_{i_k\ldots i_1}f(X_{t_q})\diamond\mathbf{X}^{\mathbf{k}}_{t_q,t_{q+1}}(i_k,\ldots,i_1)$ by Wick calculus
- Show that in the limit $|\Pi|\to 0$, only a term $\frac{1}{2}\int_s^t \Delta f(x_u)\,dR_u$ is relevant