On rough PDEs

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Sketch of the talk

1 Introduction
   - Pathwise type stochastic PDEs
   - Main aim

2 Description of the results
   - Abstract setting
   - Applications
   - Heuristic computations in the Young case
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Equation under consideration

Equation:
- \( dy_t = Ay_t dt + f(y_t)dx_t \), with \( y_0 = \psi \)
- \( t \in [0, T] \), \( y_t \in \mathcal{B} \), with \( \mathcal{B} \) of the form \( L^p \)
- \( A = \) Laplace operator, with semi-group \( (S_t)_{t \geq 0} \)
- \( f \) from \( \mathcal{B} \) to an operator space
- \( x \) general noise, \( \gamma \)-Hölder continuous, \( \gamma > 1/3 \)

Mild formulation:

\[
y_t = S_t \psi + \int_0^t S_{ts} f(y_s) dx_s, \hspace{1cm} \text{with} \hspace{1cm} S_{ts} := S_{t-s}
\]

Motivations: Engineering, Finance, Biophysics
Methods of resolution (1)

- **Brownian case**
  - Peszat-Zabczyk, Dalang (90s)
  - Well understood by probabilistic methods

- **Additive noise, \( x \) inf-dim fBm, for any \( H \)**
  - Tindel-Tudor-Viens ('03)
  - Wiener integrals estimates for fBm

- **Non-linear case, inf-dim fBm, \( H > 1/2 \)**
  - Maslowski-Nualart ('03), smooth noise in space
  - Fractional integrals

- **Linear cases, \( x \) finite-dim**
  - Caruana-Friz ('08-'09)
  - Abstract setting of rough paths

- **Nonlinear cases, \( x \) finite-dim, \( A \) generates a group**
  - Teichmann ('09)
  - Abstract setting of rough paths
Methods of resolution (2)

- **Non-linear case, $x \gamma$-Hölder path with $\gamma > 1/2**
  - Non-smooth noise in space
  - Young integration in infinite dimension
  - **Local solution**, due to the fact that $f$ is only *locally Lipschitz*
  - Lejay-Gubinelli-Tindel ('06)

- **Wave equation, $d = 1$, $x$ space-time noise with $\gamma > 1/2**
  - Young integration in the plane
  - Quer-Tindel ('07)

- **Genuine rough paths setting for SPDEs**
  - Applications: global solution for fBm, $H > 5/6$
  - Brownian case, *specific cases of $f$*
  - Gubinelli-Tindel ('10)
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General aim

- Existence and uniqueness results for
  \[ dy_t = Ay_t dt + f(y_t)dx_t \]  

  \( y_t \in \) function in space, Hölder continuous in time

  - \( A \) Laplace operator, \( f \) non-linear coefficient
  - \( x_t \) finite dimensional, \( \gamma \)-Hölder continuous in time, \( \gamma > 1/3 \)
  - Application: \( x \equiv fBm \), with \( H > 1/3 \)
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The rough paths black box for SDEs

**Hypothesis:** \( x \in C^\gamma(\mathbb{R}^d) \) with \( \gamma > 1/3 \)

\( x \) allows to define a Levy area \( x^2 \in C^{2\gamma}(\mathbb{R}^{d \times d}) \equiv \int dx \int dx \)

Coefficient \( \sigma \in C^3_b \)

**Main rough paths theorem:** Let \( y \) be the solution to \( dy = \sigma(y) \, dx \).

Then (Lyons-Qian, Friz-Victoir, Gubinelli)

\[
F : \mathbb{R}^n \times C^\gamma(\mathbb{R}^d) \times C^{2\gamma}(\mathbb{R}^{d \times d}) \rightarrow C^\gamma(\mathbb{R}^n), \quad (a, x, x^2) \mapsto y
\]

is a continuous map
State space for the solution (SPDEs)

$L^p$ space: $\mathcal{B}_p := L^p(\mathbb{R}^n)$

Fractional Sobolev spaces: for $\alpha \in [0, 1/2)$,

$$\mathcal{B}_{\alpha,p} := \mathcal{W}^{2\alpha,p}(\mathbb{R}^n) = [\text{Id} - \Delta]^{-\alpha}(L^p(\mathbb{R}^n))$$

Action of the heat semigroup:

Contraction: $S_t$ contraction on $\mathcal{B}_{\alpha,p}$

Regularization: $\|S_t\varphi\|_{\mathcal{B}_{\alpha,p}} \leq c_\alpha t^{-\alpha} \|\varphi\|_{\mathcal{B}_p}$. 
A (slightly) unusual formulation

- We solve equation (1) under the form:

\[ y_t = S_t \psi + \sum_{i=1}^{N} \int_{0}^{t} S_{ts} \, dx^i_s \, f_i(y_s), \]  

(2)

- \( f_i \) function from \( B_p \) to \( B_p \)
- \( x^i \) scalar noise
- Formulation (2) fits better to rough path type expansions

Example of nonlinear term: \([f(\varphi)](\xi) = \sigma(\xi, \varphi(\xi)), \) where \( \sigma : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) rapidly decreasing function in \( \xi. \)
Operator valued Levy area

First order integrals: \( a_{us} = S_{us} - \text{Id} \)

\[
X_{ts}^{x,i}(\varphi) = \int_s^t S_{tu}(\varphi) \, dx_u^i \\
X_{ts}^{xa,i}(\varphi, \psi) = \int_s^t S_{tu} [a_{us}(\varphi) \cdot \psi] \, dx_u^i
\]

Second order integral: \( \delta x_{us} = x_u - x_s \)

\[
X_{ts}^{xx,ij}(\varphi) = \int_s^t S_{tu}(\varphi) \, \delta x_{us}^j \, dx_u^i
\]

More specifically: if \( g := \) heat kernel

\[
\left[ X_{ts}^{xx,ij}(\varphi) \right](\xi) = \int_s^t \left( \int_{\mathbb{R}^n} g_{t-u}(\xi - \eta) \varphi(\eta) \, d\eta \right) \, \delta x_{us}^j \, dx_u^i
\]
General result (loose formulation)

Theorem

Assume

- \( x \) allows to define \( X^x, X^{xa} \) and \( X^{xx} \) (operator valued on \( \mathcal{B}_{\gamma,p} \))
- \( X^x \in C^\gamma \) and \( X^{xa}, X^{xx} \in C^{2\gamma} \)
- \( \gamma > 1/3 \)
- \( f \) regular enough

Then the equation

\[
dy_t = Ay_t dt + dx_t f(y_t)
\]

has a unique local solution \( y \) on \( [0, T] \) (global if \( \gamma > 1/2 \))

- \( y \) is a continuous function of \( X^x, X^{xa} \) and \( X^{xx} \)
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Examples of application

Space-time dependence:
\{\sigma^j; j \leq N\} collection of rapidly decreasing smooth functions

Generic form of the noise: \( x_t = \sum_{j=1}^{N} \sigma^j B_t^j \)

Application in the Young setting: \( B_j \equiv fBm \) with \( H > 1/2 \)

Application in the rough setting: \( B_j \equiv fBm \) with \( H > 1/3 \)

Equation which can be solved:

\[
\partial_t y_t(\xi) = \Delta y_t(\xi) + \sum_{j=1}^{N} \sigma^j(\xi) f(y_t(\xi)) dB_t^j
\]
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Setting

Notational simplification:
- \( x \in C^\gamma \) with \( \gamma > 1/2 \), 1-dimensional
- \([f(y)](\xi) = \sigma(\xi, y(\xi))\).

Equation we wish to solve:

\[
y_t = S_t \psi + \int_0^t S_{ts} \, dx_s \, f(y_s)
\]

Hypothesis:
The solution \( y_t \) exists in a space \( C^\gamma([0, T]; B_p) \), with \( \gamma > 1/2 \)

Main step: define the integral \( \int_0^t S_{ts} \, dx_s \, f(y_s) \)
\( \rightarrow \) fixed point argument
Formal computations

One way to catch the regularity of the solution $y$:

$$(\hat{\delta}y)_{ts} := y_t - S_{ts}y_s = \int_s^t S_{tu} \, dx_u \, f(y_u)$$

$$= \left(\int_s^t S_{tu} \, dx_u\right) f(y_s) + \int_s^t S_{tu} \, dx_u \, \delta(f(y))_{us}$$

$$= X^x_{ts} f(y_s) + y^\sharp_{ts}$$

Definition of the terms:

- $X^x_{ts} f(y_s)$ well-defined as long as $X^x$ is well-defined as an operator acting on $B_p$
- $y^\sharp_{ts}$ defined as a Young integral if $\gamma > 1/2$
  with Hölder regularity $> 1$
Integration result

Proposition

Assume that $x$ allows to build $X^x$ such that for $\gamma > 1/2$,

- For any $\alpha \in [0, 1/2)$ s.t. $2\alpha p > 1$, $X^{x,i} \in C^\gamma(\mathcal{L}(B_\alpha,p, B_\alpha,p))$
- The algebraic relation $\hat{\delta}X^{x,i} = 0$ is satisfied.

Consider $z \in C^0_1(B_{\gamma,p}) \cap C^\gamma_1(B_p)$.

Then the element $\int_s^t S_{tu}dx_uz_u$

1. Is well-defined as a Young integral
2. Defines an element of $C^\gamma$, linearly bounded in terms of $z$
3. Is limit of $\sum_{(t_k) \in \Pi} S_{tt_{k+1}}X_{t_{k+1}t_k}z_{t_k}^{i}$ along partitions $\Pi$