

# The $p$ -spin interaction model with external field

by

Xavier Bardina<sup>1</sup>, David Márquez-Carreras<sup>2</sup>, Carles Rovira<sup>2</sup>

and

Samy Tindel<sup>3</sup>

<sup>1</sup> *Departament de Matemàtiques, Universitat Autònoma de Barcelona  
08193 Bellaterra, Barcelona, Spain*

*e-mail: bardina@mat.uab.es*

<sup>2</sup> *Facultat de Matemàtiques, Universitat de Barcelona,  
Gran Via 585, 08007-Barcelona, Spain*

*e-mail: marquez@mat.ub.es, rovira@mat.ub.es*

<sup>3</sup> *Département de Mathématiques, Institut Galilée - Université Paris 13,  
Avenue J. B. Clément, 93430-Villetaneuse, France*

*e-mail: tindel@math.univ-paris13.fr*

## Abstract

This paper is devoted to a detailed study of a  $p$ -spins interaction model with external field, including some sharp bounds on the speed of self averaging of the overlap as well as a central limit theorem for its fluctuations, the thermodynamical limit for the free energy and the definition of an Almeida-Thouless type line. Those results show that the external field dominates the tendency to disorder induced by the increasing level of interaction between spins, and our system will share many of its features with the SK model, which is certainly not the case when the external magnetic field vanishes.

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# 1 Introduction

The high temperature regime of the Sherrington-Kirkpatrick model of spin glasses, with or without external field, is now understood in many of its essential features: the overlap  $R_{1,2}$  of two configurations has been shown to be a central object of study for the whole system (see [5]), the thermodynamical limit of  $R_{1,2}$  and of the free energy  $Z_N$  have been computed (see e.g. [6]), and a number of Central Limit Theorems for the fluctuations of those quantities have also been established in different contexts (see [1], [4], [9]), giving a rather complete picture of the model.

On the other hand, the results concerning a natural generalization of the SK model, namely the  $p$ -spins interaction model, are scarce (see however [7] on the low temperature regime and [3] for some fluctuation results for the free energy), especially when an external field is considered. The purpose of the present paper is then to fill this gap: we will consider a spin glass model, whose configuration space is  $\Sigma_N = \{-1, 1\}^N$ . Let  $\mu_N$  be the uniform measure on  $\Sigma_N$ . The energy of a given configuration  $\sigma \in \Sigma_N$  will be represented by a Hamiltonian  $H(\sigma)$ , and we are concerned with the Gibbs measure  $G = G_N$ , whose density with respect to  $\mu_N$  is  $Z_N^{-1} e^{-H}$ , where  $Z_N$  is the normalization factor

$$Z_N = \sum_{\sigma \in \Sigma_N} \exp(-H(\sigma)).$$

The Hamiltonian under consideration here will be defined by

$$-H_{N,\beta,h}(\sigma) = \beta u_N \sum_{(i_1, \dots, i_p) \in A_N^p} g_{i_1, \dots, i_p} \sigma_{i_1} \dots \sigma_{i_p} + h \sum_{i=1}^N \sigma_i,$$

with

$$u_N = \left( \frac{p!}{2N^{p-1}} \right)^{\frac{1}{2}},$$

$$A_N^p = \{(i_1, \dots, i_p) \in \mathbb{N}^p; 1 \leq i_1 < \dots < i_p \leq N\},$$

where the parameter  $\beta$  represents the inverse of the temperature and where  $g = \{g_{i_1, \dots, i_p}; (i_1, \dots, i_p) \in A_N^p\}$  is a family of independent standard Gaussian random variables. The strictly positive parameter  $h$  stands for the external magnetic field, under which the spins tend to take the same value  $+1$ . We will denote by  $\langle f \rangle$  the average of a function  $f : \Sigma_N \rightarrow \mathbb{R}$  with respect to  $G_N$ , as well as the average of a function  $f : \Sigma_N^n \rightarrow \mathbb{R}$  with respect to  $G_N^{\otimes n}$ , without

mentioning the number  $n$  of independent copies of the spins configurations, i.e.

$$\langle f \rangle = Z_N^{-n} \sum_{(\sigma^1, \dots, \sigma^n) \in \Sigma_N^n} f(\sigma^1, \dots, \sigma^n) \exp \left( - \sum_{l \leq n} H_{N, \beta, h}(\sigma^l) \right).$$

We write  $\nu(f) = \mathbf{E}\langle f \rangle$ . Our aim here is then to give a detailed account on the limiting behavior of this system when  $N \rightarrow \infty$ , when  $\beta$  is bounded from above by a constant  $\beta_p$ .

Notice that some of the features of the SK model are shared by our  $p$ -spins interaction model. For instance, the study of the overlap of two configurations, defined by

$$R_{1,2} = \frac{1}{N} \sum_{i \leq N} \sigma_i^1 \sigma_i^2,$$

where  $\sigma^1, \sigma^2$  are understood as two independent configurations under  $G_N$ , will be again one of the main steps to understand the limiting behavior of the system, though it generally appears under the form  $R_{1,2}^{p-1}$  (for instance in our first occurrence of the cavity method, yielding Proposition 2.1), leading to some technical complications. Our first result will then be to show that, for  $\beta$  small enough,  $R_{1,2}$  will self average into a constant  $q = q_p$ , implicitly given as the unique solution to

$$q = \mathbf{E} \left[ \tanh^2 \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right],$$

where  $Y$  stands for a standard Gaussian random variable. In particular, it will be easily shown that  $q_p$  will tend to  $\tanh^2(h)$  as  $p$  grows to  $\infty$ , showing that the natural tendency to the disorder induced by the increasing level of interaction between the spins will be dominated by the presence of the external field  $h$ .

It will be natural then to obtain some extra information on the exponential moments of  $N(R_{1,2} - q)$ , from which we will be able to get the estimate

$$\nu(R_{1,2} - q) \leq \frac{L(p, \beta)}{N}, \quad (1)$$

giving a sharp bound on the speed of self averaging of  $R_{1,2}$ . All those considerations on the overlap will yield the following replica-type formula

for  $Z_N$ :

$$\lim_{N \uparrow \infty} p_N(\beta, h, p) = \frac{\beta^2}{4} [1 - pq^{p-1} + (p-1)q^p] + \log 2 + \mathbf{E} \left[ \log \cosh \left[ \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right].$$

Some further computations on the second moments of  $R_{1,2}^{p-1}$  will then lead us to the definition of an Almeida-Thouless line, which should give the limit of the high temperature region for our model, and is defined by

$$1 - \frac{\beta^2 p(p-1)q^{p-2}}{2} \mathbf{E} \left[ \cosh^{-4} \left( \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right] > 0.$$

Notice that the fact that  $q_p \rightarrow \tanh^2(h)$  when  $p \rightarrow \infty$  will immediately imply that, if we denote by  $\beta_{\text{at}}$  the boundary of this Almeida-Thouless line, then  $\beta_{\text{at}} \rightarrow \infty$  when  $p \rightarrow \infty$  (see Remark 5.2).

Our last result will be a central limit theorem for  $R_{1,2}$ : we will show that, for the typical disorder  $g$ , the quantity  $N^{1/2}(R_{1,2}^{p-1} - q^{p-1})$  will converge to a Gaussian random variable whose variance will be identified explicitly. Notice that this behavior is quite different from the picture given by [3]. Indeed, when  $h = 0$ , the rate of fluctuation of  $Z_N$  is shown to be of order  $N^{(p-2)/2}$ , increasing thus with the number of interactions. In our case, the presence of the external field  $h$  will stabilize the behavior of the self averaging, which will occur at the same speed as in the SK case.

Of course, our methods of proofs are much indebted to the great influence of [9], through the rigorous introduction of the cavity method as well as for some key ideas for further computations of moments and limit theorems. However, the presence of an increasing number of interactions requires a careful analysis of the different quantities considered at each step of our calculations, especially in the identification of all the negligible terms involved. This is why we include almost all the details of the computations in our proofs, which, we hope, will make the lecture of the paper easier, though certainly cumbersome.

Our paper is organized as follows: at section 2, we will give some preliminary results on the cavity method for the  $p$ -spin model, allowing to reduce our system of size  $N$  into a system of size  $N - k$  for arbitrary  $1 \leq k \leq N$ . Section 3 is devoted to a preliminary study of  $R_{1,2}$ , including the self averaging result, the existence of exponential moments, and the bound (1). Section 4 will then give the limiting behavior of  $\frac{1}{N} \mathbf{E}[\log(Z_N)]$ . Section 5 will focus

on the definition of the Almeida-Thouless type line, while at Sections 6 and 7 we will establish the CLT for  $R_{1,2}$ . Finally, in the Appendix we recall the definitions of all the sets appearing throughout the paper.

In the sequel, the size of a given finite set  $D$  will be denoted by  $|D|$ . Throughout the paper,  $P_m(N)$  denotes a polynomial of order  $m$  in  $N$ . We will also denote by  $K$  almost all the constants, although their value may change from line to line. We will omit their dependence on  $k$  (the size of the cavity we will create) and  $n$  (the number of copies of  $G_N$  considered).

## 2 The cavity method

In this Section, we will introduce one of the basic tools we will use all along the paper, namely the  $k$ -cavity method, that allows to quantify in a certain way the difference between our original system and a system where the  $k$  last spins are independent from the other ones. We will first introduce the basic notations we will need further on, then get some general results for the  $k$ -cavity, and eventually a simplified version of some of these results for the particular case of a 1-cavity.

### 2.1 Notations and definitions

For  $k \in \{1, \dots, N-1\}$  and  $\beta > 0$ , let

$$\beta_k = \left( \frac{N-k}{N} \right)^{\frac{p-1}{2}} \beta,$$

that will play the role of  $\beta$  for our reduced system. Define the following set:

$$Q_{N,k}^p = \{J = (i_1, \dots, i_p) \in \mathbb{N}^p; 1 \leq i_1 < \dots < i_p \leq N, i_p > N-k\},$$

and, for  $J \in Q_{N,k}^p$ , set  $m = \max\{j, i_j \leq N-k\}$ , and let  $I, I^c$  be defined by

$$I = (i_1, \dots, i_m), \quad I^c = (i_{m+1}, \dots, i_p). \quad (2)$$

Observe that we should write  $I = I(J), I^c = I^c(J)$ , but we will omit this dependence for sake of readability. Using these sets, we define

$$\eta_J = \prod_{i_j \in I} \sigma_{i_j}, \quad \varepsilon_J = \prod_{i_j \in I^c} \sigma_{i_j},$$

and

$$\begin{aligned} g_{(k)}(\eta, \varepsilon) &= \beta u_N \sum_{i_p > N-k} g_{i_1, \dots, i_p} \sigma_{i_1} \cdots \sigma_{i_p} \\ &= \beta u_N \sum_{J \in Q_{N,k}^p} g_J \eta_J \varepsilon_J. \end{aligned}$$

The basic idea of the  $k$ -cavity method is to regroup the Hamiltonian as follows:

$$-H_{N,\beta,h}(\sigma) = -H_{N-k,\beta_k,h}(\sigma) + g_{(k)}(\eta, \varepsilon) + h \sum_{i=1}^k \varepsilon_i,$$

with  $\varepsilon_i = \sigma_{N-i+1}$ . We will denote then by  $\langle \cdot \rangle_k$  the average with respect to the Gibbs measure on  $\Sigma_{N-k}$  relative to the Hamiltonian  $H_{N-k,\beta_k,h}$ . As usual in the spin glasses theory, the cavity method will become a powerful tool through a construction of a continuous path between the original configuration, and a configuration where the  $k$  last spins are independent of the others. Set then, for  $t \in [0, 1]$  and a constant  $q \in [0, 1]$  to be precised later,

$$g_{(k),t}(\eta, \varepsilon) = t^{\frac{1}{2}} g_{(k)}(\eta, \varepsilon) + \beta u_N q^{\frac{p-1}{2}} (1-t)^{\frac{1}{2}} \sum_{J \in Q_{N,k}^p} z_J \varepsilon_J, \quad (3)$$

where  $\{z_J; J \in Q_{N,k}^p\}$  is a family of independent standard Gaussian random variables, also independent of all the disorder  $g$ .

Let  $n \geq 1$  and  $\sigma^1, \dots, \sigma^n$  be  $n$  independent copies of a  $N$ -spins configuration. Let us write

$$\begin{aligned} \mathcal{E}_{n,k,t} &= \exp \left\{ \sum_{l=1}^n (g_{(k),t}(\eta^l, \varepsilon^l) + h \sum_{i=1}^k \varepsilon_i^l) \right\}, \\ Z_{(k),t} &= \langle \mathbf{A} \mathbf{v} \mathcal{E}_{1,k,t} \rangle_k, \end{aligned}$$

where  $\mathbf{A} \mathbf{v}$  means average over  $\{\varepsilon_i^l = \pm 1, i = 1, \dots, k, l = 1, \dots, n\}$ . Then, for  $f : \Sigma_N^n \rightarrow \mathbb{R}$ , we define

$$\begin{aligned} \langle f \rangle_{k,t} &= \frac{\langle \mathbf{A} \mathbf{v} f \mathcal{E}_{n,k,t} \rangle_k}{Z_{(k),t}^n}, \\ \nu_{k,t}(f) &= \mathbf{E} \langle f \rangle_{k,t}. \end{aligned}$$

Observe that  $\nu(f) = \nu_{k,1}(f)$  for any  $k$ .

The idea of what are going to state is that  $\nu_{k,0}(f)$  (or a slight modification of this quantity) should be simpler to compute than  $\nu_{k,1}(f)$  in some interesting cases of functions  $f$ . On the other hand, we will relate these two quantities by means of

$$\nu_{k,1}(f) - \nu_{k,0}(f) = \int_0^1 \frac{d}{dt} \nu_{k,t}(f) dt, \quad (4)$$

or higher-order versions. We will generally write

$$\nu'_{k,t}(f) = \frac{d}{dt} \nu_{k,t}(f).$$

## 2.2 The $k$ -cavity

We will give here some basic relations, allowing to estimate quantities like (4) in great generality. First, we will compute the derivative of  $\nu_{k,t}(f)$  with respect to  $t$  in the following way:

**Proposition 2.1** *For  $t \in [0, 1]$  and  $f : \Sigma_N^n \rightarrow \mathbb{R}$ , we have*

$$\begin{aligned} \nu'_{k,t}(f) = & \beta^2 u_N^2 \sum_{J \in Q_{N,k}^p} \left[ \nu_{k,t} \left( f \sum_{1 \leq l < l' \leq n} (\eta_J^l \eta_J^{l'} - q^{p-1}) \varepsilon_J^l \varepsilon_J^{l'} \right) \right. \\ & - n \nu_{k,t} \left( f \sum_{l=1}^n (\eta_J^l \eta_J^{n+1} - q^{p-1}) \varepsilon_J^l \varepsilon_J^{n+1} \right) \\ & \left. + \frac{n(n+1)}{2} \nu_{k,t} \left( f (\eta_J^{n+1} \eta_J^{n+2} - q^{p-1}) \varepsilon_J^{n+1} \varepsilon_J^{n+2} \right) \right]. \quad (5) \end{aligned}$$

**Proof:** This proof is an extension of [9, Proposition 2.4.2], whose details are given for sake of completeness. We have

$$\frac{d\mathcal{E}_{n,k,t}}{dt} = \left[ \frac{1}{2t^{\frac{1}{2}}} \sum_{l=1}^n g_{(k)}(\eta^l, \varepsilon^l) - \frac{\beta u_N q^{\frac{p-1}{2}}}{2(1-t)^{\frac{1}{2}}} \sum_{J \in Q_{N,k}^p} z_J \sum_{l=1}^n \varepsilon_J^l \right] \mathcal{E}_{n,k,t},$$

and hence  $\nu'_{k,t}(f) = A_1 - A_2$ , with

$$\begin{aligned} A_1 = \frac{1}{2t^{\frac{1}{2}}} \mathbf{E} \left[ Z_{(k),t}^{-n} \sum_{l=1}^n \left\langle \mathbf{A} \mathbf{v} f g_{(k)}(\eta^l, \varepsilon^l) \mathcal{E}_{n,k,t} \right\rangle_k \right. \\ \left. - n Z_{(k),t}^{-(n+1)} \left\langle \mathbf{A} \mathbf{v} f g_{(k)}(\eta^{n+1}, \varepsilon^{n+1}) \mathcal{E}_{n+1,k,t} \right\rangle_k \right], \end{aligned}$$

and

$$A_2 = \frac{\beta u_N q^{\frac{p-1}{2}}}{2(1-t)^{\frac{1}{2}}} \sum_{J \in Q_{N,k}^p} \mathbf{E} \left[ \begin{aligned} & Z_{(k),t}^{-n} \sum_{l=1}^n \left\langle \mathbf{A} \mathbf{v} f z_J \varepsilon_J^l \mathcal{E}_{n,k,t} \right\rangle_k \\ & - n Z_{(k),t}^{-(n+1)} \left\langle \mathbf{A} \mathbf{v} f z_J \varepsilon_J^{n+1} \mathcal{E}_{n+1,k,t} \right\rangle_k \end{aligned} \right].$$

Notice that, in order to obtain the last formula, we have used the basic fact that, for  $\phi : \Sigma_N^m \rightarrow \mathbb{R}$  and  $\hat{\phi} : \Sigma_N^{\hat{m}} \rightarrow \mathbb{R}$ , we have

$$\begin{aligned} \left\langle \phi(\sigma^1, \dots, \sigma^m) \right\rangle_k \left\langle \hat{\phi}(\sigma^1, \dots, \sigma^{\hat{m}}) \right\rangle_k \\ = \left\langle \phi(\sigma^1, \dots, \sigma^m) \hat{\phi}(\sigma^{m+1}, \dots, \sigma^{m+\hat{m}}) \right\rangle_k. \end{aligned}$$

Let us first study  $A_2$ : integrating this expression with respect to  $z_J$ , and invoking the fact that  $\mathbf{E}[zF(z)] = \mathbf{E}[F'(z)]$  for a standard Gaussian random variable  $z$ , we get

$$\partial_{z_J} \mathcal{E}_{n,k,t} = \beta u_N q^{\frac{p-1}{2}} (1-t)^{\frac{1}{2}} \left( \sum_{l'=1}^n \varepsilon_J^{l'} \right) \mathcal{E}_{n,k,t},$$

and hence

$$A_2 = \frac{\beta^2 u_N^2 q^{p-1}}{2} \sum_{J \in Q_{N,k}^p} \mathbf{E} \left[ \begin{aligned} & Z_{(k),t}^{-n} \sum_{l=1}^n \sum_{l'=1}^n \left\langle \mathbf{A} \mathbf{v} f \varepsilon_J^l \varepsilon_J^{l'} \mathcal{E}_{n,k,t} \right\rangle_k \\ & - n Z_{(k),t}^{-(n+1)} \sum_{l=1}^n \left\langle \mathbf{A} \mathbf{v} f \varepsilon_J^l \varepsilon_J^{n+1} \mathcal{E}_{n+1,k,t} \right\rangle_k \\ & - n Z_{(k),t}^{-(n+1)} \sum_{l=1}^{n+1} \left\langle \mathbf{A} \mathbf{v} f \varepsilon_J^l \varepsilon_J^{n+1} \mathcal{E}_{n+1,k,t} \right\rangle_k \\ & + \kappa_n Z_{(k),t}^{-(n+2)} \left\langle \mathbf{A} \mathbf{v} f \varepsilon_J^{n+1} \varepsilon_J^{n+2} \mathcal{E}_{n+2,k,t} \right\rangle_k \end{aligned} \right],$$

with  $\kappa_n = n(n+1)$ . Note that, for any  $l \in \{1, \dots, n\}$ ,

$$Z_{(k),t}^{-n} \left\langle \mathbf{A} \mathbf{v} f \varepsilon_J^l \varepsilon_J^l \mathcal{E}_{n,k,t} \right\rangle_k = Z_{(k),t}^{-n} \left\langle \mathbf{A} \mathbf{v} f \mathcal{E}_{n,k,t} \right\rangle_k,$$

and, since  $f$  depends only on  $(\sigma^1, \dots, \sigma^n)$ , we have

$$\begin{aligned} nZ_{(k),t}^{-(n+1)} \langle \mathbf{A} \mathbf{v} f \varepsilon_J^{n+1} \varepsilon_J^{n+1} \mathcal{E}_{n+1,k,t} \rangle_k &= nZ_{(k),t}^{-(n+1)} \langle \mathbf{A} \mathbf{v} f \mathcal{E}_{n+1,k,t} \rangle_k \\ &= nZ_{(k),t}^{-n} \langle \mathbf{A} \mathbf{v} f \mathcal{E}_{n,k,t} \rangle_k. \end{aligned}$$

Thus

$$\begin{aligned} A_2 = \beta^2 u_N^2 q^{p-1} \sum_{J \in Q_{N,k}^p} \mathbf{E} \left[ \right. & Z_{(k),t}^{-n} \sum_{1 \leq l < l' \leq n} \langle \mathbf{A} \mathbf{v} f \varepsilon_J^l \varepsilon_J^{l'} \mathcal{E}_{n,k,t} \rangle_k \\ & - nZ_{(k),t}^{-(n+1)} \sum_{l=1}^n \langle \mathbf{A} \mathbf{v} f \varepsilon_J^l \varepsilon_J^{n+1} \mathcal{E}_{n+1,k,t} \rangle_k \\ & \left. + \frac{\kappa_n}{2} Z_{(k),t}^{-(n+2)} \langle \mathbf{A} \mathbf{v} f \varepsilon_J^{n+1} \varepsilon_J^{n+2} \mathcal{E}_{n+2,k,t} \rangle_k \right], \end{aligned}$$

where we recall that  $\kappa_n = n(n+1)$ . This can be read as

$$\begin{aligned} A_2 = \beta^2 u_N^2 q^{p-1} \sum_{J \in Q_{N,k}^p} \left[ \right. & \nu_{k,t} \left( f \sum_{1 \leq l < l' \leq n} \varepsilon_J^l \varepsilon_J^{l'} \right) - n\nu_{k,t} \left( f \sum_{l=1}^n \varepsilon_J^l \varepsilon_J^{n+1} \right) \\ & \left. + \frac{n(n+1)}{2} \nu_{k,t} (f \varepsilon_J^{n+1} \varepsilon_J^{n+2}) \right]. \end{aligned}$$

The same kind of computations can be lead for  $A_1$ , integrating first by parts with respect to the variables  $g_I$ . In this case, we obtain

$$\begin{aligned} A_1 = \beta^2 u_N^2 \sum_{J \in Q_{N,k}^p} \left[ \right. & \nu_{k,t} \left( f \sum_{1 \leq l < l' \leq n} \eta_J^l \eta_J^{l'} \varepsilon_J^l \varepsilon_J^{l'} \right) \\ & - n\nu_{k,t} \left( f \sum_{l=1}^n \eta_J^l \eta_J^{n+1} \varepsilon_J^l \varepsilon_J^{n+1} \right) \\ & \left. + \frac{n(n+1)}{2} \nu_{k,t} (f \eta_J^{n+1} \eta_J^{n+2} \varepsilon_J^{n+1} \varepsilon_J^{n+2}) \right]. \end{aligned}$$

Subtracting  $A_1$  and  $A_2$ , we get the desired result.  $\square$

As a consequence of the last proposition, we can bound  $\nu_{k,t}(f)$  by  $\nu_{k,1}(f)$  as follows:

**Proposition 2.2** *Let  $f : \Sigma_N^n \rightarrow \mathbb{R}_+$  be a non-negative function. Then, for  $N$  large enough, we have*

$$\nu_{k,t}(f) \leq \exp \{2\beta^2 n^2 p(k+1)\} \nu(f) \quad (6)$$

**Proof:** Appealing to relation (5), we obtain, for a non-negative  $f$ , and since  $|\eta_j^l \eta_j^{l'} - q^{p-1}| \leq 2$ ,

$$\nu'_{k,t}(f) \geq -4\beta^2 n^2 u_N^2 |Q_{N,k}^p| \nu_{k,t}(f).$$

Using the expression of  $u_N$  and the estimate of Lemma 8.4 (see the Appendix) on  $|Q_{N,k}^p|$ , we get, for a constant  $K(p, k) > 0$  depending only on  $p$  and  $k$ ,

$$\nu'_{k,t}(f) \geq -2\beta^2 n^2 p \left( k + \frac{K(p, k)}{N} \right) \nu_{k,t}(f).$$

Hence, for  $N$  large enough,

$$\nu'_{k,t}(f) \geq -2\beta^2 n^2 p(k+1) \nu_{k,t}(f).$$

Integrating this relation between  $t$  and 1, we get

$$\log(\nu(f)) - \log(\nu_{k,t}(f)) \geq -2\beta^2 n^2 p(k+1)(1-t) \geq -2\beta^2 n^2 p(k+1),$$

which yields the announced relation.  $\square$

We will finish this subsection with a useful result for the  $p-1$ -cavity. This lemma gives an idea of how our computations will become explicit when  $\nu_{p-1,0}$  is considered instead of  $\nu$ .

**Lemma 2.3** *Let  $f^- : \Sigma_N^n \rightarrow \mathbb{R}$  be a function depending on  $\{\sigma_1^l, \dots, \sigma_{N-p+1}^l; l \leq n\}$ . Let  $Y$  be a standard Gaussian random variable. For  $l \leq n$ , we designate by  $\mathcal{M}_l$  an arbitrary subset of  $\{1, \dots, p-1\}$ . Then, for a constant  $L_1(\beta) > 0$ ,*

$$\left| \nu_{p-1,0} \left( f^- \prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l \right) - \nu_{p-1,0}(f^-) \right. \\ \left. \times \prod_{j \leq p-1} \mathbf{E} \left[ \tanh^{\sum_{l=1}^n \mathbb{1}_{\{j \in \mathcal{M}_l\}}} \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right] \right| \leq \frac{L_1(\beta)}{N} \|f^-\|_\infty.$$

**Remark 2.4** *In the sequel we use the following notation:*

$$\hat{q}_n = \mathbf{E} \left[ \tanh^n \left( \beta q^{\frac{p-1}{2}} \left( \frac{p}{2} \right)^{\frac{1}{2}} Y + h \right) \right]. \quad (7)$$

In order to prove this lemma, it will be useful to split  $Q_{N,k}^p$  into

$$Q_{N,k}^p = \bar{Q}_{N,k}^p \cup \tilde{Q}_{N,k}^p, \quad (8)$$

where

$$\begin{aligned} \bar{Q}_{N,k}^p &= \{J \in Q_{N,k}^p ; m < p - 1\}, \\ \tilde{Q}_{N,k}^p &= \{J \in Q_{N,k}^p ; m = p - 1\}. \end{aligned}$$

Recall that  $m$  is defined in (2) as the maximum of  $\{j, i_j \leq N - k\}$ . Hence,  $\bar{Q}_{N,k}^p$  can also be defined as

$$\bar{Q}_{N,k}^p = \{J = (i_1, \dots, i_p) \in Q_{N,k}^p ; i_1 < \dots < i_p, i_{p-1} > N - k\}.$$

Finally, we need to introduce some additional notation and to give a technical lemma that we will only use in the proof of Lemma 2.3, and that expresses the fact that, in (3), the main part of  $\sum_{J \in Q_{N,k}^p} z_J \varepsilon_J$  is given by  $\sum_{J \in \tilde{Q}_{N,k}^p} z_J \varepsilon_J$ , which is easier to handle: let us write

$$\hat{\mathcal{E}}_{n,k,0} = \exp \left\{ \sum_{l \leq n} (\beta u_N q^{\frac{p-1}{2}} \sum_{i \leq k} \hat{z}_i \varepsilon_i^l + h \sum_{i \leq k} \varepsilon_i^l) \right\},$$

where  $\{\hat{z}_i, i = 1, \dots, k\}$  are independent zero mean Gaussian random variables with variance  $\binom{N-k}{p-1}$ . For  $f \equiv f(\varepsilon_1^l, \dots, \varepsilon_k^l, l \leq n)$ , we define

$$\hat{\nu}_{k,0}(f) = \mathbf{E} \left[ \frac{\mathbf{A}\mathbf{v}(f \hat{\mathcal{E}}_{n,k,0})}{\hat{Z}_{(k)}^n} \right],$$

with  $\hat{Z}_{(k)} = \mathbf{A}\mathbf{v} \hat{\mathcal{E}}_{1,k,0}$ .

**Lemma 2.5** *Let  $f \equiv f(\varepsilon_1^l, \dots, \varepsilon_k^l, l \leq n)$ . Then*

$$|\nu_{k,0}(f) - \hat{\nu}_{k,0}(f)| \leq \frac{K(\beta)}{N} \|f\|_\infty.$$

**Proof:** The arguments are similar to Proposition 2.1. Consider

$$\hat{\mathcal{E}}_{n,k,t} = \hat{\mathcal{E}}_{n,k,0} \times \exp \left\{ \sum_{l=1}^n \beta u_N q^{\frac{p-1}{2}} t^{\frac{1}{2}} \sum_{J \in \tilde{Q}_{N,k}^p} z_J \varepsilon_J^l \right\},$$

where  $\{z_J, J \in \bar{Q}_{N,k}^p\}$  are independent standard Gaussian random variables, and define

$$\hat{\nu}_{k,t}(f) = \mathbf{E} \left[ \frac{\mathbf{A}\mathbf{v}(f \hat{\mathcal{E}}_{n,k,t})}{\hat{Z}_{(k),t}^n} \right],$$

with  $\hat{Z}_{(k),t} = \mathbf{A}\mathbf{v} \hat{\mathcal{E}}_{1,k,t}$ .

Note that  $\hat{\nu}_{k,1}(f) = \nu_{k,0}(f)$ . Indeed,  $\hat{z}_i \sim N(0, \binom{N-k}{p-1})$  and thus

$$\sum_{i=1}^k \hat{z}_i \varepsilon_i^l \stackrel{(d)}{=} \sum_{J \in \bar{Q}_{N,k}^p} z_J \varepsilon_J^l.$$

The quantity  $\hat{\nu}_{k,t}(f)$  can be differentiated once again in  $t$ , and we have

$$\frac{d\hat{\mathcal{E}}_{n,k,t}}{dt} = \frac{1}{2t^{\frac{1}{2}}} \sum_{l=1}^n \left( \beta u_N q^{\frac{p-1}{2}} \sum_{J \in \bar{Q}_{N,k}^p} z_J \varepsilon_J^l \right) \hat{\mathcal{E}}_{n,k,t}.$$

Then,

$$\begin{aligned} \hat{\nu}'_{k,t}(f) = \frac{\beta u_N q^{\frac{p-1}{2}}}{2t^{\frac{1}{2}}} \mathbf{E} \left[ \hat{Z}_{(k),t}^{-n} \sum_{l=1}^n \mathbf{A}\mathbf{v} \left( f \sum_{J \in \bar{Q}_{N,k}^p} z_J \varepsilon_J^l \hat{\mathcal{E}}_{n,k,t} \right) \right. \\ \left. - n \hat{Z}_{(k),t}^{-(n+1)} \mathbf{A}\mathbf{v} \left( f \sum_{J \in \bar{Q}_{N,k}^p} z_J \varepsilon_J^{n+1} \hat{\mathcal{E}}_{n+1,k,t} \right) \right]. \end{aligned}$$

An integration by parts formula with respect to the random variable  $z_J$  implies

$$\begin{aligned} \hat{\nu}'_{k,t}(f) = \frac{\beta^2 u_N^2 q^{p-1}}{2} \sum_{J \in \bar{Q}_{N,k}^p} \left[ \sum_{l=1}^n \sum_{l'=1}^n \nu_{k,t} \left( f \varepsilon_J^l \varepsilon_J^{l'} \right) - \sum_{l=1}^n \nu_{k,t} \left( f \varepsilon_J^l \varepsilon_J^{n+1} \right) \right. \\ \left. - \sum_{l=1}^{n+1} \nu_{k,t} \left( f \varepsilon_J^l \varepsilon_J^{n+1} \right) + n(n+1) \nu_{k,t} \left( f \varepsilon_J^{n+1} \varepsilon_J^{n+2} \right) \right] \end{aligned}$$

Now, from Lemma 8.4, we obtain easily

$$u_N^2 |\bar{Q}_{N,k}^p| \leq \frac{K}{N},$$

that gives us the desired result.  $\square$

**Proof of Lemma 2.3:** Since  $f^-$  depends on  $\{\sigma_1^l, \dots, \sigma_{N-p}^l; l \leq n\}$ , invoking Lemma 2.5, we only need to work with  $\hat{\nu}_{p-1,0}(\prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l)$ . Indeed,

$$\nu_{p-1,0}(f^- \prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l) = \nu_{p-1,0}(f^-) \nu_{p-1,0}\left(\prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l\right),$$

and

$$\left| \nu_{p-1,0}\left(\prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l\right) - \hat{\nu}_{p-1,0}\left(\prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l\right) \right| \leq \frac{K(\beta)}{N}.$$

We will now divide our proof in two steps:

**Step 1:** By the construction of  $\hat{\nu}_{p-1,0}$ ,  $\hat{\mathcal{E}}_{n,p-1,0}$  and  $\mathcal{M}_l$ , using independence (of the  $\varepsilon_j$  with respect to the uniform measure on  $\{-1; 1\}^{np}$  and of the random variables  $\hat{z}_j$ ), we get

$$\begin{aligned} \hat{\nu}_{p-1,0}\left(\prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l\right) &= \mathbf{E} \left[ \frac{\mathbf{A}\mathbf{v}(\prod_{l \leq n, m_l \in \mathcal{M}_l} \varepsilon_{m_l}^l \hat{\mathcal{E}}_{n,p-1,0})}{\hat{Z}_{(p-1)}^n} \right] \\ &= \mathbf{E} \left[ \prod_{l \leq n, m_l \in \mathcal{M}_l} \tanh(\beta u_N q^{\frac{p-1}{2}} \hat{z}_{m_l} + h) \right] \\ &= \mathbf{E} \left[ \prod_{j \leq p-1} \left\{ \tanh(\beta u_N q^{\frac{p-1}{2}} \hat{z}_j + h) \right\}^{\sum_{l=1}^n \mathbb{I}_{\{j \in \mathcal{M}_l\}}} \right] \\ &= \prod_{j \leq p-1} \mathbf{E} \left[ \tanh^{\sum_{l=1}^n \mathbb{I}_{\{j \in \mathcal{M}_l\}}}(\beta u_N q^{\frac{p-1}{2}} \hat{z}_j + h) \right]. \end{aligned}$$

**Step 2:** By Lemma 8.4 and the fact that  $\hat{z}_j$  is a centered Gaussian random variable with variance  $\binom{N-k}{p-1}$ , we have

$$E[u_N^2 \hat{z}_j^2] = \frac{p}{2} + \mathcal{O}\left(\frac{1}{N}\right). \quad (9)$$

For  $s > 0$ , set now  $\psi(s) = \mathbf{E}[\tanh^m(X_s + h)]$ , where  $X_s$  is a centered Gaussian random variable with variance  $s^2$ . Then

$$\begin{aligned} \psi(s) &= \frac{1}{\sqrt{2\pi s^2}} \int_{-\infty}^{\infty} \tanh^m(u + h) e^{-\frac{1}{2s^2} u^2} du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tanh^m(vs + h) e^{-\frac{1}{2} v^2} dv. \end{aligned}$$

Note that  $|\psi'(s)| \leq K(m)$ . Then, using the fact that  $Y$  is a standard Gaussian random variable and (9), we have

$$\begin{aligned}
& \left| \mathbf{E} \left[ \tanh^m \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right] - \mathbf{E} \left[ \tanh^m \left( \beta u_N q^{\frac{p-1}{2}} \hat{z}_j + h \right) \right] \right| \\
&= \left| \psi \left( \beta q^{\frac{p-1}{2}} \sqrt{\frac{p}{2}} \right) - \psi \left( \beta q^{\frac{p-1}{2}} u_N \sqrt{\mathbf{E}(\hat{z}_j^2)} \right) \right| \\
&\leq K(m) \beta q^{\frac{p-1}{2}} \left| \sqrt{\frac{p}{2}} - \sqrt{u_N^2 \mathbf{E}(\hat{z}_j^2)} \right| \\
&\leq \frac{K(m, \beta)}{N},
\end{aligned}$$

which shows our claim.  $\square$

### 2.3 Particular case: The 1-cavity

The results of the previous subsection sometimes take a simpler form when expressed for a cavity of order 1. It is then useful to summarize them in this particular case: let

$$\beta_- = \left( \frac{N-1}{N} \right)^{\frac{p-1}{2}} \beta,$$

and  $\langle \cdot \rangle_-$  the averaging with respect to the Gibbs measure on  $\Sigma_{N-1}$  at inverse temperature  $\beta_-$ .

Let  $n \geq 1$  and  $\sigma^1, \dots, \sigma^n$  be  $n$  independent copies of a  $N$ -spins configuration. For any  $j \in \{1, \dots, n\}$ , we denote  $\sigma^j = (\rho^j, \varepsilon^j)$ , where  $\rho^j \in \Sigma_{N-1}$  and  $\varepsilon^j \equiv \varepsilon_1^j \in \{-1, 1\}$ . Set

$$Q_{N,1}^p = \{J = (i_1, \dots, i_{p-1}, N) \in \mathbb{N}^p; 1 \leq i_1 < \dots < i_{p-1} \leq N-1\},$$

and, in this case, for  $J \in Q_{N,1}^p$ ,

$$\eta_J = \sigma_{i_1} \cdots \sigma_{i_{p-1}}, \quad \varepsilon_J = \sigma_N = \varepsilon,$$

and

$$g_{(1)}(\eta, \varepsilon) = \varepsilon g(T(\rho)),$$

being

$$g(T(\rho)) = \beta u_N \sum_{J \in Q_{N,1}^p} g_J \eta_J.$$

Here, for one configuration, we have

$$-H_{N,\beta,h}(\sigma) = -H_{N-1,\beta_-,h}(\rho) + \varepsilon [g(T(\rho)) + h].$$

And we can also define

$$g_{(1),t}(\eta, \varepsilon) = \varepsilon g_t(T(\rho)),$$

with

$$g_t(T(\rho)) = t^{\frac{1}{2}}g(T(\rho)) + \beta u_N q^{\frac{p-1}{2}}(1-t)^{\frac{1}{2}} \sum_{J \in Q_{N,1}^p} z_J$$

where  $t \in [0, 1]$ ,  $q \in [0, 1]$  and where  $\{z_J; J \in Q_{N,1}^p\}$  is a family of independent standard Gaussian random variables, also independent of all the disorder  $g$ .

Let us write

$$\begin{aligned} \mathcal{E}_{n,1,t} &= \exp \left\{ \sum_{l=1}^n \varepsilon^l [g_t(T(\rho^l)) + h] \right\}, \\ Z_{(1),t} &= \langle \mathbf{A} \mathbf{v} \mathcal{E}_{1,1,t} \rangle_- = \langle \cosh [g_t(T(\rho^l)) + h] \rangle_-. \end{aligned}$$

For  $f : \Sigma_N^n \rightarrow \mathbb{R}$ , we can define

$$\begin{aligned} \langle f \rangle_{1,t} &= \frac{\langle \mathbf{A} \mathbf{v} f \mathcal{E}_{n,1,t} \rangle_-}{Z_{(1),t}^n}, \\ \nu_{1,t}(f) &= \mathbf{E} \langle f \rangle_{1,t}. \end{aligned}$$

Then, for  $t \in [0, 1]$  and  $f : \Sigma_N^n \rightarrow \mathbb{R}$ , the derivative of  $\nu_{1,t}(f)$  with respect to  $t$  takes the exact form of relation (4) with  $p = 1$ . Moreover, as a particular case of Proposition 2.2, we also get, for a non-negative function  $f$  and  $N$  large enough, that

$$\nu_{1,t}(f) \leq \exp \{4\beta^2 n^2 p\} \nu(f). \quad (10)$$

An important remark is the fact that in order to prove the equivalent to Lemma 2.3 (which will also be given in a simpler form), we do not need Lemma 2.5: let  $f^- : \Sigma_{N-1}^n \rightarrow \mathbb{R}$  be a function depending on  $\{\sigma_1^l, \dots, \sigma_{N-1}^l; l \leq n\}$ . Let  $Y$  be a standard Gaussian random variable. Then, for a constant  $L_1(\beta) > 0$ ,

$$\left| \nu_{1,0}(f^- \varepsilon^1 \cdots \varepsilon^n) - \nu_{1,0}(f^-) \mathbf{E} \left[ \tanh^n \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right] \right| \leq \frac{L_1(\beta)}{N} \|f^-\|_\infty. \quad (11)$$

### 3 Behavior of the overlap

In this section we will study the limiting behavior of the overlap of two configurations, namely

$$R_{l,l'} = \frac{1}{N} \sum_{i=1}^N \sigma_i^l \sigma_i^{l'},$$

where  $\sigma_i^l$  and  $\sigma_i^{l'}$  are understood as two independent configurations under  $G_N$ . In the sequel, the following assumption on  $\beta$ , that determines our high temperature region, will have to be made:

**(H)** The parameter  $\beta > 0$  is smaller than a constant  $\beta_p$  defined by

$$8p^2 \beta_p^2 \exp(16\beta_p^2 p) = \frac{1}{2}.$$

We will see then that the constant  $q = q_p$  which will be the  $L^2$  limit of  $R_{1,2}$ , is the unique solution to the equation

$$q = \mathbf{E} \left[ \tanh^2 \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right]. \quad (12)$$

Observe that  $\hat{q}_2 = q$ , where  $\hat{q}_n$  is defined by relation (7).

First, at Subsection 3.1, we will obtain the self averaging result for  $R_{1,2}$ , one of the main steps towards the replica symmetric formula. Then, at Subsection 3.2, using an elaboration of the arguments of Subsection 3.1, we will get the existence of exponential moments for  $N(R_{1,2} - q)$ . This will allow us, at Subsection 3.3, using higher order expansions, to get a sharp bound for the quantity  $\nu(R_{1,2}) - q$ .

Along this paper we will use the following two deterministic results on the overlap. The first one is taken from Talagrand [7, Lemma 5.11]:

$$\left| u_N^2 \sum_{J \in Q_{N,1}^p} \eta_J^l \eta_J^{l'} - \frac{p}{2} R_{l,l'}^{p-1} \right| \leq \frac{K}{N}, \quad (13)$$

while the second one is an easy consequence of Lemma 8.4:

$$\left| u_N^2 |Q_{N,1}^p| q^{p-1} - \frac{p}{2} q^{p-1} \right| \leq \frac{K}{N}. \quad (14)$$

Let us first state an elementary Proposition, that will give some useful information about the whole  $p$  spins system.

**Proposition 3.1** *Under assumption (H), equation (12) has a unique solution  $q_p$  in  $[0, 1]$ . Moreover*

$$\lim_{p \rightarrow \infty} q_p = \tanh^2(h).$$

**Proof:** Let  $\varphi, \phi_p : [0, 1] \rightarrow [0, 1]$  be defined by

$$\varphi(x) = x, \quad \phi_p(x) = \mathbf{E} \left[ \tanh^2 \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} x^{\frac{p-1}{2}} Y + h \right) \right].$$

It is easily seen that  $\varphi(0) = 0$ ,  $\phi_p(0) = \tanh^2(h)$  on one hand, and that  $\varphi(1) = 1$ ,  $\phi_p(1) < 1$  on the other hand. Furthermore, a simple Gaussian integration by parts argument (see also the proof of Lemma 4.2) shows that

$$\phi_p'(x) = \frac{\beta^2 p(p-1)}{2} x^{p-2} \mathbf{E} \left[ \psi \left( \beta \left( \frac{p}{2} \right)^{1/2} x^{\frac{p-1}{2}} Y + h \right) \right],$$

where

$$\psi(u) = \frac{1 - 2 \sinh^2(u)}{\cosh^4(u)}.$$

A quick study of  $\psi$  shows that  $\|\psi\|_\infty = 1$ , and hence, for any  $x \in [0, 1]$ ,

$$|\phi_p'(x)| \leq \frac{\beta^2 p(p-1)}{2} x^{p-2} \leq 8p^2 \beta_p^2 \exp(16\beta_p^2 p).$$

Hence, if  $\beta$  satisfies condition (H), the existence and uniqueness of the solution to (12) is trivially obtained. The second claim easily follows from the fact that  $\phi_p(0) = \tanh^2(h)$ , and that for any  $a \in (0, 1)$

$$\lim_{p \rightarrow \infty} \sup_{x \in [0, a]} |\phi_p'(x)| = 0.$$

□

### 3.1 Self averaging property

This part of the paper is devoted to prove that  $R_{1,2}$  converges to  $q$  in a  $L^2$  sense. A 1-cavity will be enough to reach the conclusion of this Section, and we refer to Section 2.3 for further notations and results on this method. First of all recall that  $\nu_{1,1}(f) = \nu(f)$ .

**Proposition 3.2** *Let  $f$  be a function from  $\Sigma_N^n$  to  $\mathbb{R}$ , and  $\alpha_1, \alpha_2 > 1$  such that  $\alpha_1^{-1} + \alpha_2^{-1} = 1$ . Then, there exists a positive constant  $L_2$  such that*

$$|\nu(f) - \nu_{1,0}(f)| \leq (np\beta)^2 \exp(4\beta^2 n^2 p) \nu^{1/\alpha_1}(|f|^{\alpha_1}) \left( \nu^{1/\alpha_2}(|R_{1,2} - q|^{\alpha_2}) + \frac{L_2}{N} \right).$$

**Proof:** Consider the term

$$U_t = \beta^2 u_N^2 \sum_{J \in Q_{N,1}^p} \nu_{1,t} \left( f \varepsilon^l \varepsilon^{l'} (\eta_J^l \eta_J^{l'} - q^{p-1}) \right).$$

Then, by Hölder's inequality,

$$U_t \leq \beta^2 \nu_{1,t}^{1/\alpha_1} (|f|^{\alpha_1}) \nu_{1,t}^{1/\alpha_2} \left( \left| u_N^2 \sum_{J \in Q_{N,1}^p} \eta_J^l \eta_J^{l'} - u_N^2 |Q_{N,1}^p| q^{p-1} \right|^{\alpha_2} \right).$$

Using (13) and (14), we obtain

$$U_t \leq \frac{p\beta^2}{2} \nu_{1,t}^{1/\alpha_1} (|f|^{\alpha_1}) \left( \nu_{1,t}^{1/\alpha_2} \left( |R_{l,l'}^{p-1} - q^{p-1}|^{\alpha_2} \right) + \frac{L_2}{N} \right),$$

and since  $|R_{l,l'}^{p-1} - q^{p-1}| \leq p|R_{l,l'} - q|$ , we get

$$U_t \leq \frac{p^2\beta^2}{2} \nu_{1,t}^{1/\alpha_1} (|f|^{\alpha_1}) \left( \nu_{1,t}^{1/\alpha_2} (|R_{1,2} - q|^{\alpha_2}) + \frac{L_2}{N} \right).$$

By relation (6), we then have

$$U_t \leq \frac{p^2\beta^2}{2} \exp(4\beta^2 n^2 p) \nu^{1/\alpha_1}(|f|^{\alpha_1}) \left( \nu^{1/\alpha_2} (|R_{1,2} - q|^{\alpha_2}) + \frac{L_2}{N} \right).$$

Our result is then obtained by iteration of this kind of calculations for the other terms in (5). □

**Proposition 3.3** *Let  $q$  be the solution to (12). If  $\beta$  satisfies (H), then*

$$\nu((R_{1,2} - q)^2) = \mathbf{E}\langle (R_{1,2} - q)^2 \rangle \leq \frac{K}{N}.$$

**Proof:** The symmetry between sites implies that

$$\nu((R_{1,2} - q)^2) = \nu(\bar{f}), \quad (15)$$

where

$$\bar{f} = (\varepsilon^1 \varepsilon^2 - q)(R_{1,2} - q) = A_1 + A_2,$$

with

$$\begin{aligned} A_1 &= \frac{1}{N}(\varepsilon^1 \varepsilon^2 - q)^2, \\ A_2 &= (\varepsilon^1 \varepsilon^2 - q) \left( R_{1,2}^- - \frac{N-1}{N} q \right), \end{aligned}$$

and

$$R_{1,2}^- = \frac{1}{N} \sum_{i=1}^{N-1} \sigma_i^1 \sigma_i^2.$$

Since  $|\varepsilon^1 \varepsilon^2 - q| \leq 2$ , it is obvious that

$$\nu_{1,0}(A_1) \leq \frac{4}{N}.$$

On the other hand, by relation (11) and the fact that  $q$  is the solution to (12), we get

$$\begin{aligned} \nu_{1,0}(A_2) &= \nu_{1,0} \left( R_{1,2}^- - \frac{N-1}{N} q \right) \\ &\quad \times \left[ \mathbf{E} \left[ \tanh^2 \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right] - q \right] + \mathcal{O} \left( \frac{1}{N} \right) \\ &= \mathcal{O} \left( \frac{1}{N} \right), \end{aligned}$$

and Proposition 3.2 for  $n = \alpha_1 = \alpha_2 = 2$  yields

$$|\nu(\bar{f}) - \nu_{1,0}(\bar{f})| \leq (2p\beta)^2 \exp\{16\beta^2 p\} \nu^{1/2}(|\bar{f}|^2) \left( \nu^{1/2}(|R_{1,2} - q|^2) + \frac{L_2}{N} \right).$$

Then (15) and the estimates for  $A_1$  and  $A_2$  imply

$$\nu((R_{1,2} - q)^2) \leq 8p^2 \beta^2 \exp\{16\beta^2 p\} \nu((R_{1,2} - q)^2) + \frac{K}{N}.$$

Thus, if  $\beta_p$  satisfies (H), we obtain the desired inequality.  $\square$

### 3.2 Exponential moments

The aim of this subsection is to bound the higher moments of  $R_{1,2} - q$ . Notice that these bounds will be used in the next subsection in order to control  $\nu(R_{1,2} - q)$ .

**Theorem 3.4** *Let  $q$  be the solution to (12). If  $\beta$  satisfies (H), we have*

$$\nu((R_{1,2} - q)^{2l}) = \mathbf{E}\langle (R_{1,2} - q)^{2l} \rangle \leq \left(\frac{Ll}{N}\right)^l,$$

where  $L$  does not depend on  $l$ .

**Remark 3.5** *Theorem 3.4 implies that there exists  $M > 0$  such that*

$$\nu\left(\exp\left\{\frac{N}{M}(R_{1,2} - q)^2\right\}\right) \leq M,$$

and hence the title of this section. Indeed, this is an immediate consequence of the equality  $e^{x^2} = \sum_{l \geq 0} \frac{x^{2l}}{l!}$  and the fact  $\left(\frac{l}{3}\right)^l \leq l! \leq l^l$ .

The proof of Theorem 3.4 goes along the same lines as Theorem 2.5.1 in [9], except for the introduction of a two steps induction due to the high number of interactions between spins. We will try to stress mainly on this difference. We will proceed by induction over  $l$ , and the induction hypothesis will be

$$\nu((R_{1,2} - q)^{2\bar{l}}) \leq \left(\frac{L_0 \bar{l}}{N}\right)^{\bar{l}}, \quad \text{for any } \bar{l} \in \{1, \dots, l\}, \quad (16)$$

being  $L_0$  a fixed number. The case  $l = 1$  has been proved in Proposition 3.3, and if  $L_0$  is large enough, we will show that

$$\nu((R_{1,2} - q)^{2l+2}) \leq \left(\frac{L_0(l+1)}{N}\right)^{l+1}. \quad (17)$$

First of all, since  $|R_{1,2} - q| \leq 2$ , for any  $\bar{l} \geq N$  assuming  $L_0 \geq 4$ , we have

$$\nu((R_{1,2} - q)^{2\bar{l}}) \leq 4^{\bar{l}} \leq L_0^{\bar{l}} \left(\frac{\bar{l}}{N}\right)^{\bar{l}}.$$

So, we can suppose  $l \leq N - 1$ .

In order to prove (17) we will need the following lemma.

**Lemma 3.6** Assume (16) and  $l \leq N - 1$ . Then, if  $L_0 \geq 4$  we have

$$\begin{aligned}\nu(|R_{1,2} - q|^j) &\leq \left(\frac{L_0(j+1)}{N}\right)^{j/2}, \quad \forall j \leq 2l, \\ \nu((R_{1,2}^- - q)^{2l}) &\leq 3\left(\frac{L_0(l+1)}{N}\right)^l.\end{aligned}$$

**Proof:** See the proof of Lemma 2.5.1 in [9].  $\square$

**Proof of Theorem 3.4:** Our goal is to prove (17) assuming (16). By symmetry we have

$$\nu((R_{1,2} - q)^{2l+2}) = \nu(\bar{f}) = \nu_{1,0}(\bar{f}) + [\nu(\bar{f}) - \nu_{1,0}(\bar{f})], \quad (18)$$

where

$$\bar{f} = (\varepsilon^1 \varepsilon^2 - q)(R_{1,2} - q)^{2l+1}.$$

Applying Proposition 3.2 with  $n = 2$ ,  $\alpha_1 = \frac{2l+2}{2l+1}$  and  $\alpha_2 = 2l + 2$ , and using  $|\varepsilon^1 \varepsilon^2 - q| \leq 2$  we obtain

$$|\nu(\bar{f}) - \nu_{1,0}(\bar{f})| \leq 8p^2 \beta^2 e^{16\beta^2 p} \left[ \nu((R_{1,2} - q)^{2l+2}) + \frac{L_2}{N} \nu^{\frac{2l+1}{2l+2}}((R_{1,2} - q)^{2l+2}) \right]. \quad (19)$$

Assuming condition (H) and plugging (19) into (18) we get, for  $\beta \leq \beta_p$ ,

$$\nu((R_{1,2} - q)^{2l+2}) \leq 2\nu_{1,0}(\bar{f}) + \frac{L_2}{N} \nu^{\frac{2l+1}{2l+2}}((R_{1,2} - q)^{2l+2}). \quad (20)$$

This inequality, which was sufficient in the case of the SK model (see [9]), does not allow us to reach our conclusion here, and we will have to perform a second step in our induction: using that

$$(x + y)^\alpha \leq x^\alpha + y^\alpha,$$

for  $x, y \geq 0$  and  $\alpha \in (0, 1)$ , from (20) we easily obtain, for  $\beta \leq \beta_p$ ,

$$\nu((R_{1,2} - q)^{2l+2}) \leq A_1 + A_2 + A_3, \quad (21)$$

where

$$\begin{aligned}A_1 &= 2\nu_{1,0}(\bar{f}), \\ A_2 &= \frac{L_2}{N} 2^{\frac{2l+1}{2l+2}} \nu_{1,0}^{\frac{2l+1}{2l+2}}(\bar{f}), \\ A_3 &= \frac{L_2}{N} \left( \frac{L_2}{N} \nu^{\frac{2l+1}{2l+2}}((R_{1,2} - q)^{2l+2}) \right)^{\frac{2l+1}{2l+2}}.\end{aligned}$$

Let us study first  $A_3$ . Using (16) we get for  $\beta \leq \beta_p$

$$\begin{aligned}
A_3 &\leq \frac{4 L_2^2 \left[ \nu((R_{1,2} - q)^{2l}) \right]^{\left(\frac{2l+1}{2l+2}\right)^2}}{N^{1+\frac{2l+1}{2l+2}}} \\
&\leq \frac{4 L_2^2 (L_0 l)^l}{N^{1+\frac{2l+1}{2l+2}+l\left(\frac{2l+1}{2l+2}\right)^2}} \\
&\leq \frac{4 L_2^2 (L_0 l)^l}{N^{l+1}}. \tag{22}
\end{aligned}$$

In order to study  $A_1$  note that

$$\begin{aligned}
|\nu_{1,0}(\bar{f})| &\leq \left| \nu_{1,0} \left( (\varepsilon^1 \varepsilon^2 - q) [(R_{1,2} - q)^{2l+1} - (R_{1,2}^- - q)^{2l+1}] \right) \right| \\
&\quad + \left| \nu_{1,0} \left( (\varepsilon^1 \varepsilon^2 - q) (R_{1,2}^- - q)^{2l+1} \right) \right|.
\end{aligned}$$

The independence between  $\varepsilon^1 \varepsilon^2$  and  $R_{1,2}^-$  under  $\nu_{1,0}$ , inequalities (11) and (10) and Lemma 3.6 yield

$$\begin{aligned}
\left| \nu_{1,0} \left( (\varepsilon^1 \varepsilon^2 - q) (R_{1,2}^- - q)^{2l+1} \right) \right| &\leq \left| \nu_{1,0}(\varepsilon^1 \varepsilon^2 - q) \right| \left| \nu_{1,0} \left( (R_{1,2}^- - q)^{2l+1} \right) \right| \\
&\leq \frac{L_1 e^{16\beta_p^2 p}}{2N} \left| \nu \left( (R_{1,2}^- - q)^{2l+1} \right) \right| \\
&\leq 3 L_1 e^{16\beta_p^2 p} \frac{(L_0(l+1))^l}{N^{l+1}},
\end{aligned}$$

with  $L_1 := L_1(\beta_p)$  given by (11). On the other hand, using the inequality  $|x^{2l+1} - y^{2l+1}| \leq (2l+1)|x-y|(x^{2l} + y^{2l})$  and similar arguments as before, we get

$$\begin{aligned}
& \left| \nu_{1,0} \left( (\varepsilon^1 \varepsilon^2 - q) [(R_{1,2} - q)^{2l+1} - (R_{1,2}^- - q)^{2l+1}] \right) \right| \\
& \leq 2 \left| \nu_{1,0} \left( (R_{1,2} - q)^{2l+1} - (R_{1,2}^- - q)^{2l+1} \right) \right| \\
& \leq \frac{2(2l+1)}{N} \left[ \nu_{1,0} \left( (R_{1,2} - q)^{2l} \right) + \nu_{1,0} \left( (R_{1,2}^- - q)^{2l} \right) \right] \\
& \leq \frac{2(2l+1)}{N} e^{16\beta_p^2 p} \left[ \nu \left( (R_{1,2} - q)^{2l} \right) + \nu \left( (R_{1,2}^- - q)^{2l} \right) \right] \\
& \leq \frac{2(2l+1)}{N} e^{16\beta_p^2 p} \left[ \left( \frac{L_0 l}{N} \right)^l + 3 \left( \frac{L_0(l+1)}{N} \right)^l \right] \\
& \leq \frac{4 e^{16\beta_p^2 p} [(L_0 l)^l + 3(L_0(l+1))^l] (l+1)}{N^{l+1}} \\
& \leq 8 e^{16\beta_p^2 p} \frac{L_0^l (l+1)^{l+1}}{N^{l+1}}.
\end{aligned}$$

So, it follows that

$$\begin{aligned}
A_1 & \leq 2 e^{16\beta_p^2 p} (3L_1 + 8) \frac{L_0^l (l+1)^{l+1}}{N^{l+1}} \\
& \leq 2 e^{16p} (3L_1 + 8) \frac{L_0^l (l+1)^{l+1}}{N^{l+1}}. \tag{23}
\end{aligned}$$

It is also easy to check that

$$\begin{aligned}
A_2 & \leq 2 \frac{L_2}{N} \left[ 2 e^{16p} (3L_1 + 8) \frac{L_0^l (l+1)^{l+1}}{N^{l+1}} \right]^{\frac{2l+1}{2l+2}} \\
& \leq \frac{4 L_2 e^{16p} (3L_1 + 8) L_0^l (l+1)^{l+1}}{N^{1+(l+1)\left(\frac{2l+1}{2l+2}\right)}} \\
& \leq \frac{4 L_2 e^{16p} (3L_1 + 8) L_0^l (l+1)^{l+1}}{N^{l+1}}. \tag{24}
\end{aligned}$$

Putting together (21), (22), (23) and (24) we obtain, for  $\beta \leq \beta_p$ ,

$$\nu((R_{1,2} - q)^{2l+2}) \leq \bar{K} \frac{L_0^l (l+1)^{l+1}}{N^{l+1}},$$

with

$$\bar{K} = 4 L_2^2 + 2 e^{16p} (3L_1 + 8) (1 + 2L_2).$$

So, if  $L_0 \geq \bar{K}$  the proof is completed.  $\square$

An easy consequence of Theorem 3.4 is the following

**Corollary 3.7** *Let  $q$  be the solution of (12). If  $\beta$  satisfies (H), we have*

$$\mathbf{E}\langle (R_{1,2}^{p-1} - q^{p-1})^{2l} \rangle \leq \left( \frac{\hat{L}l}{N} \right)^l.$$

**Proof:** This result is an obvious consequence of Theorem 3.4. Indeed,

$$\begin{aligned} \mathbf{E}\langle (R_{1,2}^{p-1} - q^{p-1})^{2l} \rangle &\leq \mathbf{E}\left\langle (R_{1,2} - q)^{2l} \left[ \sum_{k=0}^{p-2} R_{1,2}^{p-2-k} q^k \right]^{2l} \right\rangle \\ &\leq (p-1)^{2l} \mathbf{E}\langle (R_{1,2} - q)^{2l} \rangle \\ &\leq \left( \frac{L(p-1)^{2l}}{N} \right)^l, \end{aligned}$$

which is the desired result.  $\square$

Another immediate Corollary of the exponential inequalities for the overlap is a useful result on the expansions of  $\nu(f)$ , that we label for further use.

**Corollary 3.8** *If  $\beta$  satisfies (H) we have, for a function  $f$  on  $\Sigma_N^r$  and  $k \geq 1$ ,*

$$|\nu(f) - \nu_{k,0}(f)| \leq \frac{K}{N^{\frac{1}{2}}} \nu^{\frac{1}{2}}(f^2), \quad (25)$$

$$|\nu(f) - \nu_{k,0}(f) - \nu'_{k,0}(f)| \leq \frac{K}{N} \nu^{\frac{1}{2}}(f^2). \quad (26)$$

**Proof:** We refer to [9] for the proof of this corollary.  $\square$

### 3.3 Upper bound for $\nu(R_{1,2} - q)$

The main goal of this part of the paper will be to prove the following Theorem, that gives a sharp rate of convergence of  $R_{1,2}$  towards  $q$ .

**Theorem 3.9** *Let  $q$  be the solution of (12). Then, if  $\beta$  satisfies (H), we have*

$$|\nu(R_{1,2} - q)| \leq \frac{K}{N}.$$

An immediate consequence of this theorem is the following result.

**Corollary 3.10** *Let  $q$  be the solution of (12). If  $\beta$  satisfies (H), we have*

$$|\nu(R_{1,2}^m - q^m)| \leq \frac{K(m)}{N},$$

for all  $m \geq 1$ .

**Proof:** For a fixed  $m \geq 1$ , by Taylor's expansion, we have

$$R_{1,2}^m = q^m + m q^{m-1}(R_{1,2} - q) + \frac{m(m-1)}{2} \xi^{m-2}(R_{1,2} - q)^2, \quad (27)$$

where  $\xi \in (R_{1,2} \wedge q, R_{1,2} \vee q)$ . So

$$\nu(R_{1,2}^m - q^m) = m q^{m-1} \nu(R_{1,2} - q) + \frac{m(m-1)}{2} \nu(\xi^{m-2}(R_{1,2} - q)^2).$$

Since  $|\xi| \leq 1$  and using Theorem 3.9 and Proposition 3.3 we obtain

$$|\nu(R_{1,2}^m - q^m)| \leq \frac{K(m)}{N}.$$

□

Conversely, the next proposition will show that, in order to prove Theorem 3.9, it will be enough to establish the following upper bound:

$$\nu(R_{1,2}^{p-1} - q^{p-1}) \leq \frac{K}{N}. \quad (28)$$

**Proposition 3.11** *For  $N$  large enough, there exist positive constants  $L_3$  and  $L_4$  such that, for any  $m \geq 1$ ,*

$$|\nu(R_{1,2} - q)| \leq L_3 |\nu(R_{1,2}^m - q^m)| + \frac{L_4}{N}.$$

**Proof:** By (27), and since  $q$  is a strictly positive number, we have

$$(R_{1,2} - q) = \frac{1}{mq^{m-1}}(R_{1,2}^m - q^m) - \frac{m-1}{2q^{m-1}} \xi^{m-2}(R_{1,2} - q)^2,$$

where  $\xi \in (R_{1,2} \wedge q, R_{1,2} \vee q)$ . Using Proposition 3.3 we can bound  $\nu[(R_{1,2} - q)^2]$  by  $\frac{L}{N}$ , finishing the proof. □

We will now prepare the proof of (28) by a series of lemmas, beginning with some deterministic estimates for the overlap.

**Lemma 3.12**

$$R_{1,2}^{p-1} = \frac{(p-1)!}{N^{p-1}} \sum_{\hat{j} \in A_N^{p-1}} \eta_j^1 \eta_j^2 + \mathcal{O}\left(\frac{1}{N}\right) \quad (29)$$

$$= \frac{(p-1)!}{N^{p-1}} \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} \eta_j^1 \eta_j^2 + \mathcal{O}\left(\frac{1}{N}\right), \quad (30)$$

**Proof:** Let  $N_{p-1} = \{(i_1, \dots, i_{p-1}) \in \{1, \dots, N\}^{p-1}\}$ . We can easily check

$$\begin{aligned}
R_{1,2}^{p-1} &= \frac{1}{N^{p-1}} \sum_{(i_1, \dots, i_{p-1}) \in N_{p-1}} \sigma_{i_1}^1 \cdots \sigma_{i_{p-1}}^1 \sigma_{i_1}^2 \cdots \sigma_{i_{p-1}}^2 \quad (31) \\
&= \frac{1}{N^{p-1}} \left[ \sum_{(i_1, \dots, i_{p-1}) \in \bar{N}_{p-1}} \sigma_{i_1}^1 \cdots \sigma_{i_{p-1}}^1 \sigma_{i_1}^2 \cdots \sigma_{i_{p-1}}^2 \right. \\
&\quad \left. + \sum_{(i_1, \dots, i_{p-1}) \in \bar{N}_{p-1}^c} \sigma_{i_1}^1 \cdots \sigma_{i_{p-1}}^1 \sigma_{i_1}^2 \cdots \sigma_{i_{p-1}}^2 \right],
\end{aligned}$$

where  $\bar{N}_{p-1}$  is the set of elements  $(i_1, \dots, i_{p-1})$  belonging to  $N_{p-1}$  such that all the elements  $i_1, \dots, i_{p-1}$  are different and  $\bar{N}_{p-1}^c$  is the complementary set of  $\bar{N}_{p-1}$  with respect to  $N_{p-1}$ , that means, the set of elements  $(i_1, \dots, i_{p-1})$  belonging to  $N_{p-1}$  such that at least two values  $i_k, i_{k'}, k \neq k'$ , are equal. Then, Lemma 8.6 imply (29). Moreover, the equality

$$|A_N^{p-1} \cap (A_{N-(p-1)}^{p-1})^c| = P_{p-2}(N)$$

gives us (30).  $\square$

**Corollary 3.13**

$$u_N^2 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} \eta_j^1 \eta_j^2 = \frac{p}{2} R_{1,2}^{p-1} + \mathcal{O}\left(\frac{1}{N}\right).$$

**Proof:** Trivial from the definition of  $u_N$ .  $\square$

**Lemma 3.14** *Let  $f$  be a function from  $\Sigma_N^n$  to  $\mathbb{R}$ ,  $k$  a positive integer, and  $t \in [0, 1]$ . Then*

$$\begin{aligned}
&\left| u_N^2 \sum_{J \in Q_{N,p-1}^p} \nu_{k,t} \left( f(\eta_J^1 \eta_J^2 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^2 \right) \right. \\
&\quad \left. - u_N^2 \sum_{l=1}^p \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} \nu_{k,t} \left( f(\eta_j^1 \eta_j^2 - q^{p-1}) \varepsilon_l^1 \varepsilon_l^2 \right) \right| \leq \frac{K}{N} (\nu_{k,t}(f^2))^{\frac{1}{2}}.
\end{aligned}$$

**Proof:** Using Lemma 8.4 we have

$$u_N^2 |\bar{Q}_{N,p-1}^p| \leq \frac{K}{N}.$$

Then, the decomposition of the set  $Q_{N,p-1}^p$  given in (8) (see also Definition 8.3) yields

$$\begin{aligned} u_N^2 \sum_{J \in Q_{N,p-1}^p} \nu_{k,t} \left( f(\eta_J^1 \eta_J^2 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^2 \right) \\ = u_N^2 \sum_{J \in \tilde{Q}_{N,p-1}^p} \nu_{k,t} \left( f(\eta_J^1 \eta_J^2 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^2 \right) + \frac{K}{N} (\nu_{k,t}(f^2))^{\frac{1}{2}}. \end{aligned}$$

Note that we can write

$$\tilde{Q}_{N,p-1}^p = A_{N-(p-1)}^{p-1} \times \{\varepsilon_1, \dots, \varepsilon_k\}.$$

For  $J \in \tilde{Q}_{N,p-1}$ ,  $\eta_J$  only depends on  $A_{N-(p-1)}^{p-1}$ , and  $\varepsilon_J$  is of the form  $\varepsilon_l$  for  $l \leq k$ . So we can write  $\eta_J = \eta_{\hat{J}}$  for  $\hat{J} \in A_{N-(p-1)}^{p-1}$  instead of  $\eta_J$  for  $J \in \tilde{Q}_{N,p-1}$ . Then, using

$$\begin{aligned} \sum_{J \in \tilde{Q}_{N,p-1}^p} \nu_{k,t} \left( f(\eta_J^1 \eta_J^2 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^2 \right) \\ = \sum_{l=1}^p \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu_{k,t} \left( f(\eta_{\hat{J}}^1 \eta_{\hat{J}}^2 - q^{p-1}) \varepsilon_l^1 \varepsilon_l^2 \right), \end{aligned}$$

the proof is completed.  $\square$

**Theorem 3.15** *Let  $q$  be the solution to (12). If  $\beta$  satisfies (H), we have*

$$|\nu(R_{1,2}^{p-1} - q^{p-1})| \leq \frac{K}{N}. \quad (32)$$

**Proof:** Using (13), the symmetry among sites Lemma 8.4 we get

$$\begin{aligned} \nu(R_{1,2}^{p-1} - q^{p-1}) &= \nu \left( \frac{2}{p} u_N^2 \sum_{J \in Q_{N,1}^p} \eta_J^1 \eta_J^2 - q^{p-1} \right) + \mathcal{O} \left( \frac{1}{N} \right) \\ &= \nu \left( \frac{2}{p} u_N^2 |Q_{N,1}^p| \varepsilon_1^1 \cdots \varepsilon_{p-1}^1 \varepsilon_1^2 \cdots \varepsilon_{p-1}^2 - q^{p-1} \right) + \mathcal{O} \left( \frac{1}{N} \right) \\ &= \nu(\bar{f}) + \mathcal{O} \left( \frac{1}{N} \right), \end{aligned}$$

where

$$\bar{f} = \prod_{j \leq p-1} \varepsilon_j^1 \varepsilon_j^2 - q^{p-1}.$$

The estimate (26) for  $k = p - 1$  yields

$$|\nu(\bar{f}) - \nu_{p-1,0}(\bar{f}) - \nu'_{p-1,0}(\bar{f})| \leq \frac{K}{N} \nu^{\frac{1}{2}}(\bar{f}^2). \quad (33)$$

Moreover, Lemma 2.3 and the equation satisfied by  $q$  imply

$$\begin{aligned} \nu_{p-1,0}(\bar{f}) &= \prod_{j \leq p-1} \mathbf{E} \left[ \tanh^2 \left( \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right) \right] - q^{p-1} + \mathcal{O}\left(\frac{1}{N}\right) \\ &= q^{p-1} - q^{p-1} + \mathcal{O}\left(\frac{1}{N}\right) \\ &= \mathcal{O}\left(\frac{1}{N}\right). \end{aligned} \quad (34)$$

Let us now study  $\nu'_{p-1,0}(\bar{f})$ . Applying (5) for  $k = p - 1$ , Lemma 3.14 and the symmetry property among the  $\varepsilon_j$ , we get

$$\begin{aligned} \nu'_{p-1,0}(\bar{f}) &= \beta^2 u_N^2 \sum_{J \in Q_{N,p-1}^p} \left[ \nu_{p-1,0}(\bar{f}(\eta_J^1 \eta_J^2 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^2) \right. \\ &\quad - 4\nu_{p-1,0}(\bar{f}(\eta_J^1 \eta_J^3 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^3) \\ &\quad \left. + 3\nu_{p-1,0}(\bar{f}(\eta_J^3 \eta_J^4 - q^{p-1}) \varepsilon_J^3 \varepsilon_J^4) \right] \\ &= \beta^2 u_N^2 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} \sum_{l=1}^{p-1} \left[ \nu_{p-1,0}(\bar{f}(\eta_j^1 \eta_j^2 - q^{p-1}) \varepsilon_l^1 \varepsilon_l^2) \right. \\ &\quad - 4\nu_{p-1,0}(\bar{f}(\eta_j^1 \eta_j^3 - q^{p-1}) \varepsilon_l^1 \varepsilon_l^3) \\ &\quad \left. + 3\nu_{p-1,0}(\bar{f}(\eta_j^3 \eta_j^4 - q^{p-1}) \varepsilon_l^3 \varepsilon_l^4) \right] + \mathcal{O}\left(\frac{1}{N}\right) \\ &= \beta^2 u_N^2 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (p-1) [W_1 - 4W_2 + 3W_3] + \mathcal{O}\left(\frac{1}{N}\right), \end{aligned}$$

with

$$\begin{aligned} W_1 &= \nu_{p-1,0}(\bar{f}(\eta_j^1 \eta_j^2 - q^{p-1}) \varepsilon_{p-1}^1 \varepsilon_{p-1}^2), \\ W_2 &= \nu_{p-1,0}(\bar{f}(\eta_j^1 \eta_j^3 - q^{p-1}) \varepsilon_{p-1}^1 \varepsilon_{p-1}^3), \\ W_3 &= \nu_{p-1,0}(\bar{f}(\eta_j^3 \eta_j^4 - q^{p-1}) \varepsilon_{p-1}^3 \varepsilon_{p-1}^4). \end{aligned}$$

By means of independence we obtain

$$\nu'_{p-1,0}(\bar{f}) = \beta^2 u_N^2 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (p-1)[V_1 - 4V_2 + 3V_3] + \mathcal{O}\left(\frac{1}{N}\right),$$

with

$$\begin{aligned} V_1 &= \nu_{p-1,0}(\eta_j^1 \eta_j^2 - q^{p-1}) \left[ \nu_{p-1,0} \left( \prod_{j \leq p-2} \varepsilon_j^1 \varepsilon_j^2 \right) - q^{p-1} \nu_{p-1,0}(\varepsilon_{p-1}^1 \varepsilon_{p-1}^2) \right], \\ V_2 &= \nu_{p-1,0}(\eta_j^1 \eta_j^3 - q^{p-1}) \left[ \nu_{p-1,0} \left( \left( \prod_{j \leq p-2} \varepsilon_j^1 \varepsilon_j^2 \right) \varepsilon_{p-1}^2 \varepsilon_{p-1}^3 \right) \right. \\ &\quad \left. - q^{p-1} \nu_{p-1,0}(\varepsilon_{p-1}^1 \varepsilon_{p-1}^3) \right], \\ V_3 &= \nu_{p-1,0}(\eta_j^3 \eta_j^4 - q^{p-1}) \left[ \nu_{p-1,0} \left( \left( \prod_{j \leq p-1} \varepsilon_j^1 \varepsilon_j^2 \right) \varepsilon_{p-1}^3 \varepsilon_{p-1}^4 \right) \right. \\ &\quad \left. - q^{p-1} \nu_{p-1,0}(\varepsilon_{p-1}^3 \varepsilon_{p-1}^4) \right]. \end{aligned}$$

Now, clearly,

$$\nu_{p-1,0}(\varepsilon_{p-1}^1 \varepsilon_{p-1}^2) - 4\nu_{p-1,0}(\varepsilon_{p-1}^1 \varepsilon_{p-1}^3) + 3\nu_{p-1,0}(\varepsilon_{p-1}^3 \varepsilon_{p-1}^4) = 0. \quad (35)$$

So, using Lemma 2.3 together with (35) we have

$$\begin{aligned} \nu'_{p-1,0}(\bar{f}) &= \beta^2 u_N^2 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (p-1) \nu_{p-1,0}(\eta_j^1 \eta_j^2 - q^{p-1}) \\ &\quad \times \left[ q^{p-2}(1 - 4q + 3\hat{q}_4) \right] + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned}$$

On the other hand, Lemma 8.2 implies

$$u_N^2 |A_{N-(p-1)}^{p-1}| = \frac{p}{2} + \mathcal{O}\left(\frac{1}{N}\right).$$

Then, Corollary 3.13 gives us

$$\nu'_{p-1,0}(\bar{f}) = \frac{\beta^2 p(p-1)}{2} q^{p-2} (1 - 4q + 3\hat{q}_4) \nu_{p-1,0}(R_{1,2}^{p-1} - q^{p-1}) + \mathcal{O}\left(\frac{1}{N}\right). \quad (36)$$

Invoking inequalities (33), (34) and (36), we now get that

$$\left| \nu(R_{1,2}^{p-1} - q^{p-1}) - \frac{\beta^2 p(p-1)}{2} \right. \\ \left. \times q^{p-2} [1 - 4q + 3\hat{q}_4] \nu_{p-1,0}(R_{1,2}^{p-1} - q^{p-1}) \right| \leq \frac{K}{N}. \quad (37)$$

Finally, using (25) and Corollary 3.7 for  $l = 1$ , we have

$$|\nu(R_{1,2}^{p-1} - q^{p-1}) - \nu_{p-1,0}(R_{1,2}^{p-1} - q^{p-1})| \leq \frac{K}{N}. \quad (38)$$

Then, (37) and (38) ensure that for  $\beta \leq \beta_p$ , (32) is satisfied.  $\square$

## 4 Study of the free energy

Set

$$p_N(\beta, h, p) = \frac{1}{N} \mathbf{E}[\log Z_N(\beta, h)] = \frac{1}{N} \mathbf{E} \left[ \log \left( \sum_{\sigma \in \Sigma_N} \exp(-H_{N,\beta,h}(\sigma)) \right) \right].$$

This quantity is the *expected density of the logarithm of the partition function*, and sometimes, we will write  $p_N$  instead of  $p_N(\beta, h, p)$ . The quantity  $p_N$  is closely related to the *free energy* considered by physicists, up to a scaling factor, and we will call it the free energy of our system by usual analogy.

The main aim of this section is to prove the following result.

**Theorem 4.1** *If  $\beta$  satisfies condition (H), we have*

$$\lim_{N \uparrow \infty} p_N(\beta, h, p) = \frac{\beta^2}{4} [1 - pq^{p-1} + (p-1)q^p] \\ + \log 2 + \mathbf{E} \left[ \log \cosh \left[ \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right],$$

where  $Y$  is a standard Gaussian random variable and  $q$  is the unique solution to the equation (12).

Before proving Theorem 4.1, we will need to introduce some notation and to prove some preliminary results: consider the function  $F : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \times \mathbb{N}^*$

defined by

$$F(\beta, h, q, p) = \frac{\beta^2}{4} [1 - pq^{p-1} + (p-1)q^p] + \log 2 + \mathbf{E} \left[ \log \cosh \left[ \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right].$$

We set

$$\Phi(\beta, h, p) = F(\beta, h, q, p),$$

where  $q$  satisfies (12).

**Lemma 4.2** *We have the following two facts*

$$q \text{ is solution of (12)} \Rightarrow \frac{\partial F}{\partial q}(\beta, h, q, p) = 0, \quad (39)$$

$$\frac{\partial \Phi}{\partial \beta}(\beta, h, p) = \frac{\beta}{2}(1 - q^p). \quad (40)$$

**Proof:** We first prove (39). Using integration by parts formula and (12) we obtain

$$\begin{aligned} \frac{\partial F}{\partial q}(\beta, h, q, p) &= \frac{\beta^2}{4} [-p(p-1)q^{p-2} + p(p-1)q^{p-1}] \\ &\quad + \beta \frac{p-1}{2} \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-3}{2}} \mathbf{E} \left[ Y \tanh \left[ \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right] \\ &= \frac{\beta^2}{4} [-p(p-1)q^{p-2} + p(p-1)q^{p-1}] \\ &\quad + \beta^2 \frac{p(p-1)}{4} q^{p-2} \mathbf{E} \left[ \cosh^{-2} \left[ \beta \left( \frac{p}{2} \right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right] \\ &= 0, \end{aligned}$$

which proves (39). We now show (40). This previous result together with integration by parts and (12) yield

$$\begin{aligned}
\frac{\partial \Phi}{\partial \beta}(\beta, h, p) &= \frac{\partial F}{\partial \beta}(\beta, h, q(\beta, h, p), p) + \frac{\partial F}{\partial q}(\beta, h, q(\beta, h, p), p) \frac{\partial q}{\partial \beta}(\beta, h, p) \\
&= \frac{\beta}{2} [1 - pq^{p-1} + (p-1)q^p] \\
&\quad + \left(\frac{p}{2}\right)^{\frac{1}{2}} q^{\frac{p-1}{2}} \mathbf{E} \left[ Y \tanh \left[ \beta \left(\frac{p}{2}\right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right] \\
&= \frac{\beta}{2} [1 - pq^{p-1} + (p-1)q^p] \\
&\quad + \beta \frac{p}{2} q^{p-1} \mathbf{E} \left[ \cosh^{-2} \left[ \beta \left(\frac{p}{2}\right)^{\frac{1}{2}} q^{\frac{p-1}{2}} Y + h \right] \right] \\
&= \frac{\beta}{2} (1 - q^p).
\end{aligned}$$

□

The following result, that relates the free energy and the overlap, has been proved by Talagrand [7, Proposition 2.9].

**Lemma 4.3** *We have*

$$\left| \mathbf{E} \left[ \frac{1}{N} \frac{\partial \log Z_N}{\partial \beta} \right] - \frac{\beta}{2} [1 - \mathbf{E} \langle R_{1,2}^p \rangle] \right| \leq \frac{K}{N}.$$

Now we are going to prove the following theorem which implies Theorem 4.1.

**Theorem 4.4** *Whenever  $\beta$  satisfies (H), we have*

$$|p_N(\beta, h, p) - \Phi(\beta, h, p)| \leq \frac{K}{N}.$$

**Proof:** We only need to prove that  $p_N(0, h, p) = \Phi(0, h, p)$  and

$$\left| \frac{\partial p_N}{\partial \beta}(\beta, h, p) - \frac{\partial \Phi}{\partial \beta}(\beta, h, p) \right| \leq \frac{K}{N}, \tag{41}$$

for any  $\beta \leq \beta_p$ . For the case  $\beta = 0$ , it is obvious since  $p_N(0, h, p) = \Phi(0, h, p) = \log(2 \cosh h)$ . On the other hand, Lemmas 4.2, 4.3 and Corollary 3.10 imply (41). □

## 5 Almeida-Thouless Theorem

In this section we prove a result given in [2] for the Sherrington-Kirkpatrick model. Since the quantity  $\Delta_{p-1}^2$  defined above is almost surely positive, it gives a straightforward condition on  $\beta$  for the self averaging to hold, namely that

$$1 - \frac{p(p-1)}{2}q^{p-2}(1-2q+\hat{q}_4)\beta^2 > 0. \quad (42)$$

In fact, this inequality should give the physical limit of the high temperature region.

**Proposition 5.1** *If  $\beta$  satisfies (H), we have*

$$\left| \nu\left(\Delta_{p-1}^2\right) - \frac{4(p-1)^2q^{2(p-2)}(1-2q+\hat{q}_4)}{N\left(1 - \frac{p(p-1)}{2}q^{p-2}(1-2q+\hat{q}_4)\beta^2\right)} \right| \leq \frac{K}{N^{\frac{3}{2}}},$$

where

$$\Delta_{p-1} = R_{1,3}^{p-1} - R_{1,4}^{p-1} - R_{2,3}^{p-1} + R_{2,4}^{p-1}.$$

**Remark 5.2** *Denote by  $\beta_{at}$  the limit of the region defined by (42), that is*

$$\beta_{at} = \beta_{at}(p) = \frac{2}{p(p-1)q^{p-2}(1-2q+\hat{q}_4)} = \frac{2}{p(p-1)q^{p-2}E[\cosh^{-4}(Z)]},$$

where

$$Z = \beta\left(\frac{p}{2}\right)^{\frac{1}{2}}q^{\frac{p-1}{2}}Y + h.$$

Then, since  $q$  tends to  $\tanh^2(h)$  as  $p \rightarrow \infty$ , the exponential decay of  $q^{p-2}$  implies that

$$\lim_{p \rightarrow \infty} \beta_{at}(p) = \infty.$$

**Proof of Proposition 5.1:** Invoking (13) it follows that

$$\nu\left(\Delta_{p-1}^2\right) = \frac{2}{p}\nu\left(\left[u_N^2 \sum_{J \in Q_{N,1}^p} (\eta_J^1 - \eta_J^2)(\eta_J^3 - \eta_J^4) + V_N\right] \Delta_{p-1}\right),$$

with

$$|V_N| \leq \frac{K}{N}. \quad (43)$$

The symmetry property implies now that

$$\nu\left(\Delta_{p-1}^2\right) = \frac{2}{p}\nu\left([u_N^2|Q_{N,1}^p|(\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) + V_N] \Delta_{p-1}\right),$$

where we have used the notation  $\varepsilon^l = \varepsilon_1^l \times \cdots \times \varepsilon_{p-1}^l$  for  $l = 1, 2, 3, 4$ .

First of all we will check that  $V_N$  gives raise to a negligible term: indeed, Corollary 3.7 yields

$$\nu\left(\Delta_{p-1}^2\right) \leq \frac{K}{N}. \quad (44)$$

Then, estimates (43), (44) together with Lemma 8.4 give

$$\nu\left(\Delta_{p-1}^2\right) = \nu\left((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\Delta_{p-1}\right) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right).$$

For  $(l, l') \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ , we will now decompose  $R_{l, l'}$  into a part involving only the  $N - (p - 1)$  first spins on one hand, and a remaining term on the other hand, as follows:

$$R_{l, l'}^{p-1} = \left[ R_{l, l'}^* + \frac{1}{N} \sum_{j=1}^{p-1} \varepsilon_j^l \varepsilon_j^{l'} \right]^{p-1} = \sum_{k=0}^{p-1} \binom{p-1}{k} (R_{l, l'}^*)^k \left[ \frac{1}{N} \sum_{j=1}^{p-1} \varepsilon_j^l \varepsilon_j^{l'} \right]^{p-1-k},$$

where

$$R_{l, l'}^* = \frac{1}{N} \sum_{j=1}^{N-(p-1)} \sigma_j^l \sigma_j^{l'}.$$

Now, set

$$\Delta_{p-1}^* = (R_{1,3}^*)^{p-1} - (R_{1,4}^*)^{p-1} - (R_{2,3}^*)^{p-1} + (R_{2,4}^*)^{p-1}.$$

Using this decomposition we have

$$\nu\left(\Delta_{p-1}^2\right) = \nu\left((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \left[ \Delta_{p-1}^* + \sum_{k=0}^{p-2} \binom{p-1}{k} y_k \right]\right) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right), \quad (45)$$

with

$$\begin{aligned} y_k &= (R_{1,3}^*)^k \left[ \frac{1}{N} \sum_{j=1}^{p-1} \varepsilon_j^1 \varepsilon_j^3 \right]^{p-1-k} - (R_{1,4}^*)^k \left[ \frac{1}{N} \sum_{j=1}^{p-1} \varepsilon_j^1 \varepsilon_j^4 \right]^{p-1-k} \\ &\quad - (R_{2,3}^*)^k \left[ \frac{1}{N} \sum_{j=1}^{p-1} \varepsilon_j^2 \varepsilon_j^3 \right]^{p-1-k} + (R_{2,4}^*)^k \left[ \frac{1}{N} \sum_{j=1}^{p-1} \varepsilon_j^2 \varepsilon_j^4 \right]^{p-1-k}. \end{aligned}$$

We will deal now with the different terms appearing in (45) separately, and we start with all the terms containing  $y_k$ . By the construction of  $y_k$  it is easily checked that for any  $k \in \{0, \dots, p-2\}$ ,

$$|y_k| \leq \frac{K}{N^{p-1-k}}.$$

These bounds and inequality (25) in Corollary 3.8 yield, for any  $k \in \{0, \dots, p-2\}$ ,

$$\left| \nu \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)y_k \right) - \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)y_k \right) \right| \leq \frac{K}{N^{\frac{3}{2}}}. \quad (46)$$

Hence, we are reduced to study terms of the form  $\nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)y_k)$ . Since for any  $k \in \{0, \dots, p-3\}$ ,

$$\left| \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)y_k \right) \right| \leq \frac{K}{N^2},$$

we only have to deal with the term involving  $y_{p-2}$ . But the definition of  $y_{p-2}$  implies that

$$\begin{aligned} \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)y_{p-2} \right) &= \frac{1}{N} \sum_{j=1}^{p-1} \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \right. \\ &\quad \left. \times \left[ (R_{1,3}^*)^{p-2} \varepsilon_j^1 \varepsilon_j^3 - (R_{1,4}^*)^{p-2} \varepsilon_j^1 \varepsilon_j^4 - (R_{2,3}^*)^{p-2} \varepsilon_j^2 \varepsilon_j^3 + (R_{2,4}^*)^{p-2} \varepsilon_j^2 \varepsilon_j^4 \right] \right). \end{aligned} \quad (47)$$

Observe now the first term of the right hand side of (47). Using the independence, the meaning of  $\varepsilon^l$  and Lemma 2.3, we obtain

$$\begin{aligned} &\nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)(R_{1,3}^*)^{p-2} \varepsilon_j^1 \varepsilon_j^3 \right) \\ &= \nu_{p-1,0} \left( (R_{1,3}^*)^{p-2} \right) \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \varepsilon_{p-1}^1 \varepsilon_{p-1}^3 \right) \\ &= \nu_{p-1,0} \left( (R_{1,3}^*)^{p-2} \right) \\ &\quad \times \nu_{p-1,0} \left( \prod_{j=1}^{p-2} \varepsilon_j^1 \varepsilon_j^3 - 2 \left[ \prod_{j=1}^{p-2} \varepsilon_j^1 \varepsilon_j^4 \right] \varepsilon_{p-1}^3 \varepsilon_{p-1}^4 + \varepsilon^2 \varepsilon^4 \varepsilon_{p-1}^1 \varepsilon_{p-1}^3 \right) \\ &= \nu_{p-1,0} \left( (R_{1,3}^*)^{p-2} \right) q^{p-2} (1 - 2q + \hat{q}_4) + \mathcal{O} \left( \frac{1}{N} \right). \end{aligned} \quad (48)$$

Let us study now  $\nu_{p-1,0} \left( (R_{1,3}^*)^{p-2} \right)$ . The inequality (25) in Corollary 3.8 yields

$$\left| \nu \left( (R_{1,3}^*)^{p-2} \right) - \nu_{p-1,0} \left( (R_{1,3}^*)^{p-2} \right) \right| \leq \frac{K}{N^{\frac{1}{2}}}, \quad (49)$$

the definition of  $R_{1,3}^*$  implies

$$\left| (R_{1,3}^*)^{p-2} - R_{1,3}^{p-2} \right| \leq \frac{K}{N}, \quad (50)$$

and on the other hand, Corollary 3.10 gives us

$$\left| \nu \left( R_{1,3}^{p-2} - q^{p-2} \right) \right| \leq \frac{K}{N}. \quad (51)$$

So, putting together (49)-(51) we have

$$\nu_{p-1,0} \left( (R_{1,3}^*)^{p-2} \right) = q^{p-2} + \mathcal{O} \left( \frac{1}{N^{\frac{1}{2}}} \right). \quad (52)$$

The other terms in (47) can be studied in a similar way. Then, putting together (46), (47), (48) and (52) we easily get

$$\begin{aligned} \nu \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \sum_{k=0}^{p-2} \binom{p-1}{k} y_k \right) \\ = \frac{4(p-1)^2 q^{2(p-2)} (1 - 2q + \hat{q}_4)}{N} + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right). \end{aligned} \quad (53)$$

Now we will deal with the term containing  $\Delta_{p-1}^*$  in (45), that means with

$$\nu \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \Delta_{p-1}^* \right).$$

To that purpose, we will use (26) in Corollary 3.8. First of all, note that

$$|\Delta_{p-1} - \Delta_{p-1}^*| \leq \frac{K}{N}. \quad (54)$$

Then estimate (44) yields

$$\nu \left( (\Delta_{p-1}^*)^2 \right) \leq \frac{K}{N}. \quad (55)$$

Since the independence ensures

$$\nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \Delta_{p-1}^* \right) = \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \right) \nu_{p-1,0} \left( \Delta_{p-1}^* \right) = 0, \quad (56)$$

we have, using (55)

$$\nu \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \Delta_{p-1}^* \right) = \nu'_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \Delta_{p-1}^* \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right). \quad (57)$$

Moreover, Proposition 2.1 yields

$$\nu'_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \Delta_{p-1}^* \right) = \beta^2 u_N^2 \sum_{J \in Q_{N,p-1}^p} (D_1^J + D_2^J + D_3^J), \quad (58)$$

with

$$\begin{aligned}
D_1^J &= \nu_{p-1,0} \left( \Delta_{p-1}^* (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \sum_{1 \leq l < l' \leq 4} (\eta_J^l \eta_J^{l'} - q^{p-1}) \varepsilon_J^l \varepsilon_J^{l'} \right), \\
D_2^J &= -4 \nu_{p-1,0} \left( \Delta_{p-1}^* (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \sum_{l \leq 4} (\eta_J^l \eta_J^5 - q^{p-1}) \varepsilon_J^l \varepsilon_J^5 \right), \\
D_3^J &= 10 \nu_{p-1,0} \left( \Delta_{p-1}^* (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) (\eta_J^5 \eta_J^6 - q^{p-1}) \varepsilon_J^5 \varepsilon_J^6 \right).
\end{aligned}$$

We can check that

$$D_2^J \equiv 0, \quad \text{for any } J \in Q_{N,p-1}^{p-1}.$$

Indeed, for instance, when  $l = 1$ , we get that

$$\begin{aligned}
&\nu_{p-1,0} \left( \Delta_{p-1}^* (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) (\eta_J^1 \eta_J^5 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^5 \right) \\
&= \nu_{p-1,0} \left( \Delta_{p-1}^* (\eta_J^1 \eta_J^5 - q^{p-1}) \right) \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \varepsilon_J^1 \varepsilon_J^5 \right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \varepsilon_J^1 \varepsilon_J^5 \right) \\
&= \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) \varepsilon^3 \varepsilon_J^1 \varepsilon_J^5 \right) - \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) \varepsilon^4 \varepsilon_J^1 \varepsilon_J^5 \right) = 0.
\end{aligned}$$

This kind of argument, that will be repeated all along the remainder of the paper, will be referred to as *symmetry among the different copies of  $G_N$* . Now the cases  $l = 2, 3, 4$  in  $D_2^J$  can be studied with the same method, and furthermore, by similar arguments,

$$D_3^J \equiv 0, \quad \text{for any } J \in Q_{N,p-1}^{p-1}.$$

Thus, it only remains to deal with  $D_1^J$ . The summatory of  $D_1^J$  contains the couples

$$(l, l') \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

We start studying the couple (1, 2). Along the same lines as before, using independence and symmetry between copies of  $G_N$ , we easily get that

$$\begin{aligned}
&\nu_{p-1,0} \left( \Delta_{p-1}^* (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) (\eta_J^1 \eta_J^2 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^2 \right) \\
&= \nu_{p-1,0} \left( \Delta_{p-1}^* (\eta_J^1 \eta_J^2 - q^{p-1}) \right) \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \varepsilon_J^1 \varepsilon_J^2 \right) \\
&= 0,
\end{aligned}$$

and the same argument can be applied to (3, 4). Consider now the couple (1, 3) and the summatory of  $\sum_{J \in Q_{N,p-1}^p} D_1^J$ . Lemma 3.14, (55), the properties of independence and symmetry among all the  $\varepsilon_j$  imply

$$\begin{aligned}
& u_N^2 \sum_{J \in Q_{N,p-1}^p} \nu_{p-1,0} \left( \Delta_{p-1}^* (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) (\eta_J^1 \eta_J^3 - q^{p-1}) \varepsilon_J^1 \varepsilon_J^3 \right), \\
&= u_N^2 \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \sum_{l=1}^{p-1} \nu_{p-1,0} \left( \Delta_{p-1}^* (\eta_{\hat{J}}^1 \eta_{\hat{J}}^3 - q^{p-1}) \right) \\
&\quad \times \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \varepsilon_l^1 \varepsilon_l^3 \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right), \\
&= u_N^2 \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} (p-1) \nu_{p-1,0} \left( \Delta_{p-1}^* (\eta_{\hat{J}}^1 \eta_{\hat{J}}^3 - q^{p-1}) \right) \\
&\quad \times \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \varepsilon_{p-1}^1 \varepsilon_{p-1}^3 \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right).
\end{aligned}$$

Notice also that by means of Lemma 2.3, we have

$$\nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) (\varepsilon^3 - \varepsilon^4) \varepsilon_{p-1}^1 \varepsilon_{p-1}^3 \right) = q^{p-2} (1 - 2q + \hat{q}_4) + \mathcal{O} \left( \frac{1}{N} \right).$$

Let us handle now the remaining couples in  $D_1^J$ : it is easily seen that the couple (2, 4) has the same structure as (1, 3) and the couples (2, 3) and (1, 4) also have the same structure as (1, 3) but with the opposite sign. Plugging now (53), (56), (57), and the study of  $D_1^J, D_2^J, D_3^J$  in (58) into (45), we end up with

$$\begin{aligned}
\nu \left( \Delta_{p-1}^2 \right) &= \frac{4(p-1)^2 q^{2(p-2)} (1 - 2q + \hat{q}_4)}{N} + (p-1) q^{p-2} (1 - 2q + \hat{q}_4) \\
&\quad \times \beta^2 u_N^2 \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu_{p-1,0} \left( \Delta_{p-1}^* (\eta_{\hat{J}}^1 \eta_{\hat{J}}^3 - \eta_{\hat{J}}^1 \eta_{\hat{J}}^4 - \eta_{\hat{J}}^2 \eta_{\hat{J}}^3 + \eta_{\hat{J}}^2 \eta_{\hat{J}}^4) \right) \\
&\quad + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right).
\end{aligned}$$

Moreover, (54), Corollary 3.13 and estimate (44) imply that

$$\begin{aligned}
\nu \left( \Delta_{p-1}^2 \right) &= \frac{4(p-1)^2 q^{2(p-2)} (1 - 2q + \hat{q}_4)}{N} + \beta^2 \frac{p}{2} (p-1) q^{p-2} \\
&\quad \times (1 - 2q + \hat{q}_4) \nu_{p-1,0} \left( \Delta_{p-1}^2 \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right).
\end{aligned}$$

Finally, inequality (25) in Corollary 3.8 and an upper bound for  $\Delta_{p-1}^4$  obtained from Corollary 3.7, and similar to the one obtained at (44), ensure

$$\begin{aligned} \nu\left(\Delta_{p-1}^2\right) &= \frac{4(p-1)^2 q^{2(p-2)}(1-2q+\hat{q}_4)}{N} + \beta^2 \frac{p}{2}(p-1)q^{p-2} \\ &\quad \times (1-2q+\hat{q}_4)\nu\left(\Delta_{p-1}^2\right) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right). \end{aligned}$$

This last estimate immediately gives the desired result.  $\square$

## 6 Second Moment Computations

The results of this section will be crucial to obtain our Central Limit Theorems: in fact, the relations we will obtain in the next Section for any power  $k$  will be a mere elaboration of the ones obtained here for  $k = 2$ .

For any  $\hat{J} = (i_1, \dots, i_{p-1}) \in N_{p-1}$  (see Definition 8.5) we define  $\eta_j$  as

$$\eta_j = \prod_{j=1}^{p-1} \sigma_{i_j}. \quad (59)$$

Then set

$$T(\eta) = \{\eta_j; \hat{J} \in N_{p-1}\}$$

and

$$b = \langle T(\eta) \rangle = \left\{ \langle T(\eta_j) \rangle; \hat{J} \in N_{p-1} \right\}.$$

Let us define now

$$\begin{aligned} T_{l,l'} &= \frac{1}{N^{p-1}} (T(\eta^l) - b) \cdot (T(\eta^{l'}) - b) = \frac{1}{N^{p-1}} \sum_{\hat{J} \in N_{p-1}} (\eta_j^l - b_j)(\eta_j^{l'} - b_j) \\ T_l &= \frac{1}{N^{p-1}} (T(\eta^l) - b) \cdot b = \frac{1}{N^{p-1}} \sum_{\hat{J} \in N_{p-1}} (\eta_j^l - b_j)b_j \\ T &= \frac{b \cdot b}{N^{p-1}} - q^{p-1}. \end{aligned}$$

Equality (31) allows us to reconstruct the quantities  $R_{l,l'}^{p-1}$  by the formula

$$T_{l,l'} + T_l + T_{l'} + T = R_{l,l'}^{p-1} - q^{p-1}. \quad (61)$$

We finish this introduction with some more notation that we will use all along this section. First, set

$$\hat{T}(\eta^l) = \{\eta_j^l, \hat{J} \in A_N^{p-1}\}.$$

On the other hand, note that the set  $A_N^{p-1}$  can be decomposed into three disjoint sets as follows

$$A_N^{p-1} = A_{N-(p-1)}^{p-1} \cup \tilde{Q}_{N,p-1}^{p-1} \cup \bar{Q}_{N,p-1}^{p-1}. \quad (62)$$

We are now ready to estimate the second moment of  $T_{l,l'}$ ,  $T_l$  and  $T$ .

**Proposition 6.1** *If  $\beta$  satisfies (H), we have*

$$\left| \nu(T_{1,2}^2) - \frac{A^2}{N} \right| \leq \frac{K}{N^{\frac{3}{2}}} \quad (63)$$

where

$$A^2 = \frac{(p-1)^2 q^{2(p-1)} (1-2q+\hat{q}_4)}{1 - \frac{p(p-1)}{2} q^{p-2} (1-2q+\hat{q}_4) \beta^2}. \quad (64)$$

**Proof:** As in the proof of Lemma 3.12, we will decompose  $T_{1,2}$  into two terms

$$T_{1,2} = S_{1,2} + V_N^{1,2}, \quad (65)$$

with

$$\begin{aligned} S_{1,2} &= \frac{(p-1)!}{N^{p-1}} \sum_{\hat{J} \in A_N^{p-1}} (\eta_{\hat{J}}^1 - b_{\hat{J}})(\eta_{\hat{J}}^2 - b_{\hat{J}}), \\ V_N^{1,2} &= \frac{1}{N^{p-1}} \sum_{\hat{J} \in \bar{N}_{p-1}^c} (\eta_{\hat{J}}^1 - b_{\hat{J}})(\eta_{\hat{J}}^2 - b_{\hat{J}}), \end{aligned}$$

where  $\eta_{\hat{J}}$  for  $\hat{J} \in A_N^{p-1}$  or  $\hat{J} \in \bar{N}_{p-1}^c$  (see the Appendix) are defined as in (59). Then, it suffices to study  $\nu(S_{1,2}^2)$ . Indeed, from Lemma 8.6 we have

$$|V_N^{1,2}| \leq \frac{K}{N}. \quad (66)$$

On the other hand, using the symmetry property, we get

$$\begin{aligned} \nu(S_{1,2}^2) &= \left( \frac{(p-1)!}{N^{p-1}} \right)^2 \nu \left[ \left( \hat{T}(\eta^1) - \hat{T}(\eta^2) \right) \cdot \left( \hat{T}(\eta^3) - \hat{T}(\eta^4) \right) \right. \\ &\quad \left. \times \left( \hat{T}(\eta^1) - \hat{T}(\eta^5) \right) \cdot \left( \hat{T}(\eta^3) - \hat{T}(\eta^6) \right) \right] \\ &= \frac{[(p-1)!]^2}{N^{2(p-1)}} |A_N^{p-1}| \nu \left[ (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \left( \hat{T}(\eta^1) - \hat{T}(\eta^5) \right) \right. \\ &\quad \left. \cdot \left( \hat{T}(\eta^3) - \hat{T}(\eta^6) \right) \right]. \quad (67) \end{aligned}$$

So, Lemma 8.2, Corollary 3.8, Lemma 3.12 and Proposition 5.1 imply

$$\nu(S_{1,2}^2) = \mathcal{O}\left(\frac{1}{N}\right).$$

Then, we clearly have

$$\nu(T_{1,2}^2) = \nu(S_{1,2}^2) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right). \quad (68)$$

Let us study now  $\nu(S_{1,2}^2)$ . Relation (67), Lemma 8.2 and the decomposition of  $A_N^{p-1}$  (see (62)) allow us to write

$$\nu(S_{1,2}^2) = M_1 + M_2 + M_3 + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right), \quad (69)$$

with

$$\begin{aligned} M_1 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6) \right], \\ M_2 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \sum_{\hat{j} \in \bar{Q}_{N,p-1}^{p-1}} (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6) \right], \\ M_3 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \sum_{\hat{j} \in \bar{Q}_{N,p-1}^{p-1}} (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6) \right]. \end{aligned}$$

The term  $M_3$  is easily handled: Lemma 8.4 clearly yields

$$|M_3| \leq \frac{K}{N^2}. \quad (70)$$

We deal now with  $M_2$ . By symmetry, we have

$$\begin{aligned} M_2 &= \frac{(p-1)!}{N^{p-1}} (p-1) \nu \left[ (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \right. \\ &\quad \left. \times \sum_{\hat{j} \in A_{N-(p-1)}^{p-2}} (\eta_j^1 \varepsilon_{p-1}^1 - \eta_j^5 \varepsilon_{p-1}^5)(\eta_j^3 \varepsilon_{p-1}^3 - \eta_j^6 \varepsilon_{p-1}^6) \right]. \end{aligned}$$

Corollary 3.8 and the independence ensure that, up to a  $N^{-3/2}$  term, we have

$$\begin{aligned}
M_2 &= \frac{(p-1)!}{N^{p-1}}(p-1) \\
&\times \left[ \nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\varepsilon_{p-1}^1\varepsilon_{p-1}^3)\nu_{p-1,0}\left(\sum_{\hat{j} \in A_{N-(p-1)}^{p-2}} \eta_{\hat{j}}^1\eta_{\hat{j}}^3\right) \right. \\
&- 2\nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\varepsilon_{p-1}^1\varepsilon_{p-1}^6)\nu_{p-1,0}\left(\sum_{\hat{j} \in A_{N-(p-1)}^{p-2}} \eta_{\hat{j}}^1\eta_{\hat{j}}^6\right) \\
&\left. + \nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\varepsilon_{p-1}^5\varepsilon_{p-1}^6)\nu_{p-1,0}\left(\sum_{\hat{j} \in A_{N-(p-1)}^{p-2}} \eta_{\hat{j}}^5\eta_{\hat{j}}^6\right) \right]
\end{aligned}$$

Let us compute all the terms of the last equality: on one hand, Lemma 2.3 implies

$$\nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\varepsilon_{p-1}^1\varepsilon_{p-1}^3) = q^{p-2}(1 - 2q + \hat{q}_4) + \mathcal{O}\left(\frac{1}{N}\right),$$

and an easy symmetry argument yields

$$\nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\varepsilon_{p-1}^1\varepsilon_{p-1}^6) = \nu_{p-1,0}((\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4)\varepsilon_{p-1}^5\varepsilon_{p-1}^6) = 0.$$

On the other hand, an obvious extension of (30) in Lemma 3.12 shows that

$$R_{1,2}^{p-2} = \frac{(p-2)!}{N^{p-2}} \sum_{\hat{j} \in A_{N-(p-1)}^{p-2}} \eta_{\hat{j}}^1\eta_{\hat{j}}^2 + \mathcal{O}\left(\frac{1}{N}\right).$$

Then, Corollaries 3.8 and 3.10 ensure that

$$M_2 = \frac{1}{N}(p-1)^2 q^{2(p-2)}(1 - 2q + \hat{q}_4) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right). \quad (71)$$

It only remains to study  $M_1$ . Set

$$f = \frac{(p-1)!}{N^{p-1}}(\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (\eta_{\hat{j}}^1 - \eta_{\hat{j}}^5)(\eta_{\hat{j}}^3 - \eta_{\hat{j}}^6).$$

Then

$$M_1 = \nu(f).$$

Corollary 3.13 and Proposition 5.1 implies

$$\nu^{\frac{1}{2}}(f^2) \leq \frac{K}{N^{\frac{1}{2}}}.$$

Furthermore, since by symmetry between the different copies of  $G_N$ ,  $\nu_{p-1,0}(f) = 0$ , Corollary 3.8 implies

$$\nu(f) = \nu'_{p-1,0}(f) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right).$$

Let us now compute  $\nu'_{p-1,0}(f)$ . Proposition 2.1 yields

$$\nu'_{p-1,0}(f) = \beta^2 u_N^2 \sum_{J' \in Q_{N,p-1}^p} (F_1^{J'} + F_2^{J'} + F_3^{J'}),$$

with

$$\begin{aligned} F_1^{J'} &= \nu_{p-1,0} \left( f \sum_{1 \leq l < l' \leq 6} (\eta_{J'}^l \eta_{J'}^{l'} - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^{l'} \right), \\ F_2^{J'} &= -6\nu_{p-1,0} \left( f \sum_{l \leq 6} (\eta_{J'}^l \eta_{J'}^7 - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^7 \right), \\ F_3^{J'} &= 21\nu_{p-1,0} \left( f (\eta_{J'}^7 \eta_{J'}^8 - q^{p-1}) \varepsilon_{J'}^7 \varepsilon_{J'}^8 \right). \end{aligned}$$

Using again the symmetry argument (again among the different copies of  $G_N$ ), the quantities  $F_2^{J'}$ ,  $F_3^{J'}$  and all the couples of  $F_1^{J'}$  except for  $\{(1,3), (1,4), (2,3), (2,4)\}$  vanish. Now, we will analyze the couple  $(1,3)$  of  $F_1^{J'}$ . We follow the ideas given in the proof of Proposition 5.1. Properties of independence and symmetry, Lemma 3.14, Proposition 3.3 and Lemma 2.3 ensure

$$\begin{aligned}
& \frac{(p-1)!}{N^{p-1}} \beta^2 u_N^2 \sum_{J' \in Q_{N,p-1}^p} \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \varepsilon_{J'}^1 \varepsilon_{\hat{J}}^3 \right. \\
& \quad \left. \times (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6)(\eta_{j'}^1 \eta_{j'}^3 - q^{p-1}) \right) \\
&= \frac{(p-1)!}{N^{p-1}} (p-1) \beta^2 u_N^2 \\
& \quad \times \sum_{\hat{J}' \in A_{N-(p-1)}^{p-1}} \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2)(\varepsilon^3 - \varepsilon^4) \varepsilon_{\hat{J}'}^1 \varepsilon_{\hat{J}}^3 \right) \\
& \quad \times \nu_{p-1,0} \left( (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6)(\eta_{j'}^1 \eta_{j'}^3 - q^{p-1}) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right) \\
&= \frac{(p-1)!}{N^{p-1}} (p-1) \beta^2 u_N^2 q^{p-2} (1 - 2q + \hat{q}_4) \\
& \quad \times \sum_{\hat{J}' \in A_{N-(p-1)}^{p-1}} \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu_{p-1,0} \left( (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6)(\eta_{j'}^1 \eta_{j'}^3 - q^{p-1}) \right) \\
& \quad + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right),
\end{aligned}$$

Then, realizing similar operations for the other three couples and putting together all the results we have, up to a  $N^{-3/2}$  term,

$$\begin{aligned}
M_1 &= \frac{(p-1)!}{N^{p-1}} (p-1) \beta^2 u_N^2 q^{p-2} (1 - 2q + \hat{q}_4) \\
& \times \sum_{\hat{J}' \in A_{N-(p-1)}^{p-1}} \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu_{p-1,0} \left( (\eta_j^1 - \eta_j^5)(\eta_j^3 - \eta_j^6)(\eta_{j'}^1 - \eta_{j'}^2)(\eta_{j'}^3 - \eta_{j'}^4) \right).
\end{aligned}$$

Finally, from Corollary 3.13, (67), and Corollary 3.8, we can obtain

$$M_1 = \beta^2 \frac{p(p-1)}{2} q^{p-2} (1 - 2q + \hat{q}_4) \nu(S_{1,2}^2) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right). \quad (72)$$

Then, putting together (68)-(72) we finish the proof.  $\square$

It will be useful in the sequel to have some information about the correlations between  $T_{l,l'}$ ,  $T_l$  and  $T$ . This is easily obtained in the following Proposition:

**Proposition 6.2** *The following cancellations hold true.*

1. *If  $l < l'$  and  $(l, l') \neq (1, 2)$ , we have*

$$\nu(T_{1,2} T_{l,l'}) = 0.$$

2. For any  $l$ , we have

$$\nu(T_{1,2} T_l) = 0, \quad \nu(T_{1,2} T) = 0.$$

3. For any  $l \neq 1$ , we have

$$\nu(T_1 T_l) = 0, \quad \nu(T_1 T) = 0.$$

**Proof:** This is trivially obtained by some symmetry considerations among the different copies of  $G_N$ .  $\square$

Let us turn now to the second moment estimate of  $T_1$ .

**Proposition 6.3** *Whenever  $\beta$  satisfies (H), we have*

$$\left| \nu(T_1^2) - \frac{B^2}{N} \right| \leq \frac{K}{N^{\frac{3}{2}}} \quad (73)$$

where

$$B^2 = (p-1)q^{p-2}(q - \hat{q}_4) \left[ \frac{(p-1)q^{p-2} + \beta^2 \frac{p}{2} A^2}{1 - \beta^2 \frac{p(p-1)}{2} q^{p-2}(1 - 4q + 3\hat{q}_4)} \right]. \quad (74)$$

**Proof:** We can decompose  $T_1$  as follows

$$T_1 = S_1 + V_{N,T}^1,$$

with

$$S_1 = \frac{(p-1)!}{N^{p-1}} \sum_{j \in A_N^{p-1}} (\eta_j^1 - b_j) b_j,$$

$$V_{N,T}^1 = \frac{1}{N^{p-1}} \sum_{j \in \bar{N}_{p-1}^c} (\eta_j^1 - b_j) b_j.$$

The same kind of arguments as in Proposition 6.1 allow to state that

$$\nu(S_1^2) = \mathcal{O}\left(\frac{1}{N}\right), \quad (75)$$

$$\nu(T_1^2) = \nu(S_1^2) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right), \quad (76)$$

and

$$\nu(S_1^2) = \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right), \quad (77)$$

with

$$\begin{aligned}\tilde{M}_1 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 - \varepsilon^2) \varepsilon^3 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (\eta_{\hat{j}}^1 - \eta_{\hat{j}}^4) \eta_{\hat{j}}^5 \right], \\ \tilde{M}_2 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 - \varepsilon^2) \varepsilon^3 \sum_{\hat{j} \in \bar{Q}_{N,p-1}^{p-1}} (\eta_{\hat{j}}^1 - \eta_{\hat{j}}^4) \eta_{\hat{j}}^5 \right], \\ \tilde{M}_3 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 - \varepsilon^2) \varepsilon^3 \sum_{\hat{j} \in \bar{Q}_{N,p-1}^{p-1}} (\eta_{\hat{j}}^1 - \eta_{\hat{j}}^4) \eta_{\hat{j}}^5 \right].\end{aligned}$$

By Lemma 8.4 we easily get

$$|\tilde{M}_3| \leq \frac{L}{N^2}. \quad (78)$$

Using exactly the same arguments as in the study of  $M_2$ , we obtain

$$\tilde{M}_2 = \frac{1}{N} (p-1)^2 q^{2(p-2)} (q - \hat{q}_4) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right). \quad (79)$$

Finally, we deal with  $\tilde{M}_1$ . Let

$$f = \frac{(p-1)!}{N^{p-1}} (\varepsilon^1 - \varepsilon^2) \varepsilon^3 \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (\eta_{\hat{j}}^1 - \eta_{\hat{j}}^4) \eta_{\hat{j}}^5.$$

Then, Proposition 5.1, Corollary 3.8 and one symmetry consideration yield

$$\begin{aligned}\tilde{M}_1 &= \left[ \nu_{p-1,0}(f) + \nu'_{p-1,0}(f) \right] + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right) \\ &= \nu'_{p-1,0}(f) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right).\end{aligned}$$

Using Proposition 2.1 we can write

$$\nu'_{p-1,0}(f) = \beta^2 u_N^2 \sum_{J' \in Q_{N,p-1}^p} (\tilde{F}_1^{J'} + \tilde{F}_2^{J'} + \tilde{F}_3^{J'}),$$

with

$$\begin{aligned}\tilde{F}_1^{J'} &= \nu_{p-1,0} \left( f \sum_{1 \leq l < l' \leq 5} (\eta_{J'}^l \eta_{J'}^{l'} - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^{l'} \right), \\ \tilde{F}_2^{J'} &= -5\nu_{p-1,0} \left( f \sum_{l \leq 5} (\eta_{J'}^l \eta_{J'}^6 - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^6 \right), \\ \tilde{F}_3^{J'} &= 15\nu_{p-1,0} \left( f (\eta_{J'}^6 \eta_{J'}^7 - q^{p-1}) \varepsilon_{J'}^6 \varepsilon_{J'}^7 \right).\end{aligned}$$

The only non-vanishing terms in these expressions are the one induced by the couples in  $W_1$ , where

$$W_1 = \{(1, 3), (2, 3), (1, 4), (2, 4), (1, 5), (2, 5), (1, 6), (2, 6)\},$$

We can then rewrite  $\tilde{M}_1$  as follows:

$$\tilde{M}_1 = \sum_{(l,l') \in W_1} \tilde{F}_{(l,l')} + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right),$$

with

$$\begin{aligned}\tilde{F}_{(l,l')} &= \beta^2 u_N^2 \sum_{J' \in Q_{N,p-1}^p} \nu_{p-1,0} \left( f (\eta_{J'}^l \eta_{J'}^{l'} - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^{l'} \right), \quad l' \neq 6, \\ \tilde{F}_{(l,6)} &= -5\beta^2 u_N^2 \sum_{J' \in Q_{N,p-1}^p} \nu_{p-1,0} \left( f (\eta_{J'}^l \eta_{J'}^6 - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^6 \right).\end{aligned}$$

We first analyze together the couples (1, 6), (2, 6). The most important remark is the following consequence of Lemma 2.3:

$$\nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) \varepsilon_{p-1}^3 \varepsilon_{p-1}^6 \right) = -\nu_{p-1,0} \left( (\varepsilon^1 - \varepsilon^2) \varepsilon_{p-1}^3 \varepsilon_{p-1}^2 \varepsilon_{p-1}^6 \right) = q^{p-2} (q - \hat{q}_4).$$

Then, by means of the arguments used in Proposition 6.1 we can prove that

$$\tilde{F}_{(1,6)} + \tilde{F}_{(2,6)} = -5\beta^2 \frac{p(p-1)}{2} q^{p-2} (q - \hat{q}_4) \nu(S_1^2) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right).$$

Performing the same sort of computations, we can get

$$\tilde{F}_{(1,3)} + \tilde{F}_{(2,3)} = \beta^2 \frac{p(p-1)}{2} q^{p-2} (1 - q) \nu(S_1^2) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right).$$

We finish the study of  $\tilde{M}_1$  by considering the last four couples. As usual, independence, symmetry, relations between the different sets, Proposition 3.3, Lemma 2.3 and Corollary 3.8 imply that, up to a term of order  $N^{-3/2}$ ,

$$\begin{aligned} \tilde{F}_{(1,4)} + \tilde{F}_{(2,4)} + \tilde{F}_{(1,5)} + \tilde{F}_{(2,5)} &= \beta^2 \frac{p(p-1)}{2} q^{p-2} (q - \hat{q}_4) \left( \frac{(p-1)!}{N^{p-1}} \right)^2 \\ &\times \sum_{\hat{J}' \in A_{N-(p-1)}^{p-1}} \sum_{\hat{J} \in A_{N-(p-1)}^{p-1}} \nu \left( (\eta_j^1 - \eta_j^4) \eta_j^5 (\eta_{j'}^1 - \eta_{j'}^2) (\eta_{j'}^4 + \eta_{j'}^5) \right) \end{aligned}$$

Then, (30), (75), (61), Propositions 6.2, 3.3, 5.1, (63), (64) and (76) yield

$$\begin{aligned} &\tilde{F}_{(1,4)} + \tilde{F}_{(2,4)} + \tilde{F}_{(1,5)} + \tilde{F}_{(2,5)} \\ &= \beta^2 \frac{p(p-1)}{2} q^{p-2} (q - \hat{q}_4) \nu \left( \left( R_{1,5}^{p-1} - R_{4,5}^{p-1} \right) \right. \\ &\quad \left. \times \left( R_{1,4}^{p-1} - R_{2,4}^{p-1} - R_{2,5}^{p-1} + R_{1,5}^{p-1} \right) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right) \\ &= \beta^2 \frac{p(p-1)}{2} q^{p-2} (q - \hat{q}_4) \nu \left( \left( T_{1,5} - T_{4,5} + T_1 - T_4 \right) \right. \\ &\quad \left. \times \left( T_{1,4} - T_{2,4} - T_{2,5} + T_{1,5} + 2T_1 - 2T_2 \right) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right) \\ &= \beta^2 \frac{p(p-1)}{2} q^{p-2} (q - \hat{q}_4) \left( \frac{A^2}{N} + 2\nu(S_1^2) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right). \end{aligned}$$

So,

$$\tilde{M}_1 = \beta^2 \frac{p(p-1)}{2} q^{p-2} \left[ (1 - 4q + 3\hat{q}_4) \nu(S_1^2) + (q - \hat{q}_4) \frac{A^2}{N} \right] + \mathcal{O} \left( \frac{1}{N^{\frac{3}{2}}} \right). \quad (80)$$

Finally, putting together (76)-(80) we get (73) and (74).  $\square$

The second moment of  $T$  can also be estimated in the following way:

**Proposition 6.4** *If  $\beta$  satisfies (H), we have*

$$\left| \nu(T^2) - \frac{C^2}{N} \right| \leq \frac{L}{N^{\frac{3}{2}}}$$

where

$$\begin{aligned} C^2 &= \\ &(p-1)q^{p-2} \left[ \frac{(\hat{q}_4 - q^2) [(p-1)q^{p-2} + \beta^2 \frac{p}{2} A^2] + \beta^2 p(2q + q^2 - 3\hat{q}_4) B^2}{1 - \beta^2 \frac{p(p-1)}{2} q^{p-2} (1 - 4q + 3\hat{q}_4)} \right]. \end{aligned}$$

**Proof:** The proof of this relation goes along the same lines as Propositions 6.1 and 6.3, and we will only point out the main differences with the latter.

Observe that

$$\nu(T^2) = \hat{M}_1 + \hat{M}_2 + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right), \quad (81)$$

with

$$\begin{aligned} \hat{M}_1 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 \varepsilon^2 - q^{p-1}) \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (\eta_{\hat{j}}^3 \eta_{\hat{j}}^4 - q^{p-1}) \right], \\ \hat{M}_2 &= \frac{(p-1)!}{N^{p-1}} \nu \left[ (\varepsilon^1 \varepsilon^2 - q^{p-1}) \sum_{\hat{j} \in \tilde{Q}_{N,p-1}^{p-1}} (\eta_{\hat{j}}^3 \eta_{\hat{j}}^4 - q^{p-1}) \right]. \end{aligned}$$

The same computation as in Proposition 6.3 yield, for the term  $\tilde{M}_2$ , the following equality:

$$\hat{M}_2 = \frac{1}{N} (p-1)^2 q^{2(p-2)} (\hat{q}_4 - q^2) + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right). \quad (82)$$

In order to deal with  $\hat{M}_1$  we have to evaluate the derivative of  $\nu_{p-1,0}(f)$ , where

$$f = \frac{(p-1)!}{N^{p-1}} (\varepsilon^1 \varepsilon^2 - q^{p-1}) \sum_{\hat{j} \in A_{N-(p-1)}^{p-1}} (\eta_{\hat{j}}^3 \eta_{\hat{j}}^4 - q^{p-1}).$$

As in Proposition 6.3, by arguments of symmetry, we can rewrite  $\hat{M}_1$  as

$$\hat{M}_1 = \sum_{(l,l') \in W_2} \hat{F}_{(l,l')} + \mathcal{O}\left(\frac{1}{N^{\frac{3}{2}}}\right), \quad (83)$$

where

$$W_2 = \{(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (4, 5), (5, 6)\},$$

and, for  $(l, l') \in W_2$ ,

$$\begin{aligned}
\hat{F}_{(l,\nu)} &= \beta^2 u_N^2 (p-1)! \sum_{J' \in Q_{N,p-1}} \nu_{p-1,0} \left( f(\eta_{J'}^l \eta_{J'}^{\nu} - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^{\nu} \right), \quad l' \neq 5, 6, \\
\hat{F}_{(l,5)} &= -4\beta^2 u_N^2 (p-1)! \sum_{J' \in Q_{N,p-1}} \nu_{p-1,0} \left( f(\eta_{J'}^l \eta_{J'}^5 - q^{p-1}) \varepsilon_{J'}^l \varepsilon_{J'}^5 \right), \\
\hat{F}_{(5,6)} &= 10\beta^2 u_N^2 (p-1)! \sum_{J' \in Q_{N,p-1}} \nu_{p-1,0} \left( f(\eta_{J'}^5 \eta_{J'}^6 - q^{p-1}) \varepsilon_{J'}^5 \varepsilon_{J'}^6 \right).
\end{aligned}$$

Let

$$K_{\beta,p,q} = \beta^2 \frac{p(p-1)}{2} q^{p-2}.$$

Operating as in Propositions 6.1 and 6.3, we can obtain

$$\begin{aligned}
\hat{F}_{(1,2)} &= K_{\beta,p,q} (1 - q^2) \nu (T^2), \\
\hat{F}_{(l,l')} &= K_{\beta,p,q} q(1 - q) [\nu (T^2) + B^2], \quad \text{for } (l, l') = (1, 3), (1, 4), (2, 3), (2, 4), \\
\hat{F}_{(3,4)} &= K_{\beta,p,q} (\hat{q}_4 - q^2) [\nu (T^2) + 2B^2 + A^2], \\
\hat{F}_{(l,\nu)} &= -4K_{\beta,p,q} q(1 - q) \nu (T^2), \quad \text{for } (l, l') = (1, 5), (2, 5), \\
\hat{F}_{(l,l')} &= -4K_{\beta,p,q} (\hat{q}_4 - q^2) [\nu (T^2) + B^2], \quad \text{for } (l, l') = (3, 5), (4, 5), \\
\hat{F}_{(5,6)} &= 10K_{\beta,p,q} (\hat{q}_4 - q^2) \nu (T^2).
\end{aligned} \tag{84}$$

Then, from (81)-(84) we can conclude the proof of this proposition.  $\square$

## 7 Central Limit Theorems

The main result of this section will be a CLT for the fluctuations of  $R_{1,2}$ , though we will get, on our way to the proof of this theorem, some general limit relations for the joint fluctuations of  $T_{l,l'}$ ,  $T_l$  and  $T$ . First, observe that, as an immediate consequence of Propositions 6.1, 6.2, 6.3, 6.4 and equality (61) we have the following

**Proposition 7.1** *If  $\beta$  verifies (H), we have*

$$\left| \nu \left( (R_{1,2}^{p-1} - q^{p-1})^2 \right) - \frac{1}{N} (A^2 + 2B^2 + C^2) \right| \leq \frac{K}{N^{\frac{3}{2}}}.$$

Our aim here will be to generalize this estimate, obtaining a similar relation for

$$\nu \left( \left( R_{1,2}^{p-1} - q^{p-1} \right)^k \right).$$

Let us first state the result we obtain for the fluctuations of  $T_{l,l'}$ : define  $a(k) = \mathbf{E}g^k$ , for a standard Gaussian random variable  $g$ .

**Theorem 7.2** *Let  $\beta$  satisfying (H),  $n \in \mathbb{N}$ . For any couple  $1 \leq l < l' \leq n$ , consider an integer  $k(l, l') \geq 0$ . Set  $k = \sum_{l < l'} k(l, l')$ . Then, we have*

$$\left| \nu \left( \prod_{l < l'} T_{l,l'}^{k(l,l')} \right) - \frac{1}{N^{\frac{k}{2}}} \prod_{l < l'} a(k(l, l')) A^k \right| \leq \frac{L(k)}{N^{\frac{k+1}{2}}}.$$

**Proof:** We will follow the proof of Theorem 2.7.1 in [9], and we prove this theorem by induction over  $k$ . The case  $k = 2$  has been proved in Section 6, Proposition 6.1.

We assume now that the result is true up to the order  $k - 1$ . Then, using (65) and (66), we can easily check that if  $\sum_{l < l'} k(l, l') = j$  where  $j \leq k - 1$ , we have

$$\nu \left( \prod_{l < l'} S_{l,l'}^{k(l,l')} \right) = \mathcal{O} \left( \frac{1}{N^{\frac{j}{2}}} \right). \quad (85)$$

Furthermore, using the induction, (85), and relations (65) and (66), we also have

$$\nu \left( \prod_{l < l'} T_{l,l'}^{k(l,l')} \right) = \nu \left( \prod_{l < l'} S_{l,l'}^{k(l,l')} \right) + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right). \quad (86)$$

Observe that  $\prod_{l < l'} S_{l,l'}^{k(l,l')}$  can be decomposed as

$$\prod_{l < l'} S_{l,l'}^{k(l,l')} = \prod_{1 \leq v \leq k} S_{l(v), l'(v)},$$

where, for any integer  $v \leq k$ ,  $l(v), l'(v)$  are two integers such that

$$(l(v), l'(v)) = (1, 2) \iff v \leq k(1, 2).$$

We can assume without loss of generality that  $k(1, 2) \geq 1$ . We consider, for  $1 \leq v \leq k$ , integers  $j(v), j'(v)$ , all different, and greater than  $n$ . Thus

$$\nu \left( \prod_{l < l'} S_{l,l'}^{k(l,l')} \right) = \nu \left( \prod_{1 \leq v \leq k} S_{l(v), l'(v)} \right) = \nu \left( \prod_{1 \leq v \leq k} U(v) \right),$$

with

$$U(v) = \frac{(p-1)!}{N^{p-1}} \left( \hat{T}(\eta^{l(v)}) - \hat{T}(\eta^{j(v)}) \right) \cdot \left( \hat{T}(\eta^{l'(v)}) - \hat{T}(\eta^{j'(v)}) \right).$$

Set

$$\varepsilon(v) = \left( \varepsilon^{l(v)} - \varepsilon^{j(v)} \right) \left( \varepsilon^{l'(v)} - \varepsilon^{j'(v)} \right).$$

By the usual symmetry argument, and using (85) and Lemma 8.2 we have

$$\nu \left( \prod_{l < l'} S_{l, l'}^{k(l, l')} \right) = \nu \left( \varepsilon(1) \prod_{2 \leq v \leq k} U(v) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right). \quad (87)$$

For any  $v \geq k$ ,  $U(v)$  can be written as

$$U(v) = U_1(v) + U_2(v) + U_3(v),$$

where  $U_1(v), U_2(v), U_3(v)$  are defined by means of  $A_{N-(p-1)}^{p-1}, \tilde{Q}_{N, p-1}^{p-1}, \bar{Q}_{N, p-1}^{p-1}$ , respectively. Similarly to Proposition 6.1, we can get

$$\begin{aligned} \nu \left( \varepsilon(1) \prod_{2 \leq v \leq k} U(v) \right) &= \nu \left( \varepsilon(1) \prod_{2 \leq v \leq k} (U_1(v) + U_2(v)) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right) \\ &= \nu \left( \varepsilon(1) \prod_{2 \leq v \leq k} U_1(v) \right) + I + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right), \end{aligned} \quad (88)$$

with

$$I = \sum_{2 \leq u \leq k} \nu \left( \varepsilon(1) U_2(u) \prod_{v \neq u} U_1(v) \right),$$

where  $\prod_{v \neq u}$  means that the product is over  $2 \leq v \leq k, v \neq u$ .

We now study  $I$ . As in [9], Corollary 3.8 and the usual procedure imply, if  $k(1, 2) \geq 2$ ,

$$I = \frac{(k(1, 2) - 1)}{N} (p-1)^2 q^{2(p-2)} (1 - 2q + \hat{q}_4) \nu \left( \prod_{3 \leq v \leq k} U_1(v) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right). \quad (89)$$

If  $k(1, 2) = 1$ ,  $I = 0$ .

Now we deal with the other term of (88). Using again Corollary 3.8, we should study particularly the derivative of

$$\nu_{p-1,0} \left( \varepsilon(1) \prod_{2 \leq v \leq k} U_1(v) \right).$$

Indeed, the following relation is not difficult to obtain:

$$\begin{aligned} & \nu \left( \varepsilon(1) \prod_{2 \leq v \leq k} U_1(v) \right) \\ &= \nu'_{p-1,0} \left( \varepsilon(1) \prod_{2 \leq v \leq k} U_1(v) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right) \\ &= \beta^2 \frac{p(p-1)}{2} q^{p-2} (1-2q+\hat{q}_4) \nu \left( U_1(1) \prod_{2 \leq v \leq k} U(v) \right) + \mathcal{O} \left( \frac{1}{N^{\frac{k+1}{2}}} \right). \end{aligned} \tag{90}$$

Putting together (86), (87)-(90) and reasoning by induction over  $k$  we can conclude the proof of this proposition.  $\square$

In a similar way to the previous theorem we can prove the following relations on the joint behavior of  $T_{l,l'}$ ,  $T_l$  and  $T$  (we do not include the proofs here, since they follow closely the lines of [9]).

**Theorem 7.3** *Let  $\beta$  satisfying (H),  $n \in \mathbb{N}$ . For  $1 \leq l < l' \leq n$ , consider integers  $k(l, l') \geq 0$  and  $k_1 = \sum_{1 \leq l < l' \leq n} k(l, l')$ . For  $1 \leq l \leq n$ , let  $k(l)$  be a positive integer and set  $k_2 = \sum_{1 \leq l \leq n} k(l)$ . Then, if  $k = k_1 + k_2$ , we have*

$$\left| \nu \left( \prod_{1 \leq l < l' \leq n} T_{l,l'}^{k(l,l')} \prod_{1 \leq l \leq n} T_l^{k(l)} \right) - \frac{1}{N^{\frac{k}{2}}} \prod_{1 \leq l < l' \leq n} a(k(l, l')) \prod_{1 \leq l \leq n} a(k(l)) A^{k_1} B^{k_2} \right| \leq \frac{L(k)}{N^{\frac{k+1}{2}}}.$$

**Theorem 7.4** *Let  $\beta$  satisfying (H),  $n \in \mathbb{N}$ . For  $1 \leq l < l' \leq n$ , consider integers  $k(l, l') \geq 0$  and  $k_1 = \sum_{1 \leq l < l' \leq n} k(l, l')$ . For  $1 \leq l \leq n$ , let  $k(l) \geq 0$*

and  $k_2 = \sum_{1 \leq l \leq n} k(l)$ . Let  $k_3 \in \mathbb{N}$ . Then, if  $k = k_1 + k_2 + k_3$ , we have

$$\left| \nu \left( \prod_{1 \leq l < l' \leq n} T_{l,l'}^{k(l,l')} \prod_{1 \leq l \leq n} T_l^{k(l)} T^{k_3} \right) - \frac{1}{N^{\frac{k}{2}}} \prod_{1 \leq l < l' \leq n} a(k(l,l')) \prod_{1 \leq l \leq n} a(k(l)) a(k_3) A^{k_1} B^{k_2} C^{k_3} \right| \leq \frac{L(k)}{N^{\frac{k+1}{2}}}.$$

All the preceding considerations allow us to get the following Central Limit Theorem.

**Theorem 7.5** *Let  $\beta$  satisfying (H),  $\hat{k} \in \mathbb{N}$ . Then,*

$$\left| \nu \left( \left( R_{1,2}^{p-1} - q^{p-1} \right)^{\hat{k}} \right) - \frac{1}{N^{\frac{\hat{k}}{2}}} a(\hat{k}) (A^2 + 2B^2 + C^2)^{\frac{\hat{k}}{2}} \right| \leq \frac{L(\hat{k})}{N^{\frac{\hat{k}+1}{2}}}.$$

**Proof:** It is well-known that for any  $\tilde{k} \in \mathbb{N}$ ,

$$\begin{aligned} a(2\tilde{k}) &= \frac{(2\tilde{k})!}{2^{\tilde{k}} \tilde{k}!}, \\ a(2\tilde{k} + 1) &= 0. \end{aligned} \tag{91}$$

By (61), a combinatorial property, Theorems 7.2, 7.3 and 7.4 and (91), we have

$$\begin{aligned} & \nu \left( \left( R_{1,2}^{p-1} - q^{p-1} \right)^{\hat{k}} \right) = \nu \left( (T_{1,2} + T_1 + T_2 + T)^{\hat{k}} \right) \\ &= \sum \frac{\hat{k}!}{k_{1,2}! k_1! k_2! k!} \nu \left( T_{1,2}^{k_{1,2}} T_1^{k_1} T_2^{k_2} T^k \right) \\ &= \frac{1}{N^{\frac{\hat{k}}{2}}} \sum \frac{a(k_{1,2}) a(k_1) a(k_2) a(k) \hat{k}!}{k_{1,2}! k_1! k_2! k!} A^{k_{1,2}} B^{k_1+k_2} C^k + \mathcal{O} \left( \frac{1}{N^{\frac{\hat{k}+1}{2}}} \right) \\ &= \frac{1}{N^{\frac{\hat{k}}{2}}} \sum_{\text{even}} \frac{a(\hat{k}) \left( \frac{\hat{k}}{2} \right)! (A^2)^{\frac{k_{1,2}}{2}} (B^2)^{\frac{k_1+k_2}{2}} (C^2)^{\frac{k}{2}}}{\left( \frac{k_{1,2}}{2} \right)! \left( \frac{k_1}{2} \right)! \left( \frac{k_2}{2} \right)! \left( \frac{k}{2} \right)!} + \mathcal{O} \left( \frac{1}{N^{\frac{\hat{k}+1}{2}}} \right) \\ &= \frac{1}{N^{\frac{\hat{k}}{2}}} a(\hat{k}) (A^2 + 2B^2 + C^2)^{\frac{\hat{k}}{2}} + \mathcal{O} \left( \frac{1}{N^{\frac{\hat{k}+1}{2}}} \right), \end{aligned}$$

where  $\sum$  means the summatory of  $k_{1,2}, k_1, k_2, k \in \mathbb{N}$  such that  $k_{1,2} + k_1 + k_2 + k = \hat{k}$  and  $\sum_{\text{even}}$  means the summatory of  $k_{1,2}, k_1, k_2, k \in \mathbb{N}$  such that all these numbers are even and  $k_{1,2} + k_1 + k_2 + k = \hat{k}$ .  $\square$

Eventually, a CLT for  $R_{1,2}$ , that can be considered as the main result of this section, is easily obtained from the last theorem.

**Corollary 7.6** *Let  $\beta$  satisfying (H),  $\hat{k} \in \mathbb{N}$ . Then, we have*

$$\left| \nu \left( (R_{1,2} - q)^{\hat{k}} \right) - \frac{1}{N^{\frac{\hat{k}}{2}}} a(\hat{k}) \frac{(A^2 + 2B^2 + C^2)^{\frac{\hat{k}}{2}}}{((p-1)q^{p-2})^{\hat{k}}} \right| \leq \frac{L(\hat{k})}{N^{\frac{\hat{k}+1}{2}}}. \quad (92)$$

**Proof:** From (27) and since  $q$  is a strictly positive number we have

$$(R_{1,2} - q) = \frac{1}{(p-1)q^{p-2}} (R_{1,2}^{p-1} - q^{p-1}) - \frac{p-2}{2q^{p-2}} \xi^{p-3} (R_{1,2} - q)^2,$$

where  $\xi \in (R_{1,2} \wedge q, R_{1,2} \vee q)$ .

Since  $\xi \leq 1$  we obtain (92) by means of Theorems 7.5 and 3.4.  $\square$

## 8 Appendix

In this appendix we will recall the definitions of all the sets appearing in the paper, as well as some results about their size that will be used throughout this paper. Since the method and the tools needed to prove these results are always the same, we will only give some examples. Recall that  $P_m(N)$  denotes a polynomial of order  $m$  in  $N$ .

**Definition 8.1** *For  $w \geq 1$  and  $1 \leq r \leq w$ , set*

$$A_w^r = \{(i_1, \dots, i_r) \in \mathbb{N}^r; 1 \leq i_1 < \dots < i_r \leq w\}.$$

**Lemma 8.2** *For  $N \geq 1$  and  $p \geq 1$ , we have*

$$|A_N^p| = \binom{N}{p} = \frac{N^p}{p!} + P_{p-1}(N).$$

**Definition 8.3** *For  $w \geq 1$  and  $1 \leq r \leq w$ , set*

$$Q_{w,j}^r = \{(i_1, \dots, i_r) \in \mathbb{N}^r; 1 \leq i_1 < \dots < i_r \leq w, i_r > w - j\}.$$

The set  $Q_{w,j}^r$  can be split into

$$Q_{w,j}^r = \bar{Q}_{w,j}^r \cup \tilde{Q}_{w,j}^r,$$

where

$$\begin{aligned} \bar{Q}_{w,j}^r &= \{(i_1, \dots, i_r) \in Q_{w,j}^r; i_1 < \dots < i_r, i_{r-1} > w - j\}, \\ \tilde{Q}_{w,j}^r &= \{(i_1, \dots, i_r) \in Q_{w,j}^r; i_1 < \dots < i_r, i_{r-1} \leq w - j\}. \end{aligned}$$

**Lemma 8.4** For  $N \geq 1$  and  $k \geq 1$ , we have

$$\begin{aligned} (p-1)!|Q_{N,1}^p| &= N^{p-1} - \frac{p(p-1)}{2}N^{p-2} + P_{p-3}(N), \\ (p-1)!|Q_{N,k}^p| &= kN^{p-1} + P_{p-2}(N), \\ |\bar{Q}_{N,k}^p| &= P_{p-2}(N), \\ |\tilde{Q}_{N,p}^p| &= P_{p-1}(N). \end{aligned}$$

**Proof:** It is easily seen that

$$\begin{aligned} (p-1)!|Q_{N,1}^p| &= (N-1) \cdots (N-(p-1)) \\ &= N^{p-1} - \left( \sum_{j=1}^{p-1} j \right) N^{p-2} + P_{p-3}(N), \end{aligned}$$

which implies our first claim. In order to prove the second one, we use the following fact

$$\begin{aligned} |Q_{N,k}^p| &= \binom{N}{p} - \binom{N-k}{p} \\ &= \frac{1}{p!} \left( \sum_{j=k}^{k+p-1} j - \sum_{j=1}^{p-1} j \right) N^{p-1} + P_{p-2}(N) \\ &= \frac{k}{(p-1)!} N^{p-1} + P_{p-2}(N). \end{aligned}$$

Finally

$$|\bar{Q}_{N,k}^p| = \binom{N}{p} - \binom{N-k}{p} - k \binom{N-k}{p-1} = P_{p-2}(N),$$

and this finishes the proof of the third claim. The last one is an easy consequence of the previous results.  $\square$

**Definition 8.5** For  $r \geq 1$ , set

$$N_r = \{(i_1, \dots, i_r) \in \{1, \dots, N\}^r\}.$$

The set  $N_r$  can be split into

$$N_r = \bar{N}_r \cup \bar{N}_r^c,$$

where

$$\bar{N}_r = \{(i_1, \dots, i_r) \in N_r ; i_j \neq i_k \text{ for all } j \neq k\}.$$

**Lemma 8.6** For  $N \geq 1$  and  $p \geq 2$ , we have

$$|\bar{N}_{p-1}^c| = \frac{(p-1)(p-2)}{2} N^{p-2} + P_{p-3}(N).$$

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