On the multiple overlap function of the SK model

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Abstract

In this note, we prove an asymptotic expansion and a central limit theorem for the multiple overlap $R_{1,\ldots,s}$ of the SK model, defined for given $N, s \geq 1$ by $R_{1,\ldots,s} = N^{-1}\sum_{i\leq N}\sigma_i^1\ldots\sigma_i^s$. These results are obtained by a careful analysis of the terms appearing in the cavity derivation formula, as well as some graph induction procedures. Our method could hopefully be applied to other spin glasses models.

Keywords: SK model, overlap function, cavity method.

MSC:82D30, 60G15.

1 Introduction

The celebrated SK model, which can be seen as a generic spin model with random interactions, also happened to model (together with some of its generalizations) different situations, such as disordered particle systems or neural capacity (see [5], [7]). Briefly speaking, the canonical space of the model is the set $\Sigma_N = \{-1,1\}^N$, called space of configurations, where N is a positive integer which represents the number of spins. A configuration $\sigma = (\sigma_1,\ldots,\sigma_N) \in \Sigma_N$ specifies the values of all spins and the probabilistic feature of the model emerges when we suppose that the spin interactions occur randomly and the energy of each configuration, sum of all the interactions, can be written as

$$-H_N(\sigma) = \frac{1}{N^{\frac{1}{2}}}\sum_{1\leq i<j\leq N} g_{i,j}\sigma_i\sigma_j, \tag{1}$$

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where $g_{i,j}$ is a family of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $1/N^{\frac{1}{2}}$ is a normalization factor.

As usual in statistical mechanics, we associate a Gibbs measure G_N on Σ_N to the Hamiltonian H_N, and this Gibbs measure depends on a parameter β whose meaning is the inverse of the system temperature. The model constructed then starting from (1) has been introduced first [7] in order to describe spin glass systems, i.e. magnetic systems in which the interaction between the magnetic moments are 'in conflict' with each other. Since then, the Physicists have been mostly interested in the behavior of the SK model for large values of β, but let us mention at this point that during all this work, we assume to be in the region of high temperature (i.e. $\beta < 1$) for which a huge amount of information is available (see [8, 2, 6], and the path-breaking papers [3, 9] for the SK model with external field). Let us introduce also some classical notation, which will allow us to state our main results: given a positive integer number n (number of system replicas) and f a function on Σ_N^n, we define $\langle f \rangle$ as the expected value of f with respect to the product measure $dG_N^{\otimes n}$ and $\nu(f)$ as the expected value of $\langle f \rangle$ with respect to the randomness contained in the coefficients $g_{i,j}$, that is $\nu(f) = E[\langle f \rangle]$.

The problem we will deal with starts from the following observation: a large proportion of the structural information about the behavior of Σ_N under G_N is usually obtained by studying the so-called overlap between two configurations σ^1 and σ^2, defined by

$$R_{1,2} \triangleq \frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \ ,$$

which can be also related to the Hamming distance between σ^1 and σ^2 (understood as two independent configurations under $G_N^{\otimes 2}$). And a natural extension of $R_{1,2}$ would be a quantity that measures the correlation among s configurations, for example:

$$R_{1,2,\ldots,s} \triangleq \frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \ldots \sigma_i^s. \tag{2}$$

Clearly, the asymptotic behavior of such a quantity would give us some additional information about the limiting spin system when N goes to infinity. However, in spite of the sharp asymptotic estimates available for $R_{1,2}$, the study of $R_{1,2,\ldots,s}$ for $s > 2$ is still poorly developed, and this paper proposes to make a step in that direction: we will prove the following CLT (central limit theorem) for a family $(R_{\ell_1,\ell_2,\ldots,\ell_s})_{1 \le \ell_1 < \ldots < \ell_s \le n}$ for any $s > 2$.

Theorem 1.1. *Consider two integers* $3 \leq s \leq n$, *and for* $1 \leq \ell_1 < \ldots < \ell_s \leq n$ *some non-negative integers* $k(\ell_1, \ldots, \ell_s)$. *Set* $k = \sum_{\ell_1,\ldots,\ell_s} k(\ell_1, \ldots, \ell_s)$. *Then*

$$\nu\Big(\prod_{\ell_1<\ldots<\ell_s} R_{\ell_1,\ldots,\ell_s}^{k(\ell_1,\ldots,\ell_s)} \Big) - \prod_{\ell_1<\ldots<\ell_s} \frac{a(k(\ell_1, \ldots, \ell_s))}{N^{\frac{k(\ell_1,\ldots,\ell_s)}{2}}} = O(k+1), \tag{3}$$

where we denote by $a(k)$ *the* k^{th} *moment of a standard Gaussian random variable, and where the relation* $g = \mathrm{O}(k)$ *means the existence of a constant* c *such that* $|g| < \frac{c}{N^{k/2}}$.

This Theorem implies that for a typical disorder, a finite family of functions,

$$\hat{R}_{\ell_1,\ldots,\ell_s} = \sqrt{N} R_{\ell_1,\ldots,\ell_s}$$

defined on $(\Sigma_N^n, G_N^{\otimes n})$, with $s \geq 3$, asymptotically looks like an independent family of standard centered Gaussian random variables. It is worth observing that, contrarily to the case $s = 2$ treated in [8], for $s > 2$, the dependence on β in the normalization of $R_{\ell_1,\ldots,\ell_s}$ disappears. This has been a surprise for us. Let us also mention at this point that, from our point of view, the study of multiple overlaps is a natural question, which illustrates the fact that the understanding of the SK model is still far from being complete.

On our way to the proof of Theorem 1.1, we will have to compute the first two terms in the expansion of $\nu(R_{1,2\ldots,s}^2)$, and we will obtain a result which generalizes a result obtained by Talagrand [8, Proposition 2.3.5] for $s = 2$:

Theorem 1.2. *Given* $s \in \mathbb{N}$ *and* $\beta < 1$, *the following relations hold true:*

i) If s *is odd* $(s \geq 3)$, *then*

$$\nu\left(R_{1,2\ldots,s}^2\right) = \frac{1}{N} + O(2p)\ , \ \textit{for all } p \geq 2. \tag{4}$$

ii) If s *is even* $(s = 2k)$, *we have*

$$\nu\left(R_{1,2,\ldots,s}^2\right) = \frac{1}{N} + \frac{c(\beta, s)}{N^k} + O(2k+1), \tag{5}$$

where $c(\beta, k) = \frac{(2k)!}{k!}\left(\frac{\beta^2}{2(1-\beta^2)}\right)^k$.

Theorem 1.2 can be seen in fact as the main contribution of this paper. Indeed, on one hand, once these relations are proven, the announced CLT can be deduced by means of the standard methods introduced e.g. in [8], and one could also argue that it is implicitly contained in [4] (or at least, that the techniques involved in [4] could yield the proof of our Theorem 1.1); notice however that this latter reference relies heavily on the fact that the SK model without external field is considered. On the other hand, our expansion of $\nu(R^2_{1,2\ldots,s})$ is new; it will be achieved thanks to some graph-type methods, which have their own interest in the SK context, and are introduced here for the first time (as far as we know). Furthermore, it seems that our computations don't depend too much on the specific model we have considered, and thus we hope to extend this kind of methodology to other situations, like the p spins models with external field or the perceptron model.

Our paper is organized as follows: In the next section, we introduce some notations and definitions. In the third section, performing a Taylor expansion, we obtain a general expression of $\nu_t(f)$ where f is a function defined on Σ_N^n, and we evaluate the leading term of $\nu(\prod_{i=1}^m \epsilon_{\ell_i}\epsilon_{j_i} R_{\ell_i,j_i})$ for some specific $\ell_i's$ and $j_i's$ (where $\epsilon_l = \sigma_N^l$). The fourth section will be devoted to the computation of $\nu_0(f)$ for a certain class of functions f. Eventually, in the last two sections, we conclude with the proof of Theorems 1.2 and 1.1.

2 Preliminaries

In this section, we will first introduce some notations, and then give briefly some definitions which will be used in the sequel of the paper. Eventually, we will explain the strategy of the proof of Theorem 1.2.

2.1 Smart path and overlap products

In order to expand $\nu(R^2_{1,2\ldots,s})$ in terms of N, the use of Taylor series is certainly a natural idea. So, for a given configuration $\sigma \in \Sigma_N$ and a parameter $t \in [0,1]$, define a new energy function

$$H_{N,t}(\sigma) = \frac{1}{N^{1/2}} \sum_{1\le i<j\le N-1} g_{i,j}\sigma_i\sigma_j + \left(\frac{t}{N}\right)^{1/2} \sigma_N \sum_{1\le i\le N-1} \sigma_i g_{i,N},$$

where the coefficients $g_{i,j}$ are, as before, independent Gaussian standard random variables. Set now

$$G_{N,t}(\{\sigma\}) = \frac{\exp(-\beta H_{N,t}(\sigma))}{Z_{N,t}}, \quad \text{where} \quad Z_{N,t} = \sum_{\sigma\in\Sigma_N} \exp(-\beta H_{N,t}(\sigma)).$$

These random measures induce some averages $\langle f\rangle_t$ and $\nu_t(f)$ defined, for a function $f:\Sigma_N^n\to\mathbb{R}$, by

$$\langle f\rangle_t=\frac{\sum_{\sigma^1,\dots,\sigma^n\in\Sigma_N} f(\sigma^1,\dots,\sigma^n)\exp\left(\sum_{i=1}^n -\beta H_{N,t}(\sigma^i)\right)}{Z_{N,t}^n}$$

and $\nu_t(f)=E[\langle f\rangle_t]$. Define also the overlap functions R_{ℓ_i,j_i} and $R^-_{\ell_i,j_i}$ by

$$R_{\ell_i,j_i}\triangleq\frac{1}{N}\sum_{k\le N}\sigma_k^{\ell_i}\sigma_k^{j_i},\qquad\text{and}\qquad R^-_{\ell_i,j_i}\triangleq\frac{1}{N}\sum_{k\le N-1}\sigma_k^{\ell_i}\sigma_k^{j_i}.$$

Then the function $t\mapsto\nu_t(f)$ can be differentiated in the following way (see [8]):

Proposition 2.1. *Given a function f on Σ_N^n and $t\ge 0$, we have*

$$\nu_t'(f)=\beta^2\sum_{1\le l<l'\le n}\nu_t(f\epsilon_l\epsilon_{l'}R^-_{l,l'})$$
$$-\beta^2 n\sum_{l\le n}\nu_t(f\epsilon_l\epsilon_{n+1}R^-_{l,n+1})+\beta^2\frac{n(n+1)}{2}\nu_t(f\epsilon_{n+1}\epsilon_{n+2}R^-_{n+1,n+2}).$$

This Proposition will the basis of our future expansions.

Apart from the usual overlap function $R_{1,2}$, we will have to introduce a specific notation for some products of overlaps which will appear throughout our computations: given some arbitrary positive integer numbers $\ell_1,j_1,\dots,\ell_m,j_m$ such that $\ell_i\le j_i$ for all $i\le m$, we set

$$S_{\ell_1,j_1,\dots,\ell_m,j_m}\triangleq\prod_{i=1}^m\epsilon_{\ell_i}\epsilon_{j_i}R_{\ell_i,j_i},\quad S^-_{\ell_1,j_1,\dots,\ell_m,j_m}\triangleq\prod_{i=1}^m\epsilon_{\ell_i}\epsilon_{j_i}R^-_{\ell_i,j_i}.\tag{6}$$

Remark 2.2. *The importance of the products $\epsilon_{\ell_i}\epsilon_{j_i}R_{\ell_i,j_i}$ stems basically from Proposition 2.1, in which they appear naturally.*

2.2 Sets and graphs

Our proofs will also make use of two subsets of tuples of positive integers: given a positive integer k, set

$$\Omega^{2k}\triangleq\{(r_1,\dots,r_{2k})\in\mathbb{N}^{2k}|\,r_i\le N,\,r_i\ne r_j\ \text{ if } 1\le i<j\le 2k \text{ and } r_{2u-1}<r_{2u}\text{ for all } u\le k\}\tag{7}$$

and

$$\mathcal{C}_k\triangleq\{\alpha=(\ell_1,j_1,\dots,\ell_m,j_m)\,|\,(H)\text{ holds true }\},\tag{8}$$

where (H) is the following assumption:

- ℓ_i is smaller than j_i for any $i \le m$;
- If α also designates the set $\{\ell_1, j_1, \dots, \ell_m, j_m\}$, then $\{1, 2, \dots, 2k\} \subset \alpha$;
- The only elements of α which appear an odd number of times are $1, 2, \dots, 2k$.

Obviously, the definition of the quantity $S_{\ell_1, j_1, \dots, \ell_m, j_m}$ depends on the sequence $(\ell_1, j_1, \dots, \ell_m, j_m)$. For sake of clarity, we will associate a graph to such kind of sequence, where a graph is understood for us in the following sense:

Definition 2.3. *Let I be a set of positive integers and E be a subset of $I \times I$. We refer to I as the vertex set and to E as the edge set. In addition, if $(i, j) \in E$, assume that $i < j$ and let $\Upsilon : E \to \mathbb{N}^*$ be a function which counts the number of edges of type (i, j). Then, the triple (I, E, Υ) is called a graph. Given a graph (I, E, Υ), for each $J \subseteq I$, $F \subseteq J \times J$ with $F \subseteq E$ and $V : F \to \mathbb{N}^*$ such that for all $e \in F$, $V(e) \le \Upsilon(e)$, we call (J, F, V) a subgraph of (I, E, Υ). Obviously, a subgraph is also a graph.*

Here is now the procedure we will use for our graph construction: pick a sequence $(\ell_1, j_1, \dots, \ell_m, j_m)$ of $2m$ numbers, and assume, for sake of simplicity, that $\ell_i < j_i$ for all $1 \le i \le m$. Define then

- $I = \{\ell_1, j_1, \dots, \ell_m, j_m\}$;
- $E = \{(\ell_i, j_i) | i \le m\}$;
- $\Upsilon((\ell_i, j_i)) = \#\{r \le m | (\ell_i, j_i) = (\ell_r, j_r)\}$.

We denote this graph by $\mathrm{G}((\ell_1, j_1, \dots, \ell_m, j_m))$. In particular, given our set $\mathcal{C}_k$ we can associate the family of graphs $\mathcal{G}_k = \{\mathrm{G(c)} | c \in \mathcal{C}_k\}$.

Let us define some local and global objects on a graph $g = (I, E, \Upsilon)$. Set first

$$N_g(i) \triangleq \sum_{e \in E:\, i \in e} \Upsilon(e) \quad \text{and} \quad N(g) = \sum_{e \in E} \Upsilon(e).$$

Obviously, $N_g(i)$ represents the number of edges having i as an endpoint, and $N(g)$ the total number of edges of g. Furthermore, it is easily checked that $N(g) = \frac{1}{2} \sum_{i \in I} N_g(i)$. Let us also define a quantity, associated to each vertex i, indicating if $N_g(i)$ is an odd number or not:

$$\mathrm{Od}(i) = \frac{1}{2}[N_g(i) \bmod(2)] \quad \text{and} \quad \mathrm{Od}(g) \triangleq \sum_{i \in I} \mathrm{Od}(i).$$

Associated to these notions, some subgraphs of graphs in $\mathcal{G}_k$ will play a special role in the sequel: for each $g \in \mathcal{G}_k$ with $N(g) = m$ and any $u \leq m$, we define

$$S_u(g) \triangleq \{h | h \text{ is a subgraph of } g, \text{ and } N(h) = u\}.$$

Notice that the definitions of the current subsection won't be used until Proposition 4.4. However, we have already introduced them at this point, since they are at the core of our method.

2.3 Strategy of the proof for Theorem 1.2

The proof of our main result Theorem 1.2 is built upon a series of Lemmas and Propositions which will be stated and proved throughout Sections 3 and 4. Since the reader may get lost during these preliminary steps, here is a brief sketch of the methodology we will follow in order to estimate $\nu(R^2_{1,\dots,s})$.

(1) Using the symmetry property among sites, we will check that

$$\nu\left(R^2_{1,2,\dots,s}\right) = \frac{1}{N} + \nu\left(\epsilon_1\epsilon_2\dots\epsilon_s R^-_{1,2,\dots,s}\right).$$

With this relation in mind, our main task will be obviously to estimate the term $\nu(\epsilon_1\epsilon_2\dots\epsilon_s R^-_{1,2,\dots,s})$. We will see that, whenever s is an odd number, the estimation is quite easy, and thus, we will concentrate mainly on the case $s = 2k$.

(2) In order to get an equivalent of $\nu(\epsilon_1\epsilon_2\dots\epsilon_{2k} R^-_{1,2,\dots,2k})$, we will perform a Taylor expansion of this quantity along the smart path defined by ν_t. Then, due to the presence of the products of ε's, we will be able to show that many terms of the expansion vanish, or can be neglected. These preliminary considerations will be developed at Section 3.1, and will lead us to focus essentially on some terms of the form

$$\nu_0\left(U^-_k S^-_\alpha\right) \quad \text{with} \quad U^-_k = \epsilon_1\epsilon_2\dots\epsilon_{2k} R^-_{1,2,\dots,2k},$$

where the multi-index $\alpha = (\ell_1, j_1, \dots, \ell_m, j_m)$ lies in a certain class which will be determined throughout Section 3.

(3) Recall that $\mathcal{C}_k$ has been defined in (8). Then we will prove that, whenever the multi-index $\alpha = (\ell_1, j_1, \dots, \ell_m, j_m)$ belongs to $\mathcal{C}_k$, we have

$$\nu_0(U^-_k S^-_\alpha) = \nu(S_\alpha) + O(2k+1). \tag{9}$$

This will be achieved at Section 4.2, through the introduction of a family of functions, called R-systems, allowing an operational backward induction on the order of multi-indexes defined in (8).

(4) By looking at relation (9), we see that we are left with with the evaluation of the quantities S_α^-. Equivalently, since the random variables S_α^- are stable by multiplication, we have to deal with their covariance structure. This depend mainly on the form of the multi-index α, and after some rather standard computations, we will base our estimates on:

1. An equivalence relation between multi-indexes (see Proposition 3.8).
2. A graph structure on these multi-indexes, which will be used mainly at Section 4.1.

Thanks to the two tools mentioned above, we will be able to analyze precisely the covariance structure of the random variables S_α^-, leading then to the conclusion of our proof by a series of elementary considerations.

3 Some general Taylor expansions

In this section, we will first establish a general expression for the Taylor expansion of the function $t \mapsto \nu_t(f)$ around 0, for a given $f : \Sigma_N^n \to \mathbb{R}$. Then we will identify some negligible terms and give a more explicit expression for the typical term of this expansion. Eventually, we will examine the special case where f is the function $S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}$, and using an induction argument, we will evaluate $\nu(S_{\ell_1,j_1,\ldots,\ell_m,j_m})$.

3.1 General and error term

Let us start this section by giving an extension of Proposition 2.1: for $k, n \geq 1$, define the set $\mathcal{D}_{n,k}$ as

$$\mathcal{D}_{n,k} = \{\alpha = (\ell_1, j_1, \ldots, \ell_k, j_k);\ \ell_i, j_i \leq n + 2k,\ \ell_i < j_i \text{ for all } i \leq k\}. \quad (10)$$

Proposition 3.1. *Let f be a function on Σ_N^n and consider $t \geq 0$. Then the kth derivative of $\nu_t(f)$ can be written as*

$$\nu_t^{(k)}(f) = \sum_{\alpha=(\ell_1,j_1,\ldots,\ell_k,j_k)\in\mathcal{D}_{n,k}} c\,(n,k,\alpha)\,\beta^{2k}\nu_t\left(f S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}\right), \quad (11)$$

where the family $\{c\,(n,k,\alpha)\,;\alpha \in \mathcal{D}_{n,k}\}$ is just a family of $\mathbb{Z}$-valued coefficients.

Proof. The approach used to show the result is an induction argument on k: the case $k = 1$ can be easily shown thanks to Proposition 2.1, and in order to advance the induction, we assume that the result holds for $k = u - 1$. Let us differentiate now a typical term of $\nu_t^{(u-1)}(f)$, of the form

$$c\beta^{2(u-1)}\nu_t\left(g\right), \quad \text{with} \quad g = fS^-_{\ell_1,j_1,\dots,\ell_{u-1},j_{u-1}},$$

where $\ell_1, j_1, \dots, \ell_{u-1}, j_{u-1} \leq n'$, with $n' \equiv n + 2(u-1)$. Thus we get, by means of Proposition 2.1, that

$$\begin{aligned}\nu_t'(g) = \beta^2 \sum_{1\leq l<l'\leq n'} \nu_t(gS^-_{l,l'}) \\ - \beta^2 n' \sum_{l\leq n'} \nu_t(gS^-_{l,n'+1}) + \beta^2 \frac{n'(n'+1)}{2}\nu_t(gS^-_{n'+1,n'+2}),\end{aligned}$$

which is easily seen to be of the form given by (11). □

Let us recall now an estimate for $\nu_t^{(i)}(f)$ which can be found in [8]:

Proposition 3.2. *If f is a function defined on Σ_N^n and $\beta < 1$, then for all $t \in [0,1)$ we have*

$$\left|\nu_t^{(i)}(f)\right| \leq \frac{K(\beta,i,n)}{N^{\frac{i}{2}}}\nu(f^2)^{\frac{1}{2}}.$$

The following estimations for the variables $S_{\ell_1,j_1,\dots,\ell_s,j_s}$ will be also be used several times along the article:

Proposition 3.3. *Given $s \geq 1$ and a family of integers $\ell_1, j_1, \dots, \ell_s, j_s$, we have, for all $\beta < 1$:*

(a) $\nu(S^-_{\ell_1,j_1,\dots,\ell_s,j_s}) = O(s)$

(b) $\nu(S_{\ell_1,j_1,\dots,\ell_s,j_s}) = O(s)$

(c) $\nu_0(S^-_{\ell_1,j_1,\dots,\ell_s,j_s}) = O(s)$.

(d) $\nu_t^{(u)}(S^-_{\ell_1,j_1,\dots,\ell_s,j_s}) = O(u+s)$ *for all* $t \in [0,1]$.

Proof. Relations (a)-(c) are proved in [1]. The last relation follows easily from the previous ones, together with Proposition 3.2. □

3.2 Negligible terms

We will try now to find a class of terms in (11) for which the coefficient $c(n,k,\alpha)$ vanishes. And a basic tool for this kind of identification can be found again in [8]:

Proposition 3.4. *Let f be a function defined on Σ_N^n. Assume $f = f^- f'$ where f^- is a function of the $N-1$-spin system, and f' depends only on $\epsilon_1, \ldots, \epsilon_n$. If $Av f' = 0$ (where Av means average on $\epsilon_1 = \pm 1, \ldots, \epsilon_N = \pm 1$) then*

$$\nu_0(f) = 0.$$

As an application of this proposition, we easily get the following result:

Proposition 3.5. *Let f be a function on Σ_N^n. Assume $f = \epsilon_{m_1} \ldots \epsilon_{m_\jmath} f^-$ with $\jmath$ a positive integer number, f^- a function on the $N-1$-spin system and where all of the m's are different positive integers. Then $\nu_0^{(u)}(f) = 0$ in the following two cases:*

i) $\jmath = 2k$ for $k \in \mathbb{N}$ and $u < k$.

ii) $\jmath$ is an odd number, without any restriction on $u \in \mathbb{N}$.

Proof. First of all, according to Proposition 3.1, $\nu_0^{(u)}(f)$ can be written as a sum of terms of the type

$$c\beta^{2u}\nu_0\left(f^- \prod_{i=1}^{\jmath} \epsilon_{m_i} S^-_{\ell_1,j_1,\ldots,\ell_u,j_u}\right).$$

Now, either if $\jmath = 2k$ and $u < k$, or if $\jmath$ is an odd number, there exist some distinct positive integers $\tilde{m}_1, \ldots, \tilde{m}_v$ such that

$$\nu_0\left(f^- \prod_{i=1}^{t} \epsilon_{m_i} S^-_{\ell_1,j_1,\ldots,\ell_u,j_u}\right) = \nu_0\left(\prod_{i=1}^{v} \epsilon_{\tilde{m}_i} \tilde{f}\right),$$

where $\tilde{f}$ is a function of the $N-1$-spin system. Invoking Proposition 3.4, the previous term vanishes, which ends the proof. □

3.3 A more explicit general term

Our next task here will be to compute the values of some of the constants c's appearing in Proposition 3.1. This will lead us to introduce a relation defined on the $2k$-tuples of positive integers: given two $2k$-tuples of positive integers r and s we will say that $r \sim s$ if for all $1 \le i \le k$ there exists a $1 \le j \le k$ such that $(r_{2i-1}, r_{2i}) = (s_{2j-1}, s_{2j})$, and reciprocally, if for all j there exists a i such that $(s_{2j-1}, s_{2j}) = (r_{2i-1}, r_{2i})$.

Proposition 3.6. *Given $k \ge 1$, recall that Ω^{2k} has been defined at relation (7), and consider a function f defined on Σ_N^{2k}. For $u \le k$, pick an element $w \triangleq (w_1, \dots, w_{2u}) \in \Omega^{2u}$, such that $w_i \le 2k$ for all $i \le 2u$. Then, for all $\beta < 1, t \in [0,1]$, we have*

$$\nu_t^{(u)}(f) = u!\beta^{2u}\nu_t\left(fS^-_{w_1,w_2,\dots,w_{2u-1},w_{2u}}\right) + \beta^{2u}\sum_{r \nsim w} c(r)\nu_t\left(fS^-_{r_1,r_2,\dots,r_{2u-1},r_{2u}}\right), \tag{12}$$

for a family of $\mathbb{Z}$-valued constants $\{c(r), r \nsim w\}$ which vanish except for a finite number of $r \nsim w$.

Proof. Here again, we will use an induction argument on u: the case $u = 1$ is a direct application of Proposition 2.1, since in this case $\Omega^{2u} = \Omega^2 = \{(r_1, r_2); r_1 < r_2\}$, and the only elements $w \in \Omega^2$ satisfying $w_i \le 2k$ are of the form $w = (l, l')$ with $1 \le l < l' \le 2k$.

Now, let us assume (12) holds true for $u = v - 1$. For a given $w = (w_1, \dots, w_{2v}) \in \Omega^{2v}$, let W be the set defined by:

$$W = \{\tilde{w} \in \Omega^{2v-2} \mid \text{ for all } i \le v-1, \text{ there exists } j \le v \text{ such that } (\tilde{w}_{2i-1}, \tilde{w}_{2i}) = (w_{2j-1}, w_{2j})\}.$$

Let us denote also by $\tilde{W}$ the set $W/\sim$. For each $\tilde{w}$ which represents a class in $\tilde{W}$, our induction hypothesis yields

$$\nu_t^{(v-1)}(f) = (v-1)!\beta^{2v-2}\nu_t\left(fS^-_{\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_{2v-3},\tilde{w}_{2v-2}}\right) + \beta^{2v-2}\sum_{\tilde{r} \nsim \tilde{w}} c(\tilde{r})\nu_t\left(fS^-_{\tilde{r}_1,\tilde{r}_2,\dots,\tilde{r}_{2v-3},\tilde{r}_{2v-2}}\right). \tag{13}$$

When we take into account all the possible classes in $\tilde{W}$, we conclude that the sum (13) can be decomposed as:

$$\nu_t^{(v-1)}(f) = (v-1)!\beta^{2v-2} \sum_{\tilde{w}\in\tilde{W}} \nu_t\left(fS^-_{\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_{2v-3},\tilde{w}_{2v-2}}\right) + \beta^{2v-2} \sum_{\tilde{r}\notin\tilde{W}} c(\tilde{r})\nu_t\left(fS^-_{\tilde{r}_1,\tilde{r}_2,\dots,\tilde{r}_{2v-3},\tilde{r}_{2v-2}}\right). \quad (14)$$

Now, for each $\tilde{w}\in\tilde{W}$, we choose the only couple $(\ell_1(\tilde{w}),\ell_2(\tilde{w}))$ such that $(\tilde{w},\ell_1(\tilde{w}),\ell_2(\tilde{w}))\sim w$. Our assumptions also imply that $\ell_1,\ell_2\le 2k$. Furthermore, using the case u equals to one, we can calculate the derivative of each term on the right hand side of (14). Hence, differentiating Equation (14), we get:

$$\begin{aligned}\nu_t^{(v)}(f) &= (v-1)!\beta^{2v} \sum_{\tilde{w}\in W} \nu_t\left(fS^-_{\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_{2v-3},\tilde{w}_{2v-2}}S^-_{\ell_1(\tilde{w}),\ell_2(\tilde{w})}\right)\\ &+ (v-1)!\beta^{2v} \sum_{\tilde{w}\in W}\sum_{l\neq(\ell_1(\tilde{w}),\ell_2(\tilde{w}))} c(l)\nu_t\left(fS^-_{\tilde{w}_1,\tilde{w}_2,\dots,\tilde{w}_{2v-3},\tilde{w}_{2v-2}}S^-_{\ell_1,\ell_2}\right)\\ &+ \beta^{2v}\sum_{\tilde{r}\notin\tilde{W}}\sum_{l=(\ell_1,\ell_2)} c(\tilde{r})\nu_t\left(fS^-_{\tilde{r}_1,\tilde{r}_2,\dots,\tilde{r}_{2v-3},\tilde{r}_{2v-2}}S^-_{\ell_1,\ell_2}\right). \quad (15)\end{aligned}$$

Recall now that, for each $\tilde{w}\in\tilde{W}$, there is a unique couple $(\ell_1(\tilde{w}),\ell_2(\tilde{w}))$ such that $(\tilde{w},\ell_1(\tilde{w}),\ell_2(\tilde{w}))\sim w$. Since $|\tilde{W}|=v$, we conclude that the first term on the right side of (15) is equal to $u!\beta^{2u}\nu_t(fS^-_{w_1,w_2,\dots,w_{2u-1},w_{2u}})$. Moreover, it is easily checked that the other terms in (15) give some contributions of the form

$$\beta^{2u}\sum_{r\nsim w} c(r)\nu_t\left(fS^-_{r_1,r_2,\dots,r_{2u-1},r_{2u}}\right),$$

concluding the proof. □

3.4 The product of overlap functions

Let us focus now on the special case $f=S^-_{\ell_1,j_1,\dots,\ell_k,j_k}$, and let us try to identify some additional negligible terms in our expansion: according to Proposition 3.6, the k^{th} order differentiation of $\nu_t(f)$ brings out some terms of the form

$$\nu_t\left(S^-_{\ell_1,j_1,\dots,\ell_k,j_k}S^-_{r_1,r_2,\dots,r_{2k-1},r_{2k}}\right) = \nu_t\left(S^-_{\ell_1,j_1,\dots,\ell_k,j_k,r_1,r_2,\dots,r_{2k-1},r_{2k}}\right). \quad (16)$$

Recall that Theorem 1.2 claims an expansion up to order $O(2k+1)$, and thus, a natural concern for us will be to establish if the terms of the form (16) are of order $O(2k)$ or not. A first step in that direction will be to replace $S^-_{\ell_1,j_1,\dots,\ell_k,j_k}$ by $S_{\ell_1,j_1,\dots,\ell_k,j_k}$, which can be done thanks to the following:

Proposition 3.7. *Given $\beta < 1$, $s \geq 1$ and a collection of integers $\ell_1, j_1, \ldots, \ell_s, j_s$, we have*

i) $\nu(S^-_{\ell_1,j_1,\ldots,\ell_s,j_s}) = \nu_0(S^-_{\ell_1,j_1,\ldots,\ell_s,j_s}) + O(s+1)$.

ii) $\nu(S^-_{\ell_1,j_1,\ldots,\ell_s,j_s}) = \nu(S_{\ell_1,j_1,\ldots,\ell_s,j_s}) + O(s+1)$.

Proof. i) This relation is an easy consequence of the general expansion given at Proposition 2.1, and of Proposition 3.3 item (d).

ii) Notice that

$$S^-_{\ell_1,j_1,\ldots,\ell_s,j_s} = \prod_{j=1}^{s} S^-_{\ell_i,j_i}.$$

Thus, using the relation $S^-_{l,l'} = S_{l,l'} - \frac{1}{N}$, it is readily checked that

$$\nu\left(S^-_{\ell_1,j_1,\ldots,\ell_s,j_s}\right) = \sum_{u=1}^{s} \sum_{1\leq i_1<\ldots<i_u\leq s} \frac{(-1)^{s-u}}{N^{s-u}} \nu\left(S_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}\right) + \frac{(-1)^s}{N^s}. \tag{17}$$

Applying now Proposition 3.3 for each tuple $(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u})$, it follows that we have, for $u \leq s-1$,

$$\frac{(-1)^{s-u}}{N^{s-u}} \nu\left(S_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}\right) = O(2(s-u)+u). \tag{18}$$

Furthermore, for $1 \leq u \leq s-1$, we have $2(s-u)+u \geq s+1$. Thus, the announced result is easily obtained by plugging (18) into (17). □

Let us compute now the terms of the form (16):

Proposition 3.8. *For $\beta < 1$ and $r, w \in \Omega^{2k}$, if $r \nsim w$, the following relation holds true:*

$$\nu\left(S^-_{r_1,r_2,\ldots,r_{2k-1},r_{2k}} S^-_{w_1,w_2,\ldots,w_{2k-1},w_{2k}}\right) = O(2k+1). \tag{19}$$

On the other hand, if $r \sim w$, we have

$$\begin{aligned} \nu\left(S^-_{r_1,r_2,\ldots,r_{2k-1},r_{2k}} S^-_{w_1,w_2,\ldots,w_{2k-1},w_{2k}}\right) &= \nu\left(\left(S^-_{r_1,r_2,\ldots,r_{2k-1},r_{2k}}\right)^2\right) \\ &= \tfrac{1}{[N(1-\beta^2)]^k} + O(2k+1). \end{aligned} \tag{20}$$

Proof. Let $(\ell_1, j_1, \dots, \ell_{2k}, j_{2k})$ be by definition the tuple $(r_1, \dots, r_{2k}, w_1, \dots, w_{2k})$. Then

$$\nu\left(S^-_{r_1,r_2,\dots,r_{2k-1},r_{2k}} S^-_{w_1,w_2,\dots,w_{2k-1},w_{2k}}\right) = \nu\left(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right).$$

Step 0: we can assume w is a permutation of r.
Proposition 3.7 item i) yields

$$\nu\left(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) = \nu_0\left(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) + O(2k+1).$$

Moreover, whenever w is not a permutation of r, there exists a v-tuple $(t_1, \dots, t_v)$ such that all the indices $t_1, \dots, t_v$ are distinct and such that

$$\prod_{i=1}^{2k} \epsilon_{\ell_i} \epsilon_{j_i} = \prod_{i=1}^{v} \epsilon_{t_i}.$$

Thanks to Proposition 3.5 it follows that the term $\nu_0(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}})$ vanishes, and we get $\nu(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}) = O(2k+1)$. Hence, in the sequel of the proof, w will be assumed to be a permutation of r. Notice also that the remainder of the proof is a little cumbersome, and the reader may wish to follow it with an example as a guideline. One possible simple choice is $k = 3$ and

$$r = (1, 2, 4, 7, 3, 5) \quad \text{and} \quad w = (4, 7, 1, 2, 3, 5) \tag{21}$$

Step 1: Decomposition of $\nu(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}})$ into three terms
Applying Proposition 3.7 item ii), we obtain that

$$\nu\left(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) = \nu\left(S_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) + O(2k+1). \tag{22}$$

Let us analyze now the term $\nu(S_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}})$: by the very definition of $S_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}$, we have

$$\nu\left(S_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) = \nu\left(\prod_{i=1}^{2k} R_{\ell_i,j_i}\right). \tag{23}$$

Furthermore, we have

$$\begin{aligned}\nu\left(S_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) &= \frac{1}{N}\nu\left(\sum_{k=1}^{N} \sigma_k^{\ell_1} \sigma_k^{j_1} \prod_{2\le i\le 2k} R_{\ell_i,j_i}\right) \\ &= \nu\left(\epsilon_{\ell_1}\epsilon_{j_1} \prod_{2\le i\le 2k} R_{\ell_i,j_i}\right),\end{aligned} \tag{24}$$

where the last step has just been obtained as an easy consequence of the symmetry property among sites. Observe also that, since r is a permutation of w, then $\prod_{i=1}^{2k} \epsilon_{\ell_i}\epsilon_{j_i} = 1$ and we have

$$\epsilon_{\ell_1}\epsilon_{j_1}\prod_{i=1}^{2k}\epsilon_{\ell_i}\epsilon_{j_i} = \epsilon_{\ell_1}\epsilon_{j_1} \quad \text{and} \quad \epsilon_{\ell_1}\epsilon_{j_1}\prod_{i=1}^{2k}\epsilon_{\ell_i}\epsilon_{j_i} = \prod_{2\le i\le 2k}\epsilon_{\ell_i}\epsilon_{j_i},$$

which yields

$$\epsilon_{\ell_1}\epsilon_{j_1} = \prod_{2\le i\le 2k}\epsilon_{\ell_i}\epsilon_{j_i}. \tag{25}$$

By plugging the expressions (24) and (25) into (22), we thus get

$$\begin{aligned}\nu\left(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) &= \nu\left(\epsilon_{\ell_1}\epsilon_{j_1}\prod_{2\le i\le 2k}R_{\ell_i,j_i}\right) + O(2k+1)\\ &= \nu\left(S_{\ell_2,j_2,\dots,\ell_{2k},j_{2k}}\right) + O(2k+1).\end{aligned}$$

Repeating now the process of decomposition of Equation (17), we obtain

$$\nu\left(S^-_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) = K_1 + K_2 + K_3, \tag{26}$$

where

$$K_1 = \nu\left(S^-_{\ell_2,j_2,\dots,\ell_{2k},j_{2k}}\right), \tag{27}$$

$$K_2 = \frac{1}{N}\sum_{i_2\le a_1<\dots<a_{2k-2}\le i_{2k}}\nu\left(S^-_{\ell_{a_1},j_{a_1},\dots,\ell_{a_{2k-2}},j_{a_{2k-2}}}\right), \tag{28}$$

$$K_3 = \sum_{u=1}^{2k-3}\frac{1}{N^{2k-u-1}}\sum_{i_2\le a_1<\dots<a_u\le i_{2k}}\nu\left(S^-_{\ell_{a_1},j_{a_1},\dots,\ell_{a_u},j_{a_u}}\right) + \frac{1}{N^{2k-1}}. \tag{29}$$

We will now estimate each term K_1, K_2 and K_3 separately.

Step 2: We will show that

$$K_1 = \beta^2\nu\left(S_{\ell_1,j_1,\dots,\ell_{2k},j_{2k}}\right) + O(2k+1). \tag{30}$$

Indeed, performing a Taylor expansion for K_1, we have

$$\begin{aligned}K_1 = \nu_0\left(S^-_{\ell_2,j_2,\dots,\ell_{2k},j_{2k}}\right) &+ \nu_0'\left(S^-_{\ell_2,j_2,\dots,\ell_{2k},j_{2k}}\right)\\ &+ \frac{1}{2}\nu_\xi^{(2)}\left(S^-_{\ell_2,j_2,\dots,\ell_{2k},j_{2k}}\right),\end{aligned} \tag{31}$$

for a certain $\xi \in [0,1]$. Moreover, it is easily seen from Proposition 3.5 that the first term, $\nu_0(S^-_{\ell_2,j_2,\ldots,\ell_{2k},j_{2k}})$, vanishes, and according to Proposition 3.3 item (d), the last term of relation (31) is of order $O(2k+1)$. Eventually, by Propositions 2.1 and 3.5, it is readily checked that

$$\nu_0'\left(S^-_{\ell_2,j_2,\ldots,\ell_{2k},j_{2k}}\right) = \beta^2\nu_0\left(S^-_{\ell_1,j_1,\ldots,\ell_{2k},j_{2k}}\right),$$

and thanks to Proposition 3.7 item ii) it follows that

$$\nu_0'\left(S^-_{\ell_2,j_2,\ldots,\ell_{2k},j_{2k}}\right) = \beta^2\nu\left(S_{\ell_1,j_1,\ldots,\ell_{2k},j_{2k}}\right) + O(2k+1),$$

which gives our claim (30).

Step 3: We will prove that $K_3 = O(2k+1)$.
Notice that for each $a_1,\ldots,a_u$ we have

$$\frac{1}{N^{2k-u-1}}\nu\left(S^-_{\ell_{a_1},j_{a_1},\ldots,\ell_{a_u},j_{a_u}}\right) = O(2(2k-u-1)+u),$$

where we have just applied Proposition 3.3. Since $2(2k-u-1)+u$ is greater than $2k+1$ whenever $u \le 2k-3$, it follows that

$$K_3 = O(2k+1). \tag{32}$$

Step 4: Study of K_2.
We claim that

$$K_2 = \begin{cases} \frac{1}{N}\nu\left(S_{\tilde{l}_1,\tilde{k}_1,\ldots,\tilde{l}_{2k-2},\tilde{k}_{2k-2}}\right) + O(2k+1), & \text{if } r \sim w\ ; \\ O(2k+1), & \text{otherwise,} \end{cases} \tag{33}$$

where $(\tilde{l}_1,\tilde{k}_1,\ldots,\tilde{l}_{2k-2},\tilde{k}_{2k-2}) = (\tilde{r},\tilde{w})$ for a certain couple with $\tilde{r},\tilde{w} \in \Omega^{2(k-1)}$ satisfying $\tilde{r} \sim \tilde{w}$.

Indeed, using Proposition 3.7, for any family $(a_1,\ldots,a_{2k-2})$ such that $i_2 \le a_1 < \ldots < a_{2k-2} \le i_{2k}$, we have

$$\frac{1}{N}\nu\left(S^-_{\ell_{a_1},j_{a_1},\ldots,\ell_{a_{2k-2}},j_{a_{2k-2}}}\right) = \frac{1}{N}\nu_0\left(S^-_{\ell_{a_1},j_{a_1},\ldots,\ell_{a_{2k-2}},j_{a_{2k-2}}}\right) + O(2k+1). \tag{34}$$

In the case $r \nsim w$, since $r \in \Omega^{2k}$, there exists an index $i \in \{1,\ldots,2k\}$ such that for all $u \in \{1,\ldots,2k\}\backslash\{i\}$, we have $(\ell_i,j_i) \neq (\ell_u,j_u)$ (remember that, by definition, the l's and j's are the elements of r and w). We denote this index i by i_1. In this case the index i_2 such that the product $\prod_{v=1}^{2k-2} \epsilon_{\ell_{a_v}} \epsilon_{j_{a_v}} = \epsilon_{\ell_{i_1}} \epsilon_{j_{i_1}} \epsilon_{\ell_{i_2}} \epsilon_{j_{i_2}}$ satisfies $\{\epsilon_{\ell_{i_1}}, \epsilon_{j_{i_1}}\} \cap \{\epsilon_{\ell_{i_2}}, \epsilon_{j_{i_2}}\} \neq \{\epsilon_{\ell_{i_1}}, \epsilon_{j_{i_1}}\}$. Thus, applying Proposition 3.5, the first term on the right side in (34) vanishes and we obtain

$$\frac{1}{N} \nu \left(S^-_{\ell_{a_1}, j_{a_1}, \ldots, \ell_{a_{2k-2}}, j_{a_{2k-2}}} \right) = O(2k+1).$$

On the other hand, in the case $r \sim w$, there exists a unique sequence $(\hat{a}_1, \ldots, \hat{a}_{2k-2})$ which satisfies $\hat{a}_1 < \hat{a}_2 < \ldots < \hat{a}_{2k-2}$ and

$$\prod_{i=1}^{2k-2} \epsilon_{\ell_{\hat{a}_i}} \epsilon_{j_{\hat{a}_i}} = 1.$$

Notice that in our example (21), we have $(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) = (2, 3, 4, 6)$. Then it is easily seen, with the same kind of arguments as in the previous steps, that

$$K_2 = \frac{1}{N} \nu \left(S^-_{\ell_{\hat{a}_1}, j_{\hat{a}_1}, \ldots, \ell_{\hat{a}_{2k-2}}, j_{\hat{a}_{2k-2}}} \right) + O(2k+1).$$

Thanks to Proposition 3.7, we thus get

$$K_2 = \frac{1}{N} \nu \left(S_{\ell_{\hat{a}_1}, j_{\hat{a}_1}, \ldots, \ell_{\hat{a}_{2k-2}}, j_{\hat{a}_{2k-2}}} \right) + O(2k+1),$$

and we remark that $(\ell_{\hat{a}_1}, j_{\hat{a}_1}, \ldots, \ell_{\hat{a}_{2k-2}}, j_{\hat{a}_{2k-2}}) = (\tilde{r}, \tilde{w})$ with $\tilde{r}, \tilde{w} \in \Omega^{2k-2}$ and $\tilde{r} \sim \tilde{w}$, since $r \sim w$ (in our example (21), $r = w = (4, 7, 3, 5)$). Our claim is now proved.

Step 5: Conclusion.
Plugging (30), (32) and (33) into (26), and invoking Proposition 3.7 item ii), we obtain, for any $\beta < 1$,

$$\nu \left(S^-_{\ell_1, j_1, \ldots, \ell_{2k}, j_{2k}} \right) = \begin{cases} \frac{1}{(1-\beta^2)N} \nu \left(S_{\tilde{l}_1, \tilde{k}_1, \ldots, \tilde{l}_{2k-2}, \tilde{k}_{2k-2}} \right) + O(2k+1), & \text{if } r \sim w; \\ O(2k+1), & \text{otherwise,} \end{cases}$$

and equation (20) follows now easily by induction on k. Indeed, the case $k = 1$ has been shown by Talagrand in [8], under the following form: for $\beta < 1$, we have

$$\nu(R_{1,2}^2) = \frac{1}{N(1-\beta^2)} + O(3).$$

The induction is now a trivial fact. □

As a consequence of the previous properties, we can evaluate the following general term:

Proposition 3.9. *Let $(\ell_1, j_1, \ldots, \ell_k, j_k) \in \Omega^{2k}$. Then, for all $\beta < 1$, we have*

$$\frac{1}{k!}\nu_0^{(k)}\left(S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) = \left(\frac{\beta^2}{N(1-\beta)^2}\right)^k + O(2k+1). \tag{35}$$

Proof. Applying Proposition 3.6 with $f = S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}$ and $w = (\ell_1, j_1, \ldots, \ell_k, j_k)$, we obtain that

$$\begin{aligned}\nu_0^{(k)}\left(S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) &= k!\beta^{2k}\nu_0\left(S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}\right)\\ &\quad + \beta^{2k}\sum_{r\nsim w} c(r)\nu_0\left(S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}S^-_{r_1,r_2,\ldots,r_{2k-1},r_{2k}}\right).\end{aligned}$$

Hence, according to Proposition 3.8, we can conclude that

$$\frac{1}{k!}\nu_0^{(k)}\left(S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) = \beta^{2k}\left(\frac{1}{N(1-\beta)^2}\right)^k + O(2k+1).$$

□

Eventually, we will end the section by the evaluation of the first term in the expansion of $\nu\left(S_{\ell_1,j_1,\ldots,\ell_k,j_k}\right)$:

Lemma 3.10. *Let $r = (\ell_1, j_1, \ldots, \ell_k, j_k) \in \Omega^{2k}$. Then, for all $\beta < 1$, the following relation holds true:*

$$\nu\left(S_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) = \frac{1}{N^k}\left(\frac{1}{1-\beta^2}\right)^k + O(2k+1).$$

Proof. First remark that $S_{\ell_1,j_1,\ldots,\ell_k,j_k} = \prod_{i=1}^k S_{\ell_i,j_i}$, and thanks to the relation $S_{l,l'} = S^-_{l,l'} + \frac{1}{N}$, we obtain

$$\begin{aligned}S_{\ell_1,j_1,\ldots,\ell_k,j_k} &= \sum_{u=1}^k \sum_{1\le i_1<\ldots<i_u\le k} \frac{1}{N^{k-u}}\prod_{v=1}^u S^-_{\ell_{i_v},j_{i_v}} + \frac{1}{N^k}\\ &= \sum_{u=1}^k \sum_{1\le i_1<\ldots<i_u\le k} \frac{1}{N^{k-u}} S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}} + \frac{1}{N^k}.\end{aligned} \tag{36}$$

Hence

$$\nu\left(S_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) = \sum_{u=1}^k \sum_{1\le i_1<\ldots<i_u\le k} \frac{1}{N^{k-u}}\nu\left(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}\right) + \frac{1}{N^k}. \tag{37}$$

Notice that $r \in \Omega^{2k}$ iff for any $u \leq k$ and any sequence $(i_1, \ldots, i_u)$ such that $1 \leq i_1 < \ldots < i_u \leq k$, we have $(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}) \in \Omega^{2u}$. Whence, expanding the Taylor series, we get

$$\frac{1}{N^{k-u}} \nu \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right) = \frac{1}{N^{k-u}} \sum_{v=0}^{u} \frac{1}{v!} \nu_0^{(v)} \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right) + \frac{1}{N^{k-u}} \frac{1}{(u+1)!} \nu_\xi^{(u+1)} \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right), \quad (38)$$

for a certain $\xi \in [0, 1]$. Now, Invoking Proposition 3.5, all the derivative terms of order smaller than u vanish, and by Proposition 3.3 item (d), the error term can be estimated as follows:

$$\frac{1}{N^{k-u}} \frac{1}{(u+1)!} \nu_\xi^{(u+1)} \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right) = O(2(k-u) + 2u + 1) = O(2k+1).$$

Hence, we get the following expression:

$$\frac{1}{N^{k-u}} \nu \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right) = \frac{1}{u! N^{k-u}} \nu_0^{(u)} \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right) + O(2k+1). \quad (39)$$

On the other hand, the derivative term of order u can be evaluated by means of Proposition 3.9: since $(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}) \in \Omega^{2u}$, by plugging (35) into (39) , we get

$$\begin{aligned} \frac{1}{N^{k-u}} \nu \left(S^-_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}} \right) &= \frac{1}{N^{k-u}} \frac{1}{u!} \left[u! \left(\frac{\beta^2}{N(1-\beta^2)} \right)^u + O(2u+1) \right] \\ &= \frac{1}{N^k} \left(\frac{\beta^2}{1-\beta^2} \right)^u + O(2k+1). \end{aligned} \quad (40)$$

Moreover,

$$\text{Card} \{ (i_1, \ldots, i_u) | 1 \leq i_1 < \ldots < i_u \leq k \} = \binom{k}{u},$$

and thus we can recast Equation (37) into

$$\begin{aligned} \nu (S_{\ell_1, j_1, \ldots, \ell_k, j_k}) &= \sum_{u=1}^{k} \frac{1}{N^k} \binom{k}{u} \left(\frac{\beta^2}{(1-\beta^2)} \right)^u + \frac{1}{N^k} + O(2k+1) \\ &= \frac{1}{N^k} \left(1 + \frac{\beta^2}{1-\beta^2} \right)^k + O(2k+1). \end{aligned}$$

This completes the proof. □

4 R-systems and Graphs

In this section, we will make an essential step towards the evaluation of multiple overlaps of the form $R_{1,\dots,s}$ defined at (2). Indeed, we will prove an important preliminary result involving the functional $U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}$, where $U_k^- = \epsilon_1\epsilon_2\dots\epsilon_{2k}R^-_{1,2,\dots,2k}$. We will also evaluate $\nu(S_{\ell_1,j_1,\dots,\ell_m,j_m})$ for some special cases of indexes $(\ell_1,j_1,\dots,\ell_m,j_m)$. More specifically, this section is devoted to the proof of the following result:

Proposition 4.1. *Let k be a positive integer, $\beta<1$ and recall that $\mathcal{C}_k$ has been defined at (8). Then, for any $m\ge k$ and $(\ell_1,j_1,\dots,\ell_m,j_m)\in\mathcal{C}_k$, we have:*

i) $\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) = \nu(S_{\ell_1,j_1,\dots,\ell_m,j_m}) + O(2k+1)$.

ii) $\nu(S_{\ell_1,j_1,\dots,\ell_m,j_m}) = O(2k+1)$ *if* $m\ge k+1$.

Notice that the proof of this result will require two kind of tools: first a graph representation that will help us to identify the main contribution in our expansions, and then the introduction of some families of functions whose role is to avoid a cumbersome recursive procedure.

4.1 Graph tools: proof of Proposition 4.1 item (ii)

We will include in fact Proposition 4.1 item (ii) into a more general statement:

Proposition 4.2. *Consider a positive integer k and $\beta<1$. Assume that the sequence $(\ell_1,j_1,\dots,\ell_m,j_m)$ belongs to $\mathcal{C}_k$ with $m\ge k+1$. Then the following estimations hold true:*

i) $\nu(S_{\ell_1,j_1,\dots,\ell_m,j_m}) = O(2k+1)$.

ii) For all $u\ge 1$ *and* $1\le i_1<\dots<i_u\le m$, *we have*

$$\frac{1}{N^{m-u}}\nu(S^-_{\ell_{i_1},j_{i_1},\dots,\ell_{i_u},j_{i_u}}) = O(2k+1).$$

iii) For all $u\ge 1$ *and* $1\le i_1<\dots<i_u\le m$, *we have*

$$\frac{1}{N^{m-u}}\nu(S_{\ell_{i_1},j_{i_1},\dots,\ell_{i_u},j_{i_u}}) = O(2k+1).$$

Proof. Let $(\ell_1, j_1, \ldots, \ell_m, j_m) \in \mathcal{C}_k$. Using the same kind of calculation as in relation (36), we obtain

$$\nu(S_{\ell_1,j_1,\ldots,\ell_m,j_m}) = \sum_{u=1}^{m} \sum_{1\le i_1<\ldots<i_u\le m} \frac{1}{N^{m-u}} \nu(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) + \frac{1}{N^m}. \tag{41}$$

For each $u \le m$ and $1 \le i_1 < \ldots < i_u \le m$, let us expand the term $\nu(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}})$ up to an order $v \in \mathbb{N}$. We get

$$\nu(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) = \sum_{r=1}^{v} \frac{1}{r!} \nu_0^{(r)}(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) + \frac{1}{(v+1)!} \nu_\zeta^{(v+1)}(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) \tag{42}$$

for a certain $\zeta \in \mathbb{R}$. Let us admit for the moment the following proposition, whose proof will require the introduction of the graph tools mentioned above:

Proposition 4.3. *Given a positive integer k and $(\ell_1, j_1, \ldots, \ell_m, j_m) \in \mathcal{C}_k$, the following holds true for any $u \ge 1$ and $1 \le i_1 < \ldots < i_u \le m$:*

i) There exists a positive integer

$$\hat{a} = \hat{a}(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u})$$

such that $\prod_{p=1}^{u} \epsilon_{\ell_{i_p}} \epsilon_{j_{i_p}} = \epsilon_{c_1} \ldots \epsilon_{c_{2\hat{a}}}$, where all the indexes c's are different.

ii) $u - \hat{a}$ is bounded by $m - k$.

Let us apply now this last proposition: set $v = \hat{a}$ in equation (42). Then, invoking Proposition 3.5 item (i), it is easily seen that

$$\nu(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) = \frac{1}{\hat{a}!} \nu_0^{(\hat{a})}(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) + \frac{1}{(\hat{a}+1)!} \nu_\zeta^{(\hat{a}+1)}(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}).$$

Furthermore, according to Proposition 3.3 item (d), $\nu(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}})$ is of order $O(\hat{a} + u)$. We thus get the following estimation:

$$\frac{1}{N^{m-u}} \nu(S^-_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) = O\left(2m - (u - \hat{a})\right).$$

Eventually, thanks to Item ii) in Proposition 4.3, and since we have assumed $m \ge k+1$, we get

$$2m - (u - \hat{a}) \ge 2m - (m - k) = m + k \ge 2k + 1,$$

and hence

$$\frac{1}{N^{m-u}}\nu(S^-_{\ell_{i_1},j_{i_1},\dots,\ell_{i_u},j_{i_u}}) = O(2k+1), \tag{43}$$

which proves Item ii) of our Proposition 4.2. Moreover, putting together (43) and (41), Item i) of Proposition 4.2 is also easily shown.

In order to obtain iii) in Proposition 4.2 we perform again the same expansion as in (36), and we get

$$\begin{aligned}\frac{1}{N^{m-u}}\nu(S_{\ell_{i_1},j_{i_1},\dots,\ell_{i_u},j_{i_u}}) &= \frac{1}{N^{m-u}}\sum_{q=0}^{u}\sum_{i_1\le a_1<\dots<a_q\le i_u}\frac{1}{N^{u-q}}\nu(S^-_{\ell_{a_1},j_{a_1},\dots,\ell_{a_q},j_{a_q}})\\ &= O(2k+1),\end{aligned}$$

where we used ii) for each q and $(a_1,\dots,a_q)$ and $m \ge k+1$ □

The remainder of this section will now be devoted to prove Proposition 4.3, starting with item (i), for which we will use the graph definitions of Section 2.2:

Proposition 4.4. *Let k, u be two positive integers such that $u \le k$. Let also $(\ell_1, j_1, \dots, \ell_m, j_m) \in \mathcal{C}_k$ with $m \ge k+1$ and $1 \le i_1 < \dots < i_u \le m$. Consider $g = G((\ell_1, j_1, \dots, \ell_m, j_m))$ and $h \triangleq G((\ell_{i_1}, j_{i_1}, \dots, \ell_{i_u}, j_{i_u})$. Then*

i) h belongs to $S_u(g)$.

ii) There exists an integer t such that $\prod_{i=1}^{u}\epsilon_{\ell_{i_i}}\epsilon_{j_{i_i}} = \epsilon_{c_1}\dots\epsilon_{c_t}$ for any value of ϵ, where all the indexes $(c_1,\dots,c_t)$ are different. Furthermore, $\mathrm{Od}(h) = \frac{t}{2}$.

Remark 4.5. *Item i) justifies our interest for the class $S_u(g)$, while Item ii) implies Item i) of Proposition 4.3, with $\hat{a}(\ell_{i_1},\dots,j_{i_u}) = \mathrm{Od}(h)$.*

Proof of Proposition 4.4. i) This is a straightforward consequence of the definitions given at Section 2.2.

ii) The quantities ϵ_{c_i} are just the elements which appear an odd number of times in $\prod_{p=1}^{u}\epsilon_{\ell_{i_p}}\epsilon_{j_{i_p}}$, and as a consequence,

$$\mathrm{Od}(h) = \sum_{i\in I;\, N_h(i)\text{ is odd}}\frac{1}{2} = \frac{t}{2}.$$

□

Given a graph g such that $N(g) = m$, another quantity of interest for us will be an upper bound on $\max_{h\in S_u(g)} u - \mathrm{Od}(h)$. Define then, for each $u \in \{1, \dots, m\}$, the function

$$\begin{array}{rccc} M_u^g : & S_u(g) & \longrightarrow & \mathbb{N} \\ & h & \longmapsto & u - \mathrm{Od}(h) \end{array}$$

In order to simplify the notations we will use during the proof of the next proposition, we define an operation with graphs that we call the *juxtaposition*: given two graphs $g_1 = (I_1, E_1, \Upsilon_1)$ and $g_2 = (I_2, E_2, \Upsilon_2)$, we denote by $g = g_1 + g_2$ the graph defined by $g = (I, E, \Upsilon)$, such that $I = I_1 \cup I_2$, $E = E_1 \cup E_2$ and $\Upsilon = \Upsilon_1 + \Upsilon_2$ (we consider that $\Upsilon_1(e)$ and $\Upsilon_2(e)$ are equal to zero when they are not defined in Υ_1 and Υ_2 separately).

Recall that the class of graphs $\mathcal{G}_k$ is defined by relation (8). Then the next Lemma asserts an inner characteristic of monotonicity for $\mathcal{G}_k$.

Lemma 4.6. *Given $k \in \mathbb{N}$ and a graph $g = (I, E, \Upsilon) \in \mathcal{G}_k$, the following holds true:*

i) $\max_{h\in S_u(g)} M_u^g$ *is increasing with u.*

ii) $\max_{u\le N(g)} \max_{h\in S_u(g)} M_u^g \le N(g) - k$.

Remark 4.7. *Item (ii) of Proposition 4.3 is an easy consequence of item (ii) in Lemma 4.6.*

Proof of Lemma 4.6. i) Let $h = (I_1, E_1, \Upsilon_1) \in S_u(g)$ such that $u < N(g)$ and

$$\max_{h\in S_u(g)} M_u^g = u - \mathrm{Od}(h).$$

Let $e = (p, q) \in E \setminus E_1$ (e exists because $u < m$). One defines $h_1 = (\{p, q\}, \{e\}, \Upsilon_2)$ such that $\Upsilon_2(e) = 1$, and let $\tilde{h}$ be the graph $h + h_1$. Then $\tilde{h} \in S_{u+1}$ because $N(\tilde{h}) = N(h) + N(h_1) = u+1$, $I_1 \cup \{p, q\} \subseteq I$, $E_1 \cup \{e\} \subseteq E$ and $\Upsilon_1(e) + \Upsilon_2(e) \le \Upsilon(e)$. We will show that $M_{u+1}^g(\tilde{h}) \ge M_u^g(h)$ which, in turn, implies statement i).

There are three possible cases for p and q:

- $p, q \notin I_1$
  In this case $N_{\tilde{h}}(p) = N_{\tilde{h}}(q) = 1$, and then $\mathrm{Od}(\tilde{h}) = \mathrm{Od}(h) + \frac{1}{2} + \frac{1}{2}$, which gives $M_{u+1}^g(\tilde{h}) = u + 1 - (\mathrm{Od}(h) + 1) = M_u^g(h)$.

- $p \in I_1,\ q \notin I_1$ (or $q \in I_1,\ p \notin I_1$)

  One has $N_{\tilde{h}}(q) = 1$, and if $N_{\tilde{h}}(p)$ is odd, then $\mathrm{Od}(\tilde{h}) = \mathrm{Od}(h) + \frac{1}{2} + \frac{1}{2}$ and thus one obtains the same result than in the previous item. If $N_{\tilde{h}}(p)$ is even, which gives $\mathrm{Od}(\tilde{h}) = \mathrm{Od}(h) + \frac{1}{2} - \frac{1}{2}$ then $M^g_{u+1}(\tilde{h}) = u + 1 - \mathrm{Od}(h) > M^g_u(h)$.

- $p, q \in I_1$

  If both $N_{\tilde{h}}(p)$ and $N_{\tilde{h}}(q)$ are odd, then $\mathrm{Od}(\tilde{h}) = \mathrm{Od}(h) + 1$; if $N_{\tilde{h}}(p)$ is even and $N_{\tilde{h}}(q)$ is odd, then $\mathrm{Od}(\tilde{h}) = \mathrm{Od}(h)$, and these two cases have already been studied. In the case where both $N_{\tilde{h}}(p)$ and $N_{\tilde{h}}(q)$ are even, then $\mathrm{Od}(\tilde{h}) = \mathrm{Od}(h) - 1$, and $M^g_{u+1}(\tilde{h}) = u + 1 - (\mathrm{Od}(h) - 1) > M^g_u(h)$. The proof of point i) is now clear.

The statement ii) follows from item i), because $\max_u \max_{h \in S_u(g)} M^g_u = M^g_m = m - k$, where in the last step, we have used the fact that g is the only subgraph of g with m edges such that $g \in \mathcal{G}_k$. □

Let us recall that, at that point, we have proved Proposition 4.3, and thus Proposition 4.1 item (ii).

4.2 R-systems: proof of Proposition 4.1 item (i)

The aim of this subsection is to finish the proof of Proposition 4.1 item (i), which amounts to prove

$$\nu_0(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) = \nu(S_{\ell_1,j_1,\ldots,\ell_m,j_m}) + O(2k+1). \tag{44}$$

The general strategy we will use here is a backward induction principle on m. However, in order to simplify the cumbersome procedure one is faced with at first sight, we will introduce a family of function that we call R-systems. Let us delve now into the details of the proof:

Step 1: First step of the induction.
In the case $m \geq 2k+2$, thanks to Schwarz inequality and Proposition 3.2, we easily get

$$\nu_0(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) \leq K(\beta, m)\, \nu\left(\left(S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}\right)^2\right)^{\frac{1}{2}} = O(m)$$

for a positive constant $K(\beta, m)$, where in the last step, we have used Proposition 3.3. The statement follows because $m \geq 2k+2 > 2k+1$ and Proposition 4.2 yields $\nu(S_{\ell_1,j_1,\ldots,\ell_m,j_m}) = O(2k+1)$. Whence the difference

$\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) - \nu(S_{\ell_1,j_1,\dots,\ell_m,j_m})$ is also $O(2k+1)$, which finishes the proof.

Step 2: We will start our induction procedure. Let us pick a $m < 2k+2$, and we assume the result holds true for all $r > m$. First, we will show that

$$\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) = \nu(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) + O(2k+1).$$

Indeed, performing an inverse Taylor expansion, we get, for a certain $\zeta \in [0,1]$,

$$\begin{aligned}\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) &= \nu(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) \\ &\quad - \sum_{r=1}^{k} \frac{1}{r!}\nu_0^{(r)}(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) - \frac{1}{(k+1)!}\nu_\zeta^{(k+1)}(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}). \quad (45)\end{aligned}$$

Let us bound now the last term of this inequality: applying Schwarz' inequality and Propositions 3.2 and 3.3, we have

$$\begin{aligned}\nu_\zeta^{(k+1)}(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) &\le \frac{1}{N^{\frac{k+1}{2}}}\nu\left(\left(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}\right)^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{N^{\frac{k+1}{2}}}\nu\left(\left(S^-_{\ell_1,j_1,\dots,\ell_m,j_m}\right)^2\right)^{\frac{1}{2}} = O(2k+1),\end{aligned}$$

according to the fact that $m \ge k$. Consequently, equation (45) becomes

$$\begin{aligned}\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) = \nu(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) - \sum_{r=1}^{k}\frac{1}{r!}\nu_0^{(r)}(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m}) \\ + O(2k+1).\end{aligned}$$

However, Proposition 3.1 asserts that each term

$$\nu_0^{(r)}(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m})$$

can be evaluated as a finite sum of terms of the form

$$c(\beta,r)\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m} S^-_{\ell_1,j_1,\dots,\ell_r,j_r}),$$

which can be rewritten as

$$c(\beta,r)\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_m,j_m} S^-_{\ell_1,j_1,\dots,\ell_r,j_r}) = c(\beta,r)\nu_0(U_k^- S^-_{\ell_1,j_1,\dots,\ell_{m+r},j_{m+r}}). \quad (46)$$

By Proposition 3.4, if $(\ell_1, j_1, \ldots, \ell_{m+r}, j_{m+r}) \notin \mathcal{C}_k$, the expression (46) vanishes. Otherwise, by backward induction hypothesis, we get

$$c(\beta, r)\nu_0(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_{m+r},j_{m+r}}) = c(\beta, r)\nu(S_{\ell_1,j_1,\ldots,\ell_{m+r},j_{m+r}}) + O(2k+1).$$

Thus, since $r \geq 1$, it is readily checked from Proposition 4.2 that

$$\nu_0(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) = \nu(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) + O(2k+1), \tag{47}$$

which was the claim to be proved.

Step 3: Decomposition of $\nu_0(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m})$.
Set $U_k = \epsilon_1 \cdots \epsilon_{2k} R_{1\ldots 2k}$. Then, obviously, $U_k^- = -\frac{1}{N} + U_k$. Using this fact, and performing the same kind of computations as in (36), we obtain

$$\nu(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) = \sum_{u=1}^{m} \sum_{1\leq i_1<\ldots<i_u\leq m} \frac{(-1)^{m-u+1}}{N^{m-u+1}} \nu(S_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) + \frac{1}{N^{m+1}}$$
$$+ \nu(U_k S_{\ell_1,j_1,\ldots,\ell_m,j_m}) + \sum_{u=1}^{m-1} \sum_{1\leq i_1<\ldots<i_u\leq m} \frac{(-1)^{m-u}}{N^{m-u}} \nu(U_k S_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}).$$

Furthermore, thanks to Proposition 4.2, we obtain

$$\nu(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) = \nu(U_k S_{\ell_1,j_1,\ldots,\ell_m,j_m})$$
$$+ \sum_{u=1}^{m-1} \sum_{1\leq i_1<\ldots<i_u\leq m} \frac{(-1)^{m-u}}{N^{m-u}} \nu(U_k S_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) + O(2k+1).$$

Now, the fact that $(\ell_1, j_1 \ldots, \ell_m, j_m) \in \mathcal{C}_k$ and the symmetry property yield

$$\nu\Big(U_k S_{\ell_1,j_1,\ldots,\ell_m,j_m}\Big) = \nu\Big(\epsilon_1 \ldots \epsilon_{2k} R_{1,\ldots,2k} \prod_{i\leq m} \epsilon_{\ell_i}\epsilon_{j_i} R_{\ell_i,j_i}\Big)$$
$$= \nu\Big(R_{1,\ldots,2k} \prod_{i\leq m} R_{\ell_i,j_i}\Big) = \nu\Big(\epsilon_1 \ldots \epsilon_{2k} \prod_{i\leq m} R_{\ell_i,j_i}\Big) = \nu\Big(\prod_{i\leq m} \epsilon_{\ell_i}\epsilon_{j_i} R_{\ell_i,j_i}\Big)$$
$$= \nu\Big(S_{\ell_1,j_1,\ldots,\ell_m,j_m}\Big).$$

Thus

$$\nu(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_m,j_m}) = \nu(S_{\ell_1,j_1,\ldots,\ell_m,j_m})$$
$$+ \sum_{u=1}^{m-1} \sum_{1\leq i_1<\ldots<i_u\leq m} \frac{(-1)^{m-u}}{N^{m-u}} \nu(U_k S_{\ell_{i_1},j_{i_1},\ldots,\ell_{i_u},j_{i_u}}) + O(2k+1). \tag{48}$$

In order to finish our decomposition, let us introduce a little more notation: for $1 \le i_1 < \ldots < i_u \le m$, set $\alpha = (\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u})$ and we define I as the set of all possible α when u varies in $\{1, \ldots, m-1\}$. For any $\alpha \in I$, denote by $\tilde{G}_\alpha$ the function defined by

$$\tilde{G}_\alpha = \frac{(-1)^{m-u}}{N^{m-u}} \nu(U_k S_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}}).$$

Putting together Equations (47) and (48) we have proved that

$$\nu_0(U_k^- S^-_{\ell_1, j_1, \ldots, \ell_m, j_m}) = \nu(S_{\ell_1, j_1, \ldots, \ell_m, j_m}) + \sum_{\alpha \in I} \tilde{G}_\alpha + O(2k+1). \tag{49}$$

Step 4: Pick a given $1 \le u \le m-1$. For $\alpha = (\ell_{i_1}, \ldots, j_{i_u})$ we will write $\hat{\alpha} \subseteq \alpha$ if $\hat{\alpha}$ is a subset of indexes contained in α. A generic element $\hat{\alpha} \subseteq \alpha$ will be of the form $\hat{\alpha} = (\ell_{\hat{\imath}_1}, \ldots, j_{\hat{\imath}_v})$. We will show that

$$\tilde{G}_\alpha = \sum_{\hat{\alpha} \in I_\alpha} G_{\hat{\alpha}} \nu_0(U_k^- S^-_{\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v}}) + O(2k+1), \tag{50}$$

where for each $\hat{\alpha}$, $G_{\hat{\alpha}}$ is a coefficient of order $O(1)$, and where I_α is a set of indexes which will be defined later on. Indeed, repeating the process that lead to (36) and using $U_k = \frac{1}{N} + U_k^-$, we get

$$\begin{aligned}
\frac{1}{N^{m-u}} \nu(U_k S_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}}) &= \frac{1}{N^m} \nu(U_k^-) \\
&+ \frac{1}{N^{m-u}} \sum_{v=1}^{u} \sum_{\hat{\alpha} \subseteq \alpha} \frac{1}{N^{u-v}} \nu(U_k^- S^-_{\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v}}) + \frac{1}{N^{m-u+u+1}} \\
&+ \frac{1}{N^{m-u}} \sum_{v=1}^{u} \sum_{\hat{\alpha} \subseteq \alpha} \frac{1}{N^{u-v+1}} \nu(S^-_{\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v}}).
\end{aligned} \tag{51}$$

Let us analyze now some of the terms in the sum above. First, it is easily seen that

$$\frac{1}{N^m} \nu(U_k^-) = O(2k+1). \tag{52}$$

On the other hand, a first idea one could have, in order to handle the term $N^{-(m-v+1)} \nu(S^-_{\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v}})$ for $1 \le v \le u \le m-1$, would be to apply Proposition 4.2. However, notice that we have only assumed $m \ge k$, while the latter proposition requires $m \ge k+1$. This induces us to use an additional trick: the tuple $(\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v})$ has been generated from $(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u})$. However, since $u \le m-1$, we could also have generated $(\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v})$ from the following sequence of size $2(m+1)$ in

$\mathcal{C}_k$: take all the couples in $(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u})$ except for one, say (ℓ_{i^*}, j_{i^*}), and assume $(\ell_{i^*}, j_{i^*}) \notin (\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v})$. Now, we will split (ℓ_{i^*}, j_{i^*}) into two couples $(\ell_{i^*}, 1)$ and $(1, j_{i^*})$, and form the desired sequence by aggregating all these couples. As an illustrating example of this procedure, take $k = m = 3$, $(\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}) = (1, 3, 2, 4, 5, 6) \in \mathcal{C}_3$, $u = v = 2$, and $(\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}) = (1, 3, 5, 6)$. Then, this last sequence could have been generated as well from

$$(1, 3, 5, 6, 2, 1, 4, 1) \in \mathcal{C}_3.$$

Now, Proposition 4.2 item (ii) can be applied to the new sequence of size $2(m+1)$ we have just constructed, and thus, for each $1 \le u \le m-1$, $\hat{\alpha} \subset \alpha$ we get:

$$\frac{1}{N^{m-v+1}} \nu(S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{i_v}, j_{\hat{i}_v}}) = O(2k+1). \tag{53}$$

Hence, plugging (52) and (53) into (51), we obtain:

$$\frac{1}{N^{m-u}} \nu(U_k S_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}}) = \sum_{v=1}^{u} \sum_{\hat{\alpha} \subseteq \alpha} \frac{1}{N^{m-v}} \nu(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}) + O(2k+1). \tag{54}$$

Now, for each $\hat{\alpha} = (\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v})$ in equation (54) we perform a Taylor expansion, which gives

$$\nu(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}) = \sum_{r=0}^{2k} \frac{1}{r!} \nu_0^{(r)}(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}) + \frac{1}{(2k+1)!} \nu_\zeta^{(2k+1)}(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}),$$

and from Schwarz inequality and Propositions 3.2 and 3.3,

$$\nu(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}) = \sum_{r=0}^{2k} \frac{1}{r!} \nu_0^{(r)}(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}) + O(2k+1).$$

Thanks to Proposition 3.1, we get

$$\nu(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}}) = \sum_{r=0}^{2k} \sum_{(\ell_{\tilde{i}_1}, \ldots, j_{\tilde{i}_r}) \in J_{\hat{\alpha}}} c(\hat{\alpha}, r, \beta) \nu_0(U_k^- S^-_{\ell_{\hat{i}_1}, j_{\hat{i}_1}, \ldots, \ell_{\hat{i}_v}, j_{\hat{i}_v}} S^-_{\ell_{\tilde{i}_1}, j_{\tilde{i}_1}, \ldots, \ell_{\tilde{i}_r}, j_{\tilde{i}_r}}) + O(2k+1), \tag{55}$$

where we denote by $J_{\hat{\alpha}}$ the (finite) set of all possible values of $(\ell_{\tilde{\imath}_1}, \ldots, j_{\tilde{\imath}_r})$ given by Proposition 3.1. Hence, putting together Equations (55) and (54), we obtain:

$$\tilde{G}_\alpha = \frac{(-1)^{m-u}}{N^{m-u}} \nu(U_k S_{\ell_{i_1}, j_{i_1}, \ldots, \ell_{i_u}, j_{i_u}}) =$$
$$\sum_{v=1}^{u} \sum_{\hat{\alpha} \subseteq \alpha} \sum_{r=0}^{2k} \sum_{\tilde{\alpha} \in J_{\hat{\alpha}}} \frac{(-1)^{m-u}}{N^{m-v}} c \nu_0(U_k^- S^-_{\ell_{\hat{\imath}_1}, j_{\hat{\imath}_1}, \ldots, \ell_{\hat{\imath}_v}, j_{\hat{\imath}_v}, \ell_{\tilde{\imath}_1}, j_{\tilde{\imath}_1}, \ldots, \ell_{\tilde{\imath}_r}, j_{\tilde{\imath}_r}}) + O(2k+1),$$

which proves our claim (50), by just setting

$$I_\alpha = \{\alpha^* = (\hat{\imath}_1, \hat{\imath}_1, \ldots, \hat{\imath}_v, \hat{\imath}_v, \tilde{\imath}_1, \tilde{\imath}_1, \ldots, \tilde{\imath}_r, \tilde{\imath}_r) | \hat{\alpha} \subseteq \alpha \text{ and } \tilde{\alpha} \in J_{\hat{\alpha}}\},$$

and $G_{\alpha^*} = \frac{(-1)^{m-u}}{N^{m-v}}$. As $v < m$ clearly, we also obtain $G_{\alpha^*} = O(1)$.

Step 5: Conclusion.
Steps 3 and 4 lead us to the following

Definition 4.8. *Given a positive integer k, a collection of functions $(T_\alpha)_{\alpha \in I}$ is called a R-system iff for each $\alpha \in I$ there exists a finite set $I_\alpha \subseteq I$ and functions $H_\alpha, (G_{\alpha_1})_{\alpha_1 \in I_\alpha}$ such that:*

- *$H_\alpha = O(2k)$.*
- *For all $\alpha_1 \in I_\alpha$, $G_{\alpha_1} = O(1)$.*
- *$T_\alpha = H_\alpha + \sum_{\alpha_1 \in I_\alpha} T_{\alpha_1} G_{\alpha_1} + O(2k+1)$.*

We remark that, putting together relations (49) and (50), we get that

$$\nu_0(U_k^- S^-_{\ell_1, j_1, \ldots, \ell_m, j_m}) = \nu(S_{\ell_1, j_1, \ldots, \ell_m, j_m}) + \sum_{\alpha \in I} \sum_{\hat{\alpha} \in I_\alpha} G_{\hat{\alpha}} \nu_0 \left(U_k S^-_{\hat{\alpha}}\right) + O(2k+1),$$

where $G_{\hat{\alpha}} = O(1)$. Thus, if we associate to each $\alpha = (\ell_1, j_1, \ldots, \ell_m, j_m) \in \mathcal{C}_k$ the function $T_\alpha \triangleq \nu_0(U_k^- S^-_{\ell_1, j_1, \ldots, \ell_m, j_m})$ and $H_\alpha \triangleq \nu(S_{\ell_1, j_1, \ldots, \ell_m, j_m})$ then we have that $(T_\alpha)_{\alpha \in \mathcal{C}_k}$ is a R-system. Hence, according to the lemma below, the proof of (44), and thus of Proposition 4.1 Item i) will be complete.

Lemma 4.9. *Given a R-system of functions $(T_\alpha)_{\alpha \in I}$ then for each $\alpha \in I$ we have*

i) *$T_\alpha = H_\alpha + O(2k+1)$.*

ii) *In particular, $T_\alpha = O(2k)$.*

Proof. By Definition 4.8 for each $\alpha \in I$ one has

$$T_\alpha = H_\alpha + \sum_{\alpha_1 \in I_\alpha} T_{\alpha_1} G_{\alpha_1} + O(2k+1),$$

If we use definition 4.8 for each $\alpha_1 \in I_{\alpha_0}$ then there exists a set I_{α_1} such that

$$T_\alpha = H_\alpha + \sum_{\alpha_1 \in I_\alpha} \left(H_{\alpha_1} + \sum_{\alpha_2 \in I_{\alpha_1}} T_{\alpha_2} G_{\alpha_2} + O(2k+1) \right) G_{\alpha_1} + O(2k+1). \tag{56}$$

However, by definition 4.8 we can conclude that $H_{\alpha_1} G_{\alpha_1} = O(2k+1)$, and hence (56) becomes:

$$T_\alpha = H_\alpha + \sum_{\alpha_1 \in I_\alpha} \sum_{\alpha_2 \in I_{\alpha_1}} T_{\alpha_2} G_{\alpha_2} G_{\alpha_1} + O(2k+1).$$

Repeating this process $2k$ times, we obtain:

$$T_\alpha = H_\alpha + \sum_{\alpha_1 \in I_\alpha} \sum_{\alpha_2 \in I_{\alpha_1}} \cdots \sum_{\alpha_{k+1} \in I_{\alpha_k}} T_{\alpha_{2k+2}} G_{\alpha_{2k+2}} \ldots G_{\alpha_2} G_{\alpha_1} + O(2k+1).$$

Using Definition 4.8, each term $T_{\alpha_{k+1}} G_{\alpha_{k+1}} \ldots G_{\alpha_2} G_{\alpha_1}$ is of order $O(2k+2)$, concluding the proof. Item ii) is a trivial consequence of Item i). □

5 Expansion for the second moment

We conclude in this section the proof of our asymptotic expansions (4) and (5).

Proof of Theorem 1.2. We first make use of the definition of $R_{1,\ldots,s}$ and the symmetry property among sites, which yield

$$\nu\left(R^2_{1,2,\ldots,s}\right) = \frac{1}{N}\nu\left(\sum_{i=1}^{N} \sigma_i^1 \sigma_i^2 \ldots \sigma_i^s R_{1,2,\ldots,s}\right) = \nu\left(\epsilon_1 \ldots \epsilon_s R_{1,2,\ldots,s}\right).$$

Apply now the relation

$$\epsilon_1 \epsilon_2 \ldots \epsilon_s R_{1,2,\ldots,s} = \frac{1}{N} + \epsilon_1 \epsilon_2 \ldots \epsilon_s R^-_{1,2,\ldots,s},$$

where we denote by $R^-_{1,2,\ldots,s}$ the quantity $\frac{1}{N}\sum_{i=1}^{N-1} \sigma_i^1 \sigma_i^2 \ldots \sigma_i^s$. We thus get

$$\nu\left(R^2_{1,2,\ldots,s}\right) = \frac{1}{N} + \nu\left(\epsilon_1 \epsilon_2 \ldots \epsilon_s R^-_{1,2,\ldots,s}\right). \tag{57}$$

Notice that in the right side of equation (57), we have to handle the function $R^-_{1,2,\ldots,s}$, which depends on the $N-1$ spin system. The Taylor expansion of this term gives

$$\begin{aligned}&\nu\left(\epsilon_1\epsilon_2\ldots\epsilon_s R^-_{1,2,\ldots,s}\right) = \nu_0\left(\epsilon_1\epsilon_2\ldots\epsilon_s R^-_{1,2,\ldots,s}\right)\\&+\sum_{u=1}^{r}\frac{1}{u!}\nu_0^{(u)}\left(\epsilon_1\epsilon_2\ldots\epsilon_s R^-_{1,2,\ldots,s}\right)+\frac{1}{(r+1)!}\nu_\zeta^{(r+1)}\left(\epsilon_1\epsilon_2\ldots\epsilon_s R^-_{1,2,\ldots,s}\right), \quad (58)\end{aligned}$$

for any positive integer r and for a certain real number $\zeta\in[0,1]$.

Suppose now that s is odd. Invoking Proposition 3.5, then all the derivative terms in Equation (58) vanish, and we obtain

$$\begin{aligned}\nu\left(R^2_{1,2,\ldots,s}\right) &= \frac{1}{N}+\frac{1}{(r+1)!}\nu_\zeta^{(r+1)}\left(\epsilon_1\epsilon_2\ldots\epsilon_s R^-_{1,2,\ldots,s}\right)\\&= \frac{1}{N}+O(r),\end{aligned}$$

where we have used Proposition 3.2. Consequently, relation (4) holds with $r=2p$.

Let us treat now the case when s is even ($s=2k$). Recall that, given a positive integer k, we set

$$U_k^- = \epsilon_1\epsilon_2\ldots\epsilon_{2k}R^-_{1,2,\ldots,2k}$$

and

$$U_k = \epsilon_1\epsilon_2\ldots\epsilon_{2k}R_{1,2,\ldots,2k}.$$

When we choose $r=2k$, equation (58) becomes

$$\nu\left(U_k^-\right)=\nu_0\left(U_k^-\right)+\sum_{u=1}^{2k}\frac{1}{u!}\nu_0^{(u)}\left(U_k^-\right)+\frac{1}{(2k+1)!}\nu_\zeta^{(2k+1)}\left(U_k^-\right).$$

Furthermore, applying Propositions 3.5 and 3.2, we obtain

$$\begin{aligned}\nu\left(U_k^-\right) &= \sum_{u=k}^{2k}\frac{1}{u!}\nu_0^{(u)}\left(U_k^-\right)+\frac{1}{(2k+1)!}\nu_\zeta^{(2k+1)}\left(U_k^-\right)\\&= \sum_{u=k}^{2k}\frac{1}{u!}\nu_0^{(u)}\left(U_k^-\right)+O(2k+1).\end{aligned}$$

However, Proposition 3.1 asserts that $\nu_0^{(u)}(U_k^-)$ can be written as

$$\nu_0^{(u)}(U_k^-)=\sum_{\alpha=(l_1,j_1,\ldots,l_u,j_u)\in\mathcal{D}_{2k,u}} c(2k,u,\alpha)\beta^{2u}\nu_0\left(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_u,j_u}\right), \quad (59)$$

and we are now in a position to identify the negligible terms in the above sum. Indeed, setting α for a tuple $(l_1, j_1, \ldots, l_u, j_u)$, we have:

(1) According to Proposition 3.4, if $\prod_{i=1}^{u} \epsilon_{\ell_i}\epsilon_{j_i} \neq \prod_{i=1}^{k} \epsilon_{2i-1}\epsilon_{2i}$, the term $\nu_0(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_u,j_u})$ vanishes. This means in particular that, in relation (59), $c(2k, u, \alpha) = 0$ unless $\alpha \in \mathcal{C}_k$, and

$$\nu_0^{(u)}(U_k^-) = \sum_{\alpha=(l_1,j_1,\ldots,l_u,j_u)\in\mathcal{D}_{2k,u}\cap\mathcal{C}_k} c(2k,u,\alpha)\beta^{2u}\nu_0\left(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_u,j_u}\right).$$

(2) If $u \geq m+1$, Proposition 4.1 yields

$$\nu_0\left(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_u,j_u}\right) = \nu\left(S_{\ell_1,j_1,\ldots,\ell_u,j_u}\right) + O(2k+1) = O(2k+1).$$

Hence, the terms $\nu_0^{(u)}(U_k^-)$ can be neglected for $u > k$, and we obtain

$$\begin{aligned}\nu(U_k^-) &= \frac{1}{k!}\sum_{\alpha\in\mathcal{D}_{2k,k}\cap\mathcal{C}_k} c(2k,k,\alpha)\beta^{2k}\nu_0\left(U_k^- S^-_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) + O(2k+1)\\ &= \frac{1}{k!}\sum_{\alpha\in\mathcal{D}_{2k,k}\cap\mathcal{C}_k} c(2k,k,\alpha)\beta^{2k}\nu\left(S_{\ell_1,j_1,\ldots,\ell_k,j_k}\right) + O(2k+1),\end{aligned}$$

where we have applied again Proposition 4.1 item (i) for the last equality.

(3) Let us go back now to the definitions (8) and (10) of $\mathcal{C}_k$ and $\mathcal{D}_{2k,k}$, to see that

$$\begin{aligned}\mathcal{C}_k \cap \mathcal{D}_{2k,k} = \{\alpha = (l_1, j_1, \ldots, l_k, j_k);&\\ \alpha \text{ is a permutation of } (1,\ldots,2k),\ l_i < j_i \text{ for all } i \leq k\}.&\end{aligned}$$

In particular, it is easily seen that, if $\alpha \in \mathcal{C}_k \cap \mathcal{D}_{2k,k}$, α is also an element of Ω^{2k}. Thus, owing to Lemma 3.10, we obtain

$$\nu(U_k^-) = \frac{1}{k!N^k}\left(\frac{\beta^2}{1-\beta^2}\right)^k \sum_{\alpha\in\mathcal{D}_{2k,k}\cap\mathcal{C}_k} c(2k,k,\alpha) + O(2k+1). \tag{60}$$

(4) Eventually, we will finish the proof by calculating the sum

$$\sum_{\alpha\in\mathcal{D}_{2k,k}\cap\mathcal{C}_k} c(2k,k,\alpha).$$

A first step in that direction is to notice that

$$\text{Card}\left(\mathcal{D}_{2k,k}\cap\mathcal{C}_k\right) = \binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} = \frac{(2k)!}{2^k}.$$

Furthermore, it is easily seen that $\mathcal{D}_{2k,k} \cap \mathcal{C}_k$ contains exactly $\frac{(2k)!}{2^k k!}$ classes for the relation $\sim$ defined just before Proposition 3.6. Thus, Proposition 3.6 yields

$$\sum_{\alpha \in \mathcal{D}_{2k,k} \cap \mathcal{C}_k} c(2k,k,\alpha) = k! \frac{(2k)!}{2^k k!} = \frac{(2k)!}{2^k},$$

and plugging this relation into (60), we get

$$\nu(U_k^-) = \frac{(2k)!}{2^k k!} \left(\frac{\beta^2}{(1-\beta^2)N} \right)^k + O(2k+1).$$

Putting this relation together with (57), we obtain

$$\nu\left(R_{1,2,\ldots,s}^2\right) = \frac{1}{N} + \frac{(2k)!}{2^k k!} \left(\frac{\beta^2}{(1-\beta^2)N} \right)^k + O(2k+1),$$

which is the announced result (5). □

6 CLT generalization for the overlap function

We will now prove Theorem 1.1. This will be done along the same lines as in [8], except for the use of our asymptotic expansions (4) and (5). We include the proof here for sake of readability: first, we need to establish a result for the moments of $R_{1,\ldots,s}$ which is a natural consequence of Theorem 1.2.

Proposition 6.1. *If $k \geq 0$, $\beta < 1$ and $s \geq 3$ then*

$$\nu(R_{1,\ldots,s}^k) = \frac{a(k)}{N^{\frac{k}{2}}} + O(k+1),$$

where $a(k)$ is the k^{th}-moment of a standard Gaussian random variable.

Proof. We use symmetry between sites to get

$$\begin{aligned} \nu(R_{1,\ldots,s}^k) &= \nu(\epsilon_1 \ldots \epsilon_s R_{1,\ldots,s}^{k-1}) \\ &= \nu(\epsilon_1 \ldots \epsilon_s (R_{1,\ldots,s}^-)^{k-1}) + \frac{k-1}{N} \nu((R_{1,\ldots,s}^-)^{k-2}) \\ &\quad + \sum_{l \geq 2} \frac{1}{N^l} \nu((\epsilon_1 \ldots \epsilon_s)^{(l+1)} (R_{1,\ldots,s}^-)^{k-l-1}), \quad (61) \end{aligned}$$

by writing $R_{1,\ldots,s} = R_{1,\ldots,s}^- + N^{-1}\epsilon_1 \ldots \epsilon_s$ and expanding the power. However, using Theorem 1.2 and Schwarz inequality, we obtain

$$\frac{1}{N^l} \nu((\epsilon_1 \ldots \epsilon_s)^{(l+1)} (R_{1,\ldots,s})^{k-l-1}) = O(k+1),$$

for $l \geq 2$. Writing now $R^-_{1,\dots,s} = R_{1,\dots,s} - \epsilon_1 \dots \epsilon_s/N$ and expanding the quantity $(a+b)^{k-2}$, we see in a similar manner that

$$\nu((R^-_{1,\dots,s})^{k-2}) = \nu(R^{k-2}_{1,\dots,s}) + O(k-1)$$

and thus

$$\frac{1}{N}\nu((R^-_{1,\dots,s})^{k-2}) = \frac{1}{N}\nu(R^{k-2}_{1,\dots,s}) + O(k+1),$$

so that (61) gives

$$\nu(R^k_{1,\dots,s}) = \nu(\epsilon_1 \dots \epsilon_s (R_{1,\dots,s})^{k-1}) + \frac{k-1}{N}\nu((R_{1,\dots,s})^{k-2}) + O(k+1). \quad (62)$$

Now, we perform a Taylor expansion for the first term on the right hand side of (62), which yields

$$\nu(\epsilon_1 \dots \epsilon_s (R_{1,\dots,s})^{k-1}) = \nu_0(\epsilon_1 \dots \epsilon_s (R_{1,\dots,s})^{k-1}) + \nu_0'(\epsilon_1 \dots \epsilon_s (R_{1,\dots,s})^{k-1}) + \frac{1}{2}\nu_\zeta^2(\epsilon_1 \dots \epsilon_s (R_{1,\dots,s})^{k-1}). \quad (63)$$

Since $s \geq 3$, the first two terms on the right hand of (63) vanish, and from Theorem 1.2 the error term is of order $O(k+1)$. So, we can conclude that

$$\nu(R^k_{1,\dots,s}) = \frac{k-1}{N}\nu((R_{1,\dots,s})^{k-2}) + O(k+1),$$

and our claim follows by induction on k. □

Now, let us prove Theorem 1.1:

Proof of Theorem 1.1. Without loss of generality, we can assume $k(1,\dots,s) \geq 1$. For each integer $1 \leq v \leq k$ we consider integers $\ell_1(v), \dots, \ell_s(v)$ such that

$$\prod_{\ell_1 < \dots < \ell_s} R^{k(\ell_1,\dots,\ell_s)}_{\ell_1,\dots,\ell_s} = \prod_{v \leq k} R_{\ell_1(v),\dots,\ell_s(v)},$$

and we set

$$R(v) = R_{\ell_1(v),\dots,\ell_s(v)}, \quad R^-(v) = R^-_{\ell_1(v),\dots,\ell_s(v)}, \quad \epsilon(v) = \epsilon_{\ell_1(v)} \dots \epsilon_{\ell_s(v)},$$

so that $R(v) = R(v)^- + \epsilon(v)/N$. Now we use symmetry between sites to write

$$\nu\Big(\prod_{\ell_1 < \dots < \ell_s} R^{k(\ell_1,\dots,\ell_s)}_{\ell_1,\dots,\ell_s}\Big) = \nu\Big(\prod_{v \leq k} R(v)\Big) = \nu\Big(\epsilon(1) \prod_{2 \leq v \leq k} R(v)\Big), \quad (64)$$

and we expand the product

$$\prod_{2\leq v\leq k} R(v) = \prod_{2\leq v\leq k}\left(R^{-}(v) + \frac{\epsilon(v)}{N}\right).$$

In each of the $k-1$ factors, we can choose either the term $R^{-}(v)$ (henceforth called the big term) or the term $\epsilon(v)/N$ (henceforth called the small term). These $k-1$ choices result into 2^{k-1} terms.

When we choose the small term in at least two factors the resulting contribution is $O(k+1)$, which is easily seen by repeating the argument of Proposition 6.1 and invoking Hölder's inequality. If we choose the small term in exactly l factors, the resulting contribution is

$$O(2l)O(k-1-l) = O(k+1)$$

for $l \geq 2$.

Thus, we only need to consider the contributions where we have chosen the small term in at most one factor, and this gives

$$\begin{aligned}\nu(\epsilon(1)\prod_{2\leq v\leq k} R(v)) &= \nu(\epsilon(1)\prod_{2\leq v\leq k} R^{-}(v)) \\ &\quad + \frac{1}{N}\sum_{2\leq v\leq k}\nu(\epsilon(1)\epsilon(v)\prod_{u} R^{-}(u)) + O(k+1),\end{aligned}$$

where the product is for $2 \leq u \leq k,\ u \neq v$. Since there are $k-2$ terms in the product $\prod_u R^{-}(u)$ and performing another expansion, it follows that

$$\nu(\epsilon(1)\epsilon(v)\prod_{u} R^{-}(u)) = \nu_0(\epsilon(1)\epsilon(v)\prod_{u} R^{-}(u)) + O(k-1). \tag{65}$$

However, $\nu_0(\epsilon(1)\epsilon(v)\prod_u R^{-}(u))$ is zero unless $\epsilon(1)\epsilon(v) = 1$ i.e. $\{1,\ldots,s\} = \{\ell_1(v),\ldots,\ell_s(v)\}$. So, the expression (65) is of order $O(k-1)$ unless $v \leq k(1,\ldots,s)$, and thus

$$\begin{aligned}\nu\Big(\epsilon(1)\prod_{2\leq v\leq k} R(v)\Big) &= \nu\Big(\epsilon(1)\prod_{2\leq v\leq k} R^{-}(v)\Big) \\ &\quad + \frac{k(1,\ldots,s)-1}{N}\nu\Big(\prod_{u} R^{-}(u)\Big) + O(k+1).\end{aligned}$$

We then proceed as in Proposition 6.1 to get, using similar expansions as in (63),

$$\nu(\epsilon(1)\prod_{2\leq v\leq k} R^{-}(v)) = O(k+1),$$

because $s > 2$.

Thus, putting together (64) and (65), we get

$$\nu\Big(\prod_{\ell_1<\ldots<\ell_s} R_{\ell_1,\ldots,\ell_s}^{k(\ell_1,\ldots,\ell_s)}\Big) = \frac{k(1,\ldots,s)-1}{N}\nu\Big(\prod_u R^-(u)\Big) + O(k+1).$$

We can now establish our claim (3) by induction on k. □

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