STOCHASTIC HEAT AND WAVE EQUATIONS ON A LIE GROUP

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Abstract. We study nonlinear heat and wave equations on a Lie group. The noise is assumed to be a spatially homogeneous Wiener process. We give necessary and sufficient conditions for the existence of a function-valued solution in terms of the covariance kernel of the noise.

1. Introduction

In the paper we are concerned with the existence of a solution to the Cauchy problem for the stochastic heat equation

$$\frac{\partial u}{\partial t}(t) = \mathfrak{L}u + F(t, u(t)) + B(t, u(t))\dot{W}(t), \qquad u(0) = u_0 \tag{1.1}$$

and the stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2}(t) = \mathfrak{L}u(t) + R\left(t, u(t)\right) + B\left(t, u(t)\right)\dot{W}(t), \qquad u(0) = u_0,\ \frac{\partial u}{\partial t}(0) = v_0. \tag{1.2}$$

In (1.1) and (1.2), $\mathfrak{L} = \sum_{i=1}^{l} \mathfrak{X}_i^2$ is a sub-elliptic operator on a Lie group G, see Section 1.1, W is a spatially homogeneous Wiener process taking values in the space of tempered distributions $\mathcal{S}'(G)$ on G, see Section 1.2, and

$$F\left(t, u(t)\right)(x) = f\left(t, x, u(t,x)\right), \quad \left(B\left(t, u(t)\right)\mathfrak{f}\right)(x) = b\left(t, x, u(t,x)\right)\mathfrak{f}(x),$$

$$R\left(t, u(t)\right)(t,x) = f\left(t, x, u(t,x)\right) + \sum_{i=1}^{l} \mathfrak{X}_i\left(f_i\left(t, x, u(t,x)\right)\right)$$

are generalized Nemytskii operators corresponding to functions $f, f_i, b : [0,\infty) \times G \times \mathbb{R} \to \mathbb{R}$ and vector fields $\{\mathfrak{X}_i\}$ on G. In the definition of B, $\mathfrak{f}$ belongs to the

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Reproducing Hilbert Kernel Space of the Wiener process, see Section 4 for more details.

This article is part of a global attempt, initiated during the last past years, of studying stochastic heat and wave equations on spaces of dimension greater than 1. The existence, uniqueness and regularity problems are addressed in [D], [DF] and [MS] using Walsh's martingale measure technique, while this program is taken up in [Brz1], [Brz2], [BP], [KZ1], [KZ2], [P1], [P2], [PZ1] and [PZ2] considering solutions to the corresponding stochastic evolution system. On the other hand, our work is also a continuation of [TV], where an effort was made in order find conditions that had to be imposed on the spatial covariance of the noise to ensure the existence of a function-valued solution to the stochastic heat equation on a compact Lie group. Since we chose to work with Markovian solutions to our equations, we will be inspired by methods given in [P2], [PZ1], and [PZ2].

Although we refer the reader to [P3] for bibliographical information on the equations on $\mathbb{R}^d$, we recall, see [KZ1], [KZ2], and [PZ2], that the stochastic heat equation on $\mathbb{R}^d$ admits a function-valued solution provided that the coefficients f and b are Lipschitz, and the space correlation Γ of W is a measure bounded from below and satisfying the following condition

$$\begin{cases} \int_{\{|y|\leq 1\}} \log(|y|^{-1})\Gamma(\,\mathrm{d}y) < \infty & \text{if } d = 2, \\ \int_{\{|y|\leq 1\}} |y|^{-d+2}\Gamma(\,\mathrm{d}y) < \infty & \text{if } d > 2. \end{cases} \tag{1.3}$$

If $d = 1$ then the existence of a solution follows from the fact that Γ is a tempered measure. Furthermore, (1.3) is a necessary condition provided that b is non-degenerate. It turns out, see [DF], [MS], [KZ1], [KZ2], and [PZ2] that (1.3) is also a sufficient, and in a sense necessary condition for the existence of a function-valued solutions to the stochastic wave equation.

In this paper we are concerned with Markovian solution. Thus (1.1), or (1.2) will define a Markov family on a given function state space. We consider scales of state spaces. Namely let $\mathfrak{d}$ be the Carnot–Cathéodory distance associated with $\mathfrak{L}$, see Section 1.1, and let $\vartheta_\rho(x) = \mathrm{e}^{-\rho\mathfrak{d}(x,e)}$. Let $L^p_\rho = L^p(G, \vartheta_\rho(x)\,\mathrm{d}x)$, and let $\mathcal{C}_\rho$ be the space of all continuous $\psi : G \to \mathbb{R}$ such that $|\psi(x)|\vartheta_\rho(x) \to 0$ as $\mathfrak{d}(x,e) \to \infty$. We will deal with the heat equation in spaces L^p_ρ and $\mathcal{C}_\rho$, $\rho \in \mathbb{R}$, $p \in [2,\infty)$. We consider the wave equation in $\mathbb{X}_\rho = (\mathbb{L}^p_\rho, H^{-1}_\rho)$, where $\mathbb{L}^2_\rho = L^2(G, \theta_\rho\,\mathrm{d}x)$ and θ_ρ is a certain regularization of ϑ_ρ, see Section 2 for more details. We will assume that the space correlation of the noise is a measure bounded from below, see (2.1). Then we will show that the necessary and sufficient condition (2.2) for the existence of a solution to (1.1) and (1.2) arises from (1.3) by replacing the Euclidean distance by $\mathfrak{d}$. However in general the characteristic d appearing in (1.3) is bigger than the dimension of the space G, see (1.5).

Our Theorem 2.2 dealing with the regularity of a solution to the stochastic heat equation is knew even in the Euclidean case. It gives conditions for the space-time continuity of a solution. Using the same method one can obtain conditions for its Hölder continuity. The fact that we deal with general Lie groups causes the following additional difficulties:

(1) The complexity of Fourier's analysis, which was an important tool in the works mentioned above, on general non-compact Lie groups makes it difficult to use in a fashionable way for our purposes. We will try to avoid most references to this tool in the sequel, though a characterization of our main hypothesis (2.2) in terms of the Fourier transform of the covariance of the noise will be given in case of the Heisenberg group, see Section 11.
(2) As a consequence of (1), the reproducing Hilbert space kernel of the noise W will not be given in an explicit way.
(3) To our knowledge, the fact that the wave operator generates a C_0-semigroup on the weighted space $\mathbb{X}_\rho$, defined by (2.6), is not a known fact. We will prove it in Section 8.

The paper is organized as follows. In the next two subsections we recall the definitions of a sub-elliptic operator on a Lie group G and the corresponding Carnot–Carathéodory distance, and we introduce the definition of a spatially homogeneous Wiener process on G. In Section 2 we formulate the main results; Theorems 2.1, 2.2, and 2.3, on the existence and regularity of a solution to (1.1) and (1.2). Section 3 is devoted to basic properties of the heat semigroup on weighted spaces. In Section 4 we recall main facts concerning stochastic integration in L^q-spaces. Then in Section 5 we establish some estimates for the so-called γ-radonifying norm of a multiplication operator. These estimates are the crucial ingredients of the proofs of the Theorems 2.1 and 2.2, see Sections 6 and 7. The next two sections are devoted to the proof of Theorem 2.3 dealing with the existence of a solution to (1.2). Our results present conditions ((2.1) and (2.2)) for the existence of a solution in terms of the spatial correlation Γ of the noise. In Sections 10 we deal with two examples; Γ being a bounded function, which on $\mathbb{R}^N$ corresponds to the case of W being a random field, and Γ of the type $(\mathfrak{I}-\mathfrak{L})^{-\alpha}$. In Section 11, G is the Heisenberg group. We formulate our main condition (2.2) using Fourier transform. In the appendix we present the definition and basic properties of the Fourier transform on a Lie group.

1.1 Lie group G and the sub-elliptic operator $\mathfrak{L}$. In the paper G is a locally compact connected Lie group of a dimension N, with the Lie algebra $\mathcal{G}$ and identity element e. The group G is assumed to be equipped with a left invariant metric $\mathfrak{q}$ given by a scalar product on $\mathcal{G}$, and with a left invariant volume element, denoted by $\mathrm{d}x$, which is unique up to a multiplicative constant. We assume that G is unimodular, so that the Haar measure $\mathrm{d}x$ is also right invariant. Moreover, see e.g. [F, p. 48], $\mathrm{d}x$ is invariant with respect to the inverse mapping $x \to x^{-1}$.

Let $\{\mathfrak{X}_1,\ldots,\mathfrak{X}_N\}$ be an orthonormal basis of $\mathcal{G}$, and let $\{\mathfrak{X}_1,\ldots,\mathfrak{X}_l\}$ be a fixed Hörmander system taken out of this basis. In the present paper we are concerned with the *sub-elliptic operator* $\sum_{i=1}^{l} \mathfrak{X}_i^2$. It is known, see e.g. [CSV, p. 21], that $\sum_{i=1}^{l} \mathfrak{X}_i^2$ with the domain $C_0^\infty(G)$ is a symmetric negative defined operator on $L^2 = L^2(G, \mathrm{d}x)$. We will denote by $\mathfrak{L}$ its Friedrichs extension, see [CSV, p. 20]. Then $\mathfrak{L}$ is a negative defined self-adjoint operator on L^2.

Let $C_{\mathfrak{X}}$ be the set of absolutely continuous paths $\gamma : [0,1] \to G$ satisfying $\dot{\gamma}(t) = \sum_{i=1}^{l} a_i(t)\mathfrak{X}_i(\gamma(t))$ for all $t \in [0,1]$. Set $|\gamma| = \int_0^1 |a(t)|_{\mathbb{R}^l}\, \mathrm{d}t$. Then

$$\mathfrak{d}(x,y) = \inf \{|\gamma|; \gamma \in C_{\mathfrak{X}}, \gamma(0) = x, \gamma(1) = y\}$$

defines the *Carnot–Cathéodory distance* associated with $\mathfrak{L}$. Topologically, the distances $\mathfrak{q}$ and $\mathfrak{d}$ are equivalent, see e.g. [CSV, p. 39]. We write $\mathfrak{d}(x) = \mathfrak{d}(x,e)$. As $\mathfrak{d}(x,y) = \mathfrak{d}(zx,zy)$ (see [CSV, p. 40]) we have $\mathfrak{d}(x) = \mathfrak{d}\left(x^{-1}\right)$.

We assume that G is of polynomial growth, i.e. the volume of the ball $B(e,r) = \{x \in G :\ \mathfrak{d}(x) < r\}$ does not grow faster than a polynomial in r as $r \to \infty$.

Let $S = \{S(t), t \geq 0\}$ be the semigroup on L^2 generated by $\mathfrak{L}$. Thus for any $u(0,\cdot) \in L^2$, $u(t,\cdot) = S(t)u(0,\cdot)$ is a unique solution to the equation

$$\frac{\partial u}{\partial t}(t,x) = \mathfrak{L}\, u(t,x), \quad (t,x) \in (0,\infty) \times G.$$

Then S is a symmetric C_0-contraction semigroup. In the paper we deal with the weighted L^p-spaces $L^p_\rho = L^p\left(G, \vartheta_\rho(x)\, \mathrm{d}x\right)$, $p \in [1,\infty)$, $\rho \in \mathbb{R}$, where

$$\vartheta_\rho(x) = \mathrm{e}^{-\rho \mathfrak{d}(x)}. \tag{1.4}$$

Obviously $\vartheta_0 = 1$. For brevity we write L^p instead of L^p_0. We will show, see Lemma 3.3, that $\vartheta_\rho \in L^1 \cap L^2$ for $\rho > 0$, and the heat semigroup S is a C_0-semigroup on any L^p_ρ-space.

Let $\mathcal{I}(l)$ be the set of multi-indexes I with values in $\{1,\ldots,l\}$. For $I = \{i_1,\ldots,i_\alpha\} \in \mathcal{I}(l)$ we define $\mathfrak{X}^I = \left[\mathfrak{X}_{i_1}, [\mathfrak{X}_{i_2}, \ldots, [\mathfrak{X}_{i_{\alpha-1}}, \mathfrak{X}_{i_\alpha}]\ldots]\right]$. Given $j > 0$ we set $K_j = \mathrm{Span}\,\{\mathfrak{X}_I; I \in \mathcal{I}(l), |I| \leq j\}$. Then, the Hörmander condition implies the existence of a minimal $s \in \mathbb{N}$ such that $K_s = \mathcal{G}$. Define $n_0 = 0$ and $n_j = \dim\, K_j$, $j > 0$, and

$$d = \sum_{j=0}^{s} (N - n_j). \tag{1.5}$$

1.2. Spatially homogeneous Wiener process on G.

Let us denote by $\mathcal{S}(G)$ the space of all infinitely differentiable functions ψ on G for which the seminorms

$$p_{m,n}(\psi) = \sup_{x\in G}\ \sup_{I\in\mathcal{I}(l):\,|I|\leq m} \left|\, \mathfrak{d}(x)^n \mathfrak{X}_{i_1} \ldots \mathfrak{X}_{i_\alpha} \psi(x)\, \right|, \qquad m,n \in \mathbb{N}$$

are finite. The dual $\mathcal{S}'(G)$ of $\mathcal{S}(G)$ is then the *space of tempered distributions* on G. In what follows we denote by $\langle \xi, \psi \rangle$ the value of $\xi \in \mathcal{S}'(G)$ on $\psi \in \mathcal{S}(G)$.

Definition 1.1. Let $\mathfrak{A} = (\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathfrak{F}_t)_{t\geq 0}$. We say that an $\mathcal{S}'(G)$-valued process W defined on $\mathfrak{A}$ is *Wiener* iff

(i) for arbitrary finite sets $\{\psi_1, \dots, \psi_n\} \subset \mathcal{S}(G)$ and $\{t_1, \dots, t_n\} \subset [0, \infty)$ the random vector $(\langle W(t_1), \psi_1\rangle, \dots, \langle W(t_n), \psi_n\rangle)$ is Gaussian,

(ii) for any test function $\psi \in \mathcal{S}(G)$, $\langle W(t), \psi\rangle$, $t \geq 0$ is a real-valued $(\mathfrak{F}_t)$-adapted Wiener process.

Let $\left\{\tau^L_x : x \in G\right\}$ and $\left\{\tau^R_x : x \in G\right\}$ be the *groups of left and right translations* on $\mathcal{S}'(G)$, e.a. for $\xi \in \mathcal{S}'(G)$, $\psi \in \mathcal{S}(G)$ and $x, y \in G$,

$$\left\langle \tau^L_x \xi, \psi \right\rangle = \left\langle \xi, \tau^L_{x^{-1}}\psi \right\rangle, \qquad \tau^L_{x^{-1}}\psi(y) = \psi\left(x^{-1}y\right),$$
$$\left\langle \tau^R_x \xi, \psi \right\rangle = \left\langle \xi, \tau^R_{x^{-1}}\psi \right\rangle, \qquad \tau^R_{x^{-1}}\psi(y) = \psi\left(yx^{-1}\right).$$

Definition 1.2. An $\mathcal{S}'(G)$-valued Wiener process W is called *spatially homogeneous* iff for any $t \geq 0$ the law of $W(t)$ is invariant with respect to the group of left translations $\{\tau^L_x : x \in G\}$, that is for all $x \in G$ and $\mathcal{X} \in \mathcal{B}(\mathcal{S}'(G))$,

$$\mathbb{P}\left(W(t) \in \mathcal{X}\right) = \mathbb{P}\left(W(t) \in \left(\tau^L_x\right)^{-1}(\mathcal{X})\right).$$

Given two functions ψ and φ on G we set

$$\psi * \varphi(x) = \int_G \psi\left(xy^{-1}\right)\varphi(y)\,\mathrm{d}y.$$

We call a bilinear form Λ on $\mathcal{S}(G)$ *left translation invariant* iff for all $\psi, \varphi \in \mathcal{S}(G)$, and $x \in G$,

$$\Lambda(\psi, \varphi) = \Lambda\left(\tau^L_x\psi, \tau^L_x\varphi\right).$$

Given a function ψ on G we set $\psi^*(x) = \psi\left(x^{-1}\right)$.

Remark 1.1. Let $\Lambda(\psi, \varphi) = \mathbb{E}\,\langle W(1), \psi\rangle\langle W(1), \varphi\rangle$, $\psi, \varphi \in \mathcal{S}(G)$ be the *covariance form* of an $\mathcal{S}'(G)$-valued Wiener process W. Then, since W is Gaussian, it is spatially homogeneous iff Λ is invariant with respect to the group of left translations.

Since for all $z \in G$ and $\psi, \varphi \in \mathcal{S}(G)$, $\psi^* * \varphi = \left(\tau^L_z\psi\right)^* * \left(\tau^L_z\varphi\right)$ we have the following result.

Proposition 1.1. *Assume that* $\Gamma \in \mathcal{S}'(G)$. *Then*

$$\Lambda(\psi, \varphi) = \langle \Gamma, \psi^* * \varphi\rangle, \qquad \psi, \varphi \in \mathcal{S}(G) \tag{1.6}$$

is a continuous left translation invariant bilinear form on $\mathcal{S}(G)$.

Remark 1.2. It is known, see e.g. [GV] that any translation invariant continuous bilinear form on $\mathcal{S}(\mathbb{R}^N)$ is of the form (1.6). Moreover, by Bochner's theorem it is positive definite iff Γ is the Fourier transform of a positive measure.

Remark 1.3. Assume that the kernel theorem holds true on $\mathcal{S}(G)$, that is any continuous bilinear form Λ on $\mathcal{S}(G)$ is of the form $\Lambda(\psi,\varphi)=\langle\xi,\psi\otimes\varphi\rangle$, where $\xi\in\mathcal{S}'(G\times G)$. Then one can adopt the proof from [GV], and show that any left and right translation invariant continuous bilinear form on $\mathcal{S}(G)$ is of the form (1.6) with a $\Gamma\in\mathcal{S}'(G)$ satisfying $\tau^R_x\tau^L_{x^{-1}}\Gamma=\Gamma$, $x\in G$. We note that the kernel theorem holds true, see [GV], if $\mathcal{S}(G)$ is nuclear, that is its topology is given by an increasing sequence of Hilbertian seminorms $\{q_n\}$, such that the injections $H_{n+1}\hookrightarrow H_n$ are Hilbert–Schmidt, H_n being the completion on $\mathcal{S}(G)$ with respect to q_n.

Remark 1.4. It is easy to show that if G is nilpotent, than $\mathcal{S}(G)$ is nuclear. For, see [CG], Appendix A.2, the Sobolev spaces on G are isomorphic to Sobolev spaces in $\mathbb{R}^N$. Professor Malliavin has pointed out that if G is semi-simple, then the nuclearity of $\mathcal{S}(G)$ can be obtained using the techniques developed in [Ma]. Namely, G can be first decomposed as $G=K\times V$, where K is a maximal compact subgroup of G, and $V=G/K$ is a symmetric space. Clearly, it is sufficient to prove that $\mathcal{S}(K)$ and $\mathcal{S}(V)$ have a nuclear structure, where $\mathcal{S}(K)$ and $\mathcal{S}(V)$ are defined in a standard way. The nuclear structure of $\mathcal{S}(K)$ is obvious, since K is a compact Lie group. The fact that $\mathcal{S}(V)$ is also nuclear is reduced, in [Ma], using Iwasawa coordinates, to the study of a space $\mathcal{S}(A)$, where A is an abelian group, and $\mathcal{S}(A)$ is defined via a second order elliptic operator.

Remark 1.5. Having a positive-definite translation invariant continuous bilinear form Λ one can ask if there is an $\mathcal{S}'(G)$-valued Wiener process with the covariance form Λ. This holds true if $\mathcal{S}'(G)$ is nuclear, see [I], or [KX].

2. Main results

Let $\Lambda(\psi,\varphi)=\mathbb{E}\,\langle W(1),\psi\rangle\langle W(1),\varphi\rangle$, $\psi,\varphi\in\mathcal{S}(G)$ be the covariance form of W. In what follows we assume that Λ is of the form (1.6) with a distribution Γ. We call Γ the *space correlation* of W. Recall that d is defined in (1.5). In our exposition the following assumption plays an essential role.

$$\exists C_\Gamma\ge 0:\quad \Gamma+C_\Gamma\,\mathrm{d}x\qquad\text{is a non-negative measure.}\tag{2.1}$$

Definition 2.1. Let $p\in[2,\infty)$ and $\rho\in\mathbb{R}$. We say that a function $h:[0,\infty)\times G\times\mathbb{R}\to\mathbb{R}$ belongs to the class $\mathrm{Lip}\,(p,\rho)$ iff for any $T<\infty$ there are a constant L and a function $l_0\in L^p_\rho$ such that

$$|h(t,x,z)|\le L(l_0(x)+|z|)\quad\text{and}\quad |h(t,x,z)-h(t,x,\tilde z)|\le L|z-\tilde z|.$$

Our first theorem provides conditions for the existence of a solution to the stochastic heat equation (1.1). By a solution to (1.1) we understand the so-called *mild solution*, that is, a solution to the following integral equation

$$u(t)=S(t)u(0)+\int_0^t S(t-s)F(s,u(s))\,\mathrm{d}s+\int_0^t S(t-s)B(s,u(s))\,\mathrm{d}W(s),$$

where S is the semigroup generated by $\mathfrak{L}$, and the stochastic integral is understood as Itô's integral with respect to an infinite dimensional Wiener process, for more details see Section 4.

Theorem 2.1. *Let $p \in [2, \infty)$ and $\rho \in \mathbb{R}$. Assume that* (2.1) *is satisfied, and the coefficients f, b are of the class* Lip (p, ρ)*.*
(i) If

$$\begin{cases} \int_{B(e,1)} \log\left(\mathfrak{d}(x)^{-1}\right) \Gamma(\mathrm{d}x) < \infty & \text{for } d = 2, \\ \int_{B(e,1)} \mathfrak{d}(x)^{-d+2}\, \Gamma(\mathrm{d}x) < \infty & \text{for } d \neq 2, \end{cases} \tag{2.2}$$

then for any $u_0 \in L^p_\rho$ there is a unique mild solution u to (1.1) *such that for every $T < \infty$,*

$$\sup_{t \in [0,T]} \mathbb{E}\, |u(t)|^2_{L^p_\rho} < \infty. \tag{2.3}$$

(ii) Assume that $\rho > 0$. If there are $T > 0$ and $b_0 > 0$ such that $|b(t, x, z)| \geq b_0$ for all $t \in [0, T]$, $x \in G$, and $z \in \mathbb{R}$, then (2.2) *follows from the existence of a solution to* (1.1) *satisfying* (2.3)*.*

Remark 2.1. Note that as Γ is a tempered distribution, for $d = 1$, (2.2) follows from (2.1).

Let $\mathcal{C}_\rho$ be the class of all continuous functions $\psi \colon G \to \mathbb{R}$ such that $|\psi(x)|\vartheta_\rho(x) \to 0$ as $\mathfrak{d}(x) \to \infty$. Then $\mathcal{C}_\rho$ equipped with the norm $|\psi|_{\mathcal{C}_\rho} = \sup_{x\in G} |\psi(x)\vartheta_\rho(x)|$ is a Banach space. Our next result deals with time and space-time continuous solutions to the stochastic heat equation on G.

Theorem 2.2. *Let $\rho \in \mathbb{R}$ and $p \in [2, \infty)$. Assume that* (2.1) *holds and the coefficients f, b are of the class* Lip (p, ρ)*.*
(i) If there is an $\alpha > 0$ such that

$$\int_{B(e,1)} \mathfrak{d}(x)^{-d-\alpha+2}\, \Gamma(\mathrm{d}x) < \infty, \tag{2.4}$$

then for any $u_0 \in L^p_\rho$ there is a unique mild solution u to (1.1) *having continuous trajectories in L^p_ρ and satisfying*

$$\mathbb{E} \sup_{t \in [0,T]} |u(t)|^q_{L^p_\rho} < \infty \qquad \text{for all } T < \infty,\ q \in [2, \infty).$$

(ii) If $2d/p < 1$ and there is an $\alpha > 2d/p$ such that (2.4) *holds true, then for any $u_0 \in L^p_\rho \cap \mathcal{C}_{\rho/p}$ there is a unique mild solution u to* (1.1) *having continuous trajectories in $L^p_\rho \cap \mathcal{C}_{\rho/p}$ and satisfying*

$$\mathbb{E} \sup_{t\in[0,T]} \left(|u(t)|^q_{\mathcal{C}_{\rho/p}} + |u(t)|^q_{L^p_\rho} \right) < \infty \qquad \text{for all } T < \infty,\ q \in [2, \infty).$$

We are able to show the existence of a solution to the stochastic wave equation in a space with the weight θ_ρ being a regularization of ϑ_ρ, see Lemmas 3.5 and 8.1. It is obtained in the following way

$$\theta_\rho = S(1)\vartheta_\rho, \tag{2.5}$$

S being the heat semigroup. Let $\mathbb{L}^2_\rho = L^2(G, \theta_\rho(x)\,\mathrm{d}x)$, $\rho \in \mathbb{R}$. Let $r = \pm 1$, and let $\rho \in \mathbb{R}$. Define a Sobolev space H^r_ρ as the completion of $C_0^\infty(G)$ with respect to the norm

$$|\psi|_{H^r_\rho} = \left|(\mathfrak{I} - \mathfrak{L})^{r/2}\left(\theta_\rho^{1/2}\psi\right)\right|_{L^2}.$$

Let

$$\mathbb{X}_\rho = \begin{pmatrix} \mathbb{L}^2_\rho \\ H^{-1}_\rho \end{pmatrix}, \qquad \mathbb{D}_\rho = \begin{pmatrix} H^1_\rho \\ \mathbb{L}^2_\rho \end{pmatrix} \qquad \mathcal{A} = \begin{pmatrix} 0 & \mathfrak{I} \\ \mathfrak{L} & 0 \end{pmatrix}. \tag{2.6}$$

We will show, see Lemma 8.2, that $\mathcal{A}$ with the domain $\mathrm{Dom}\,\mathcal{A} = \mathbb{D}_\rho$ generates a C_0-semigroup $U = \{U(t)\}$ on $\mathbb{X}_\rho$. We define an $\mathbb{X}_\rho$-valued solution X to (1.2) as a process satisfying the following stochastic evolution equation

$$X(t) = U(t)X(0) + \int_0^t (t-s)\boldsymbol{F}(s, X(s))\,\mathrm{d}s + \int_0^t U(t-s)\boldsymbol{B}(s, X(s))\,\mathrm{d}W(s),$$

where $X(0) = (u_0, v_0)^{\mathrm{T}}$, and for $X = (u, v)^{\mathrm{T}}$,

$$\boldsymbol{F}(t, X) = \begin{pmatrix} 0 \\ R(t, u) \end{pmatrix} \qquad \text{and} \qquad (\boldsymbol{B}(t, u)\mathfrak{f}) = \begin{pmatrix} 0 \\ B(t, u)\mathfrak{f} \end{pmatrix}. \tag{2.7}$$

The theorem below provides sufficient conditions for the existence of an $\mathbb{X}_\rho$-valued solutions to the stochastic wave equation on G. In its formulation the coefficients are required to be from the following Lipschitz class.

Definition 2.2. Let $p \in [2, \infty)$ and $\rho \in \mathbb{R}$. We say that a function $h : [0, \infty) \times G \times \mathbb{R} \to \mathbb{R}$ belongs to $\mathit{Lip}\,(2, \rho)$ iff for any $T < \infty$ there are a constant L and a function $l_0 \in \mathbb{L}^2_\rho$ such that

$$|h(t, x, z)| \le L(l_0(x) + |z|) \quad \text{and} \quad |h(t, x, z) - h(t, x, \tilde{z})| \le L|z - \tilde{z}|.$$

Note that in Definition 2.1 of $\mathrm{Lip}\,(2, \rho)$ the function l_0 belongs to L^2_ρ, whereas in the definition of $\mathit{Lip}\,(2, \rho)$ it belongs to $\mathbb{L}^2_\rho$.

Theorem 2.3. *Let $\rho \in \mathbb{R}$. Assume that* (2.1) *and* (2.2) *are fulfilled, and the coefficients $f, f_i, b \in \mathit{Lip}\,(2, \rho)$. Then for any $(u_0, v_0)^{\mathrm{T}} \in \mathbb{X}_\rho$ there is a a unique solution to* (1.2) *with continuous trajectories in $\mathbb{X}_\rho$ and such that*

$$\mathbb{E} \sup_{t \in [0,T]} |X(t)|^q_{\mathbb{X}_\rho} < \infty \qquad \textit{for all } T < \infty,\ q \in [1, \infty).$$

3. Properties of the heat semigroup

Let $\mathcal{H}_t$ be the heat kernel on G, see [CSV, p. 106]. Thus for any $\psi \in L^2$,

$$S(t)\psi(x) = \psi * \mathcal{H}_t(x) = \int_G \mathcal{H}_t\left(y^{-1}x\right)\psi(y)\,\mathrm{d}y.$$

Note as S is symmetric we have $\mathcal{H}_t^* = \mathcal{H}_t$. Set $\mathfrak{h}_t(r) = t^{-d/2}\exp\{-r^2/t\}$. For a proof of the lemma below we refer the reader to [CSV, Th. 8.2.9].

Lemma 3.1. *There are constants $c, C > 0$ such that for all $t > 0$ and $x \in G$, and $i \in \{1, \dots, l\}$,*

$$c\mathfrak{h}_t\left(\mathfrak{d}(x)/c\right) \le \mathcal{H}_t(x) \le C\mathfrak{h}_t\left(\mathfrak{d}(x)/C\right),$$

$$ct^{-\frac{1}{2}}\mathfrak{h}_t\left(\mathfrak{d}(x)/c\right) \le \mathfrak{X}_i\mathcal{H}_t(x) \le Ct^{-\frac{1}{2}}\mathfrak{h}_t\left(\mathfrak{d}(x)/C\right),$$

$$ct^{-1}\mathfrak{h}_t\left(\mathfrak{d}(x)/c\right) \le \mathfrak{L}\mathcal{H}_t(x) \le Ct^{-1}\mathfrak{h}_t\left(\mathfrak{d}(x)/C\right).$$

Obviously, $\mathcal{H}_t(x) > 0$ for all x and t. Since the constant function 1 is the unique solution to

$$\frac{\partial u}{\partial t} = \mathfrak{L}u, \qquad u(0,\cdot) = 1,$$

we have $\int_G \mathcal{H}_t\left(y^{-1}x\right)\,\mathrm{d}y = 1$, $t > 0$, $x \in G$.

Lemma 3.2. *For each $t > 0$, $\mathcal{H}_t \in \mathcal{S}(G)$, and $x \to \left\{\mathcal{H}_t(y^{-1}x);\, y \in G\right\}$ is a continuous mapping from G into $\mathcal{S}(G)$. Moreover, the restriction of the heat semigroup to $\mathcal{S}(G)$ is a C_0-semigroup on $\mathcal{S}(G)$.*

Proof. Clearly it is enough to show that for all $n \in \mathbb{N}$, $I = (i_1, \dots, i_n) \in \mathcal{I}(l)$ and $\mathfrak{X}_I = \mathfrak{X}_{i_1}\dots\mathfrak{X}_{i_n}$, and a fixed $x \in G$,

$$\lim_{\mathfrak{d}(z,x)\to 0}\sup_{y\in G}\Pi_t(y,x,z) = 0,$$

where

$$\Pi_t(y,x,z) = \mathfrak{d}(y)^n\mathfrak{X}_{i_1,\dots,i_n}\left(\mathcal{H}_t\left(y^{-1}x\right) - \mathcal{H}_t\left(y^{-1}z\right)\right).$$

Note that for any $\alpha \in (0,1)$, $\Pi_t(y,x,z) \le \Pi_t^{(\alpha)}(y,x,z)\Pi_t^{(1-\alpha)}(y,x,z)$, where

$$\Pi_t^{(\alpha)}(y,x,z) = \mathfrak{d}(y)^n\left(\left|\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}x\right)\right| + \left|\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}z\right)\right|\right)^\alpha,$$

$$\Pi_t^{(1-\alpha)}(y,x,z) = \left|\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}x\right) - \mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}z\right)\right|^{1-\alpha}.$$

Let y be such that $\mathfrak{d}(y) \ge 3\,\mathfrak{d}(x)$. Then, observing that for $z \in B\left(x, \mathfrak{d}(x)/2\right)$ we have

$$\mathfrak{d}\left(y^{-1}x\right) \ge \mathfrak{d}(y) - \mathfrak{d}(x) \ge \frac{1}{2}\mathfrak{d}(y) \qquad \text{and} \qquad \mathfrak{d}\left(y^{-1}z\right) \ge \mathfrak{d}(y) - \mathfrak{d}(z) \ge \frac{1}{2}\mathfrak{d}(y)$$

we obtain

$$\Pi_t^{(\alpha)}(y,x,z) \leq$$
$$2^n \left(\mathfrak{d}\left(y^{-1}x\right)^{n/\alpha} \left|\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}x\right)\right| + \mathfrak{d}\left(y^{-1}z\right)^{n/\alpha} \left|\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}z\right)\right| \right)$$

and hence

$$\sup\left\{\Pi_t^{(\alpha)}(y,x,z):\ \mathfrak{d}(y) \geq 3\mathfrak{d}(x),\ z \in B\left(x, \mathfrak{d}(x)/2\right)\right\} < \infty.$$

By boundedness of all the derivatives of the heat kernel, we also have

$$\sup\left\{\Pi_t^{(\alpha)}(y,x,z):\ \mathfrak{d}(y) \leq 3\,\mathfrak{d}(x),\ z \in B\left(x, \mathfrak{d}(x)/2\right)\right\} < \infty,$$

and thus

$$\sup_{y\in G,\ z\in B(x,\mathfrak{d}(x)/2)} \Pi_t^{(\alpha)}(y,x,z) =: M^{(\alpha)} < \infty. \tag{3.1}$$

Consider now a normalized Cantor geodesic γ, see Section 1.1, joining x and z. Then

$$\left(\Pi_t^{(1-\alpha)}(y,x,z)\right)^{1/(1-\alpha)} = \left|\sum_{i=1}^{l}\int_0^{\mathfrak{d}(x,z)} a(\theta)\mathfrak{X}_i\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}_t\left(y^{-1}\gamma(\theta)\right)\,\mathrm{d}\theta\right|$$

and setting

$$C = \sup_{i=1,\dots,l,\ z\in G} \left|\mathfrak{X}_i\mathfrak{X}_{i_1,\dots,i_n}\mathcal{H}\left(z\right)\right|$$

we get

$$\left(\Pi_t^{(1-\alpha)}(y,x,z)\right)^{1/(1-\alpha)} \leq C\,\mathfrak{d}(x,z). \tag{3.2}$$

Putting (3.1) and (3.2) together we obtain

$$\Pi_t(y,x,z) \leq \Pi^{(\alpha)}\left(C\,\mathfrak{d}(x,z)\right)^{1-\alpha} \qquad \text{for } z \in B\left(x, \mathfrak{d}(x)/2\right),$$

which completes the proof. □

In Lemmas 3.3 and 3.4 below we show that the heat semigroup is C_0 on the weighted spaces L^p_ρ and $\mathcal{C}_\rho$. In the proofs we will use arguments of Funaki, see [Fu, Lm. 2.1].

Lemma 3.3. *Let $\rho \in \mathbb{R}$, and let ϑ_ρ be given by (1.4). Then $\vartheta_\rho \in L^1 \cap L^2$ for $\rho > 0$, and the heat semigroup S has the unique extension from $C_0^\infty(G)$ to a C_0-semigroup on L^p_ρ for arbitrary $p \in [2, \infty)$ and $\rho \in \mathbb{R}$. Moreover, $\tau_y^R \vartheta_\rho(x) \le \mathrm{e}^{|\rho|\mathfrak{d}(y)} \vartheta_\rho(x)$ for all $x, y \in G$.*

Proof. Recall that $V(r) = \int_{B(e,r)} \mathrm{d}x$ has a polynomial growth. Hence for any $\rho > 0$ we have

$$\int_G \mathrm{e}^{-\rho\mathfrak{d}(x)}\, \mathrm{d}x = \int_0^\infty \mathrm{e}^{-\rho r} V(\,\mathrm{d}r) < \infty,$$

which proves the first part of the lemma. Since $C_0^\infty(G)$ is dense in L^p_ρ it is enough to show that for all $T > 0$, $\rho \in \mathbb{R}$, and $p \in [2, \infty)$ there is a constant C such that for all $\psi \in C_0^\infty(G)$ and $t \in [0, T]$ one has $|S(t)\psi|^p_{L^p_\rho} \le C\, |\psi|^p_{L^p_\rho}$. Let us fix T, ρ and p. Let

$$\tilde{S}(t)\psi(x) = \int_G \mathfrak{h}_t\left(\mathfrak{d}\left(y^{-1}x\right)\right) \psi(y)\, \mathrm{d}y. \tag{3.3}$$

Taking into account Lemma 3.1 it is enough to show that

$$\exists \tilde{C} :\ \forall \psi \in C_0^\infty(G),\ t \in (0, T], \qquad |\tilde{S}(t)\psi|^p_{L^p_\rho} \le \tilde{C}\, |\psi|^p_{L^p_\rho}. \tag{3.4}$$

Note that

$$\begin{aligned}
\mathfrak{h}_t\left(\mathfrak{d}\left(y^{-1}x\right)\right) \vartheta_\rho(x) &= t^{-d/2} \exp\left\{-\mathfrak{d}^2\left(x, y\right)/t - \rho\mathfrak{d}(x)\right\} \\
&= t^{-d/2} \exp\left\{-\mathfrak{d}^2\left(x, y\right)/t - \rho\mathfrak{d}(x) + \rho\mathfrak{d}(y)\right\} \vartheta_\rho(y) \\
&\le t^{-d/2} \exp\left\{-\mathfrak{d}^2\left(x, y\right)/t + |\rho|\mathfrak{d}(x, y)\right\} \vartheta_\rho(y) \\
&\le \exp\{t\rho^2/2\} \mathfrak{h}_t\left(\mathfrak{d}\left(x^{-1}y\right)/\sqrt{2}\right) \vartheta_\rho(y).
\end{aligned} \tag{3.5}$$

Since $\int_G \mathcal{H}_t\left(x^{-1}y\right)\, \mathrm{d}y = 1$, Lemma 3.1 yields that there is a constant $C_1 = C_1(T)$ such that

$$\int_G \mathfrak{h}_t\left(\mathfrak{d}\left(x^{-1}y\right)/\sqrt{2}\right) \mathrm{d}x \le C_1 \qquad \text{for all } t \in (0, T]. \tag{3.6}$$

Combining (3.5) with (3.6) we can find a constant $C_2 = C_2(T, \rho)$ such that

$$\int_G \mathfrak{h}_t\left(\mathfrak{d}\left(y^{-1}x\right)\right) \vartheta_\rho(x)\, \mathrm{d}x \le C_2 \vartheta_\rho(y) \qquad \text{for all } y \in G \text{ and } t \in (0, T]. \tag{3.7}$$

From this and (3.6) one can easily obtain (3.4). In order to show the last part of the lemma note that since $\tau_y^R \mathfrak{d}(x) = \mathfrak{d}(e, xy) = \mathfrak{d}(y^{-1}, x)$, we have

$$\begin{aligned}
-\tau_y^R \mathfrak{d}(x) &\le \mathfrak{d}(y^{-1}) - \mathfrak{d}(x) = \mathfrak{d}(y) - \mathfrak{d}(x), \\
\tau_y^R \mathfrak{d}(x) &\le \mathfrak{d}(y^{-1}) + \mathfrak{d}(x) \le \mathfrak{d}(y) + \mathfrak{d}(x).
\end{aligned}$$

Consequently $\tau_y^R \mathrm{e}^{-\rho\mathfrak{d}}(x) \le \mathrm{e}^{|\rho|\mathfrak{d}(y)} \mathrm{e}^{-\rho\mathfrak{d}}(x)$. $\square$

Lemma 3.4. *Let $\rho \in \mathbb{R}$. Then the heat semigroup S is C_0 on $\mathcal{C}_\rho$. Moreover, for all $p \in [2,\infty)$ and $t > 0$, $S(t)$ is a bounded linear operator from L^p_ρ into $\mathcal{C}_{\rho/p}$, and for any T there is a constant $C = C(T,p,\rho)$ such that*

$$||S(t)||_{L(L^p_\rho,\mathcal{C}_{\rho/p})} \leq Ct^{-d/(2p)} \qquad \text{for } t \in (0,T]. \tag{3.8}$$

Proof. Let us fix T, ρ and p. Let $\tilde{S}$ be given by (3.3). Clearly it is enough to show that there are constants $\tilde{C}_1 = \tilde{C}_1(T,\rho)$ and $\tilde{C}_2 = \tilde{C}_2(T,\rho,p)$ such that

$$|\tilde{S}(t)\psi|_{\mathcal{C}_\rho} \leq \tilde{C}_1|\psi|_{\mathcal{C}_\rho} \qquad \text{for all } t \in (0,T],\ \psi \in \mathcal{S}(G) \tag{3.9}$$

and that

$$|\tilde{S}(t)\psi|_{\mathcal{C}_{\rho/p}} \leq \tilde{C}_2 t^{-d/(2p)}|\psi|_{L^p_\rho} \qquad \text{for all } t \in (0,T],\ \psi \in \mathcal{S}(G). \tag{3.10}$$

We have

$$\begin{aligned}\left|\tilde{S}(t)\psi(x)\right| &\leq \int_G \mathfrak{h}_t\left(\mathfrak{d}\left(y^{-1}x\right)\right)|\psi(y)|\,\mathrm{d}y \\ &\leq \int_G \mathfrak{h}_t\left(\mathfrak{d}\left(y^{-1}x\right)\right)\vartheta_{-\rho}(y)\vartheta_\rho(y)|\psi(y)|\,\mathrm{d}y \\ &\leq |\psi|_{\mathcal{C}_\rho}\int_G \mathfrak{h}_t\left(\mathfrak{d}\left(y^{-1}x\right)\right)\vartheta_{-\rho}(y)\,\mathrm{d}y.\end{aligned}$$

Combining this estimate and (3.7) with ρ being replaced by $-\rho$ we obtain (3.9). To show (3.10) note that using first Hölder's inequality we get

$$\left|\tilde{S}(t)\psi(x)\right| \leq |\psi|_{L^p_\rho}\left(\int_G \mathfrak{h}_t^{p^*}\left(\mathfrak{d}\left(y^{-1}x\right)\right)\vartheta_{-p^*\rho/p}(y)\,\mathrm{d}y\right)^{1/p^*},$$

where $p^* = p/(p-1)$. Using now (3.7) for ρ being replaced by $-p^*\rho/p$, one can easily obtain

$$\left(\int_G \mathfrak{h}_t^{p^*}\left(\mathfrak{d}\left(y^{-1}x\right)\right)\vartheta_{-p^*\rho/p}(y)\,\mathrm{d}y\right)^{1/p^*} \leq Ct^{-d/(2p)}\vartheta_{-\rho/p}(x),$$

which proves (3.10). □

Lemma 3.5. *Let $\rho \in \mathbb{R}$, and let θ_ρ be given by (2.4). Then $\theta_\rho > 0$, $\theta_\rho \in L^1 \cap L^2$ for $\rho > 0$, and the heat semigroup S has the unique extension to a C_0-semigroup on $\mathbb{L}^p_\rho = L^p(G,\theta_\rho(x)\,\mathrm{d}x)$ for arbitrary $p \in [2,\infty)$ and $\rho \in \mathbb{R}$.*

Proof. Clearly $\vartheta_\rho > 0$. By Lemma 3.3, $\vartheta_\rho \in L^1 \cap L^2$ for $\rho > 0$. Hence, by Lemmas 3.1 and 3.4, $\theta_\rho > 0$ and $\theta_\rho \in L^1 \cap L^2$ for $\rho > 0$. To show that S is C_0 on any $\mathbb{L}^p_\rho$-space it is enough to show that for every $T > 0$ there is a constant C such that $S(t)\theta_\rho(x) \leq C\theta_\rho(x)$ for $t \in (0,T]$. This follows from (3.7);

$$S(t)\theta_\rho = S(t)S(1)\vartheta_\rho \leq CS(1)\vartheta_\rho = C\theta_\rho.$$

□

4. Distribution-valued Wiener processes

Let W be $\mathcal{S}'(G)$-valued Wiener process. Then, see e.g. [BJ], [I], or [KX], there is a unique real separable Hilbert space $H_W \subset \mathcal{S}'(G)$ such

$$W(t) = \sum_k W_k(t)\mathfrak{f}_k, \tag{4.1}$$

where $\{\mathfrak{f}_k\}$ is an arbitrary orthonormal basis of H_W and $\{W_k\}$ is a sequence of standard independent real-valued $(\mathfrak{F}_t)$-adapted Wiener processes. The series converges in $\mathcal{S}'(G)$, in the sense that for any test function ψ, and for any t, $\langle W(t), \psi\rangle$ is the $L^2(\Omega, \mathfrak{F}, \mathbb{P})$-limit of the series $\sum_k W_k(t)\langle \mathfrak{f}_k, \psi\rangle$. We call H_W the *Reproducing Hilbert Kernel Space* of W, RHKS in short.

4.1 Stochastic integration in Hilbert spaces. In this section we recall the construction and properties of Itô's integral with respect to an $\mathcal{S}'(G)$-valued Wiener process W defined on probability basis $\mathfrak{A} = (\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$. We denote by H_W the RHKS of W. Given $q \in [2,\infty)$ and a Banach space V we denote by $\mathcal{P}^q(V)$ the space of all measurable $(\mathfrak{F}_t)$-adapted processes σ with values in V such that the seminorms

$$|\sigma|_T = \left(\mathbb{E}\int_0^T |\sigma(t)|_V^q \,\mathrm{d}t\right)^{1/q}, \qquad T \in (0,\infty),$$

are finite. Assume that V is a separable Hilbert space. Let $L_{(\mathrm{HS})}(H_W, V)$ be the space of all of Hilbert–Schmidt operators from H_W to V. Let us fix an orthonormal basis $\{\mathfrak{f}_k\}$ of H_W, and let $\{W_k\}$ be a sequence of independent real-valued Wiener processes for which (4.1) holds true. Let us denote by Π_n the projection onto the space spanned by $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$. Let $\mathcal{P}_0\left(L_{(\mathrm{HS})}(H_W, V)\right)$ denote the class of all $\sigma \in \mathcal{P}^2\left(L_{(\mathrm{HS})}(H_W, V)\right)$ such that

$$\sigma(\omega, t) = \sum_{j=1}^n \sigma_j(\omega)\Pi_i\chi_{(t_j, t_{j+1}]}(t)$$

for some positive integers n and i, and $0 \leq t_1 < \ldots < t_{n+1} < \infty$, and $\sigma_j \in L^2\left(\Omega, \mathfrak{F}_{t_j}, \mathbb{P}; L_{(\mathrm{HS})}(H_W, V)\right)$. For $\sigma \in \mathcal{P}_0\left(L_{(\mathrm{HS})}(H_W, V)\right)$ and $t \in [0,\infty)$ we put

$$\mathcal{I}_t^W\sigma = \int_0^t \sigma(s)\,\mathrm{d}W(s) = \sum_{j=1}^n\sum_{k=1}^i \big(W_k(t_{j+1}\wedge t) - W_k(t_j \wedge t)\big)\sigma_j\mathfrak{f}_k.$$

It is easy to see that $\mathcal{I}_t^W$ can be extended continuously to $\mathcal{P}^2\left(L_{(\mathrm{HS})}(H_W, V)\right)$. Moreover, for any $\sigma \in \mathcal{P}^2\left(L_{(\mathrm{HS})}(H_W, V)\right)$ the process $\mathcal{I}_t^W\sigma$, $t \in [0,\infty)$ is a continuous square integrable martingale in V, and

$$\mathbb{E}\,|\mathcal{I}_t^W\sigma|_V^2 = \mathbb{E}\int_0^t |\sigma(s)|^2_{L_{(\mathrm{HS})}(H_W,V)}\,\mathrm{d}s, \qquad t \in [0,\infty). \tag{4.2}$$

It is easy to see that the stochastic integral does not depend on the particular choice of orthonormal basis $\{\mathfrak{f}_k\}$.

Lemma 4.1. *Let $\Lambda(\psi,\varphi)=\mathbb{E}\,\langle W(1),\psi\rangle\langle W(1),\varphi\rangle$, $\psi,\varphi\in\mathcal{S}(G)$ be the covariance form of a Wiener process W with the RHKS H_W. Then for an arbitrary orthonormal basis $\{\mathfrak{f}_k\}$ of H_W one has*

$$\Lambda(\psi,\varphi)=\sum_k\langle\mathfrak{f}_k,\psi\rangle\langle\mathfrak{f}_k,\varphi\rangle,\qquad \psi,\varphi\in\mathcal{S}(G),$$

where $\langle\cdot,\cdot\rangle$ stands for the bilinear duality form on $\mathcal{S}'(G)\times\mathcal{S}(G)$.

Proof. Let $\{\mathfrak{f}_k\}$ be an orthonormal basis of H_W, and let $\{W_k\}$ be a sequence of independent standard Wiener processes such that (4.1) holds true. Then

$$\begin{aligned}\Lambda(\psi,\varphi)&=\mathbb{E}\,\langle W(1),\psi\rangle\langle W(1),\varphi\rangle\\&=\mathbb{E}\left(\sum_k\langle\mathfrak{f}_k,\psi\rangle W_k(1)\right)\left(\sum_k\langle\mathfrak{f}_k,\varphi\rangle W_k(1)\right)=\sum_k\langle\mathfrak{f}_k,\psi\rangle\langle\mathfrak{f}_k,\varphi\rangle,\end{aligned}$$

which is the desired conclusion. □

In the next result we compute the Hilbert–Schmidt norm of an integral operator on the RHKS of W.

Lemma 4.2. *Let ν be a non-negative measure on a measurable space $(O,\mathcal{O})$, let $L^2(\nu)=L^2(O,\mathcal{O},\nu)$, and let $\mathcal{K}$ be a measurable mapping from O into $\mathcal{S}(G)$. Consider the operator $K\mathfrak{f}(x)=\langle\mathfrak{f},\mathcal{K}(x)\rangle$, $\mathfrak{f}\in H_W$, $x\in O$. Then for any orthonormal basis $\{\mathfrak{f}_k\}$ of H_W one has*

$$\sum_k|K\mathfrak{f}_k|^2_{L^2(\nu)}=\int_O\Lambda\left(\mathcal{K}(x),\mathcal{K}(x)\right)\nu(\,\mathrm{d}x).$$

Proof. Since

$$\sum_k|K\mathfrak{f}_k|^2_{L^2(\nu)}=\sum_k\int_O\langle\mathfrak{f}_k,\mathcal{K}(x)\rangle^2\,\nu(\,\mathrm{d}x),$$

Lemma 4.1 gives the desired conclusion. □

4.2 Stochastic integration in weighted L^p-spaces. Let ν be a non-negative measure on a measurable space $(O,\mathcal{O})$, and let $L^p(\nu)=L^p(O,\mathcal{O},\nu)$. In this section we present basic facts on the stochastic integral in $L^p(\nu)$-spaces with respect to a spatially homogeneous Wiener process W. Most of results presented here are particular cases of the more general theory of stochastic integration in Banach spaces, see e.g. [Brz1], [Brz2], [BP], [De], and [Ne].

In this section we fix $p\in[2,\infty)$, an orthonormal basis $\{\mathfrak{f}_k\}$ of H_W, and a sequence $\{\beta_k\}$ of independent standard real-valued normal random variables defined on a probability base $\mathfrak{A}$.

A bounded linear operator $K : H_W \to L^p(\nu)$ is called *γ-radonifying*, iff the series $\sum_{k=1}^{\infty} \beta_k K \mathfrak{f}_k$ converges in $L^2(\Omega, \mathfrak{F}, \mathbb{P}; L^p(\nu))$. We use $R(H_W, L^p(\nu))$ to denote the class of all γ-radonifying operators from H_W into $L^p(\nu)$. Given a linear operator K from H_W into $L^p(\nu)$ write

$$||K||^2_{R(H_W,L^p(\nu))} = \limsup_n \mathbb{E}\left|\sum_{k=1}^{n} \beta_k K \mathfrak{f}_k\right|^2_{L^p(\nu)} \tag{4.3}$$

Then, see e.g. [Ne], K is γ-radonifying iff $||K||_{R(H_W,L^p(\nu))}$ is finite. Note that $R(H_W, L^p(\nu))$ equipped with the norm $||K||_{R(H_W,L^p(\nu))}$ is a Banach space. Note that if $p = 2$, that is if $L^p(\nu)$ is a Hilbert space, then the spaces $R(H_W, L^2(\nu))$ and $L_{(\mathrm{HS})}(H_W, L^2(\nu))$ and the radonifying and Hilbert–Schmidt norms are equal.

The lemma below provides an useful estimate for the γ-radonifying norm of an operator given by a kernel, and it is an analogue of Lemma 4.2 from the present paper. It is a reformulation of [BP, Prop. 2.1].

Lemma 4.3. *Assume that $K \in L(H_W, L^p(\nu))$ is given by $(K\mathfrak{f})(x) = \langle \mathfrak{f}, \mathcal{K}(x)\rangle$, $x \in O$, $\mathfrak{f} \in H_W$, where $\mathcal{K}$ is a measurable mapping from O into $\mathcal{S}(G)$. Then there is a constant C independent of K such that*

$$||K||_{R(H_W,L^p(\nu))} \le C\Big(\int_G \Lambda\left(\mathcal{K}(x), \mathcal{K}(x)\right)^{p/2} \nu(\,\mathrm{d}x)\Big)^{1/p}.$$

Proof. Since for each x the real-valued random variable $\sum_{k=1}^{n} \beta_k \langle \mathcal{K}(x), \mathfrak{f}_k\rangle$ is Gaussian, there exists a constant C_1 depending only on p such that

$$\begin{aligned}
\Big(\mathbb{E}\Big|\sum_{k=1}^{n} \beta_k K\mathfrak{f}_k\Big|^2_{L^p(\nu)}\Big)^{p/2} &= \Big(\mathbb{E}\Big(\int_G\Big|\sum_{k=1}^{n}\beta_k\langle \mathfrak{f}_k, \mathcal{K}(x)\rangle\Big|^p \nu(\,\mathrm{d}x)\Big)^{2/p}\Big)^{p/2} \\
&\le \mathbb{E}\int_G\Big|\sum_{k=1}^{n}\beta_k\langle \mathfrak{f}_k, \mathcal{K}(x)\rangle\Big|^p \nu(\,\mathrm{d}x) \\
&\le C_1\int_G\Big(\mathbb{E}\Big|\sum_{k=1}^{n}\beta_k\langle \mathfrak{f}_k, \mathcal{K}(x)\rangle\Big|^2\Big)^{p/2} \nu(\,\mathrm{d}x) \\
&\le C_1\int_G\Big|\sum_{k=1}^{n}\langle \mathfrak{f}_k, \mathcal{K}(x)\rangle^2\Big|^{p/2} \nu(\,\mathrm{d}x).
\end{aligned}$$

Thus by Lemma 4.1,

$$\begin{aligned}
||K||^2_{R(H_W,L^p(\nu))} &= \limsup_n \mathbb{E}\Big|\sum_{k=1}^{\infty} \beta_k K\mathfrak{f}_k\Big|^2_{L^p(\nu)} \\
&\le C_1^{2/p}\Big(\int_G \Lambda(\mathcal{K}(x), \mathcal{K}(x))^{p/2}\nu(\,\mathrm{d}x)\Big)^{2/p},
\end{aligned}$$

which is the desired conclusion. □

The stochastic integral with respect to W can be defined first for simple processes $\mathcal{P}_0(R(H_W, L^p(\nu)))$ and then extended to the space $\mathcal{P}^2(R(H_W, L^p(\nu)))$. We have the following consequence of general theorems on stochastic integration in Banach spaces, [Brz1], [De], and [Ne].

Theorem 4.1. *For any $\sigma \in \mathcal{P}^2(R(H_W, L^p(\nu)))$ the stochastic integral*

$$\int_0^t \sigma(s)\,\mathrm{d}W(s), \qquad t \geq 0$$

is an L^p_ρ-valued square integrable martingale with continuous modification and 0 mean. Moreover, for every $q \in [2, \infty)$ there is a constant C independent of T and σ, such that

$$\mathbb{E} \sup_{0\leq t\leq T} \left| \int_0^t \sigma(s)\,\mathrm{d}W(s) \right|^q_{L^p(\nu)} \leq C\,\mathbb{E} \left(\int_0^T ||\sigma(s)||^2_{R(H_W, L^p(\nu))}\,\mathrm{d}s \right)^{q/2}.$$

5. Main estimates

Throughout this section we assume that (2.1) is fulfilled with a constant $C_\Gamma \geq 0$, that is $\Gamma + C_\Gamma\,\mathrm{d}x$ is a non-negative measure, and $\{\mathfrak{f}_k\}$ is an orthonormal basis of the RKHS H_W of a spatially homogeneous Wiener process W with the space correlation Γ. Given $\psi \in \mathcal{S}(G)$ we define the multiplication operator on H_W by $M_\psi \mathfrak{f} = \psi \mathfrak{f}$. We extent M_ψ for ψ equal to the constant function 1 taking $M_1\mathfrak{f} = \mathfrak{f}$. In fact, we will show, see Corollary 5.1 that if (2.1) is satisfied, then M has a unique extension, denoted also by M, to a bounded linear operator from any L^p_ρ-space to $R(H_W, L^p_\rho)$. Recall that $||K||_{R(H_W, L^p_\rho)} \in [0, \infty]$ is given by (4.3), and that K is γ-radonifying iff $||K||_{R(H_W, L^p_\rho)} < \infty$. Moreover, $||K||_{R(H_W, L^2_\rho)} = ||K||_{L_{(\mathrm{HS})}(H_W, L^2_\rho)}$.

Let $T > 0$, $\alpha > 0$. For $r \geq 0$ we set

$$\kappa(\alpha, T, r) = \int_0^T t^{-\alpha} \mathrm{e}^{-r/t}\,\mathrm{d}t \qquad \text{and} \qquad \kappa(\alpha, r) = \int_0^\infty t^{-\alpha} \mathrm{e}^{-t} \mathrm{e}^{-r/t}\,\mathrm{d}t. \tag{5.1}$$

Lemma 5.1. *(i) for all $\alpha \geq 0$, $n \in \mathbb{N}$, and $T > 0$, $r^n\kappa(\alpha, T, r) \to 0$ and $r^n\kappa(\alpha, r) \to 0$ as $r \to \infty$,*
(ii) if $\alpha \in [0, 1)$, then $\kappa(\alpha, T, \cdot)$ and $\kappa(\alpha, \cdot)$ are bounded functions,
(iii) there are constants $C_1, C_2 \in (0, \infty)$ such that for every $r \in (0, 1]$,

$$C_1 \log\left(|r|^{-1}\right) \leq \kappa(1, T, r) \leq C_2 \log\left(|r|^{-1}\right),$$

$$C_1 \log\left(|r|^{-1}\right) \leq \kappa(1, r) \leq C_2 \log\left(|r|^{-1}\right),$$

(iv) if $\alpha > 1$*, then there are constants* $C_1, C_2 \in (0,\infty)$ *such that for every* $r \in (0,1]$*,*

$$C_1|r|^{1-\alpha} \le \kappa(\alpha,T,r) \le C_2|r|^{1-\alpha} \quad \textit{and} \quad C_1|r|^{1-\alpha} \le \kappa(\alpha,r) \le C_2|r|^{1-\alpha}.$$

Proof. Since $\kappa(\alpha,r) \ge \mathrm{e}^{-1}\kappa(\alpha,1,r)$ and

$$\begin{aligned}\kappa(\alpha,r) &\le \kappa(\alpha,1,r) + \int_1^\infty \mathrm{e}^{-t}\mathrm{e}^{-r/t}\,\mathrm{d}t \\ &\le \kappa(\alpha,1,r) + \mathrm{e}^{-\sqrt{r}}\int_1^{\max\{\sqrt{r},1\}} \mathrm{e}^{-t}\,\mathrm{d}t + \mathrm{e}^{-\sqrt{r}/2}\int_{\max\{\sqrt{r},1\}}^\infty \mathrm{e}^{-t/2}\,\mathrm{d}t \\ &\le \kappa(\alpha,1,r) + \mathrm{e}^{-\sqrt{r}/2}\int_0^\infty \mathrm{e}^{-t/2}\,\mathrm{d}t \le \kappa(\alpha,1,r) + 2\mathrm{e}^{-\sqrt{r}/2}\end{aligned}$$

it is enough to check (i) – (iv) for $\kappa(\alpha,T,\cdot)$. After changing variables $s = t/r$ we get $\kappa(\alpha,T,r) = r^{1-\alpha}\int_0^{T/r} t^{-\alpha}\mathrm{e}^{-1/t}\,\mathrm{d}t$. Hence (i) follows from

$$\lim_{x\downarrow 0} x^{-m}\int_0^x t^{-\alpha}\mathrm{e}^{-1/t}\,\mathrm{d}t = 0 \qquad \text{for all } \alpha > 0 \text{ and } m \in \mathbb{N}.$$

If $\alpha \in (0,1)$ then $\kappa(\alpha,T,r) \le \int_0^T t^{-\alpha}\,\mathrm{d}t < \infty$, which proves (ii).

Let $\alpha = 1$. Then (iii) follows from

$$\lim_{x\to+\infty}\frac{\int_0^x t^{-1}\mathrm{e}^{-1/t}\,\mathrm{d}t}{\log x} = \lim_{x\to+\infty}\frac{x^{-1}\mathrm{e}^{-1/x}}{x^{-1}} = \lim_{x\to+\infty}\mathrm{e}^{-1/x} = 1.$$

Finally, (iv) follows from

$$\lim_{x\to+\infty}\frac{x^{\alpha-1}\int_0^x t^{-\alpha}\mathrm{e}^{-1/t}\,\mathrm{d}t}{x^{\alpha-1}} = \int_0^\infty t^{-\alpha}\mathrm{e}^{-1/t}\,\mathrm{d}t \in (0,+\infty).$$

□

Lemma 5.2. *For all* $t \in (0,\infty)$ *and* $\rho > 0$*,* $S(t)M_1 \in R(H_W, L^2_\rho)$*. Moreover, one has*

$$||S(t)M_1||^2_{R(H_W,L^2_\rho)} = \Lambda(\mathcal{H}_t,\mathcal{H}_t)\int_G \vartheta_\rho(x)\,\mathrm{d}x, \qquad t > 0.$$

Proof. Let us fix t. First note that $\Lambda(\mathcal{H}_t,\mathcal{H}_t) < \infty$. For, by Lemma 3.2, $\mathcal{H}_t \in \mathcal{S}(G)$. Let $\mathcal{K}(x)(y) = \mathcal{H}_t\left(y^{-1}x\right)$. Then, by Lemma 3.2, $\mathcal{K}$ is a measurable mapping from G into $\mathcal{S}(G)$. Hence, since $S(t)M_1\mathfrak{f}(x) = \langle\mathfrak{f},\mathcal{K}(x)\rangle$, Lemma 4.2 yields

$$I = \sum_k |S(t)M_1\mathfrak{f}_k|^2_{L^2_\rho} = \int_G \Lambda(\mathcal{K}(x),\mathcal{K}(x))\,\vartheta_\rho(x)\,\mathrm{d}x.$$

Since Λ is left translation invariant and $\mathcal{H}_t^* = \mathcal{H}_t$ we have

$$I(x) := \Lambda(\mathcal{K}(x),\mathcal{K}(x)) = \Lambda(\mathcal{K}(e),\mathcal{K}(e)) = \Lambda(\mathcal{H}_t^*,\mathcal{H}_t^*) = \Lambda(\mathcal{H}_t,\mathcal{H}_t).$$

□

Lemma 5.3. *Let $p \in [2,\infty)$ and $\rho \in \mathbb{R}$. Then for all $\psi \in \mathcal{S}(G) \cap L^p_\rho$ and $t \in (0,\infty)$, $S(t)M_\psi$ is a γ-radonifying operator from H_W into L^p_ρ. Moreover, there is a constant $C \in (0,\infty)$ such that for all $\psi \in \mathcal{S}(G) \cap L^p_\rho$ and $t \in (0,\infty)$ one has*

$$||S(t)M_\psi||_{R(H_W,L^p_\rho)} \le C\mathrm{e}^{Ct} \left\langle \Gamma + C_\Gamma \,\mathrm{d}x, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C)\right\rangle^{1/2} |\psi|_{L^p_\rho}. \tag{5.2}$$

Proof. Let us fix t and ψ, and let $\mathcal{K}(x)(y) = \mathcal{H}_t\left(y^{-1}x\right)\psi(y)$. By Lemma 4.3, $S(t)M_\psi \in R(H_W, L^p_\rho)$ and

$$I = ||S(t)M_\psi||^p_{R(H_W,L^p_\rho)} \le c\int_G \Lambda\left(\mathcal{K}(x),\mathcal{K}(x)\right)^{p/2}\vartheta_\rho(x)\,\mathrm{d}x.$$

Let $I(x) = \Lambda\left(\mathcal{K}(x),\mathcal{K}(x)\right)$, and let $\tilde{\Gamma} = \Gamma + C_\Gamma\,\mathrm{d}x$, $C_\Gamma \ge 0$. Then for and $\varphi \in \mathcal{S}(G)$,

$$\langle C_\Gamma\,\mathrm{d}x, \varphi^* * \varphi\rangle = C_\Gamma \int_G\int_G \varphi\left(yx^{-1}\right)\varphi\left(y\right)\,\mathrm{d}x\,\mathrm{d}y = C_\Gamma\left(\int_G \varphi(x)\,\mathrm{d}x\right)^2 \ge 0.$$

Hence

$$\begin{aligned} I(x) &:= \Lambda\left(\mathcal{K}(x),\mathcal{K}(x)\right) = \Lambda\left(\tau^L_x\mathcal{K}(x),\tau^L_x\mathcal{K}(x)\right) \\ &= \left\langle \Gamma, (\tau^L_x\mathcal{K}(x))^* * (\tau^L_x\mathcal{K}(x))\right\rangle \le \left\langle \tilde{\Gamma}, (\tau^L_x\mathcal{K}(x))^* * (\tau^L_x\mathcal{K}(x))\right\rangle =: \tilde{I}(x). \end{aligned}$$

Hence, as $\tilde{\mathcal{K}}(x)(y) := \tau^L_x\mathcal{K}(x)(y) = \mathcal{H}_t(y)\psi(xy)$, we have

$$\begin{aligned} \tilde{I}(x) &= \int_G\int_G \tilde{\mathcal{K}}(x)\left(yz^{-1}\right)\tilde{\mathcal{K}}(x)\left(y\right)\tilde{\Gamma}(\,\mathrm{d}z)\,\mathrm{d}y \\ &\le \int_G\int_G \mathcal{H}_t\left(yz^{-1}\right)\left|\psi\left(xyz^{-1}\right)\right|\mathcal{H}_t\left(y\right)\left|\psi\left(xy\right)\right|\tilde{\Gamma}(\,\mathrm{d}z)\,\mathrm{d}y. \end{aligned}$$

Thus

$$I \le c\int_G\left(\int_G\int_G \left|\psi\left(xyz^{-1}\right)\right|\,||\psi\left(xy\right)|\,\mu_t(\,\mathrm{d}z,\,\mathrm{d}y)\right)^{p/2}\vartheta_\rho(x)\,\mathrm{d}x,$$

where

$$\mu_t(\,\mathrm{d}z,\,\mathrm{d}y) = \mathcal{H}_t\left(yz^{-1}\right)\mathcal{H}_t(y)\tilde{\Gamma}(\,\mathrm{d}z)\,\mathrm{d}y.$$

By Jensen's inequality,

$$\begin{aligned} I \le c&\left(\int_G\int_G \mu_t(\,\mathrm{d}z,\,\mathrm{d}y)\right)^{p/2-1} \\ &\qquad\times\int_G\int_G\int_G \left|\psi\left(xyz^{-1}\right)\right|^{p/2}\left|\psi\left(xy\right)\right|^{p/2}\vartheta_\rho(x)\,\mathrm{d}x\,\mu_t(\,\mathrm{d}z,\,\mathrm{d}y). \end{aligned}$$

Using Lemma 3.1, the semigroup property $\mathcal{H}_t * \mathcal{H}_s = \mathcal{H}_{t+s}$, and the identity $\mathcal{H}_t^* = \mathcal{H}_t$ we obtain

$$\int_G \int_G \mu_t(\,\mathrm{d}z,\,\mathrm{d}y) = \left\langle \tilde{\Gamma}, \mathcal{H}_t^* * \mathcal{H}_t \right\rangle \le C_1 \left\langle \tilde{\Gamma}, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C_1) \right\rangle.$$

Note also that

$$\begin{aligned} L(z,y) &:= \int_G \left|\psi\left(xyz^{-1}\right)\right|^{p/2} \left|\psi\left(xy\right)\right|^{p/2} \vartheta_\rho(x)\,\mathrm{d}x \\ &\le \left(\int_G \left|\psi\left(xyz^{-1}\right)\right|^{p} \vartheta_\rho(x)\,\mathrm{d}x\right)^{1/2} \left(\int_G \left|\psi\left(xy\right)\right|^{p} \vartheta_\rho(x)\,\mathrm{d}x\right)^{1/2} \\ &\le \left(\int_G \left|\psi(x)\right|^{p} \vartheta_\rho(xzy^{-1})\,\mathrm{d}x\right)^{1/2} \left(\int_G \left|\psi(x)\right|^{p} \vartheta_\rho\left(xy^{-1}\right)\,\mathrm{d}x\right)^{1/2}. \end{aligned}$$

Hence, by Lemma 3.3, we have

$$L(z,y) \le |\psi|^p_{L^p_\rho} \exp\Big\{ \frac{|\rho|\left(\mathfrak{d}\left(zy^{-1}\right) + \mathfrak{d}\left(y^{-1}\right)\right)}{2} \Big\},$$

and consequently,

$$I \le C_2 |\psi|^p_{L^p_\rho} \left\langle \tilde{\Gamma}, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C_1) \right\rangle^{p/2-1} R, \tag{5.3}$$

where

$$R := \int_G \int_G \exp\Big\{ \frac{|\rho|\left(\mathfrak{d}\left(zy^{-1}\right) + \mathfrak{d}\left(y^{-1}\right)\right)}{2} \Big\} \mathcal{H}_t\left(yz^{-1}\right) \mathcal{H}_t\left(y\right) \tilde{\Gamma}(\,\mathrm{d}z)\,\mathrm{d}y.$$

Now Lemma 3.1 and the inequality

$$-\frac{\mathfrak{d}^2(u)}{t} + \frac{|\rho|\mathfrak{d}(u)}{2} = -\frac{\mathfrak{d}^2(u)}{2t} + \left(-\frac{\mathfrak{d}^2(u)}{2t} + \frac{|\rho|\mathfrak{d}(u)}{2}\right) \le -\frac{\mathfrak{d}^2(u)}{2t} + \frac{t|\rho|^2}{8}.$$

imply

$$\exp\Big\{ \frac{|\rho|\mathfrak{d}(u)}{2} \Big\} \mathcal{H}_t(u) \le C_3 \mathrm{e}^{C_3 t} \mathcal{H}_{C_4 t}(u), \qquad t \in [0,\infty),\ u \in G.$$

Hence

$$\begin{aligned} R &\le C_5 \mathrm{e}^{C_5 t} \int_G \int_G \mathcal{H}_{C_6 t}\left(yz^{-1}\right) \mathcal{H}_{C_6 t}(y) \tilde{\Gamma}(\,\mathrm{d}z)\,\mathrm{d}y \\ &\le C_5 \mathrm{e}^{C_5 t} \left\langle \tilde{\Gamma}, \mathcal{H}_{C_6 t}^* * \mathcal{H}_{C_6 t} \right\rangle \le C_6 \mathrm{e}^{C_5 t} \left\langle \tilde{\Gamma}, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C_7) \right\rangle. \end{aligned}$$

Combining this estimate with (5.3) we obtain the desired conclusion. □

Since for all p and ρ, $S(G) \cap L^p_\rho$ is dense in L^p_ρ we have the following corollary to Lemma 5.3.

Corollary 5.1. *If (2.1) is satisfies, then for arbitrary $p \in [2,\infty)$ and $\rho \in \mathbb{R}$, the multiplication operator M is bounded from L^p_ρ into the space of γ-radonifying operators $R(H_W, L^p_\rho)$. Moreover, for any $\psi \in L^p_\rho$ one has (5.2).*

Lemma 5.4. *Assume that the space correlation Γ of W is equal to the Haar measure $\mathrm{d}x$. Then for all $\psi \in L^2_\rho$ and $t \in (0,\infty)$, $S(t)M_\psi \in R(H_W, L^2_\rho) = L_{(\mathrm{HS})}(H_W, L^2_\rho)$. Moreover, there are constants C_1, C, such that for all $\psi \in L^2_\rho$ and $t \in (0,\infty)$ we have*

$$||S(t)M_\psi||^2_{R(H_W,L^2_\rho)} \le C_1 \mathrm{e}^{C_1 t}\,|\psi|^2_{L^2_\rho} \int_G \mathfrak{h}_t(\mathfrak{d}(x)/C_1)\,\mathrm{d}x \le C\mathrm{e}^{Ct}\,|\psi|^2_{L^2_\rho}.$$

Proof. By Corollary 5.1, there is a constant C_1 such that

$$||S(t)M_\psi||^2_{R(H_W,L^2_\rho)} \le C_1 \mathrm{e}^{C_1 t}\,|\psi|^2_{L^2_\rho} \int_G \mathfrak{h}_t(\mathfrak{d}(x)/C_1)\,\mathrm{d}x.$$

Applying Lemma 3.1, we obtain

$$\int_G C_1 \mathfrak{h}_t(\mathfrak{d}(x)/C_1)\,\mathrm{d}x \le C_2 \int_G \mathcal{H}_{C_3 t}(x)\,\mathrm{d}x = C_2.$$

□

6. Proof of Theorem 2.1

Assume that $\xi \in \mathcal{S}'(G)$ is a measure such that $\xi + C_\xi\,\mathrm{d}x \ge 0$ for a certain C_ξ. Let $\eta(d,\xi) \in (-\infty, +\infty]$ be given by

$$\eta(d,\xi) = \begin{cases} \int_{B(e,1)} \xi(\,\mathrm{d}x) & \text{if } d = 1, \\ \int_{B(e,1)} \log\left(\mathfrak{d}(x)^{-1}\right) \xi(\,\mathrm{d}x) & \text{if } d = 2, \\ \int_{B(e,1)} \mathfrak{d}(x)^{-d+2}\,\xi(\,\mathrm{d}x) & \text{if } d > 2. \end{cases} \tag{6.1}$$

Proof of Theorem 2.1(*i*). Let $\rho \in \mathbb{R}$ and $p \in [2,\infty)$, and let $T \in (0,\infty)$ and $q \in [2,\infty)$ be fixed. Let $\tilde{B}(t,u)(x) = b(t,x,u(x))$, $t \ge 0$, $x \in G$, $u \in L^p_\rho$. Since $f, b \in \mathrm{Lip}\,(p,\rho)$, there is a constant L such for all $t \in [0,T]$ and $\psi, \varphi \in L^p_\rho$ one has

$$\begin{aligned} |F(t,\psi) - F(t,\varphi)|_{L^p_\rho} + |\tilde{B}(t,\psi) - \tilde{B}(t,\varphi)|_{L^p_\rho} &\le L|\psi - \varphi|_{L^p_\rho}, \\ |F(t,\psi)|_{L^p_\rho} + |\tilde{B}(t,\psi)|_{L^p_\rho} &\le L\big(1 + |\varphi|_{L^p_\rho}\big). \end{aligned} \tag{6.2}$$

Note that B satisfies

$$B = M_{\tilde{B}} \tag{6.3}$$

where $M_\psi \mathfrak{f} = \psi \mathfrak{f}$ is a multiplication operator. Recall that S is a C_0-semigroup on L^p_ρ, see Lemma 3.3. We will show that there is a function $a \in L^2(0,T;\mathbb{R})$ such that for all $t,s \in [0,T]$ and $\psi, \varphi \in L^p_\rho$,

$$\begin{aligned} ||S(t)B(s,\psi)||_{R(H_W,L^p_\rho)} &\le a(t)\big(1+|\varphi|_{L^p_\rho}\big),\\ ||S(t)B(s,\psi)-S(t)B(s,\varphi)||_{R(H_W,L^p_\rho)} &\le a(t)|\psi-\varphi|_{L^p_\rho}. \end{aligned} \tag{6.4}$$

Having (6.2) and (6.4), the existence and uniqueness of a solution to (1.1) satisfying (2.3) follows by means of the contraction principle, see e.g. [PZ1], [PZ2], [BP]. Taking into account (6.3) with $\tilde{B}$ satisfying (6.2) one can easily see that to show (6.4) it is enough to prove that there is a $b \in L^2(0,T;\mathbb{R})$ such that

$$||S(t)M_\psi||_{R(H_W,L^p_\rho)} \le b(t)|\psi|_{L^p_\rho} \qquad \text{for all } \psi \in L^p_\rho \text{ and } t \in [0,T]. \tag{6.5}$$

By Corollary 5.1, we have (6.5) with $b(t) = C\left\langle \Gamma + C_\Gamma \,\mathrm{d}x, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C)\right\rangle^{1/2}$. Thus what is left is to show that

$$\begin{aligned} &\int_0^T \left\langle \Gamma + C_\Gamma \,\mathrm{d}x, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C)\right\rangle \,\mathrm{d}t\\ &\qquad = \int_0^T\int_G \mathfrak{h}_t(\mathfrak{d}(x)/C)\,(\Gamma(\,\mathrm{d}x) + C_\Gamma\,\mathrm{d}x)\,\mathrm{d}t < \infty. \end{aligned} \tag{6.6}$$

Let $\tilde{\Gamma} = \Gamma + C_\Gamma \,\mathrm{d}x$. To show (6.6) note that by Lemma 5.4,

$$\int_0^T\int_G \mathfrak{h}_t(\mathfrak{d}(x)/C)\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{6.7}$$

Recall that $\kappa(\alpha, T, \cdot)$ is given by (5.1). We have

$$\begin{aligned} \int_0^T \left\langle \tilde{\Gamma}, \mathfrak{h}_t(\mathfrak{d}(\cdot)/C)\right\rangle \,\mathrm{d}t &= \int_G\int_0^T \mathfrak{h}_t(\mathfrak{d}(x)/C)\,\mathrm{d}t\tilde{\Gamma}(\,\mathrm{d}x)\\ &= \int_G \kappa\left(d/2, \mathfrak{d}^2(x)/C\right)\tilde{\Gamma}(\,\mathrm{d}x)\\ &\le C_1 \int_G \kappa\left(d/2, C_2T, \mathfrak{d}^2(x)\right)\tilde{\Gamma}(\,\mathrm{d}x) := C_1(I_1+I_2), \end{aligned}$$

where

$$I_1 = \int_G \kappa\left(d/2, C_2T, \mathfrak{d}^2(x)\right)\Gamma(\,\mathrm{d}x) \quad \text{and} \quad I_2 = C_\Gamma\int_G \kappa\left(d/2, C_2T, \mathfrak{d}^2(x)\right)\,\mathrm{d}x.$$

By (6.7), $I_2 < \infty$. In order to show that $I_1 < \infty$ note that

$$\begin{aligned} I_1 &= \int_{B(e,1)} \kappa\left(d/2, C_2T, \mathfrak{d}^2(x)\right)\Gamma(\mathrm{d}x) + \int_{G\setminus B(e,1)} \kappa\left(d/2, C_2T, \mathfrak{d}^2(x)\right)\Gamma(\mathrm{d}x) \\ &=: I_{11} + I_{12}. \end{aligned}$$

The integral I_{12} is finite since Γ is a tempered measure and since by Lemma 5.1 for every $m > 0$, $r^m\kappa(d/2, C_2T, r) \to 0$ as $r \to \infty$. To show that $I_{11} < \infty$ note that by Lemma 5.1, there is a constant c such that $I_{11} \le c\eta(d,\Gamma)$. Hence the desired conclusion follows from (2.2). $\square$

Proof of Theorem 2.1(*ii*). In the proof we use the ideas from [PZ2]. Let $\rho > 0$, $p \in [2,\infty)$, and let b satisfies the assumptions of Theorem 1.2(ii) on a time interval $[0,T]$. Finally let u be a solution to (1.1) satisfying (2.3). Then, since $\vartheta_\rho \in L^1$, u satisfies (2.3) with $p = 2$. Since F satisfies (6.2) it is easy to see that

$$\mathbb{E}\left| S(T)u(0) + \int_0^T S(T-s)F(s,u(s))\,\mathrm{d}s \right|^2_{L^2_\rho} < \infty.$$

Thus

$$\mathbb{E}\left| \int_0^T S(T-s)B(s,u(s))\,\mathrm{d}W(s) \right|^2_{L^2_\rho} < \infty.$$

Hence (4.2) yields

$$\int_0^T \mathbb{E}\,||S(T-s)B(s,u(s))||^2_{L_{(\mathrm{HS})}(H_W, L^2_\rho)}\,\mathrm{d}s < \infty. \tag{6.8}$$

By Lemma 4.2,

$$||S(T-s)B(s,u(s))||^2_{L_{(\mathrm{HS})}(H_W,L^2_\rho)} = \int_G \langle \Gamma, \mathcal{K}_s(x)^* * \mathcal{K}_s(x)\rangle\,\vartheta_\rho(x)\,\mathrm{d}x,$$

where $\mathcal{K}_s(x)(y) = \mathcal{H}_{T-s}\left(y^{-1}x\right)\tilde{B}(s,u(s))(y)$. By Lemma 5.4 there is a constant C_1 such that

$$I = \int_0^T \mathbb{E}\int_G \mathcal{K}_s(s)^* * \mathcal{K}_s(x)(y)\,\mathrm{d}y\,\vartheta_\rho(x)\,\mathrm{d}x \le C_1 \sup_{s\in[0,T]} \mathbb{E}\,|\tilde{B}(s,u(s))|^2_{L^2_\rho}.$$

Hence by (6.2) we obtain $I < \infty$. Consequently, (6.8) yields

$$\int_0^T \mathbb{E}\int_G \left\langle \tilde{\Gamma}, \mathcal{K}_s(x)^* * \mathcal{K}_s(x)\right\rangle \vartheta_\rho(x)\,\mathrm{d}x\,\mathrm{d}s < \infty.$$

where $\tilde{\Gamma} = \Gamma + C_\Gamma \,\mathrm{d}x$ is a non-negative measure. Since $\tilde{B}(s,u(s))(z) \geq b_0 > 0$ we have

$$J = \int_0^T \int_G \left\langle \tilde{\Gamma}, \mathcal{H}_{T-s} * \mathcal{H}_{T-s}(x) \right\rangle \vartheta_\rho(x)\,\mathrm{d}x\,\mathrm{d}s < \infty.$$

Hence, since $\vartheta_\rho \in L^1$ for $\rho > 0$, we have $\int_0^T \langle \tilde{\Gamma}, \mathcal{H}_{2s}\rangle \,\mathrm{d}s < \infty$. Thus there are $T_1 > 0$ and a constant C such that

$$\int_G \int_0^{T_1} \mathfrak{h}_t\left(\mathfrak{d}(x)/C\right)\,\mathrm{d}t\,\Gamma(\,\mathrm{d}x) \leq \int_G \int_0^{T_1} \mathfrak{h}_t\left(\mathfrak{d}(x)/C\right)\,\mathrm{d}t\,\tilde{\Gamma}(\,\mathrm{d}x) < \infty. \tag{6.9}$$

Since

$$\int_0^{T_1} \mathfrak{h}_t\left(\mathfrak{d}(x)/C\right)\,\mathrm{d}t = \kappa\left(d/2, T_1, \mathfrak{d}^2(x)/C^2\right),$$

where κ is given by (5.1), (2.2) follows from (6.9) and Lemma 5.1. □

7. Proof of Theorem 2.2

Given $q \in [2,\infty)$, $T > 0$, and a Banach space V we denote by $\mathcal{K}^q_T(V)$ the Banach space of all adapted processes Z with continuous trajectories in V such that

$$|||Z|||_{\mathcal{K}^q_T(V)} := \left(\mathbb{E}\sup_{t\in[0,T]} |Z(t)|^q_V\right)^{1/q} < \infty.$$

We equip the space $L^p_\rho \cap \mathcal{C}_{\rho/p}$, with the norm $|\cdot|_{L^p_\rho} + |\cdot|_{\mathcal{C}_{\rho/p}}$. In the proof of Theorem 2.2 we will use the contraction principle for the functional $\mathcal{J}$ given by

$$\mathcal{J}(Z)(t) = S(t)X(0) + \int_0^t S(t-s)F(s,Z(s))\,\mathrm{d}s + \int_0^t S(t-s)B(s,Z(s))\,\mathrm{d}W(s).$$

Our goal will be to show that under the hypothesis of the theorem for q large enough one can chose $T = T(q) > 0$ such that $\mathcal{J}$ is a contraction from $\mathcal{K}^p_T(L^p_\rho)$ into $\mathcal{K}^p_T(L^p_\rho)$, or from $\mathcal{K}^p_T(L^p_\rho \cap \mathcal{C}_{\rho/p})$ into $\mathcal{K}^p_T(L^p_\rho \cap \mathcal{C}_{\rho/p})$. Having regular solution on a small time interval one can easily prolong it to an arbitrary time interval. Let

$$\mathcal{L}(Z)(t) = S(t)X(0) + \int_0^t S(t-s)F(s,Z(s))\,\mathrm{d}s,$$
$$\mathcal{I}(Z)(t) = \int_0^t S(t-s)B(s,Z(s))\,\mathrm{d}W(s).$$

Since the heat semigroup is C_0 on L^p_ρ and $\mathcal{C}_\rho$ spaces it is not difficult to show that $\mathcal{L}$ is a contraction on a proper space. In the proof we will concentrate on showing this for $\mathcal{I}$. Let $\tilde{B}(t,u)(x) = b(t,x,u(x))$. Note that $\tilde{B}$ is a Lipschitz mapping from $\mathcal{K}^p_T(L^p_\rho)$ into $\mathcal{K}^p_T(L^p_\rho)$. Thus it is enough to show that

$$I(Z)(t) = \int_0^t S(t-s) M_{Z(s)}\,\mathrm{d}W(s), \tag{7.1}$$

where M is a multiplication operator, is a bounded linear operator from $\mathcal{K}^p_T(L^p_\rho)$ into $\mathcal{K}^p_T(L^p_\rho)$ in the point (i), and from $\mathcal{K}^p_T(L^p_\rho)$ into $\mathcal{K}^p_T(L^p_\rho \cap \mathcal{C}_{\rho/p})$ in (ii), and that its norm goes to 0 as $T \to 0$. To do this we will use the Da Prato–Kwapień–Zabczyk factorization, see [DP];

$$I(Z)(t) = \mathcal{R}_\beta Y_\beta(Z)(t), \tag{7.2}$$

where

$$\begin{aligned} \mathcal{R}_\beta \psi(t) &= \frac{\sin \pi\beta}{\pi} \int_0^t (t-s)^{\beta-1} S(t-s)\psi(s)\,\mathrm{d}s, \\ Y_\beta(Z)(t) &= \int_0^t (t-s)^{-\beta} S(t-s) M_{Z(s)}\,\mathrm{d}W(s). \end{aligned} \tag{7.3}$$

In the proof of Theorem 2.2 we will need the following lemma.

Lemma 7.1. *Let $p \in [2,\infty)$ and $\rho \in \mathbb{R}$, and let $\mathcal{R}_\beta$ be given by* (7.3)*. Then:*
(i) for arbitrary $\beta > 0$, $T > 0$, and $q \in [2,\infty)$ such that $(\beta-1)q^* > -1$, $\mathcal{R}_\beta$ is a bounded linear operator from $L^q(0,T;L^p_\rho)$ into $C([0,T];L^p_\rho)$ and

$$||\mathcal{R}_\beta||_{L(L^q(0,T;L^p_\rho), C([0,T];L^p_\rho))} \to 0 \qquad \text{as } T \to 0.$$

(ii) For arbitrary $\beta > d/(2p)$, $T > 0$, and $q \in [2,\infty)$ such that $(\beta - 1 - d/(2p))q^* > -1$, $\mathcal{R}_\beta$ is a bounded linear operator from $L^q(0,T;L^p_\rho)$ into $C([0,T];\mathcal{C}_{\rho/p})$ and

$$||\mathcal{R}_\beta||_{L(L^q(0,T;L^p_\rho), C([0,T];\mathcal{C}_{\rho/p}))} \to 0 \qquad \text{as } T \to 0.$$

Proof of (i). It is enough to show that $\mathcal{R}_\beta$ transforms continuously $L^q(0,T;L^p_\rho)$ into $L^\infty(0,T;L^p_\rho)$ with the norm decreasing to 0 as $T \to 0$, see [DZ]. This follows from Hölder's inequality. Namely given $\widehat{T} > 0$ one can find a constant C_1 depending on S and $\widehat{T}$ such that for any $T \le \widehat{T}$,

$$\begin{aligned} ||\mathcal{R}_\beta||_{L(L^q(0,T;L^p_\rho), L^\infty(0,T;L^p_\rho))} &\le \sup_{t\in[0,T]} \left(\int_0^t (t-s)^{(\beta-1)q*} ||S(t-s)||^{q^*}_{L(L^p_\rho, L^p_\rho)}\,\mathrm{d}s \right)^{1/q^*} \\ &\le C_1 \left(\frac{T^{(\beta-1)q^*+1)}}{(\beta-1)q^*+1} \right)^{1/q^*}. \end{aligned}$$

□

Proof of (ii). Again it is enough to show that $\mathcal{R}_\beta$ maps continuously $L^q(0,T;L^p_\rho)$ into $L^\infty(0,T;\mathcal{C}_{\rho/p})$ with the norm decreasing to 0 as $T\to 0$. Let $\widehat{T}>0$. Using Lemma 3.4 and arguments from the proof of the first part of the lemma one can find a constant C such that for $T\le \widehat{T}$,

$$||\mathcal{R}_\beta||_{L(L^q(0,T;L^p_\rho),L^\infty(0,T;\mathcal{C}_{\rho/p}))}\le C\left(\frac{T^{(\beta-1-d/(2p))q^*+1)}}{(\beta-1-d/(2p))q^*+1}\right)^{1/q^*}.$$

□

Proof of Theorem 2.2(i). Let us fix p, and ρ, and $\widehat{T}\in(0,\infty)$. Let be such that (2.4) holds, and let $\beta=\alpha/4$. Clearly, we may assume that $\beta<1/2$. Let $q\in(2,\infty)$ be such that $(\beta-1)q^*>-1$. Let Y_β be given by (7.3). Taking into account (7.2) and Lemma 7.1(i) the proof will be completed as soon as we show that

$$\exists C:\forall T\le\widehat{T},\ Z\in\mathcal{K}^q_T(L^p_\rho),\qquad \mathbb{E}\int_0^T|Y_\beta(Z)(t)|^q_{L^p_\rho}\,\mathrm{d}t\le C\,\mathbb{E}\int_0^T|Z(t)|^q_{L^p_\rho}\,\mathrm{d}t. \tag{7.4}$$

By Theorem 4.1, there is a constant C_1 independent of T and Z such that for all $t\in[0,T]$,

$$\mathbb{E}\,|Y_\beta(Z)(t)|^q_{L^p_\rho}\le C_1\,\mathbb{E}\left(\int_0^t(t-s)^{-2\beta}||S(t-s)M_{Z(s)}||^2_{R(H_W,L^p_\rho)}\,\mathrm{d}s\right)^{q/2}. \tag{7.5}$$

By Lemma 5.3 the left hand side of (7.5) is less or equal to

$$C_3\,|||Z|||^q_{\mathcal{K}^q_T(L^p_\rho)}\left(\int_0^t(t-s)^{-2\beta}b(t-s)\,\mathrm{d}s\right)^{q/2},$$

where $b(s)=\langle\Gamma+C_\Gamma\,\mathrm{d}x,\mathfrak{h}_s\left(\mathfrak{d}(\cdot)/C_4\right)\rangle$. Since

$$\int_0^t(t-s)^{-2\beta}b(t-s)\,\mathrm{d}s\le\left\langle\Gamma+C_\Gamma\,\mathrm{d}x,\kappa\left(d/2+2\beta,\widehat{T},\mathfrak{d}^2(\cdot)/C_4^2\right)\right\rangle,$$

we conclude by Lemma 5.1(i). □

Proof of Theorem 2.2(ii). Let $p\ge 2$ such that $d<p$, and let $\alpha>2d/p$ be such that (2.4) holds true. Since $d/p<1$, we may assume that $\beta=\alpha/4<1/2$. Let $q\in[2,\infty)$ be such that $(\beta-1)q^*>-1$. Since $\beta>d/(2p)$, Lemma 7.2(ii) yields that for any $T>0$, $\mathcal{R}_\beta$ is a bounded operator from $L^q(0,T;L^p_\rho)$ into $C([0,T];\mathcal{C}_{\rho/p})$. Thus as in the proof of (i) it is enough to show that for any $\widehat{T}>0$ there is a C such that for all $T\le\widehat{T}$ and $Z\in\mathcal{K}^q_T(L^p_\rho)$, one has

$$\mathbb{E}\int_0^T|Y_\beta(Z)(t)|^q_{L^p_\rho}\,\mathrm{d}t\le C\,\mathbb{E}\int_0^T|Z(t)|^q_{L^p_\rho}\,\mathrm{d}t.$$

This follows from (2.4) in the same way as in the proof of (i). □

8. Wave semigroup

Recall that θ_ρ is given by (2.5), the Sobolev spaces H^1_ρ and H^{-1}_ρ are defined in Section 2, and that the spaces $\mathbb{X}_\rho$, $\mathbb{D}_\rho$, and the operator $\mathcal{A}$ are given by (2.6). For brevity we write $\mathbb{X}$ and $\mathbb{D}$ instead of $\mathbb{X}_0$ and $\mathbb{D}_0$. Let

$$j_\rho \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \theta_\rho^{-1/2}\psi \\ \theta_\rho^{-1/2}\varphi \end{pmatrix}.$$

Then j_ρ is an isometry between $\mathbb{X}$ and $\mathbb{X}_\rho$, and $\mathbb{D}$ and $\mathbb{D}_\rho$.

Lemma 8.1. *Let $\rho \in \mathbb{R}$. Then:*
(i) $\theta_\rho^{-1}\mathfrak{X}_i\theta_\rho \in L^\infty(G, \mathrm{d}x)$,
(ii) $\theta_\rho^{-1}\mathfrak{L}\theta_\rho \in L^\infty(G, \mathrm{d}x)$,
(iii) $\mathfrak{X}_i$, $i = 1, \dots, l$ are bounded operators from H^1_ρ into $\mathbb{L}^2_\rho$ and from $\mathbb{L}^2_\rho$ into H^{-1}_ρ.

Proof if (i). Note that

$$\theta_\rho^{-1}\mathfrak{X}_j\theta_\rho(x) = \frac{\vartheta_\rho(x)^{-1}\mathfrak{X}_j S(1)\vartheta_\rho(x)}{\vartheta_\rho(x)^{-1}S(1)\vartheta_\rho(x)} = \frac{I_1(x)}{I_2(x)}.$$

By Lemma 3.1 there is a constant $C > 0$ such that

$$\begin{aligned} I_1(x) &\le C \int_G \exp\left\{-\mathfrak{d}^2(x,y)/C + \rho\mathfrak{d}(x) - \rho\mathfrak{d}(y)\right\} \mathrm{d}y \\ &\le C \int_G \exp\left\{-\mathfrak{d}^2(x,y)/C + |\rho|\mathfrak{d}(x,y)\right\} \mathrm{d}y \\ &\le C \exp\{C\rho^2/2\} \int_G \exp\left\{-\mathfrak{d}^2(y)/(2C)\right\} \mathrm{d}y. \end{aligned}$$

Hence I_1 is bounded from above as $\int_G \exp\left\{-\mathfrak{d}^2(y)/(2C)\right\} \mathrm{d}y < \infty$.

Applying Lemma 3.1 again we can find a constant $c > 0$ such that

$$\begin{aligned} I_2(x) &\ge c \int_G \exp\left\{-\mathfrak{d}^2(x,y)/c + \rho\mathfrak{d}(x) - \rho\mathfrak{d}(y)\right\} \mathrm{d}y \\ &\ge c \int_G \exp\left\{-\mathfrak{d}^2(x,y)/(2c) - \mathfrak{d}^2(x,y)/(2c) - |\rho|\mathfrak{d}(x,y)\right\} \mathrm{d}y \\ &\ge c \exp\{-c\rho^2/2\} \int_G \exp\left\{-\mathfrak{d}^2(y)/(2c)\right\} \mathrm{d}y. \end{aligned}$$

Thus I_2 is bounded from below, which completes the proof of (i). □

Proof of (ii). One can easily prove (ii) using the arguments from the proof of (i).

Proof of (iii). First note that

$$\sum_i \int_G \left(\mathfrak{X}_i\psi\right)^2(x)\,\mathrm{d}x = -\int_G \psi(x)\mathfrak{L}\psi(x)\,\mathrm{d}x \le \int_G \psi(x)\left(\mathfrak{I}-\mathfrak{L}\right)\psi(x)\,\mathrm{d}x$$
$$\le \int_G \left((\mathfrak{I}-\mathfrak{L})^{1/2}\psi\right)^2(x)\,\mathrm{d}x.$$

Hence $\mathfrak{X}_i(\mathfrak{I}-\mathfrak{L})^{-1/2}$, $i=1,\ldots,l$ are bounded operators on L^2. Now for any $\psi\in C_0^\infty(G)$ we have

$$|\mathfrak{X}_i\psi|_{\mathbb{L}^2_\rho} = \left|\theta_\rho^{1/2}\mathfrak{X}_i\psi\right|_{L^2} = \left|\mathfrak{X}_i\left(\theta_\rho^{1/2}\psi\right) - \psi\mathfrak{X}_i\theta_\rho^{1/2}\right|_{L^2}$$
$$\le \left|\mathfrak{X}_i\left(\mathfrak{I}-\mathfrak{L}\right)^{-1/2}\left(\mathfrak{I}-\mathfrak{L}\right)^{1/2}\left(\theta_\rho^{1/2}\psi\right)\right|_{L^2} + \left|\psi\theta_\rho^{-1/2}\mathfrak{X}_i\theta_\rho^{1/2}\psi\right|_{\mathbb{L}^2_\rho}.$$

This gives the continuity of $\mathfrak{X}_i$ from H^1_ρ into $\mathbb{L}^2_\rho$ since $\theta_\rho^{-1/2}\mathfrak{X}_i\theta_\rho^{1/2} = 1/2\theta_\rho^{-1}\mathfrak{X}_i\theta_\rho$ is by (*i*) a bounded function. To see that $\mathfrak{X}_i$ is bounded from $\mathbb{L}^2_\rho$ into H^{-1}_ρ note that

$$H^1_\rho \hookrightarrow \mathbb{L}^2_\rho = (\mathbb{L}^2_\rho)^* \hookrightarrow (H^1_\rho)^* = H^{-1}_\rho.$$

Thus $\mathfrak{X}_i^*$ is bounded from $\mathbb{L}^2_\rho$ into H^{-1}_ρ. Since for $\psi,\varphi\in C_0^\infty(G)$ we have, see [CSV, p. 21], $\langle\mathfrak{X}_i\psi,\varphi\rangle_{L^2} = -\langle\psi,\mathfrak{X}_i\varphi\rangle_{L^2}$, we have

$$\mathfrak{X}_i^*\psi = -\theta_\rho^{-1/2}\mathfrak{X}_i\left(\theta_\rho^{1/2}\psi\right) = -\left(\theta_\rho^{-1/2}\mathfrak{X}_i\theta_\rho^{1/2}\right)\psi - \mathfrak{X}_i\psi.$$

Hence $\mathfrak{X}_i$ is bounded as $\theta_\rho^{-1/2}\mathfrak{X}_i\theta_\rho^{1/2}$ is a bounded function. □

Since $\mathfrak{L}=\sum_i\mathfrak{X}_i^2$ we have the following corollary to Lemma 8.1.

Corollary 8.1. *Let $\rho\in\mathbb{R}$. Then the operator $\mathfrak{L}$ is a bounded linear operator from H^{-1}_ρ into H^1_ρ.*

Lemma 8.2. *The operator $\mathcal{A}$ with* $\operatorname{Dom}\mathcal{A}=\mathbb{D}_\rho$ *generates C_0-semigroup on $\mathbb{X}_\rho$.*

Proof. Clearly $\mathcal{A}$ generates C_0-semigroup on $\mathbb{X}_\rho$ iff $\tilde{A} = j_\rho^{-1}\mathcal{A}j_\rho$ with $\operatorname{Dom}\tilde{A}=\mathbb{D}$ generates C_0 semigroup on $\mathbb{X}$. Note that $\tilde{\mathcal{A}}=\mathcal{A}+\mathcal{P}$, where $\mathcal{A}$ has the domain $\mathbb{D}$, and

$$\mathcal{P} = \begin{pmatrix} 0 & 0\\ P & 0\end{pmatrix},$$

where

$$P\psi = \theta_\rho^{1/2}\mathfrak{L}\left(\theta_\rho^{-1/2}\psi\right) - \mathfrak{L}\psi = \left(\theta_\rho^{1/2}\mathfrak{L}\theta_\rho^{-1/2}\right)\psi + 2\sum_i\left(\theta_\rho^{1/2}\mathfrak{X}_i\theta_\rho^{-1/2}\right)\mathfrak{X}_i\psi.$$

Now the fact that $\mathcal{A}$ generates C_0-semigroup on $\mathbb{X}$ follows directly from the fact that $\mathfrak{L}$ is sefl-adjoint, see e.g. [CP]. Thus it is enough to show that $\mathcal{P}$ is a bounded operator on $\mathbb{X}$, or equivalently, that P is a bounded linear operator acting from L^2 into H^{-1}. Since

$$\theta_\rho^{1/2}\mathfrak{L}\theta_\rho^{-1/2} = -\frac{1}{2}\sum_i \theta_\rho^{1/2}\mathfrak{X}_i\left(\theta_\rho^{-3/2}\mathfrak{X}_i\theta_\rho\right) = -\frac{1}{2}\theta_\rho^{-1}\mathfrak{L}\theta_\rho + \frac{5}{4}\sum_i\left(\theta_\rho^{-1}\mathfrak{X}_i\theta_\rho\right)^2$$

and $\theta_\rho^{1/2}\mathfrak{X}_i\theta_\rho^{-1/2} = -1/2\theta_\rho^{-1}\mathfrak{X}_i\theta_\rho$, we conclude by Lemma 8.1. $\square$

Let

$$\mathbb{S} = \begin{pmatrix}\mathcal{S}(G)\\ \mathcal{S}(G)\end{pmatrix} \qquad \text{and} \qquad \mathbb{M}_X\mathfrak{f} = \begin{pmatrix}0\\ u\mathfrak{f}\end{pmatrix}$$

for $X = (u,v)^{\mathrm{T}} \in \mathbb{S}$ and $\mathfrak{f} \in \mathcal{S}'(G)$. Recall that $\eta(d,\xi)$ is given by (6.1).

Lemma 8.3. *Assume that* (2.1) *holds with the constant* C. *Then for any* $\rho \in \mathbb{R}$ *there is a constant* C_1 *such that for arbitrary* $X \in \mathbb{S}$ *and orthonormal basis* $\{\mathfrak{f}_k\}$ *of* H_W *one has*

$$\sum_k |\mathbb{M}_X\mathfrak{f}_k|^2_{\mathbb{X}_\rho} \le C_1\,|X|^2_{\mathbb{X}_\rho}\left(\eta(d,\Gamma + C\,\mathrm{d}x) + 1\right).$$

Proof. Since $\mathbb{M}_X f = (0,\mathfrak{f}u)^{\mathrm{T}}$ it is enough to prove that there is a constant C_1 such that for arbitrary $u \in \mathcal{S}(G)$, and orthonormal basis $\{\mathfrak{f}_k\}$,

$$I = \sum_k |M_u\mathfrak{f}_k|^2_{H_\rho^{-1}} \le C_1\,|u|^2_{\mathbb{L}^2_\rho}\left(\eta(d,\Gamma + C\,\mathrm{d}x) + 1\right).$$

Set

$$\mathcal{K}(x)(y) = \int_0^\infty \mathrm{e}^{-t}\mathcal{H}_t\left(y^{-1}x\right)\theta_\rho^{1/2}(y)u(y)\,\mathrm{d}t$$

Then

$$I = \sum_k \int_G \langle \mathfrak{f}_k, \mathcal{K}(x)\rangle\,\mathrm{d}x$$

and consequently, by Lemma 4.2, $I = \int_G \Lambda\left(\mathcal{K}(x),\mathcal{K}(x)\right)\mathrm{d}x$. Let $\tilde\Gamma = \Gamma + C_\Gamma\,\mathrm{d}x$ be a non-negative measure, and let $\tilde\Lambda(\psi,\varphi) = \left\langle\tilde\Gamma, \psi^* * \varphi\right\rangle$. Then

$$I \le \int_G \tilde\Lambda\left(\tilde{\mathcal{K}}(x),\tilde{\mathcal{K}}(x)\right)\mathrm{d}x,$$

where

$$\tilde{\mathcal{K}}(x)(y) = \int_0^\infty \mathrm{e}^{-t}\mathcal{H}_t\left(y^{-1}x\right)\theta_\rho^{1/2}(y)|u(y)|\,\mathrm{d}t.$$

Thus

$$I \le \int_0^\infty \int_0^\infty \mathrm{e}^{-(t+s)} \int_G \tilde{\Lambda}\left(\mathcal{K}_t(x), \mathcal{K}_s(x)\right) \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}s,$$

where $\mathcal{K}_t(x)(y) = \mathcal{H}_t\left(y^{-1}x\right)\theta_\rho^{1/2}(y)|u(y)|$. Note that

$$\tau_x^L\left(\mathcal{K}_t(x)\right)(y) = \mathcal{H}_t(y)\theta_\rho^{1/2}(xy)|u(xy)|.$$

Hence

$$\tilde{\Lambda}\left(\mathcal{K}_t(x), \mathcal{K}_s(x)\right) = \tilde{\Lambda}\left(\tau_x^L\mathcal{K}_t(x), \tau_x^L\mathcal{K}_s(x)\right) = \\ \int_G\int_G \mathcal{H}_t\left(yz^{-1}\right)\mathcal{H}_s\left(y\right)\theta_\rho^{1/2}(xyz^{-1})\left|u(xyz^{-1})\right|\theta_\rho^{1/2}(xy)\left|u(xy)\right|\tilde{\Gamma}(\,\mathrm{d}z)\,\mathrm{d}y.$$

Note that

$$\int_G \theta_\rho^{1/2}\left(xyz^{-1}\right)\left|u\left(xyz^{-1}\right)\right|\theta_\rho^{1/2}\left(xy\right)\left|u\left(xy\right)\right| \,\mathrm{d}x \le |u|^2_{\mathbb{L}^2_\rho}.$$

Therefore

$$\tilde{\Lambda}\left(\mathcal{K}_t(x), \mathcal{K}_s(x)\right) \le c_1\left\langle\tilde{\Gamma}, \mathcal{H}_t^* * \mathcal{H}_s\right\rangle |u|^2_{\mathbb{L}^2_\rho} \le c_2\left\langle\tilde{\Gamma}, \mathfrak{h}_{(t+s)}(\mathfrak{d}(\cdot)/c_2)\right\rangle |u|^2_{\mathbb{L}^2_\rho}.$$

Summing up we have

$$I \le c_2\left\langle\tilde{\Gamma}, \int_0^\infty\int_0^\infty \mathrm{e}^{-(t+s)}\mathfrak{h}_{(t+s)}(\mathfrak{d}(\cdot)/c_2)\,\mathrm{d}s\,\mathrm{d}t\right\rangle |u|^2_{\mathbb{L}^2_\rho} = c_2\left\langle\tilde{\Gamma}, \tilde{\kappa}\left(\mathfrak{d}^2(\cdot)/c_2\right)\right\rangle |u|^2_{\mathbb{L}^2_\rho},$$

where

$$\tilde{\kappa}(r) = \int_0^\infty\int_0^\infty \mathrm{e}^{-(t+s)}(t+s)^{-d/2}\mathrm{e}^{-r/(t+s)}\,\mathrm{d}s\,\mathrm{d}t.$$

Note that

$$\tilde{\kappa}(r) = \int_0^\infty\int_t^\infty \mathrm{e}^{-s}s^{-d/2}\mathrm{e}^{-r/s}\,\mathrm{d}s\,\mathrm{d}t \le \int_0^\infty \mathrm{e}^{-t/2}\int_t^\infty \mathrm{e}^{-s/2}s^{-d/2}\mathrm{e}^{-r/s}\,\mathrm{d}s\,\mathrm{d}t \\ \le 2\int_0^\infty \mathrm{e}^{-s/2}s^{-d/2}\mathrm{e}^{-r/s}\,\mathrm{d}s \le 2^{2-d/2}\kappa(d/2, 2r),$$

κ being defined by (5.1). Thus the lemma follows from Lemma 5.1 and the fact that Γ is a tempered distribution. $\square$

9. Proof of Theorem 2.3

Let $\rho \in \mathbb{R}$. By Lemma 8.2, $\mathcal{A}$ with the domain $\mathbb{D}_\rho$ generates C_0-semigroup U on $\mathbb{X}_\rho$. Let $\boldsymbol{F}$ and $\boldsymbol{B}$ be given by (2.7). Lemma 8.1 and the assumption $f, f_i \in Lip\,(2,\rho)$ ensure that for any $T < \infty$ there is a constant L such that for all $t \in [0,T]$ and $X, Y \in \mathbb{X}_\rho$,

$$|\boldsymbol{F}(t,X) - \boldsymbol{F}(t,Y)|_{\mathbb{X}_\rho} \le L|X-Y|_{\mathbb{X}_\rho} \qquad \text{and} \qquad |\boldsymbol{F}(t,X)| \le L(1+|X|_{\mathbb{X}_\rho}).$$

Lemma 8.3 and the assumption $b \in Lip\,(2,\rho)$ guarantee that for any $T < \infty$ there is a constant L such that for all $t \in [0,T]$ and $X, Y \in \mathbb{X}_\rho$,

$$\begin{aligned} ||\boldsymbol{B}(t,X) - \boldsymbol{B}(t,Y)||_{L_{(\mathrm{HS})}(H_W,\mathbb{X}_\rho)} &\le L|X-Y|_{\mathbb{X}_\rho}, \\ ||\boldsymbol{B}(t,X)||_{L_{(\mathrm{HS})}(H_W,\mathbb{X}_\rho)} &\le L(1+|X|_{\mathbb{X}_\rho}). \end{aligned}$$

Having the Lipschitz continuity of the nonlinear coefficients one can prove the existence and uniqueness of the solution to (1.2) by means of the Banach fix point theorem, just applying known existence results, see e.g. [BP1], [PZ1], [PZ2]. Namely, given $T \in [0,\infty)$ and $q \ge 2$ define $\mathcal{K}^q_T(\mathbb{X}_\rho)$ as the class of all adapted continuous in $\mathbb{X}_\rho$ processes Z satisfying

$$\mathbb{E} \sup_{t\in[0,T]} ||Z(t)||^p_{\mathbb{X}_\rho} < \infty.$$

We define on $\mathcal{K}^p_T(\mathbb{X}_\rho)$ a functional $\mathcal{J}$,

$$\mathcal{J}(Z)(t) = U(t)X(0) + \int_0^t U(t-s)\boldsymbol{F}(s,Z(s))\,\mathrm{d}s + \int_0^t U(t-s)\boldsymbol{B}(s,Z(s))\,\mathrm{d}W(s).$$

For q large enough one can chose $T = T(q) > 0$ such that $\mathcal{J}$ is a contraction from $\mathcal{K}^p_T(\mathbb{X}_\rho)$ into $\mathcal{K}^p_T(\mathbb{X}_\rho)$. In this point the Da Prato–Kwapień–Zabczyk factorization, see (7.1)–(7.2), enables us to take supremum operator in the stochastic integral

$$\mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t U(t-s)\boldsymbol{B}(s,Z(s))\,\mathrm{d}W(s)\right\|^p_{\mathbb{X}_\rho}$$

outside the expectation operator, see e.g. [BP1], [PZ1], [PZ2], or the proof of Theorem 2.2. □

10. Examples

Example 10.1. Assume that Γ is a bounded function. Then it satisfies (2.1) and (2.2). For (2.2) is by Lemma 5.1 equivalent to $\int_0^1 \langle \Gamma, \mathcal{H}_t\rangle\,\mathrm{d}t < \infty$. This is satisfied by any bounded Γ as $\langle\,\mathrm{d}x, \mathcal{H}_t\rangle = 1$.

Example 10.2. Given $\alpha \in (-\infty, 1)$ we set $\Gamma_\alpha(x) = \int_0^\infty t^{-\alpha} \mathrm{e}^{-t} \mathcal{H}_t(x)\,\mathrm{d}t$, $x \in G$. Then we have the following result.

Theorem 10.1. *(i) Γ_α is a non-negative finite tempered measure on G, and hence it satisfies (2.1) with $C_{\Gamma_\alpha} = 0$. Moreover,*

$$\Lambda_\alpha(\psi, \varphi) = \langle \Gamma_\alpha, \psi^* * \varphi \rangle \,, \qquad \psi, \varphi \in \mathcal{S}(G)$$

is a continuous positive-definite left translation invariant bilinear form on $\mathcal{S}(G)$.
(ii) (2.2) is satisfied if $d = 1, 2$, or $\alpha < 1 - d/2$, or

$$d = N \text{ and } \alpha < 2 - N/2, \qquad \text{or} \qquad d = N + 1 \text{ and } \alpha < 1 - N/2.$$

(iii) If G is nilpotent then (2.2) is satisfied if $d = 1, 2$, or $\alpha < 2 - d/2$.

Proof of (i). First note that $\Gamma_\alpha(x) \in [0, +\infty]$ for every $x \in G$. Moreover, by Lemmas 3.1 and 5.1 one has $\Gamma_\alpha(x) < \infty$ for $x \neq e$, and for every $m \geq 0$, $\mathfrak{d}^m(x)\Gamma_\alpha(x) \to 0$ as $\mathfrak{d}(x) \to \infty$. Thus $\Gamma_\alpha \in \mathcal{S}'(G)$ follows from $\int_G \mathcal{H}_t(x)\,\mathrm{d}x = 1$. Due to Theorem 1.1, Λ_α is continuous and left translation invariant. Since

$$\langle \mathcal{H}_t, \psi^* * \varphi \rangle = \langle \mathcal{H}_t * \psi, \varphi \rangle = \langle S(t)\psi, \varphi \rangle$$

we have

$$\Lambda_\alpha(\psi, \varphi) = \int_0^\infty t^{-\alpha} \mathrm{e}^{-1} \langle S(t)\psi, \varphi \rangle \; \mathrm{d}t = \int_0^\infty t^{-\alpha} \mathrm{e}^{-1} \langle S(t/2)\psi, S(t/2)\varphi \rangle \; \mathrm{d}t,$$

and hence Λ_α is positive-definite. $\square$

Proof of (ii). Note that, by Lemmas 3.1 and 5.1 there are constants $C_1, C_2 \in (0, \infty)$ such that for all $x \in B(e, 1)$,

$$\begin{cases} C_1 \leq \Gamma_\alpha(x) \leq C_2 & \text{if } \alpha + d/2 < 1, \\ C_1 \log\left(\mathfrak{d}(x)^{-1}\right) \leq \Gamma_\alpha(x) \leq C_2 \log\left(\mathfrak{d}(x)^{-1}\right) & \text{if } \alpha + d/2 = 1, \\ C_1 \mathfrak{d}(x)^{2-2\alpha-d} \leq \Gamma_\alpha(x) \leq C_2 \mathfrak{d}(x)^{2-2\alpha-d} & \text{if } \alpha + d/2 > 1. \end{cases}$$

Thus (2.2) holds true iff $d = 1$, or $\alpha + d/2 < 1$, or

$$\begin{cases} \int_{B(e,1)} \left(\log\left(\mathfrak{d}(x)^{-1}\right)\right)^2 \,\mathrm{d}x < \infty & \text{if } \alpha = 0,\ d = 2, \\ \int_{B(e,1)} \mathfrak{d}(x)^{-2\alpha} \log\left(\mathfrak{d}(x)^{-1}\right) \,\mathrm{d}x < \infty & \text{if } \alpha > 0,\ d = 2, \\ \int_{B(e,1)} \mathfrak{d}(x)^{2-d} \log\left(\mathfrak{d}(x)^{-1}\right) \,\mathrm{d}x < \infty & \text{if } \alpha + d/2 = 1,\ d > 2, \\ \int_{B(e,1)} \mathfrak{d}(x)^{4-2d-2\alpha} \,\mathrm{d}x < \infty & \text{if } \alpha + d/2 > 1,\ d > 2. \end{cases} \tag{10.1}$$

Note that $\mathfrak{d}$ dominates the original metric $\mathfrak{q}$ on G, and the exponential map is a local isomorphism between G and $\mathbb{R}^N$. Thus (10.1) holds true if

$$\begin{cases} \int_0^1 \left(\log\left(t^{-1}\right)\right)^2 t^{N-1}\,\mathrm{d}t < \infty & \text{if } \alpha = 0,\ d = 2, \\ \int_0^1 t^{-2\alpha+N-1} \log\left(t^{-1}\right)\,\mathrm{d}t < \infty & \text{if } \alpha > 0,\ d = 2, \\ \int_0^1 t^{2-d+N-1} \log\left(t^{-1}\right)\,\mathrm{d}t < \infty & \text{if } \alpha + d/2 = 1,\ d > 2, \\ \int_0^1 t^{4-2d-2\alpha+N-1}\,\mathrm{d}t < \infty & \text{if } \alpha + d/2 > 1,\ d > 2. \end{cases} \tag{10.2}$$

Note that if $d = 2$ then $N = 2$, and consequently the first two conditions in (10.2) hold true. It is easy to see that if $\alpha < 1 - d/2$, or one of the last two conditions hold true, than either $d = N > 2$ and $\alpha < 2 - N/2$, or $N = d + 1$ and $\alpha < 1 - N/2$. $\square$

Proof of (iii). Taking into account the second part of the theorem we can assume that $d > 2$ and $\alpha \geq 1 - d/2$. Let $V(t) = \int_{B(e,t)} \mathrm{d}x$, $t > 0$. From (10.1) it is enough to show that

$$\begin{cases} \int_0^1 t^{2-d} \log\left(t^{-1}\right)\,\mathrm{d}V(t) < \infty & \text{if } \alpha + d/2 = 1,\ d > 2, \\ \int_0^1 t^{4-2d-2\alpha}\,\mathrm{d}V(t) < \infty & \text{if } \alpha + d/2 > 1,\ d > 2 \end{cases}$$

implies $\alpha \in [1 - d/2, 2 - d/2)$. This is a simple consequence of the fact that if G is nilpotent, then there is a constant C such that $V(t) \leq C t^d$, $t > 0$, see [CSV]. $\square$

11. Heisenberg group

For $n \geq 1$, the Heisenberg group G_n is the group whose underlying space is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ or equivalently $\mathbb{C}^n \times \mathbb{R}$, and whose group law is

$$(\xi_1, \eta_1, \tau_1)\,(\xi_2, \eta_2, \tau_2) = \left(\xi_1 + \xi_2, \eta_1 + \eta_2, \tau_1 + \tau_2 + \frac{1}{2}(\xi_1\eta_2 - \eta_1\xi_2)\right).$$

Note that G_n is identified with its Lie algebra $\mathcal{G}_n$. The canonical basis of $\mathcal{G}_n$ will be denoted by $(\mathfrak{X}_1, \dots, \mathfrak{X}_n, \mathfrak{Y}_1, \dots, \mathfrak{Y}_n, \mathfrak{T})$. The Haar measure on G_n is just Lebesgue's one on $\mathbb{C}^n \times \mathbb{R}$, and the distance $\mathfrak{q}$ is also the Euclidean distance.

In this section

$$\mathfrak{L} = \sum_{i=1}^{n} \left(\mathfrak{X}_i^2 + \mathfrak{Y}_i^2\right).$$

Note that since $[\mathfrak{X}_i, \mathfrak{Y}_i] = \mathfrak{T}$ for all $i = 1, \dots, n$ and $N = 2n + 1$, formula (1.5) reads $d = 2n + 2$.

Let $\{h_k; k \geq 1\}$ be the $L^2(\mathbb{R})$-orthonormal basis given by the Hermite functions

$$h_k(t) = \frac{(-1)^k}{(2^k\sqrt{\pi}k!)^{1/2}} \frac{\mathrm{d}^k}{\mathrm{d}t^k}(\mathrm{e}^{-t^2})\mathrm{e}^{t^2/2},$$

let $\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$ for $\alpha \in \mathbb{N}^n$, $x \in \mathbb{R}^n$, and let

$$\kappa_{\alpha,\lambda}(r) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{n/2} r), \quad r \in \mathbb{R}^n.$$

Denote by $\widehat{\Gamma}$ the Fourier transform of $\Gamma \in L^2(G_n)$, see Appendix A and the proof of the theorem below. In this case, $\widehat{\Gamma}$ is well defined as a function from $\mathbb{R}^*$ into the space $L(L^2(\mathbb{R}^n))$ of linear operators on $L^2(\mathbb{R}^n)$. For $\lambda \in \mathbb{R}^*$, $\alpha \in \mathbb{N}^n$, set

$$\gamma_\alpha^2(\lambda) = \left\langle \widehat{\Gamma}(\lambda)\kappa_{\alpha,\lambda}, \kappa_{\alpha,\lambda} \right\rangle_{L^2(\mathbb{R}^n)}.$$

Then, see [F, p. 137], $\gamma_\alpha^2(\lambda) \geq 0$ for all α, λ.

Theorem 11.1. *Assume that* $\Gamma \in L^2$. *Then* Γ *satisfies* (2.2) *iff*

$$\sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}} \frac{\gamma_\alpha^2(\lambda)|\lambda|^n}{1 + |\lambda||\alpha|} \, \mathrm{d}\lambda < \infty. \tag{11.1}$$

Proof. The Fourier analysis on G_n involves the set of representations $\{\pi_\lambda; \lambda \in \mathbb{R}^*\}$ defined as follows. For all $\lambda \in \mathbb{R}^*$, $x = (\xi, \eta, \tau) \in G_n$, $\pi_\lambda(x)$ is an element of $L^2(\mathbb{R}^n)$, and

$$[\pi_\lambda(x)\phi](r) = \mathrm{e}^{i(\lambda\tau + \xi r + \frac{1}{2}\xi\eta)}\phi(r + \eta) \quad \phi \in L^2(\mathbb{R}^n),\ r \in \mathbb{R}^n.$$

Thus in the terminology of the appendix $H_\pi = L^2(\mathbb{R}^n)$. If $\psi \in L^1(G_n)$, then the Fourier transform of ψ will be an $L(L^2(\mathbb{R}^n))$-valued function $\{\widehat{\psi}(\lambda); \lambda \in \mathbb{R}^*\}$. Theorem A.1 from the appendix holds then with the Plancherel measure given by

$$\mu(\mathrm{d}\lambda) = \frac{|\lambda|^n}{(2\pi)^{n+1}} \, \mathrm{d}\lambda.$$

The operator $\mathfrak{L}$ and the Fourier transform on G_n are related in the following way, see [Th, p. 51]. If $\psi \in \mathcal{S}(G_n)$, then for every $\lambda \in \mathbb{R}^*$, $(\mathfrak{L}\psi)\widehat{\ }(\lambda) = \widehat{\psi}(\lambda)U(\lambda)$, where

$$U(\lambda) = \Delta_{\mathbb{R}^n} - \lambda^2|r|^2$$

is the scaled Hermite operator. Then, for all $\lambda \in \mathbb{R}^*$, $U(\lambda)$ has the eigenvectors $\{\kappa_{\alpha,\lambda}\}$ with the corresponding eigenvalues $\sigma_{\alpha,\lambda} = -(2|\alpha| + n)|\lambda|$. In particular, $U(\lambda)$ generates a C_0-semigroup $\mathcal{U}_t(\lambda)$ on $L^2(\mathbb{R}^n)$. Let $\mathcal{H}_t$ be the heat kernel on G_n. Our first goal is to show that

$$\widehat{\mathcal{H}}_t(\lambda) = \mathcal{U}_t(\lambda) \qquad \lambda \in \mathbb{R}^*. \tag{11.2}$$

For the function $\mathcal{H}_t$ is the fundamental solution to the heat equation on G_n. In Fourier coordinates for any $\psi \in \mathcal{S}(\mathbb{R}^n)$ one has

$$\begin{cases} \partial_t v(t,r) = U(\lambda)\, v(t,r), & (t,r) \in [0,\infty) \times \mathbb{R}^n \\ v(0,r) = \psi(r), \quad r \in \mathbb{R}^n \end{cases}$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we have $\mathcal{H}_t^*(\lambda) = \mathcal{U}_t(\lambda)$, which gives (11.2).

By Lemma 5.1, (2.2) is equivalent to $\int_0^T \langle \Gamma, \mathcal{H}_s \rangle \,\mathrm{d}s < \infty$, which in Fourier coordinates yields, using Plancherel's formula,

$$\int_0^T \int_{\mathbb{R}} \operatorname{Tr}\left(\widehat{\Gamma}(\lambda)\widehat{\mathcal{H}}_s^*(\lambda)\right) \mu(\,\mathrm{d}\lambda) < \infty.$$

Then

$$\operatorname{Tr}\left(\widehat{\Gamma}(\lambda)\widehat{\mathcal{H}}_s^*(\lambda)\right) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha^2(\lambda) \exp\left(-\sigma_{\alpha,\lambda} s\right),$$

which is equivalent to (11.1). □

A. Appendix

The material included here is taken mainly from [F, Chapter 7] and references therein, and we refer to that book for further details. We say that π is a unitary representation of G on a separable Hilbert space H_π, if it is a homomorphism from G to $U(H_\pi)$, where $U(H_\pi)$ denotes the set of unitary operators on H_π. Let $\widehat{G}$ be the set of equivalence classes of unitary irreducible representations (see [F, Chapter 3] for basic definitions of representation theory). We will still write π for the generic element of $\widehat{G}$.

The Mackey Borel structure on $\widehat{G}$ is the σ-algebra $\mathcal{M}$ on $\widehat{G}$ which makes all the functions

$$\pi \mapsto \langle \pi(x)u, v\rangle_{H(\pi)}, \quad x \in G, \;\; u, v \in H(\pi)$$

measurable. Suppose that G is of type I (see the definition in [F, p. 206]), which occurs if G is either Abelian, or semisimple, or nilpotent, or a real algebraic group. Then $(\widehat{G}, \mathcal{M})$ is a standard measurable space (see [F, Th. 7.6]). For a given measure ν on $(\widehat{G}, \mathcal{M})$, and a family of separable Hilbert spaces $\{\mathcal{H}_\pi; \pi \in \widehat{G}\}$ one can associate, as in [F, Section 7.4], the direct integral of the spaces $\mathcal{H}_\pi$ with respect to ν, denoted by

$$\int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \nu(\,\mathrm{d}\pi),$$

which is the space of measurable vector fields ψ such that $\psi(\pi) \in \mathcal{H}_\pi$, and

$$\|\psi\|^2 = \int_{\widehat{G}} |\psi(\pi)|_{\mathcal{H}_\pi}^2 \nu(\,\mathrm{d}\pi) < \infty.$$

For a fixed element $\pi \in \widehat{G}$ and $f \in L^1(G)$, the Fourier transform of ψ at π is defined as the vector-valued integral

$$\widehat{\psi}(\pi) = \int_G \psi(x)\pi(x)\,\mathrm{d}x.$$

Let us denote by $L_{(\mathrm{HS})}(H)$ the space of Hilbert–Schmidt operators on a Hilbert space H.

Theorem A.1. *Suppose G is a unimodular locally compact type I group. Then there exists a unique measure μ on $(\widehat{G}, \mathcal{M})$ such that the Fourier transform can be extended into a unitary map*

$$\psi \in L^2(G) \mapsto \widehat{\psi} \in \int_{\widehat{G}}^{\oplus} L_{(\mathrm{HS})}(H_\pi)\mu(\,\mathrm{d}\pi)$$

and the following Plancherel formula holds on $L^2(G)$;

$$\int_G \psi(x)\bar{\varphi}(x)\,\mathrm{d}x = \int_{\widehat{G}} \mathrm{Tr}\left(\widehat{\psi}(\pi)\widehat{\varphi}(\pi)^*\right)\mu(\,\mathrm{d}\pi).$$

Furthermore, the Fourier transform has the following properties:

(1) *If $\psi, \varphi \in L^1(G) \cap L^2(G)$, then, for all $\pi \in \widehat{G}$,*

$$(\psi * \varphi)\widehat{\ }(\pi) = \widehat{\psi}(\pi)\widehat{\varphi}(\pi).$$

(2) *If X is a left invariant first order differential operator, and $\psi \in C_b^\infty(G) \cap L^1(G)$, then*

$$(Xf)\widehat{\ }(\pi) = \widehat{f}(\pi)A_X(\pi),$$

where $A_X(\pi)$ is the skew-symmetric operator on H_π defined by $A_X(\pi) = -d\pi_e(X)$.

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