On rough PDEs

Samy Tindel

University of Nancy

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Joint work with: Aurélien Deya and Massimiliano Gubinelli

Sketch of the talk

1. Introduction
   - Pathwise type stochastic PDEs
   - Main aim
2. Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case

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Equation under consideration

Equation:

- $dy_t = Ay_t dt + f(y_t) dx_t$, with $y_0 = \psi$
- $t \in [0, T]$, $y_t \in \mathcal{B}$, with $\mathcal{B}$ of the form L^p
- $A =$ Laplace operator, with semi-group $(S_t)_{t \geq 0}$
- f from $\mathcal{B}$ to an operator space
- x general noise, γ-Hölder continuous, $\gamma > 1/3$

Mild formulation:

$$y_t = S_t \psi + \int_0^t S_{ts} f(y_s) dx_s, \quad \text{with} \quad S_{ts} := S_{t-s}$$

Motivations: Engineering, Finance, Biophysics

Methods of resolution (1)

- Brownian case
  Peszat-Zabczyk, Dalang (90s)
  Well understood by probabilistic methods
- Additive noise, x inf-dim fBm, for any H
  Tindel-Tudor-Viens ('03)
  Wiener integrals estimates for fBm
- Non-linear case, inf-dim fBm, $H > 1/2$
  Maslowski-Nualart ('03), smooth noise in space
  Fractional integrals
- Linear cases, x finite-dim
  Caruana-Friz ('08-'09)
  Abstract setting of rough paths
- Nonlinear cases, x finite-dim, A generates a group
  Teichmann ('09)
  Abstract setting of rough paths

Methods of resolution (2)

- Non-linear case, x γ-Hölder path with $\gamma > 1/2$
  Non-smooth noise in space
  Young integration in infinite dimension
  Local solution, due to the fact that f is only locally Lipschitz
  Lejay-Gubinelli-Tindel ('06)
- Wave equation, $d = 1$, x space-time noise with $\gamma > 1/2$
  Young integration in the plane
  Quer-Tindel ('07)
- Genuine rough paths setting for SPDEs
  Applications: global solution for fBm, $H > 5/6$
  Brownian case, specific cases of f
  Gubinelli-Tindel ('10)

Sketch of the talk

General aim

- Existence and uniqueness results for

$$dy_t = Ay_t dt + f(y_t) dx_t \tag{1}$$

- $y_t \in$ function in space, Hölder continuous in time
- A Laplace operator, f non-linear coefficient
- x_t finite dimensional, γ-Hölder continuous in time, $\gamma > 1/3$
- Application: $x \equiv$ fBm, with $H > 1/3$

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Sketch of the talk

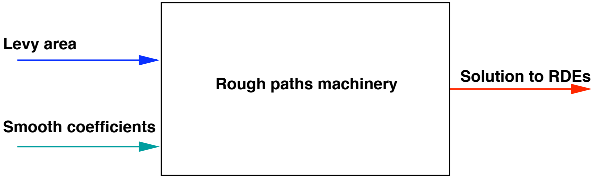
The rough paths black box for SDEs

Hypothesis: $x \in \mathcal{C}^{\gamma}(\mathbb{R}^d)$ with $\gamma > 1/3$
x allows to define a Levy area $\mathbf{x^2} \in \mathcal{C}^{2\gamma}(\mathbb{R}^{d\times d}) \equiv \int dx \int dx$
Coefficient $\sigma \in C_b^3$

Main rough paths theorem: Let y be the solution to $dy = \sigma(y)\, dx$.
Then (Lyons-Qian, Friz-Victoir, Gubinelli)

$$F : \mathbb{R}^n \times \mathcal{C}^{\gamma}(\mathbb{R}^d) \times \mathcal{C}^{2\gamma}(\mathbb{R}^{d\times d}) \longrightarrow \mathcal{C}^{\gamma}(\mathbb{R}^n), \quad (a, x, \mathbf{x^2}) \mapsto y$$

is a continuous map



State space for the solution (SPDEs)

L^p space: $\mathcal{B}_p := L^p(\mathbb{R}^n)$

Fractional Sobolev spaces: for $\alpha \in [0, 1/2)$,

$$\mathcal{B}_{\alpha,p} := \mathcal{W}^{2\alpha,p}(\mathbb{R}^n) = [\mathrm{Id} - \Delta]^{-\alpha} \left(L^p(\mathbb{R}^n) \right)$$

Action of the heat semigroup:
Contraction: S_t contraction on $\mathcal{B}_{\alpha,p}$
Regularization: $\|S_t \varphi\|_{\mathcal{B}_{\alpha,p}} \leq c_\alpha t^{-\alpha} \|\varphi\|_{\mathcal{B}_p}$.

A (slightly) unusual formulation

- We solve equation (1) under the form:

$$y_t = S_t\psi + \sum_{i=1}^{N} \int_0^t S_{ts}\, dx_s^i\, f_i(y_s), \tag{2}$$

- f_i function from $\mathcal{B}_p$ to $\mathcal{B}_p$
- x^i scalar noise
- Formulation (2) fits better to rough path type expansions

Example of nonlinear term: $[f(\varphi)](\xi) = \sigma(\xi, \varphi(\xi))$, where $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ rapidly decreasing function in ξ.

Operator valued Levy area

First order integrals: $a_{us} = S_{us} - \mathrm{Id}$

$$\begin{aligned} X_{ts}^{x,i}(\varphi) &= \int_s^t S_{tu}(\varphi)\, dx_u^i \\ X_{ts}^{xa,i}(\varphi,\psi) &= \int_s^t S_{tu}\left[a_{us}(\varphi)\cdot\psi\right]\, dx_u^i \end{aligned}$$

Second order integral: $\delta x_{us} = x_u - x_s$

$$X_{ts}^{xx,ij}(\varphi) = \int_s^t S_{tu}(\varphi)\, \delta x_{us}^j\, dx_u^i$$

More specifically: if $g :=$ heat kernel

$$\left[X_{ts}^{xx,ij}(\varphi)\right](\xi) = \int_s^t \left(\int_{\mathbb{R}^n} g_{t-u}(\xi-\eta)\varphi(\eta)\, d\eta \right) \delta x_{us}^j\, dx_u^i$$

General result (loose formulation)

Theorem

Assume

- x *allows to define* X^x, X^{xa} *and* X^{xx} *(operator valued on* $\mathcal{B}_{\gamma,p}$*)*
- $X^x \in \mathcal{C}^\gamma$ *and* $X^{xa}, X^{xx} \in \mathcal{C}^{2\gamma}$
- $\gamma > 1/3$
- f *regular enough*

Then the equation

$$dy_t = Ay_t dt + dx_t\, f(y_t)$$

has a unique local solution y *on* $[0, T]$ *(global if* $\gamma > 1/2$*)*

- y *is a continuous function of* X^x, X^{xa} *and* X^{xx}

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Examples of application

Space-time dependence:
$\{\sigma^j;\, j \leq N\}$ collection of rapidly decreasing smooth functions
Generic form of the noise: $x_t = \sum_{j=1}^{N} \sigma^j\, B_t^j$

Application in the Young setting: $B^j \equiv$ fBm with $H > 1/2$

Application in the rough setting: $B^j \equiv$ fBm with $H > 1/3$

Equation which can be solved:

$$\partial_t y_t(\xi) = \Delta y_t(\xi) + \sum_{j=1}^{N} \sigma^j(\xi)\, f(y_t(\xi))\, dB_t^j$$

Sketch of the talk

Setting

Notational simplification:

- $x \in \mathcal{C}^\gamma$ with $\gamma > 1/2$, 1-dimensional
- $[f(y)](\xi) = \sigma(\xi, y(\xi))$.

Equation we wish to solve:

$$y_t = S_t\psi + \int_0^t S_{ts}\, dx_s\, f(y_s)$$

Hypothesis:
The solution y_t exists in a space $\mathcal{C}^\gamma([0, T]; \mathcal{B}_p)$, with $\gamma > 1/2$

Main step: define the integral $\int_0^t S_{ts}\, dx_s\, f(y_s)$
$\hookrightarrow$ fixed point argument

Formal computations

One way to catch the regularity of the solution y:

$$
\begin{aligned}
(\hat{\delta} y)_{ts} &:= y_t - S_{ts} y_s = \int_s^t S_{tu}\, dx_u\, f(y_u) \\
&= \left(\int_s^t S_{tu}\, dx_u \right) f(y_s) + \int_s^t S_{tu}\, dx_u\, \delta(f(y))_{us} \\
&= X^x_{ts}\, f(y_s) + y^\sharp_{ts}
\end{aligned}
$$

Definition of the terms:

- $X^x_{ts}\, f(y_s)$ well-defined as long as
  X^x is well-defined as an operator acting on $\mathcal{B}_p$
- $y^\sharp_{ts}$ defined as a Young integral if $\gamma > 1/2$
  with Hölder regularity > 1

Integration result

Proposition

Assume that x allows to build X^x *such that for* $\gamma > 1/2$,

- *For any* $\alpha \in [0, 1/2)$ *s.t.* $2\alpha p > 1$, $X^{x,i} \in \mathcal{C}^\gamma(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p}))$
- *The algebraic relation* $\hat{\delta} X^{x,i} = 0$ *is satisfied.*

Consider $z \in \mathcal{C}_1^0(\mathcal{B}_{\gamma,p}) \cap \mathcal{C}_1^\gamma(\mathcal{B}_p)$.

Then the element $\int_s^t S_{tu} dx_u z_u$

1. *Is well-defined as a Young integral*
2. *Defines an element of* $\mathcal{C}^\gamma$, *linearly bounded in terms of z*
3. *Is limit of* $\sum_{(t_k)\in\Pi} S_{tt_{k+1}} X^{x,i}_{t_{k+1}t_k} z^i_{t_k}$ *along partitions* Π