On rough paths above fractional Brownian motion

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Joint work with: David Nualart

Sketch of the talk

1. Introduction
   - Brief rough paths review
   - Rough paths and fractional Brownian motion
   - Aim of the talk
2. Divergence of the Levy area: heuristic considerations
3. Rough paths construction for $H \leq 1/4$
   - Main result
   - Construction of the Levy area
   - Perspectives

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Rough path assumptions

Regularity of X: $X \in \mathcal{C}^{\gamma}(\mathbb{R}^{d})$ with $\gamma > 0$.

Iterated integrals: X allows to define

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \ldots, i_n) = \int_{s \le u_1 < \cdots < u_n \le t} dX_{u_1}(i_1)\, dX_{u_2}(i_2) \cdots dX_{u_n}(i_n),$$

for $0 \le s < t \le T$, $n \le \lfloor 1/\gamma \rfloor$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$.

Regularity of the iterated integrals: $\mathbf{X}^{\mathbf{n}} \in \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n})$, where

$$\mathcal{N}[g; \mathcal{C}_2^{\kappa}] \equiv \sup_{0 \le s < t \le T} \frac{|g_{st}|}{|t-s|^{\kappa}}$$

Main rough paths result

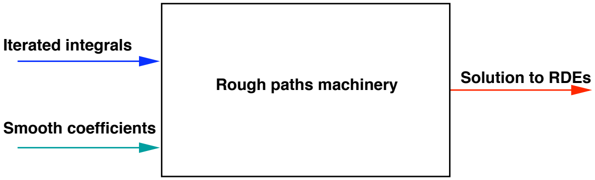
Theorem (loose formulation): Under the assumption of the previous slide, plus regularity assumptions on σ, one can

1. Obtain change of variables formula of Itô's type
2. Solve equations of the form $dY_t = \sigma(Y_t)dX_t$

Moreover, the application

$$\begin{array}{rcl} F : \mathbb{R}^n \times \mathcal{C}_2^{\gamma}(\mathbb{R}^d) \times \cdots \times \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n}) & \longrightarrow & \mathcal{C}^{\gamma}(\mathbb{R}^m) \\ (a, \mathbf{X}^1, \ldots, \mathbf{X}^n) & \longmapsto & Y \end{array}$$

is a continuous map (Lyons-Qian, Friz-Victoir, Gubinelli).



Iterated integrals

Smooth coefficients

Rough paths machinery

Solution to RDEs

Elementary notions

Definition 1: The first order increment associated to X is

$$\mathbf{X}^{\mathbf{1}}_{st} \equiv X_t - X_s.$$

Definition 2: For $g \in \mathcal{C}_1$ (functions of 1 variable), set

$$(\delta g)_{st} = g_t - g_s.$$

Definition 3: For $h \in \mathcal{C}_2$ (functions of 2 variables), set

$$(\delta h)_{sut} = h_{st} - h_{su} - h_{ut}.$$

Meaning of the Levy area

Definition

The Levy area associated to X is an element $\{\mathbf{X}^{\mathbf{2}}_{st}(i_1, i_2);\, s \leq t,\, 1 \leq i_1, i_2 \leq d\}$ satisfying:

(i) The regularity condition $\mathbf{X}^{\mathbf{2}} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$.

(ii) The multiplicative property

$$\delta \mathbf{X}^{\mathbf{2}}_{sut}(i_1, i_2) = \mathbf{X}^{\mathbf{1}}_{su}(i_1)\, \mathbf{X}^{\mathbf{1}}_{ut}(i_2) = \left[X_u(i_1) - X_s(i_1)\right] \left[X_t(i_2) - X_u(i_2)\right].$$

(iii) The geometric relation

$$\mathbf{X}^{\mathbf{2}}_{st}(i_1, i_2) + \mathbf{X}^{\mathbf{2}}_{st}(i_2, i_1) = \mathbf{X}^{\mathbf{1}}_{st}(i_1)\, \mathbf{X}^{\mathbf{1}}_{st}(i_2).$$

Remark: these relations are satisfied when X is smooth, with $\mathbf{X}^{\mathbf{2}}_{st}(i,j) = \int_s^t \delta X_{su}(i)\, dX_u(j)$.

Meaning of the n^{th} iterated integral

Definition

The n^{th} order iterated integral associated to X is an element $\{\mathbf{X}^{\mathbf{n}}_{st}(i_1,\ldots,i_n);\ s \le t,\ 1 \le i_1,\ldots,i_n \le d\}$ satisfying:

(i) The regularity condition $\mathbf{X}^{\mathbf{n}} \in \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n})$.

(ii) The multiplicative property:

$$\delta \mathbf{X}^{\mathbf{n}}_{sut}(i_1,\ldots,i_n) = \sum_{n_1=1}^{n-1} \mathbf{X}^{\mathbf{n_1}}_{su}(i_1,\ldots,i_{n_1})\mathbf{X}^{\mathbf{n-n_1}}_{ut}(i_{n_1+1},\ldots,i_n).$$

(iii) The geometric relation: $\mathbf{X}^{\mathbf{n}}_{st}(i_1,\ldots,i_n)\,\mathbf{X}^{\mathbf{m}}_{st}(j_1,\ldots,j_m)$ can be expressed in terms of higher order integrals

Geometric and weakly geometric rough paths

Remark:

- The stack $\{\mathbf{X}^{\mathbf{n}};\, n \le \lfloor 1/\gamma \rfloor\}$ as defined above is called a weakly geometric rough path above X
  $\hookrightarrow$ allows a reasonable differential calculus
- When there exists a family X^{ε} such that
  - X^{ε} is smooth
  - $\mathbf{X}^{\mathbf{n},\varepsilon}$ is the n^{th} Riemann integral of X^{ε}
  - $\mathbf{X}^{\mathbf{n}} = \lim_{\varepsilon \to 0} \mathbf{X}^{\mathbf{n},\varepsilon}$

  then one has a so-called geometric rough path above X
  $\hookrightarrow$ easier physical interpretation

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Canonical example: fractional Brownian motion

- $B = (B(1), \ldots, B(d))$
- $B(i)$ centered Gaussian process, independence of coordinates
- Variance of the increments:

$$E[|\delta B_{st}(i)|^2] \equiv E[|B_t(i) - B_s(i)|^2] = |t-s|^{2H}$$

- $H^- \equiv$ Hölder-continuity exponent of B
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$, most natural generalization of BM

Motivations: Engineering, Finance, Biophysics

Rough paths and fBm

Situation 1: $H > 1/4$
$\hookrightarrow$ 3 possible geometric rough paths constructions for B.

- Malliavin calculus tools, with Volterra representation
- Regularization or linearization of the fBm path (Coutin-Qian)
- Analytic approximation (Unterberger)

Situation 2: $d = 1$
$\hookrightarrow$ Then one can take $\mathbf{B}^{\mathbf{n}}_{st} = \frac{(B_t - B_s)^n}{n!}$

Situation 3: $H \leq 1/4$, $d > 1$
The constructions by approximation diverge
Existence result by dyadic approximation (Lyons-Victoir)
Recent advances (Unterberger, Nualart-T)
for weakly geometric rough path construction

Sketch of the talk

Aim of the talk

For the case $H \leq 1/4$, $d > 1$:

- Construction of a weakly geometric rough path above B
- Explicit formula for the (substitute to) iterated integrals
- Moments estimates

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fBm kernel

Recall: B is a d-dimensional fBm, with

$$B_t(i) = \int_{\mathbb{R}} K_t(u)\, dW_u(i), \qquad t \geq 0,$$

where W is a d-dimensional Wiener process and

$$\begin{aligned} K_t(u) &\approx (t-u)^{H-\frac{1}{2}}\, \mathbf{1}_{\{0<u<t\}} \\ \partial_t K_t(u) &\approx (t-u)^{H-\frac{3}{2}}\, \mathbf{1}_{\{0<u<t\}}. \end{aligned}$$

fBm differential

Formal differential:

we have $B_v(j) = \int_0^v K_v(u)\, dW_u(j)$ and thus formally for $H > 1/2$

$$\dot{B}_v(j) = \int_0^v \partial_v K_v(u)\, dW_u(j)$$

Formal definition of the area:

Consider $B(i)$. Then formally

$$\begin{aligned} \int_0^T B_v(i)\, dB_v(j) &= \int_0^T B_v(i) \left(\int_0^v \partial_v K_v(u)\, dW_u(j) \right) dv \\ &= \int_0^T \left(\int_u^T \partial_v K_v(u)\, B_v(i)\, dv \right) dW_u(j) \end{aligned}$$

This works for $H > 1/2$ since $H - 3/2 > -1$.

fBm differential for $H < 1/2$

Formal definition of the area for $H < 1/2$:
Use the regularity of $B(i)$ and write

$$\int_0^T B_v(i)\, dB_v(j) = \int_0^T \left(\int_u^T \partial_v K_v(u)\, B_v(i)\, dv \right) dW_u(j)$$

$$= \int_0^T \left(\int_u^T \partial_v K_v(u)\, \delta B_{uv}(i)\, dv \right) dW_u(j) + \int_0^T K_T(u)\, B_u(i)\, dW_u(j).$$

Control of singularity: $\partial_v K_v(u)\, \delta B_{uv}(i) \approx (v-u)^{H-3/2+H}$
$\hookrightarrow$ Definition works for $2H - 3/2 > -1$, i.e. $H > 1/4$!

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$$B_t(i) = \int_{\mathbb{R}} K_t(u)\, dW_u(i), \qquad t \geq 0,$$

where W is a d-dimensional Wiener process and

$$K_t(u) \approx (t-u)^{H-\frac{1}{2}}\, \mathbf{1}_{\{0<u<t\}}.$$

Set also

$$\delta K_{st}(u) = K_t(u) - K_s(u)$$

Building blocks of the rough path

Consider: $H < 1/2$, $j \le n$, $s < t$, and $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$

Some multiple Stratonovich integrals:

$$\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1, \ldots, i_n)$$
$$= (-1)^{j-1} \int_{A^n_j} \prod_{l=1}^{j-1} K_s(u_l)\, \delta K_{st}(u_j) \prod_{l=j+1}^{n} K_t(u_l)\; dW_{u_1}(i_1) \cdots dW_{u_n}(i_n),$$

where A^n_j is the subset of $[0,t]^n$ defined by

$$A^n_j = \Big\{ (u_1, \ldots, u_n) \in [0,t]^n;\; u_j = \min(u_1, \ldots, u_n),$$
$$u_1 > \cdots > u_{j-1},\ \text{and}\ u_{j+1} < \cdots < u_n \Big\}.$$

Main result

Theorem

Consider $H \in (0, 1/2)$ *and for* $2 \leq n \leq \lfloor 1/H \rfloor$, *define*

$$\mathbf{B}^{\mathbf{n}}_{st}(i_1, \ldots, i_n) = \sum_{j=1}^{n} \hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1, \ldots, i_n).$$

Then the family $\{\mathbf{B}^{\mathbf{n}};\, 1 \leq n \leq \lfloor 1/H \rfloor\}$ *defines a weakly geometric rough path over* B.

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Basic aim

Recall: we wish to produce $\mathbf{B}^{\mathbf{2}}_{st} \in \mathbb{R}^{d,d}$ satisfying

1. The algebraic relation

$$\delta \mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2) = \mathbf{B}^{\mathbf{1}}_{su}(i_1)\, \mathbf{B}^{\mathbf{1}}_{ut}(i_2) = [B_u(i_1) - B_s(i_1)]\, [B_t(i_2) - B_u(i_2)]$$

2. $\mathbf{B}^{\mathbf{2}}$ is 2γ-Hölder for any $\gamma < H$

Two interesting increments

Definition: consider the two increments,

$$A^1_{st} := \delta B_{st}(i_1) B_t(i_2), \quad \text{and} \quad A^2_{st} := -B_s(i_1)\, \delta B_{st}(i_2).$$

One can prove easily that

$$\delta A^1_{sut} = \delta A^2_{sut} = \mathbf{B}^{\mathbf{1}}_{su}(i_1)\, \mathbf{B}^{\mathbf{1}}_{ut}(i_2).$$

Problem:
The two increments above are only $\mathcal{C}^\gamma$ for $\gamma < H$ (instead of $\mathcal{C}^{2\gamma}$)

Reordering trick

Expression for A^2: we have (in the Stratonovich sense)

$$\begin{aligned} -B_s(i_1)\,\delta B_{st}(i_2) &= -\int_{\mathbb{R}} K_s(u_1) dW_{u_1}(i_1) \int_{\mathbb{R}} \delta K_{st}(u_2)\, dW_{u_2}(i_2) \\ &= -\int_{\mathbb{R}^2} K_s(u_1)\delta K_{st}(u_2)\, dW_{u_1}(i_1) dW_{u_2}(i_2) \end{aligned}$$

Reordering: replace this quantity by

$$\hat{\mathbf{B}}^{\mathbf{2},2}_{st}(i_1, i_2) \triangleq -\int_{u_2<u_1} K_s(u_1)\,\delta K_{st}(u_2)\, dW_{u_1}(i_1) dW_{u_2}(i_2)$$

In the same way for $A^1_{st} = \delta B_{st}(i_1) B_t(i_2)$, set

$$\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1, i_2) \triangleq \int_{u_1<u_2} \delta K_{st}(u_1)\, K_t(u_2)\, dW_{u_1}(i_1) dW_{u_2}(i_2)$$

Levy area

Proposition

Set

$$\mathbf{B}^{\mathbf{2}}_{st}(i_1, i_2) = \hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1, i_2) + \hat{\mathbf{B}}^{\mathbf{2},2}_{st}(i_1, i_2)$$

Then $\mathbf{B}^{\mathbf{2}}$ *satisfies:*

1. $\delta\mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2) = \mathbf{B}^{\mathbf{1}}_{su}(i_1)\,\mathbf{B}^{\mathbf{1}}_{ut}(i_2)$
2. $\mathbf{B}^{\mathbf{2}}$ *is* 2γ*-Hölder for any* $\gamma < H$

Proof:

Point 1: elementary algebraic computations.

Point 2: moment estimates for Stratonovich integrals w.r.t Wiener process

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Perspectives

Two ongoing works:

1. Construction of a family of (non Gaussian) processes B^ε yielding a geometric rough path for B, within the framework of renormalization theory (Unterberger).
2. Going from Stratonovich to Skorokhod integration theory for $H < 1/4$, in order to check the performances of the differential calculus based on our iterated integrals.