Approximations of fractional stochastic differential equations

Samy Tindel

University of Nancy

Probability Seminar - Kansas 2009

Sketch

1. Introduction
2. Some notions of rough path analysis
3. Approximation of the Levy area
4. Approximation schemes

Sketch

1. Introduction
2. Some notions of rough path analysis
3. Approximation of the Levy area
4. Approximation schemes

Equation under consideration

$$y_t = a + \int_0^t \sigma(y_s) dx_s + \int_0^t b(y_s) ds, \quad t \in [0, T] \tag{1}$$

- $a \in \mathbb{R}^n$ initial condition
- b, σ smooth coefficients
- $x = B = (B^1, \ldots, B^d)$ d-dimensional fractional Brownian motion
- B^i iid fractional Brownian motions, $H > 1/3$

Motivations: Engineering, Finance

Some motivations

- Electrical engineering
  Denk, Meintrup y Schäffler (2003):
  Modeling, Simulation and Optimization of Integrated Circuits
- Financial time series
  Bayraktar, Poor, Sircar (2004):
  Estimating the fractal dimension of the S&P 500 Index
  using wavelet analysis
- Biophysics
  Kou (2008):
  Stochastic modeling in nanoscale biophysics:
  subdiffusion within proteins
- Biophysics 2 (with $H \approx 0.34$)
  Odde, Tanaka, Hawkins, Buettner (1996):
  Stochastic dynamics of the nerve growth cone and its
  microtubules during neurite outgrowth.

Definition of the fractional Brownian motion

- B^i centered Gaussian process
- Variance of the increments:

$$E[|(\delta B)_{st}|^2] \equiv E[|B_t - B_s|^2] = |t-s|^{2H}$$

- $H^- \equiv$ Hölder-continuity exponent of B
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$, pathwise methods (rough paths) in order to solve (1)
- On rough paths theory:
    - Lyons-Qian, Coutin-Qian, Lejay, Baudoin, Friz-Victoir
    - Feyel-de la Pradelle, Gubinelli

Aim of the talk

1. Recall how Levy areas enter into rough paths analysis: Follows essentially Gubinelli's work
2. Approximations of fractional SDEs
    - Approximation of the Levy area (with A. Neuenkirch, J. Unterberger)
    - Implementable approximation of fractional SDEs (with A. Neuenkirch, A. Deya)
    - Weak approximations:
        - Stroock's type (with X. Bardina, I. Nourdin, C. Rovira)
        - Donsker's type (with X. Bardina, C. Rovira)

Sketch

1. Introduction
2. Some notions of rough path analysis
3. Approximation of the Levy area
4. Approximation schemes

Formal computations (1)

Notational simplification: x is γ-Hölder process, $y_t \in \mathbb{R}$ and

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s$$

Hypothesis:

The solution y_t exists in a space $C^\gamma([0,T])$, with $\gamma < H$

Thus:

$$\begin{aligned}
(\delta y)_{st} &\equiv y_t - y_s = \int_s^t \sigma(y_v) dx_v \\
&= \sigma(y_s)(\delta x)_{st} + \int_s^t [\sigma(y_v) - \sigma(y_s)] dx_v \\
&= \zeta_s (\delta x)_{st} + r_{st}
\end{aligned}$$

Regularity of the coefficients:

$\zeta = \sigma(y)$: bounded, γ-Hölder, r: 2γ-Hölder

Formal computations (2)

Let z be such that (controlled process):

$$(\delta z)_{st} = \zeta_s (\delta x)_{st} + r_{st}, \quad \text{with} \quad \zeta \in C^{\gamma}, \; r \in C^{2\gamma} \tag{2}$$

Formally:

$$\begin{aligned} \int_s^t z_v \, dx_v &= z_s (\delta x)_{st} + \int_s^t (\delta z)_{sv} \, dx_v \\ &= z_s (\delta x)_{st} + \zeta_s \int_s^t (\delta x)_{sv} \, dx_v + \int_s^t r_{sv} \, dx_v \\ &= z_s (\delta x)_{st} + \zeta_s \mathbf{x}^{\mathbf{2}}_{st} + \int_s^t r_{sv} \, dx_v \end{aligned}$$

Formal computations (3)

If we are allowed to write:

$$\int_s^t z_v dx_v = z_s (\delta x)_{st} + \zeta_s \mathbf{x}^{\mathbf{2}}_{st} + \int_s^t r_{sv} dx_v$$

Then we have:

- $z_s(\delta x)_{st}$ trivially defined
- $\zeta_s \mathbf{x}^{\mathbf{2}}_{st}$ easily defined, provided a Levy area $\mathbf{x}^{\mathbf{2}}$ exists
- $\int_s^t r_{sv} dx_v$ defined in a generalized Young sense if $r \in C^{2\gamma}$, $x \in C^{\gamma}$ and $3\gamma > 1$

Remark

- *These claims can be rigorously proved, and allow to define $\int_s^t z_v dx_v$ for a controlled process z*
- *The SDE can be solved thanks to a fixed point argument, in the class of processes admitting the decomposition (2)*

Levy area: important facts

Hypothesis: $x \in \mathcal{C}_1^{\gamma}(\mathbb{R}^d)$ allows to define $\mathbf{x^2} = " \int dx \int dx"$ such that:

- $\mathbf{x^2}$ is an element of $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$, where

$$\|f\|_\mu = \sup_{s,t\in[a,b]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu = \{f \in \mathcal{C}_2;\ \|f\|_\mu < \infty\}.$$

- Algebraic property of $\mathbf{x^2}$: for $0 \le s < u < t \le T$

$$\mathbf{x}^{\mathbf{2}}_{st}(i,j) - \mathbf{x}^{\mathbf{2}}_{su}(i,j) - \mathbf{x}^{\mathbf{2}}_{ut}(i,j) = (\delta x^i)_{su}(\delta x^j)_{ut}$$

The rough paths black box

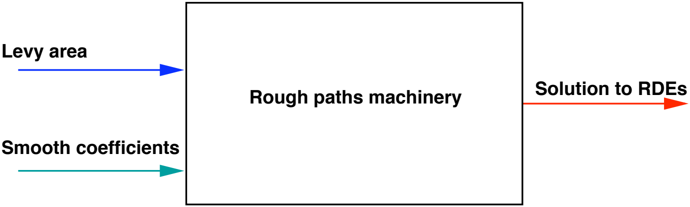
Hypothesis:

$x \in \mathcal{C}_1^{\gamma}(\mathbb{R}^d)$ with $\gamma > 1/3$ and Levy area $\mathbf{x}^2 \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$
Coefficient $\sigma \in C_b^3$

Main rough paths theorem: Let y be the solution to $dy = \sigma(y)\, dx$. Then

$$F : \mathbb{R}^n \times \mathcal{C}_1^{\gamma}(\mathbb{R}^d) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d}) \longrightarrow \mathcal{C}_1^{\gamma}(\mathbb{R}^n), \quad (a, x, \mathbf{x}^2) \mapsto y$$

is a continuous map



Levy area

Rough paths machinery

Solution to RDEs

Smooth coefficients

Sketch

1. Introduction
2. Some notions of rough path analysis
3. Approximation of the Levy area
4. Approximation schemes

Definition of the fractional Levy area

Let B be a 2-dim fBm with $H > 1/4$
The Stratonovich-Levy area $\mathbf{B^2}$ is defined by

$$\mathbf{B}^{\mathbf{2}}_{st}(1,1) = \frac{\left[B^{(1)}_t - B^{(1)}_s\right]^2}{2}, \quad \mathbf{B}^{\mathbf{2}}_{st}(2,2) = \frac{\left[B^{(2)}_t - B^{(2)}_s\right]^2}{2}$$

The term $\mathbf{B}^{\mathbf{2}}_{st}(1,2) = \int_s^t dB^{(1)}_{u_1} \int_s^{u_1} dB^{(2)}_{u_2}$ can be defined either by:

- Malliavin calculus tools
- Regularization or linearization of the fBm path (Coutin-Qian)
- Analytic approximation (Unterberger)
    - Based on a complex-valued Gaussian process Γ
    - Γ defined on the complex upper half plane $\bar{\Pi}^+$, analytic on Π^+
    - The restriction of Γ on $\mathbb{R}$ is a fBm
    - Allows contour deformations and other complex analysis tools

Approximations of the Levy area

Consider $X_T = \mathbf{B}^{\mathbf{2}}_{0T}(2,1) = \int_0^T B_s^{(1)}\, dB_s^{(2)}$.

Natural approximation: partition $\{iT/n;\ i \le n\}$, and

$$X_T^n = \sum_{i=0}^{n-1} B_{iT/n}^{(1)} \left(B_{(i+1)T/n}^{(2)} - B_{iT/n}^{(2)} \right)$$

Another natural (trapezoidal) approximation:

$$\widehat{X}_T^n = \frac{1}{2} \sum_{i=0}^{n-1} \left(B_{iT/n}^{(1)} + B_{(i+1)T/n}^{(1)} \right) \left(B_{(i+1)T/n}^{(2)} - B_{iT/n}^{(2)} \right)$$

L^2 approximation

Theorem

$\mathbf{E}|X_T - X_T^n|^2$ *can be expanded as:*

$$\begin{cases} \alpha_1(H) \cdot T^{4H} \cdot n^{-4H+1} & + & o(n^{-4H+1}) & \mathit{for} & H \in (1/4, 3/4), \\ \frac{9}{128} \cdot T^{4H} \cdot \log(n)\, n^{-2} & + & o(\log(n)\, n^{-2}) & \mathit{for} & H = 3/4, \\ \alpha_2(H) \cdot T^{4H} \cdot n^{-2} & + & o(n^{-2}) & \mathit{for} & H \in (3/4, 1). \end{cases}$$

$\mathbf{E}|X_T - \widehat{X}_T^n|^2$ *can be expanded, for* $H > 1/4$, *as:*

$$\mathbf{E}|X_T - \widehat{X}_T^n|^2 = \alpha_3(H) \cdot T^{4H} \cdot n^{-4H+1} + o(n^{-4H+1}),$$

Sketch of the proof (1)

Set $\mathcal{A}_{st} = \mathbf{B}^{\mathbf{2}}_{st}(2, 1)$.

The error for the Euler scheme can be decomposed as:

$$X_T - X^n_T = \sum_{i=1}^{n} J^n_i$$

where the random variables J^n_i are defined by

$$J^n_i = \int_{iT/n}^{(i+1)T/n} (B^{(1)}_s - B^{(1)}_{i/n})\, dB^{(2)}_s = \mathcal{A}_{(iT)/n,(i+1)T/n}$$

Hence

$$\mathbf{E}[|X_T - X^n_T|^2] = \sum_{i,j} \mathbf{E}[J^n_i J^n_j] = \frac{T^{4H}}{n^{4H}} \sum_{i,j} \mathbf{E}[\mathcal{A}_{i,i+1}\, \mathcal{A}_{j,j+1}]$$

Sketch of the proof (2)

Example of computation:

$$\mathbf{E}\left[\mathcal{A}_{01}\,\mathcal{A}_{12}\right] \asymp$$
$$\int_1^2 \int_0^1 |s_1 - s_2|^{2H-2} \left(s_1^{2H} - 1 - |s_1 - s_2|^{2H} + |s_2 - 1|^{2H}\right)\, ds_2\, ds_1,$$

which can be computed explicitly

Differences in the rates of convergence:

Due to the influence of the off diagonal terms, which grows with H

More specifically, one has to compare

$$\sum_{|i-j|>1,\, 1\leq i,j\leq n} |i-j|^{4H-4} \quad \text{and} \quad n$$

Central and non central limit theorems

Theorem

Let Z be a standard Gaussian random variable.

1. *Case $1/4 < H < 3/4$: the following central limit theorems hold:*

$$\lim_{n\to\infty} n^{2H-1/2}\,(X_T - X_T^n) \overset{(d)}{=} \sqrt{\alpha_1(H)}\,T^{2H}\cdot Z$$

2. $\lim_{n\to\infty} n(\log(n))^{-1/2}\,(X_T - X_T^n) \overset{(d)}{=} \frac{3}{4\sqrt{8}}T^{2H}\cdot Z$ *for* $H = 3/4$.
3. *Case $H > 3/4$: let R_1 and R_2 be two independent Rosenblatt processes. Then it holds*

$$\lim_{n\to\infty} n\,(X_T - X_T^n) \overset{(d)}{=} \sqrt{2\alpha_4(H)}\,T^{2H}\cdot(R_1 - R_2).$$

Sketch of the proof

Reduction of the problem: Random variables in the second chaos of B
$\hookrightarrow$ one can use Nualart-Peccati's criterion

Criterion: Let $\{Z_n;\ n \geq 1\}$ in the p^{th} chaos of B.
Assume that $\lim_{n\to\infty} \mathbf{E}[Z_n^2] = 1$.
Then $\mathcal{L} - \lim_{n\to\infty} Z_n = \mathcal{N}(0,1)$ if and only if $\lim_{n\to\infty} \mathbf{E}[Z_n^4] = 3$

Computation of the 4^{th} moments: Complex analysis tools and graphical representations

Case $H > 3/4$:
Study of the difference between Euler and trapezoidal scheme
$\hookrightarrow$ Products of increments $(B^{(1)}_{(i+1)/n} - B^{(1)}_{i/n})(B^{(2)}_{(i+1)/n} - B^{(2)}_{i/n})$ appear
Yields the Rosenblatt type behavior

Sketch

1. Introduction
2. Some notions of rough path analysis
3. Approximation of the Levy area
4. Approximation schemes

Euler scheme

Driving signal: m-dimensional fBm B, with $1/3 < H < 1/2$

Equation: $dY_t = \sum_{i=1}^m \sigma^{(i)}(Y_t)\, dB_t^{(i)}, \qquad Y_0 \in \mathbb{R}^d$

Grid: $t_k = \frac{kT}{n}$

Euler scheme: $Y^n_{t_{k+1}} = Y^n_{t_k} + \sum_{i=1}^m \sigma^{(i)}(Y^n_{t_k})(B^{(i)}_{t_{k+1}} - B^{(i)}_{t_k})$

Problem: when $1/3 < H < 1/2$, the Euler scheme is divergent!

A 1-d example: $dY_t = Y_t\, dB_t, \qquad Y_0 = 1$

Exact solution: $Y_t = \exp(B_t)$

Approximation for the example

Euler approximation $t = 1$: $Y_1^n = \prod_{k=0}^{n-1}(1 + (B_{(k+1)/n} - B_{k/n}))$

Asymptotics of the Euler approximation: for $n \in \mathbb{N}$ sufficiently large

$$Y_1^n = \exp\left(\sum_{k=0}^{n-1} \log(1 + (B_{(k+1)/n} - B_{k/n})) \right)$$
$$= \exp\left(B_1 - \frac{1}{2} \sum_{k=0}^{n-1} |B_{(k+1)/n} - B_{k/n}|^2 + \rho_n \right),$$

where $\rho_n \xrightarrow{\text{Prob.}} 0$ for $n \to \infty$ for $H > 1/3$.

But for $H < 1/2$: $\sum_{k=0}^{n-1} |B_{(k+1)/n} - B_{k/n}|^2 \xrightarrow{\text{a.s.}} \infty$

$\Longrightarrow Y_1^n \xrightarrow{\text{Prob.}} 0$

Milstein-Davie scheme

Equation: $dY_t = \sum_{i=1}^m \sigma^{(i)}(Y_t)\, dB_t^{(i)}, \qquad Y_0 \in \mathbb{R}^d$

Milstein-Davie scheme: set $\mathcal{D}^{(i)} = \sum_{l=1}^d \sigma_l^{(i)} \partial_{x_l}$ and

$$\tilde{Y}_{t_{k+1}}^n = \tilde{Y}_{t_k}^n + \sum_{i=1}^m \sigma^{(i)}(\tilde{Y}_{t_k}^n)(B_{t_{k+1}}^{(i)} - B_{t_k}^{(i)}) + \frac{1}{2} \sum_{i,j=1}^m \mathcal{D}^{(i)} \sigma^{(j)}(\tilde{Y}_{t_k}^n)\, \mathbf{B}^{\mathbf{2}}_{t_k t_{k+1}}(i,j)$$

Theorem (Davie; Friz-Victoir): $|\tilde{Y}_T^n - Y_T| \to 0$, with an a.s. rate of convergence $n^{-(3H-1-\varepsilon)}$

Problem: $\mathbf{B}^{\mathbf{2}}_{t_k t_{k+1}}$ cannot be simulated exactly
$\hookrightarrow$ Davie's scheme cannot be implemented directly

An implementable scheme

Approximation:

$$\widehat{Y}^n_{t_{k+1}} = \widehat{Y}^n_{t_k} + \sum_{i=1}^{m} \sigma^{(i)}(\widehat{Y}^n_{t_k}) \left(B^{(i)}_{t_{k+1}} - B^{(i)}_{t_k} \right) + \frac{1}{2} \sum_{i,j=1}^{m} \mathcal{D}^{(i)} \sigma^{(j)}(\widehat{Y}^n_{t_k}) \left(B^{(i)}_{t_{k+1}} - B^{(i)}_{t_k} \right) \left(B^{(j)}_{t_{k+1}} - B^{(j)}_{t_k} \right)$$

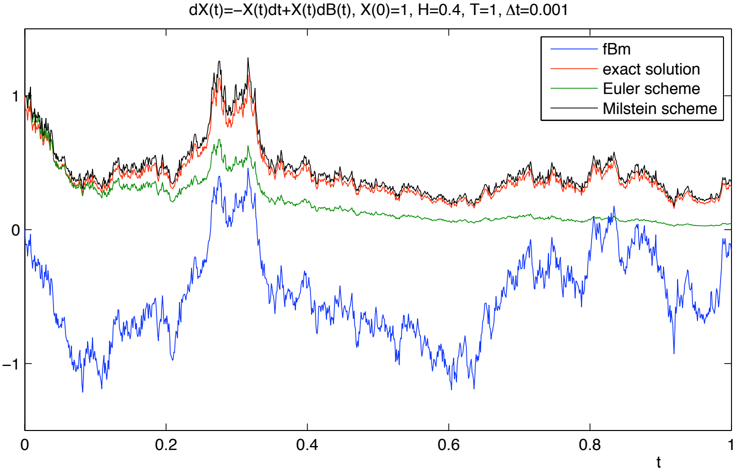
Theorem

Let $\gamma \in (1/4, H)$ *and* $\varepsilon \in (0, H - \gamma)$*.*
Then, there exists a finite r.v. $\eta_{\gamma,\varepsilon,T}$ *such that*

$$\| Y - \widehat{Y}^n \|_\gamma \le \eta_{\gamma,\varepsilon,T} \cdot n^{-(H-\gamma-\varepsilon)}.$$

Convergence illustration

Equation: $dY_t = -Y_t\, dt + Y_t\, dB_t$, hence $Y_t = e^{-t+B_t}$



Sketch of the proof

Observation:
$\bar{B}^n :=$ linear approximation of B based on the same grid kT/n
$\bar{Y} :=$ the solution to $d\bar{Y}_t = \sum_{i=1}^m \sigma^{(i)}(\bar{Y}_t)\, d\bar{B}_t^{(i),n}$
Then $\widehat{Y}^n =$ Davie-Milstein scheme of $\bar{Y}$

Splitting the proof:
(1) Convergence of $\bar{Y}$ towards Y, which amounts to the convergence of $\bar{\mathbf{B}}^{\mathbf{2},n}$ towards $\mathbf{B}^{\mathbf{2}}$
↪ elaboration of Coutin-Qian, and of Section 3
(2) Convergence of $\widehat{Y}^n$ towards $\bar{Y}$
↪ elaboration of Davie's and Friz-Victoir results

Ongoing work: Analysis of the error process
↪ a.s optimal rate of convergence $n^{-(2H-1/2)}$?

Open problem: moment estimates (and linear equations)!