Density estimates for rough differential equations driven by a fBm

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Based on some ongoing works
Outline

1. Introduction
   - SDEs driven by fBm
   - Main results

2. Methodology
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2 Methodology
Equation under consideration

Equation:
Standard differential equation driven by fBm, $\mathbb{R}^n$-valued

$$Y_t = a + \int_0^t V_0(Y_s) \, ds + \sum_{j=1}^d \int_0^t V_j(Y_s) \, dB^j_s,$$  \hspace{1cm} (1)

with

- $t \in [0, 1]$.
- Vector fields $V_0, \ldots, V_d$ in $C^\infty_b$.
- A $d$-dimensional fBm $B$ with $1/3 < H < 1$.
- Note: some results will be extended to $H > 1/4$. 
Fractional Brownian motion

- $B = (B^1, \ldots, B^d)$
- $B^j$ centered Gaussian process, independence of coordinates
- Variance of the increments:
  $$\mathbb{E}[|B^j_{t} - B^j_{s}|^2] = |t - s|^{2H}$$
- $H^- \equiv$ Hölder-continuity exponent of $B$
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$ natural generalization of BM

Remark: FBm widely used in applications
Rough paths assumptions

**Context:** Consider a Hölder path $x$ and

- For $n \geq 1$, $x^n \equiv$ linearization of $x$ with mesh $1/n$
  $\Rightarrow x^n$ piecewise linear.
- For $0 \leq s < t \leq 1$, set

$$x_{st}^{2;n,i;j} \equiv \int_{s < u < v < t} dx_{u}^{n,i} dx_{v}^{n,j}$$

**Rough paths assumption 1:**

- $x$ is a $C^\gamma$ function with $\gamma > 1/3$.
- The process $x^{2;n}$ converges to a process $x^2$ as $n \to \infty$
  $\Rightarrow$ in a $C^{2\gamma}$ space.

**Rough paths assumption 2:**

- Vector fields $V_0, \ldots, V_j$ in $C^\infty_b$. 

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Density estimates for RDEs

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Brief summary of rough paths theory

Main rough paths theorem (Lyons): Under previous assumptions

\[ y^n_t = a + \int_0^t V_0(y^n_u) \, du + \sum_{j=1}^d \int_0^t V_j(y^n_u) \, dx^{n,j}_u. \]

Then

- \( y^n \) converges to a function \( y \) in \( C^\gamma \).
- \( y \) can be seen as solution to

\[ y_t = a + \int_0^t V_0(y_u) \, du + \sum_{j=1}^d \int_0^t V_j(y_u) \, dx^{j}_u. \]
Iterated integrals and fBm

**Nice situation:** \( H > 1/4 \)

\[ \rightarrow 2 \text{ possible constructions for geometric iterated integrals of } B. \]

- Malliavin calculus tools (Ferreiro-Utzet)
- Regularization or linearization of the fBm path (Coutin-Qian)

**Conclusion:** for \( H > 1/4 \), one can solve equation

\[ dY_t = V_0(Y_t) \, dt + V_j(Y_t) \, dB^j_t, \]

in the rough paths sense.

**Remark:** Recent extensions to \( H \leq 1/4 \) (Unterberger, Nualart-T).
Density results for RDEs in the elliptic case

Elliptic assumptions: the matrix $V = (V^1, \ldots, V^d)$ satisfies

\[(EA) \quad V(z) V^*(z) \geq \varepsilon I_d, \quad \text{for all} \quad z \in \mathbb{R}^n\]
\[(TEA) \quad V(z) V^*(\hat{z}) \geq \varepsilon I_d, \quad \text{for all} \quad z, \hat{z} \in \mathbb{R}^n\]

Existing results under elliptic assumptions:

- **Case $H > 1/2$:**
  - Smooth density: Hu-Nualart
  - Further estimates by Baudoin-Ouyang.

- **Case $1/4 < H < 1/2$:**
  - Existence of the density in elliptic and Hörmander cases: Cass-Friz.
  - Smoothness of density under Hörmander’s conditions
    Work in progress with Cass-Hairer-Litterer.
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Upper bounds

Theorem (With F. Baudoin, E. Nualart, C. Ouyang)

Let $B$ be a $d$-dimensional fBm, $Y$ the solution to (1) and $V$ a smooth coefficient satisfying (TEA).

Then

- If $H > 1/2$ and $t \in (0, 1]$ the density $p_t(z)$ of $y_t$ satisfies

  $$p_t(z) \leq \frac{c_1}{t^{nH}} \exp \left( - \frac{c_2 (z - a)^2}{t^{2H}} \right)$$

- If $1/4 < H < 1/2$ and $t \in (0, 1]$ the density $p_t(z)$ of $y_t$ satisfies

  $$p_t(z) \leq \frac{c_1}{t^\alpha} \exp \left( - \frac{c_2 (z - a)^{2H+1}}{t^{2H}} \right)$$

Note: we are still working on $\alpha$. 
Lower bounds

**Theorem (With M. Besalú, A. Kohatsu)**

Let $B$ be a $d$-dimensional fBm, $Y$ the solution to (1) and $V$ a smooth coefficient satisfying (EA).

Then

- If $H > 1/2$ and $t \in (0, 1]$ the density $p_t(z)$ of $y_t$ satisfies

\[ p_t(z) \geq \frac{c_3}{t^{nH}} \exp \left( - \frac{c_4 (z - a)^2}{t^{2H}} \right) \]

- Case $1/4 < H < 1/2$: scary for the moment.
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Bally-Kohatsu’s method: Take \( t = 1 \) and consider

- A partition \( 0 = t_0 < \cdots < t_n = 1 \) of \([0, 1]\), for \( n \) large.
- Set \( F_n \equiv Y_{t_n} = Y_1 \)
- Use a decomposition \( F_{j+1} = F_j + I_{j+1} + R_{j+1} \) where
  - \( I_{j+1} \) conditionally Gaussian w.r.t \( \mathcal{F}_{t_j} \)
  - \( R_{j+1} \) is a remainder (in Malliavin calculus sense)
- Use Gaussian convolution in order to spread Gaussian bound.

Natural candidate for decomposition: \( F_j = X_{t_j} \) and

\[
F_{j+1} = F_j + V_k(Y_{t_j}) \left[ B^k_{t_{j+1}} - B^k_{t_j} \right] + \int_{t_j}^{t_{j+1}} \left( V_k(Y_t) - V_k(Y_{t_j}) \right) dB^k_t
\]

Problem: \( \lim_{n \to \infty} \sum_{j=0}^{n-1} \text{Var}(B^k_{t_{j+1}} - B^k_{t_j}) = 0 \)

\( \leftrightarrow \) This decomposition does not capture the total variance.
Lower bound (2)

SDE w.r.t Wiener process: Write stochastic term as

\[ \int_0^t \left( \int_s^t \partial_u K(u, s) V_k(Y_u) \, du \right) \circ dW_s^k \]

in the Stratonovich-Malliavin sense, where \( B_t^k = \int_0^t K(t, u) dW_u^k \).

Appropriate decomposition: Set

- \( \eta_j(u) \equiv \inf(u, t_j) \)
- \( g_{j,s}^k \equiv \int_s^t \partial_u K(u, s) V_k(Y_{\eta_j(u)}) \, du \)

Then

\[ F_{j+1} = F_j + I_{j+1} + R_{j+1} \]

with

\[ F_j = \int_0^{t_j} g_{j,s}^k \circ dW_s^k, \quad I_{j+1} = \int_{t_j}^{t_{j+1}} g_{j,s}^k \circ dW_s^k \]
Upper bound

General bound:

\[ p_t(z) \leq c \cdot P\left( |Y_t - a| \geq |z| \right)^{1/2} \cdot \left\| \frac{1}{\gamma_t^{-1}} \right\|_{n,2^{n+2}} \cdot \left\| D Y_t \right\|_{n,2^{n+2}} \]

Ingredients for the proof:

- Concentration for \( Y_t \)
- Bounds for derivatives (results by Cass-Lyons-Litterer)
- Functional inequalities for \( \mathcal{H} \)