Skorohod calculus for Gaussian processes via rough paths methods

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Ongoing joint work with: Maria Jolis and Yaozhong Hu
Happy Birthday Süleyman!
Sketch of the talk

1. Introduction
   - Brief rough paths review
   - Rough paths and fractional Brownian motion
   - Aim of the talk

2. Skorohod type formula
   - Stratonovich type formula
   - Skorohod type formula
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Rough path assumptions

Regularity of $X$: $X \in C^\gamma(\mathbb{R}^d)$ with $\gamma > 0$.

Set: $X_{st}^1 \equiv X_t - X_s$.

Iterated integrals: $X$ allows to define

$$X_{st}^n(i_1, \ldots, i_n) = \int_{s \leq u_1 < \cdots < u_n \leq t} dX_{u_1}(i_1) dX_{u_2}(i_2) \cdots dX_{u_n}(i_n),$$

for $0 \leq s < t \leq T$, $n \leq \lfloor 1/\gamma \rfloor$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$.

Regularity of the iterated integrals: $X^n \in C_2^{n\gamma}(\mathbb{R}^{dn})$, where

$$\mathcal{N}[g; C_2^\kappa] \equiv \sup_{0 \leq s < t \leq T} \frac{|g_{st}|}{|t - s|^{\kappa}}$$
Main rough paths result

Theorem (loose formulation): Under the assumption of the previous slide, plus regularity assumptions on $\sigma$, one can

1. Obtain change of variables formula of Itô’s type
2. Solve equations of the form $dY_t = \sigma(Y_t)dX_t$, $Y_0 = a$

Moreover, the application

$$F : \mathbb{R}^n \times C^\gamma_2(\mathbb{R}^d) \times \cdots \times C^\gamma_{dn}(\mathbb{R}^{dn}) \longrightarrow C^\gamma(\mathbb{R}^m)$$

$$(a, X^1, \ldots, X^n) \longmapsto Y$$

is a continuous map (Lyons-Qian, Friz-Victoir, Gubinelli).

Iterated integrals

Smooth coefficients

Rough paths machinery

Solution to RDEs
Meaning of the second order integral

Definition

The double integral associated to $X$ is an element
$\{X_{st}^2(i_1, i_2); \ s \leq t, \ 1 \leq i_1, i_2 \leq d\}$ satisfying:

(i) The regularity condition $X^2 \in C_2^{2\gamma}(\mathbb{R}^d,d)$.

(ii) The multiplicative property

$$X_{st}^2(i_1, i_2) - X_{su}^2(i_1, i_2) - X_{ut}^2(i_1, i_2) = X_{su}^1(i_1) X_{ut}^1(i_2) = [X_u(i_1) - X_s(i_1)] [X_t(i_2) - X_u(i_2)].$$

(iii) The geometric relation

$$X_{st}^2(i_1, i_2) + X_{st}^2(i_2, i_1) = X_{st}^1(i_1) X_{st}^1(i_2).$$
Meaning of iterated integrals (2)

Remark:
The above relations are satisfied when $X$ is smooth, with

$$X_{st}^2(i, j) = \int_s^t \delta X_{su}(i) \, dX_u(j).$$

Definition of $n^{th}$ order iterated integrals:
Can also be given by a set of algebraic and analytic relations
Remark:

- The stack \( \{X^n; \, n \leq \lfloor 1/\gamma \rfloor \} \) as defined above is called a weakly geometric rough path above \( X \)
  \( \leftrightarrow \) allows a reasonable differential calculus

- When there exists a family \( X^\varepsilon \) such that
  - \( X^\varepsilon \) is smooth
  - \( X^{n,\varepsilon} \) is the \( n^{\text{th}} \) Riemann integral of \( X^\varepsilon \)
  - \( X^n = \lim_{\varepsilon \to 0} X^{n,\varepsilon} \)

then one has a so-called geometric rough path above \( X \)
  \( \leftrightarrow \) easier physical interpretation
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Canonical example: fractional Brownian motion

- $B = (B(1), \ldots, B(d))$
- $B(i)$ centered Gaussian process, independence of coordinates
- Variance of the increments:
  \[ E[|B_t(i) - B_s(i)|^2] = |t - s|^{2H} \]

- $H^- \equiv$ Hölder-continuity exponent of $B$
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$, most natural generalization of BM

**Motivations:** Engineering, Finance, Biophysics
Rough paths and fBm

**Situation 1:** $H > 1/4$

$\implies$ 3 possible geometric rough paths constructions for $B$.

- Malliavin calculus tools, with Volterra representation
- Regularization of the fBm path (Coutin-Qian, Friz-Victoir)
- Analytic approximation (Unterberger)

**Situation 2:** $d = 1$

$\implies$ Then one can take $B_{st}^n = \frac{(B_t - B_s)^n}{n!}$

**Situation 3:** $H \leq 1/4$, $d > 1$

The constructions by approximation diverge

Existence result by dyadic approximation (Lyons-Victoir)

Recent advances (Unterberger, Nualart-T)

for weakly geometric rough path construction
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Aim of the talk

For the case $H \leq 1/4$, $d > 1$:

- Consider a rather general **multidimensional** Gaussian process $X$
- Assume that $X$ gives raise to a weakly geometric rough path
- Then show a Skorohod type change of variable for $X$

**Conclusion:** Weakly geometric $\implies$ Skorohod change of variable
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Assumptions

Consider:

- $\gamma > 0$ with $\lfloor 1/\gamma \rfloor = N$
- $X \in C_1^\gamma$, admitting a rough path of order $N$
- $f \in C_b^{N+1}(\mathbb{R}^d; \mathbb{R})$
- $0 \leq s < t \leq T$
- An arbitrary partition $\Pi_{st} = \{s = t_0, \ldots, t_n = t\}$ of $[s, t]$
Stratonovich type formula

**Theorem**

*Under the previous conditions:*

(1) The following limit exists:

\[
J_{st} (\nabla f(X) \, dX) := \lim_{\|\Pi_{st}\| \to 0} \left( \sum_{q=0}^{n-1} \partial_i f(X_{t,q}) X^1_{t_q, t_{q+1}} (i) + \sum_{k=2}^{N} \partial_{i_k \ldots i_1} f(X_{t,q}) X^k_{t_q, t_{q+1}} (i_k, \ldots, i_1) \right)
\]

(2) We have

\[
f(X_t) - f(X_s) = J_{st} (\nabla f(X) \, dX) = \int_s^t \langle \nabla f(X_u), dX_u \rangle \mathbb{R}^d
\]
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Additional assumptions

Suppose:

- $\gamma > 0$ with $\lfloor 1/\gamma \rfloor = N$
- $X \in C_1^\gamma$, admitting a rough path of order $N$
- $f \in C_b^{N+1}(\mathbb{R}^d; \mathbb{R})$
- $X$ is a centered Gaussian process with independent coordinates
- If $R_t := E[|X_t(1)|^2]$, then $t \mapsto R_t$ has finite variations
Skorohod type formula

**Theorem**

Under the previous conditions:

1. The following limit exists:

   \[ J_{st} (\nabla f(X) \diamond dX) := \lim_{|\Pi_{st}| \to 0} \sum_{q=0}^{n-1} \left[ \partial_i f(X_{t_q}) \diamond X_{t_q, t_{q+1}}^1(i) + \sum_{k=2}^{N} \partial_{i_k \ldots i_1} f(X_{t_q}) \diamond X_{t_q, t_{q+1}}^k(i_k, \ldots, i_1) \right] \]

2. We have

   \[ f(X_t) - f(X_s) = J_{st} (\nabla f(X) \diamond dX) + \frac{1}{2} \int_s^t \Delta f(x_u) \, dR_u. \]

3. If \( X \equiv fBm \) with \( H \in (0, 1) \), then \( J_{st} (\nabla f(X) \diamond dX) \) can be interpreted in the dual sense of Malliavin calculus.
Strategy of the proof

Brief summary:

- Start from the fact that the Stratonovich sum is convergent:

\[
\sum_{q=0}^{n-1} \left[ \partial_i f(X_{t_q}) \mathbf{X}_{t_q,t_{q+1}}^1(i) + \sum_{k=2}^{N} \partial_{i_k\ldots i_1} f(X_{t_q}) \mathbf{X}_{t_q,t_{q+1}}^{k}(i_k, \ldots, i_1) \right]
\]

- Compute the corrections between \( \partial_{i_k\ldots i_1} f(X_{t_q}) \mathbf{X}_{t_q,t_{q+1}}^{k}(i_k, \ldots, i_1) \) and \( \partial_{i_k\ldots i_1} f(X_{t_q}) \triangleleft\mathbf{X}_{t_q,t_{q+1}}^{k}(i_k, \ldots, i_1) \) by Wick calculus.

- Show that in the limit \( |\Pi| \rightarrow 0 \), only a term \( \frac{1}{2} \int_s^t \Delta f(x_u) dR_u \) is relevant.