

On the Brownian directed polymer in a Gaussian random environment

by

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Abstract

In this paper, we introduce a model of Brownian polymer in a continuous random environment. The asymptotic behavior of the partition function associated to this polymer measure is studied, and we are able to separate a weak and strong disorder regime under some reasonable assumptions on the spatial covariance of the environment. Some further developments, concerning some concentration inequalities for the partition function, are given for the weak disorder regime.

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1 Introduction

The directed polymer model in a random environment first appeared in the Mathematical Physics literature, as a canonical model of inhomogeneous systems (see e.g. [11], [15] for results in that direction). After some interesting relationships between this object and many other natural models of non-equilibrium dynamics have been established, the study of the polymer measure has been undertaken by Mathematicians, and a great amount of rigorous results is now available on the topic. These results concern basically the so-called partition function, the fluctuations and wandering exponents of the model, as well as the superdiffusive behavior of the polymer under the influence of the random media. On the other hand, a wide range of contexts have been explored: discrete random walks (see [1], [2], [3], [15], [23]), Brownian motion in a discrete potential (see [7], [8]), or Brownian motion in a Poisson-type potential ([6] or [29], [22] for an undirected polymer).

This paper proposes to begin the study of a model which, from our point of view, is also worth considering, namely the Brownian polymer in a continuous Gaussian potential. More specifically, a complete description of our model can be given as follows:

1. Our polymer will be modeled by a d -dimensional Brownian path $\{\omega_t; t \geq 0\}$, defined on a complete probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ equipped with a filtration $\{\hat{\mathcal{F}}_t; t \geq 0\}$. We will denote by E_ω the expectation with respect to \hat{P} , that will be considered as the Wiener measure. We will also set \hat{P}^x for the Wiener measure shifted by a constant $x \in \mathbb{R}^d$, which is of course the measure of a Wiener process with initial condition $x \in \mathbb{R}^d$.
2. The random environment will be defined by a Gaussian landscape B on $\mathbb{R}_+ \times \mathbb{R}^d$, with rough fluctuations in time, and homogeneous with respect to the space coordinate: B will be given, on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, as a centered Gaussian process whose covariance structure is

$$\mathbf{E}[B(t, x)B(s, y)] = (s \wedge t)Q(x - y),$$

where Q is a homogeneous covariance function such that $Q(0) < \infty$ (which implies that Q is bounded).

Notice that Q can also be represented by a Fourier transform procedure: there exists (see e.g. [5] for further details) a Gaussian independently scattered

measure M on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$B(t, x) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbf{1}_{[0, t]}(s) e^{\iota \lambda x} M(ds, d\lambda),$$

where λx stands for the inner product of λ and x in \mathbb{R}^d , and where the law of M is defined by the following covariance structure: for any test functions $f, g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{C}$, we have

$$\begin{aligned} \mathbf{E} \left[\int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s, \lambda) M(ds, d\lambda) \overline{\int_{\mathbb{R}_+ \times \mathbb{R}^d} g(s, \lambda) M(ds, d\lambda)} \right] \\ = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s, \lambda) \overline{g(s, \lambda)} \hat{Q}(d\lambda) ds, \end{aligned}$$

and the finite (real) measure \hat{Q} is the Fourier transform of Q . With this notation in mind, we can complete the description of our polymer measure by

3. For any $t > 0$, the energy of a given path (or configuration) ω on $[0, t]$ will be given by

$$-H_t(\omega) = \int_0^t B(ds, \omega_s) = \int_0^t \int_{\mathbb{R}^d} e^{\iota \lambda \omega_s} M(ds, d\lambda).$$

Notice that, for any fixed path ω , $H_t(\omega)$ is a centered Gaussian random variable with variance $tQ(0)$.

Based on this Hamiltonian, for any $x \in \mathbb{R}^d$, and a given constant β (interpreted as the inverse of the temperature of the system), we will define our (random) polymer measure by

$$dG_t^x(\omega) = \frac{e^{-\beta H_t(\omega)}}{Z_t^x} d\hat{P}^x(\omega), \quad \text{with} \quad Z_t^x = E_\omega^x [e^{-\beta H_t(\omega)}].$$

In the sequel, we will also consider the Gibbs average with respect to the polymer measure, defined as follows: for all $t \geq 0$, $n \geq 1$, and for any bounded measurable functional $f : (C([0, t]; \mathbb{R}^d))^n \rightarrow \mathbb{R}$, we set

$$\langle f \rangle_t = \frac{E_\omega^x [f(\omega^1, \dots, \omega^n) e^{-\beta \sum_{l \leq n} H_t(\omega^l)}]}{Z_t^n}, \quad (1)$$

where the $\omega^l, 1 \leq l \leq n$, are understood as independent Brownian configurations.

Our paper will be mainly concerned with the study of the partition function Z_t of the model described above, and let us mention already that, for the results we have obtained so far, the relevant parameters for our model will be the covariance function Q , and the inverse of the temperature β . Based on these parameters, we will get the following results:

- A concentration inequality and the almost sure limit for $\frac{1}{t} \log(Z_t)$.
- A natural definition of the weak and strong disorder regime for our polymer (see Definition 2.7)
- In the case of a covariance function $Q(x)$ that can be written as $\tilde{Q}(|x|)$ with $\tilde{Q} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, for $d \geq 3$, we will show that a sufficient condition in order to be in the weak disorder regime is $\int_0^\infty u \tilde{Q}(u) du < \infty$ and β small enough.
- For the weak disorder regime, we will show some refined concentration results for $\frac{1}{t} \log(Z_t)$, using some general techniques taken from the random media literature (cf [24], [3]).
- We will show that, for any $d \geq 1$, if $c_1(1 + |x|^2)^{-\lambda} \leq Q(x) \leq c_2(1 + |x|^2)^{-\hat{\lambda}}$ for some constants $c_1 > 0, c_2 > 0$ and $0 < \hat{\lambda} \leq \lambda < \frac{1}{2}$, then the polymer will be in the strong disorder regime, regardless of the value of $\beta > 0$.

Of course, many problems remain open for this model: behavior of the wandering and fluctuation exponents, existence of a covariance function Q for which a phase transition can be seen as β grows to ∞ , computations involving the overlap function associated to the model (which will be defined by equation (14)), etc. We plan to report on these issues in a subsequent paper.

It is also worth mentioning that we have chosen to deal with this specific model for two main reasons:

1. The continuous Gaussian model, which is physically a reasonable choice, allows us to use the huge amount of techniques available for this kind of processes (stochastic calculus, concentration inequalities, Malliavin calculus, among others), leading to some quite simple proofs of the main results contained in this paper.

2. It is well known that Z_t behaves, in law, like the Feynman-Kac representation of $u(t, 0)$, where $u(t, x)$ is the mild solution to the stochastic PDE

$$\partial_t u(t, x) = \Delta u(t, x) + \beta u(t, x) \dot{W}(dt, dx), \quad t \geq 0, x \in \mathbb{R}^d, \quad (2)$$

understood in the Stratonovich sense, with $u(0, x) = 1$, and thus

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(Z_t)$$

can be interpreted as the Lyapounov exponent for this equation. Our problem is thus closely related to the one considered in [5], [25], [26] (see also [4] and [9] for the discrete case), and though the questions adressed here are not exactly the same as in the latter papers, we believe that the present article gives some more insight on the (rather) old problem of the Lyapounov exponent for equation (2). For instance, to our knowledge, the existence of this Lyapounov exponent had never been proven before, and its exact computation for $d \geq 3$ had never been performed either.

Our paper will be organized as follows: at Section 2, we recall some basic notions and theorems of stochastic analysis that will be used in the sequel, we give some results on the almost sure behavior of $\frac{1}{t} \log(Z_t)$, and we define our notions of weak and strong disorder. At Section 3, we study in detail the weak disorder regime. At Section 4, we give a basic example of a strong disorder situation.

2 Almost sure limit of the partition function

In this section, we will give some basic results about the almost sure convergence of Z_t , and some rough bounds on its limit. This will allow us to define precisely a notion of weak and strong disorder for the polymer measure. First of all, we will introduce some notation on Malliavin calculus for the Gaussian measure M , that we will use throughout the paper.

2.1 Malliavin calculus preliminaries

We will give here some notations and basic results, taken mainly from [17], [18] and [27]. Let us specify first the Wiener space we will consider: for any

test functions $f, g : \mathbb{R}^d \rightarrow \mathbb{C}$, set

$$(f, g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(\lambda) \overline{g(\lambda)} \hat{Q}(d\lambda).$$

Call H the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to that positive bilinear form, and $(\cdot, \cdot)_H$ the corresponding inner product. Set also $\mathcal{H} = L^2(\mathbb{R}_+; H)$. The Gaussian process M can be seen as a zero-mean Gaussian family $\{M(h); h \in \mathcal{H}\}$ satisfying

$$\mathbf{E} \left[M(h_1) \overline{M(h_2)} \right] = (h_1, h_2)_{\mathcal{H}} \equiv \int_{\mathbb{R}_+} (h_1(t), h_2(t))_H dt, \quad h_1, h_2 \in \mathcal{H},$$

where we have set, for $h \in \mathcal{H}$,

$$M(h) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} h(\sigma, \lambda) M(d\sigma, d\lambda).$$

Furthermore, we will assume that \mathcal{F} is generated by M . Then $(M, \mathcal{H}, \mathbf{P})$ defines a Wiener space on $C(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{C})$, on which the traditional tools of Malliavin calculus can be introduced. Let us recall some of them for sake of completeness: a smooth functional of M will be of the form

$$F = f(M(u_1), \dots, M(u_m)), \quad m \geq 1, u_j \in \mathcal{H}, f \in C^\infty(\mathbb{C}^m), \quad (3)$$

and we will denote by \mathcal{S} the set of such functionals. Now, for F as in (3), the Malliavin derivative of F will be defined, as an element of \mathcal{H} , by

$$D_{t,\lambda} F = \sum_{j=1}^m \partial_{x_j} f(M(u_1), \dots, M(u_m)) u_j(t, \lambda). \quad (4)$$

Then it can be shown that the operator $D : \mathcal{S} \rightarrow \mathcal{H}$ is closable, and, as usual, for any $p \in [1, \infty]$, we will denote by $\mathbb{D}^{1,p}$ the Sobolev space obtained by completing \mathcal{S} with respect to the norm

$$\|F\|_{1,p}^p = \mathbf{E} [|F|^p] + \mathbf{E} [|DF|_{\mathcal{H}}^p].$$

Notice that the following chain rule is available for functionals F in $\mathbb{D}^{1,p}$: if $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $\varphi(F), \varphi'(F) \in L^\alpha(\Omega)$ for any $\alpha > 0$, then $\varphi(F) \in \mathbb{D}^{1,r}$ for any $r < p$, and

$$D\varphi(F) = \varphi'(F) DF. \quad (5)$$

Let us also mention that, among all the elaborated integration by parts formulae of the Malliavin calculus, we will only use the following basic one in the sequel: if $F \in \mathbb{D}^{1,2}$ and u is a deterministic element of \mathcal{H} , then

$$\mathbf{E}[F M(u)] = \mathbf{E}[(DF, u)_{\mathcal{H}}]. \quad (6)$$

Concentration inequalities are a useful tool in random system theory, and we will use the following one, taken from [27]:

Proposition 2.1 *Let $F \in \mathbb{D}^{1,p}$ for some $p > 1$, and suppose that $DF \in L^\infty(\Omega; \mathcal{H})$. Set $m = \mathbf{E}[F]$ and $\sigma^2 = \|DF\|_\infty^2$. Then we have, for any $c \geq 0$,*

$$\mathbf{P}(|F| \geq c) \leq 2 \exp\left(-\frac{(c-m)^2}{2\sigma^2}\right).$$

We will also need a refinement of Proposition 2.1, for which we have to introduce a little more notation and a 0-1 type law, that we learned from [28]:

Lemma 2.2 *Let J be a measurable set in M , such that $J + \mathcal{H} \subset J$. Then $P(J) \in \{0, 1\}$.*

Proof: It is well known (see [18, page 31]) that $F = \mathbf{1}_J$ is an element of $\mathbb{D}^{1,2}$ iff $P(J) \in \{0, 1\}$. However if $J + \mathcal{H} \subset J$, we also have $J + \mathcal{H} = J$, and this easily yields $D\mathbf{1}_J = 0$ and thus $\mathbf{1}_J \in \mathbb{D}^{1,2}$. \square

For $h \in \mathcal{H}$, set $\tilde{h}_t = \int_0^t h_s ds$. For a measurable subset A of $(M, \mathcal{H}, \mathbf{P})$ and $m \in M$, define

$$q_A(m) = \inf \left\{ |h|_{\mathcal{H}}; m + \tilde{h} \in A \right\}.$$

Then we claim that:

Lemma 2.3 *Suppose $\mathbf{P}(A) \geq p > 0$. Then, for any $u > 0$,*

$$\mathbf{P}\left(q_A > u + (2 \log(2/p))^{1/2}\right) \leq 2 \exp\left(-\frac{u^2}{2}\right).$$

Proof: Let us prove first that $|Dq_A|_{\mathcal{H}} \leq 1$ (see [13] for further details on this functional). First, by Lemma 2.2, q_A is almost surely finite. Indeed, if $J = \{\omega; q_A(\omega) < \infty\}$, then it is easily checked that $J + \mathcal{H} \subset J$, and thus $\mathbf{P}(J) \in \{0, 1\}$. On the other hand, $A \subset J$, and by assumption $\mathbf{P}(A) > 0$, which gives the finiteness of q_A . Furthermore, if $l \in \mathcal{H}$, by the usual triangular inequality, we have almost surely

$$q_A(m + \tilde{l}) \leq q_A(m) + |l|_{\mathcal{H}},$$

and hence q_A is a Lipschitz functional on M with Lipschitz constant 1, which yields, in particular, $|Dq_A|_{\mathcal{H}} \leq 1$. Then applying Proposition 2.1, we obtain that for any $u > 0$,

$$\mathbf{P}(|q_A - \mathbf{E}[q_A]| > u) \leq 2 \exp\left(-\frac{u^2}{2}\right).$$

The proof now follows the lines of Talagrand [24] (see also [3]): if $u < \mathbf{E}[q_A]$, we have

$$p \leq \mathbf{P}(A) \leq \mathbf{P}(|q_A - \mathbf{E}[q_A]| > u) \leq 2 \exp\left(-\frac{u^2}{2}\right),$$

which yields that $\mathbf{E}[q_A] \leq (2 \log(2/p))^{1/2}$, and the proof is now easily completed. □

2.2 Almost sure behavior

Let us begin with a Markov type decomposition for Z_t^x : for $x, y \in \mathbb{R}^d$, and $t, h \geq 0$, set

$$\begin{aligned} Z_{t,t+h}(x, y, B) &= E_{\omega}^x \left[e^{\beta \int_0^{t+h} B(ds, \omega_s)} \mid \omega_t = y \right], \\ Z_t(x, B) &= E_{\omega}^x \left[e^{\beta \int_0^t B(ds, \omega_s)} \right]. \end{aligned}$$

Then the following property holds true:

Lemma 2.4 *Let p_t be the heat kernel on \mathbb{R}^d at time $t \geq 0$, and set, for $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, $\theta_t B(s, x) = B(s + t, x)$. Then, for any $t, h \geq 0$, we have*

$$Z_{t+h}(x, B) = \int_{\mathbb{R}^d} Z_h(y, \theta_t B) Z_{t,t}(x, y, B) p_t(dy).$$

Proof: Notice first the relationship

$$\begin{aligned} Z_{t+h}(x, B) &= \int_{\mathbb{R}^d} Z_{t,t+h}(x, y, B) p_t(dy) \\ &= \int_{\mathbb{R}^d} E_\omega^x \left[e^{\beta \int_0^t B(ds, \omega_s)} e^{\beta \int_t^{t+h} B(ds, \omega_s)} \Big| \omega_t = y \right] p_t(dy). \end{aligned}$$

Thus, we have that

$$\begin{aligned} Z_{t+h}(x, B) &= E_\omega^x \left[E_\omega^x \left[e^{\beta \int_0^t B(ds, \omega_s)} e^{\beta \int_t^{t+h} B(ds, \omega_s)} \Big| \hat{\mathcal{F}}_t \right] \right] \\ &= E_\omega^x \left[e^{\beta \int_0^t B(ds, \omega_s)} E_{\hat{\omega}}^{\omega_t} \left[e^{\beta \int_0^h (\theta_t B)(ds, \hat{\omega}_s)} \right] \right] \\ &= \int_{\mathbb{R}^d} E_\omega^x \left[e^{\beta \int_0^t B(ds, \omega_s)} E_{\hat{\omega}}^y \left[e^{\beta \int_0^h (\theta_t B)(ds, \hat{\omega}_s)} \right] \Big| \omega_t = y \right] p_t(dy) \\ &= \int_{\mathbb{R}^d} Z_h(y, \theta_t B) Z_{t,t}(x, y, B) p_t(dy), \end{aligned}$$

where $\hat{\omega}$ denotes a d -dimensional Brownian path, independent of ω . □

As usual in disordered systems theory, the free energy, defined by

$$p_t(\beta) = \frac{1}{t} \mathbf{E} [\log (Z_t^x)],$$

will play an important role in the qualitative description of the asymptotic behavior of the system. Note that, taking into account the space homogeneity of B , $\mathbf{E}[\log(Z_t^x)]$ will be independent of the parameter $x \in \mathbb{R}^d$. This is why we will concentrate now on this quantity for $x = 0$. In fact, from now, x will be understood as 0 when not specified, and E_ω, Z_t will stand for E_ω^0, Z_t^0 , etc. We are now in position to state a first basic result about the limit of the quantity $p_t(\beta)$.

Proposition 2.5 *For all $\beta > 0$ there exists a constant $p(\beta) > 0$ such that*

$$p(\beta) \equiv \lim_{t \rightarrow \infty} p_t(\beta) = \sup_{t \geq 0} p_t(\beta).$$

Proof: This result is presumably fairly standard, but we include its proof for sake of readability: for $t, h \geq 0$, invoking Lemma 2.4, Jensen's inequality,

and the independence of the time increments of B , we get

$$\begin{aligned}
& \mathbf{E} [\log Z_{t+h}(0, B)] \\
&= \mathbf{E} \left[\log \int_{\mathbb{R}^d} Z_h(y, \theta_t B) Z_{t,t}(0, y, B) p_t(dy) \right] \\
&= \mathbf{E} [\log Z_t(0, B)] + \mathbf{E} \left[\log \int_{\mathbb{R}^d} Z_h(y, \theta_t B) \frac{Z_{t,t}(0, y, B)}{Z_t(0, B)} p_t(dy) \right] \\
&\geq \mathbf{E} [\log Z_t(0, B)] + \mathbf{E} \left[\int_{\mathbb{R}^d} (\log Z_h(y, \theta_t B)) \frac{Z_{t,t}(0, y, B)}{Z_t(0, B)} p_t(dy) \right] \\
&= \mathbf{E} [\log Z_t(0, B)] + \int_{\mathbb{R}^d} \mathbf{E} [\log Z_h(y, \theta_t B)] \mathbf{E} \left[\frac{Z_{t,t}(0, y, B)}{Z_t(0, B)} \right] p_t(dy) \\
&= \mathbf{E} [\log Z_t(0, B)] + \mathbf{E} [\log Z_h(0, \theta_t B)] \mathbf{E} \left[\int_{\mathbb{R}^d} \frac{Z_{t,t}(0, y, B)}{Z_t(0, B)} p_t(dy) \right] \\
&= \mathbf{E} [\log Z_t(0, B)] + \mathbf{E} [\log Z_h(0, B)].
\end{aligned}$$

Notice that, in the above inequality, we have also used the fact that, for any $y \in \mathbb{R}^d$,

$$\mathbf{E} [\log Z_h(y, \theta_t B)] = \mathbf{E} [\log Z_h(0, \theta_t B)],$$

thanks to the space homogeneity of B . Thus, for all $t, h \geq 0$,

$$(t+h)p_{t+h}(\beta) \geq tp_t(\beta) + hp_h(\beta).$$

This easily yields

$$\lim_{t \rightarrow \infty} p_t(\beta) = \sup_{t \geq 0} p_t(\beta) := p(\beta)$$

by a superadditivity argument. □

We will now summarize some elementary properties of $p(\beta)$:

Proposition 2.6 *The function p introduced at Proposition 2.5 satisfies:*

1. *The map $\beta \mapsto p(\beta)$ is a convex nondecreasing function on \mathbb{R}_+ .*
2. *The following upper bound holds true:*

$$p(\beta) \leq \frac{\beta^2}{2} Q(0). \tag{7}$$

3. \mathbf{P} -almost surely, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log Z_t = p(\beta). \quad (8)$$

Proof: We will divide this proof in several steps.

Step 1: The convexity of p is a trivial consequence of Hölder's inequality.

Step 2: In order to prove the third point of the proposition, let us compute first the Malliavin derivative of $U_t \equiv \frac{1}{t} \log(Z_t)$: since Z_t is \mathcal{F}_t -adapted, we have $D_{\tau, \lambda} U_t = 0$ if $\tau > t$, and if $\tau \leq t$, according to (5), we get $U_t \in \mathbb{D}^{1,2}$ and

$$\frac{D_{\tau, \lambda}(\log(Z_t))}{t} = \frac{D_{\tau, \lambda}(Z_t)}{tZ_t} = \frac{\beta E_\omega [e^{\iota \lambda \omega_\tau} e^{-\beta H_t(\omega)}]}{tZ_t} = \frac{\beta \langle e^{\iota \lambda \omega_\tau} \rangle_t}{t},$$

where we have used (4) in order to differentiate $e^{-\beta H_t(\omega)}$. Notice also that the order of D and E_ω can be interchanged by a simple uniform convergence argument. Hence, by definition of the inner product in \mathcal{H} ,

$$\begin{aligned} |DU_t|_{\mathcal{H}}^2 &= \left(\frac{\beta}{t}\right)^2 \int_0^t \int_{\mathbb{R}^d} \langle e^{\iota \lambda \omega_\tau} \rangle_t \langle e^{-\iota \lambda \omega_\tau} \rangle_t \hat{Q}(d\lambda) d\tau \\ &= \left(\frac{\beta}{t}\right)^2 \int_0^t \left\langle \int_{\mathbb{R}^d} e^{\iota \lambda (\omega_\tau^1 - \omega_\tau^2)} \hat{Q}(d\lambda) \right\rangle_t d\tau \\ &= \left(\frac{\beta}{t}\right)^2 \int_0^t \langle Q(\omega_\tau^1 - \omega_\tau^2) \rangle_t d\tau. \end{aligned}$$

Observe that, in the above expression, ω^1 and ω^2 are understood as two independent configurations under the polymer measure, and that we have also used the notation (1). In particular,

$$|DU_t|_{\mathcal{H}}^2 \leq \frac{\beta^2 Q(0)}{t},$$

almost surely. Hence, as a direct consequence of Proposition 2.1, we get

$$P\left(\left|\frac{1}{t} \log Z_t - p_t(\beta)\right| > c\right) \leq 2 \exp\left(-\frac{tc^2}{4Q(0)\beta^2}\right), \quad (9)$$

from which (8) can be deduced by a standard Borel-Cantelli argument.

Step 3: In order to prove the bound (7), let us just observe that Jensen's inequality trivially yields

$$p_t(\beta) \leq \frac{1}{t} \log \mathbf{E}[Z_t].$$

However, the computation of $\mathbf{E}[Z_t(\beta)]$ is an easy task: for any fixed ω , $-H_t(\omega)$ is a Gaussian random variable, and hence

$$\mathbf{E} [e^{-\beta H_t(\omega)}] = \exp \left(\frac{\beta^2 \mathbf{E}[(H_t(\omega))^2]}{2} \right) = e^{\frac{\beta^2 Q(0)t}{2}}. \quad (10)$$

Thus, for any $\beta \geq 0$,

$$p_t(\beta) \leq \frac{\beta^2 Q(0)}{2}. \quad (11)$$

□

2.3 Weak and strong disorder

The amount of influence of the environment B on the path ω is usually captured through the behavior of Z_t (see [6], [3]). More specifically, we can argue as follows: recall that relation (7) states that $p(\beta) \leq \frac{\beta^2 Q(0)}{2}$. The weak disorder regime is then naturally characterized by the relation

$$p(\beta) = \frac{\beta^2 Q(0)}{2} \quad \text{i.e.} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \mathbf{E} [\log(Z_t)] = \lim_{t \rightarrow \infty} \frac{1}{t} \log (\mathbf{E} [Z_t]),$$

while the strong disorder phase should be defined by $p(\beta) < \frac{\beta^2 Q(0)}{2}$. However, it will be more convenient to define the weak and strong disorder regimes through an associated process: set, for $t \geq 0$,

$$W_t = Z_t \exp \left(-\frac{\beta^2 Q(0)t}{2} \right). \quad (12)$$

Then it is easily seen that W is a positive \mathcal{F}_t -martingale, that converges almost surely. Set then

$$W_\infty = \lim_{t \rightarrow \infty} W_t.$$

By Kolmogorov's 0-1 law and an easy elaboration of [2, Lemma 2] , we have

$$\mathbf{P}(W_\infty = 0) \in \{0, 1\}.$$

Observe that if $W_\infty > 0$ almost surely, then $\log(W_\infty)$ is finite almost surely, and hence

$$\text{a.s.} - \lim_{t \rightarrow \infty} \left(\log(Z_t) - \frac{\beta^2 t Q(0)}{2} \right) = \log(W_\infty),$$

which yields

$$\text{a.s.} - \lim_{t \rightarrow \infty} \frac{\log(Z_t)}{t} = \frac{\beta^2 Q(0)}{2},$$

and hence

$$p(\beta) = \frac{\beta^2 Q(0)}{2}. \tag{13}$$

In other words, $W_\infty > 0$ implies $p(\beta) = \frac{\beta^2 Q(0)}{2}$, and hence a weak disorder type behavior of the polymer. This is why we will adopt the following definition:

Definition 2.7 *We will say that the polymer is in a strong disorder regime if $W_\infty = 0$ almost surely, while the weak disorder phase will be defined by $W_\infty > 0$ almost surely.*

Another relevant quantity for the study of disordered systems is the so-called overlap, that measures the similarity of two independent configurations under the considered random measure. In our case, this overlap is of the form

$$\frac{1}{t} \int_0^t \langle Q(\omega_s^1 - \omega_s^2) \rangle_s ds, \tag{14}$$

and observe that, since $Q(x)$ is usually a decreasing function of $|x|$, the last quantity really measures how close ω^1 is from ω^2 . One is then also allowed to relate the behavior of W_t and of the overlap in the following way:

Proposition 2.8 *Let W_t be defined by (12) for $t > 0$, and consider the statements:*

1. $W_\infty > 0$ almost surely.
2. $\int_0^\infty \langle Q(\omega_s^1 - \omega_s^2) \rangle_s ds < \infty$.
3. $L^1 - \lim_{t \rightarrow \infty} W_t = W_\infty$.

Then 1. and 2. are equivalent, and are both a consequence of statement 3.

Proof: Let us check first that 3. implies 1. The convergence in L^1 that we are assuming implies that

$$E(W_\infty) = \lim_{t \rightarrow \infty} E(W_t) = 1.$$

Using Kolmogorov's 0-1 law, we get $P(W_\infty > 0) = 1$.

Let us prove now the equivalence between 1 and 2: for $t \geq 0$, set

$$N_t = \beta \int_0^t \int_{\mathbb{R}^d} e^{\iota \lambda \omega_s} M(ds, d\lambda).$$

Then, for any fixed configuration ω , N_t is a martingale, whose quadratic variation process is given by

$$[N]_t = \beta^2 \int_0^t \int_{\mathbb{R}^d} \hat{Q}(d\lambda) = \beta^2 Q(0)t.$$

Furthermore, we have

$$W_t = E_\omega \left[\exp \left(N_t - \frac{\beta^2 Q(0)t}{2} \right) \right],$$

and Itô's formula applied to $\varphi(x) = e^x$ gives the following martingale decomposition for W :

$$W_t = 1 + \beta \int_0^t \int_{\mathbb{R}^d} E_\omega \left[e^{\iota \lambda \omega_s} \exp \left(N_s - \frac{\beta^2 Q(0)s}{2} \right) \right] M(ds, d\lambda), \quad t \geq 0. \quad (15)$$

The process W_t is also almost surely strictly positive. Hence, one can apply again Itô's formula to the function $\psi(x) = \log(x)$ to get

$$\log(W_t) = \int_0^t \frac{dW_s}{W_s} - \frac{1}{2} \int_0^t \frac{d[W]_s}{W_s^2} = M_t - \frac{1}{2} A_t,$$

with

$$\begin{aligned} M_t &= \beta \int_0^t \int_{\mathbb{R}^d} \langle e^{\iota \lambda \omega_s} \rangle_s M(ds, d\lambda) \\ A_t &= \beta^2 \int_0^t \int_{\mathbb{R}^d} \langle e^{\iota \lambda (\omega_s^1 - \omega_s^2)} \rangle_s \hat{Q}(d\lambda) ds = \beta^2 \int_0^t \langle Q(\omega_s^1 - \omega_s^2) \rangle_s ds. \end{aligned}$$

Moreover, notice that $\{M_t; t \geq 0\}$ is a martingale with quadratic variation A_t , and that we can write

$$\log(W_t) = A_t \left(\frac{M_t}{A_t} - \frac{1}{2} \right). \quad (16)$$

Now one can argue as follows:

- (a) Assume that $A_\infty = \infty$. We can now apply the strong law of large numbers for continuous martingales (see for instance [20]), that implies $\frac{M_t}{A_t} \rightarrow 0$, almost surely. Then expression (16) gives us that $W_\infty = 0$.
- (b) Assume that $A_\infty < \infty$. If $A_\infty < \infty$ then we have that M_t is a L^2 -bounded martingale which converges almost surely to M_∞ when t goes to ∞ . So $M_\infty < \infty$ almost surely, which clearly yields that $\log W_\infty > -\infty$ almost surely.

□

3 The weak disorder regime

In this section, we will give a sufficient condition under which the polymer is in the weak disorder phase. It is usually satisfied when ω is transient under \hat{P} , and Q rapidly decays at infinity, as we will see in some examples. Eventually, we will show that the concentration of $\frac{1}{t} \log(Z_t)$ below $\frac{\beta Q(0)}{2}$ occurs at a higher speed than the one indicated by (9). For sake of readability, we will make, in this section, the following assumption:

(H) Q is a symmetric function from \mathbb{R}^d to \mathbb{R} and β a positive constant satisfying

$$E_\omega \left[e^{\frac{\beta^2}{2} I_\infty(Q)} \right] < \infty, \quad \text{where} \quad I_\infty(Q) = \int_0^\infty Q(\omega_s) ds$$

Let us begin with our sufficient condition ensuring the weak disorder regime:

Proposition 3.1 *Under hypothesis (H), we have*

$$\mathbf{P}(W_\infty > 0) = 1 \quad \text{and} \quad p(\beta) = \frac{\beta^2 Q(0)}{2}.$$

Proof: We will divide this proof in two steps

Step 1: Let us compute $\mathbf{E}[Z_t^2]$: notice that

$$Z_t^2 = E_\omega \left[e^{\beta \int_0^t B(ds, \omega_s^1) + B(ds, \omega_s^2)} \right],$$

and hence, using the fact that $\int_0^t B(ds, \omega_s^1) + B(ds, \omega_s^2)$ is a Gaussian random variable for any fixed ω^1 and ω^2 , we get

$$\mathbf{E} [Z_t^2] = E_\omega \left[\exp \left(\frac{\beta^2}{2} \mathbf{E} \left[\left| \int_0^t B(ds, \omega_s^1) + B(ds, \omega_s^2) \right|^2 \right] \right) \right].$$

On the other hand, since

$$\int_0^t B(ds, \omega_s^1) + B(ds, \omega_s^2) = \int_0^t \int_{\mathbb{R}^d} \left(e^{\iota \lambda \omega_s^1} + e^{\iota \lambda \omega_s^2} \right) M(ds, d\lambda),$$

we get

$$\begin{aligned} & \mathbf{E} \left[\left| \int_0^t B(ds, \omega_s^1) + B(ds, \omega_s^2) \right|^2 \right] \\ &= \int_0^t \int_{\mathbb{R}^d} \left(e^{\iota \lambda \omega_s^1} + e^{\iota \lambda \omega_s^2} \right) \left(e^{-\iota \lambda \omega_s^1} + e^{-\iota \lambda \omega_s^2} \right) \hat{Q}(d\lambda) ds \\ &= 2 \left(Q(0)t + \int_0^t Q(\omega_s^1 - \omega_s^2) ds \right). \end{aligned}$$

Thus

$$\mathbf{E} [Z_t^2] = E_\omega \left[e^{\beta^2 (Q(0)t + \int_0^t Q(\omega_s^1 - \omega_s^2) ds)} \right]. \quad (17)$$

Step 2: Recall now that $W_t = Z_t e^{2^{-1} \beta Q(0)t}$. Thus, using the fact that $\omega^1 - \omega^2$ can be written, in law, as $2^{1/2} \omega$, where ω is again a \hat{P} -Brownian motion, we get

$$\mathbf{E} [W_t^2] = E_\omega \left[e^{\beta^2 \int_0^t Q(\omega_s^1 - \omega_s^2) ds} \right] \leq E_\omega \left[e^{\frac{\beta^2}{2} I_\infty(Q)} \right].$$

Hence, under assumption (H), W_t is a bounded martingale in L^2 with $\mathbf{E}[W_t] = 1$, which yields in particular $\mathbf{E}[W_\infty] = 1$, and thus $\mathbf{P}(W_\infty > 0) = 1$. The fact that $p(\beta) = \frac{\beta^2 Q(0)}{2}$ is now easily seen from (13).

□

Of course, Proposition 3.1 would be meaningless without some simple sufficient conditions on Q ensuring hypothesis (H). Those sufficient conditions will be given in the following

Proposition 3.2 *Assume $d \geq 3$, that Q is a positive radial function from \mathbb{R}^d to \mathbb{R} , and write $Q(x) = \tilde{Q}(|x|)$ for $x \in \mathbb{R}^d$, where \tilde{Q} is a positive function from \mathbb{R} to \mathbb{R} . Assume that β is small enough and that*

$$\int_0^\infty x \tilde{Q}(x) dx < \infty.$$

Then hypothesis (H) is satisfied.

Proof: We will recall first some results presented in [30]: let us denote by R_d the Bessel process in dimension d , by $\{l_t^x(R_d); x > 0, t \geq 0\}$ the local time of the Bessel process, and by X a standard planar Brownian Motion (we will assume that all those objects can be defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$). Then we have, for $d \geq 3$,

$$\{l_\infty^x(R_d); x > 0\} \stackrel{(\mathcal{L})}{=} \left\{ \frac{1}{(d-2)x^{d-3}} |X_{x^{d-2}}|^2; x > 0 \right\}.$$

Now, obviously, if $Q(x) = \tilde{Q}(|x|)$, we have

$$I_\infty(Q) = \int_0^\infty \hat{Q}(R_d(s)) ds = \int_0^\infty \hat{Q}(x) l_\infty^x(R_d) dx.$$

Hence, by changing variables, we get

$$\begin{aligned} E_\omega \left[e^{\frac{\beta^2}{2} I_\infty(Q)} \right] &= E_\omega \left[\exp \left(\frac{\beta^2}{2} \int_0^\infty \hat{Q}(x) l_\infty^x(R_d) dx \right) \right] \\ &= E_\omega \left[\exp \left(\int_0^\infty \beta^2 S(v) |X_v|^2 dv \right) \right] \\ &= \prod_{i \leq 2} E_\omega \left[\exp \left(\int_0^\infty \beta^2 S(v) |X_v^i|^2 dv \right) \right], \end{aligned}$$

where

$$S(v) = \frac{\tilde{Q}(v^{\frac{1}{d-2}})}{2(d-2)^2 v^{\frac{2(d-3)}{d-2}}}.$$

Observe now that, following Fernique's definitions and results (see [12]),

1. $\varphi \mapsto N(\varphi) = (\int_0^\infty S(v)\varphi^2(v)dv)^{1/2}$ is a gauge on $C(\mathbb{R}_+)$ (see [12, Definition 1.2.1]). Indeed, the only fact that has to be checked is to show that N is lower semi-continuous, ie that all the sets $\{\varphi; N(\varphi) \leq M\}$ are closed in $C(\mathbb{R}_+)$, for any value of $M \leq 0$. But this point is a direct consequence of Fatou's lemma.
2. Since, for $i = 1, 2$, X_s^i is a Gaussian process, if $N(X^i) < \infty$ almost surely and β is small enough, then $E_\omega[e^{\beta^2 N(X^i)}] < \infty$ (cf [12, Theorem 1.2.3]).

Thus, condition **(H)** is now implied, for β small enough, by the condition

$$\hat{P} \left(\int_0^\infty S(v)|X_v^1|^2 < \infty \right) = 1.$$

However, by [19, Proposition 2.2], this occurs iff $\int_0^\infty vS(v)dv < \infty$, which is equivalent to $\int_0^\infty x\tilde{Q}(x)dx < \infty$ by an elementary change of variables. \square

We can now state an improved concentration result below $\frac{\beta^2 Q(0)}{2}$ in the weak disorder regime:

Proposition 3.3 *Assume (H) is satisfied, and that*

$$E_\omega \left[I_\infty(Q) e^{\frac{\beta^2}{2} I_\infty(Q)} \right] < \infty.$$

Then there exists a positive constant K_1 depending on β and Q such that

$$\mathbf{P} \left(\log(Z_t) \leq \frac{\beta^2}{2} tQ(0) - u \right) \leq K_1 \exp \left(-\frac{u^2}{K_1} \right),$$

for all $u, t > 0$.

Proof: This proof will be again divided in two steps.

Step 1: *Some moment inequalities.*

Using (17) and (10) we have, under assumption **(H)**,

$$\frac{\mathbf{E}[Z_t^2]}{(\mathbf{E}[Z_t])^2} = E_\omega \left[e^{\beta^2 \int_0^t Q(\omega_s^1 - \omega_s^2) ds} \right] \leq K_1,$$

for some positive constant K_1 . Then Paley-Zygmund's inequality gives us

$$\sup_t \mathbf{P} \left(Z_t \geq \frac{1}{2} \mathbf{E}[Z_t] \right) \leq \sup_t \frac{1}{4} \frac{(\mathbf{E}[Z_t])^2}{\mathbf{E}(Z_t^2)} \geq K_2,$$

for some positive constant K_2 . For $t > 0$, set $I_t^{(2)}(Q) = \int_0^t Q(\omega_s^1 - \omega_s^2) ds$. Given another positive constant K_3 , and recalling notation (1), we are now able to compute

$$\begin{aligned} & \mathbf{P} \left(Z_t \geq \frac{1}{2} \mathbf{E}[Z_t], \langle I_t^{(2)}(Q) \rangle_t \leq K_3 \right) \\ & \geq \mathbf{P} \left(Z_t \geq \frac{1}{2} \mathbf{E}[Z_t], E_\omega \left[I_t^{(2)}(Q) e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right] \leq \frac{K_3 (\mathbf{E}[Z_t])^2}{4} \right) \\ & \geq \mathbf{P} \left(Z_t \geq \frac{1}{2} \mathbf{E}[Z_t] \right) - 1 \\ & \quad + \mathbf{P} \left(E_\omega \left[I_t^{(2)}(Q) e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right] \leq \frac{K_3 (\mathbf{E}[Z_t])^2}{4} \right). \end{aligned}$$

However, Chebychev's inequality yields

$$\begin{aligned} & \mathbf{P} \left(E_\omega \left[I_t^{(2)}(Q) e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right] > \frac{K_3 (\mathbf{E}[Z_t])^2}{4} \right) \\ & \leq \frac{4}{K_3 (\mathbf{E}[Z_t])^2} \mathbf{E} \left[E_\omega \left[I_t^{(2)}(Q) e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right] \right]^2 \\ & = \frac{4}{K_3 (\mathbf{E}[Z_t])^2} E_\omega \left[I_t^{(2)}(Q) \mathbf{E} \left[e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right] \right]^2. \end{aligned}$$

Since

$$\mathbf{E} \left[e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right] = (\mathbf{E}[Z_t])^2 e^{\beta^2 I_t^{(2)}(Q)},$$

we finally obtain

$$\mathbf{P} \left(Z_t \geq \frac{1}{2} \mathbf{E}[Z_t], \langle I_t^{(2)}(Q) \rangle_t \leq K_3 \right) \geq K_2 - \frac{4}{K_3} E_\omega \left[I_t^{(2)}(Q) e^{\beta^2 I_t^{(2)}(Q)} \right].$$

So, our assumptions imply that, choosing K_3 large enough, we have

$$\mathbf{P} \left(Z_t \geq \frac{1}{2} \mathbf{E}[Z_t], \langle I_t^{(2)}(Q) \rangle_t \leq K_3 \right) \geq \frac{1}{K_3}. \quad (18)$$

Step 2: *Application of the concentration inequalities.*

For a given Gaussian landscape B , that can be considered as an element of M , set

$$Z_t(B) = E_\omega \left[e^{\beta \int_0^t B(ds, \omega_s)} \right],$$

and

$$\langle f(\omega^1, \omega^2) \rangle_t^B := \frac{E_\omega \left(f(\omega^1, \omega^2) e^{\beta \sum_{i=1}^2 \int_0^t B(ds, \omega_s^i)} \right)}{Z_t^2(B)}.$$

For the constant K_3 used for inequality (18), we can now consider the set

$$A := \left\{ g \in M; Z_t(g) \geq \frac{1}{2} \mathbf{E}[Z_t], \langle I_t^{(2)}(Q) \rangle_t^g \leq K_3 \right\},$$

and we have checked that

$$\mathbf{P}(B \in A) \geq \frac{1}{K_3}.$$

Applying Lemma 2.3, this yields that, for all $u > 0$,

$$\mathbf{P}(q_A > u + K_4) \leq 2 \exp\left(-\frac{u^2}{2}\right), \quad (19)$$

with $K_4 = (2 \log(2K_3))^{\frac{1}{2}}$.

Consider now another Gaussian landscape \bar{B} , but keep the notation $Z_t = Z_t(B)$. We can write

$$\begin{aligned} Z_t &= E_\omega \left[e^{\beta \int_0^t B(ds, \omega_s)} \right] \\ &= E_\omega \left[e^{\beta \int_0^t B(ds, \omega_s) - \bar{B}(ds, \omega_s)} e^{\beta \int_0^t \bar{B}(ds, \omega_s)} \right] \\ &= Z_t(\bar{B}) \langle e^{\beta \int_0^t B(ds, \omega_s) - \bar{B}(ds, \omega_s)} \rangle_t^{\bar{B}} \\ &\geq Z_t(\bar{B}) e^{\beta \langle \int_0^t B(ds, \omega_s) - \bar{B}(ds, \omega_s) \rangle_t^{\bar{B}}}, \end{aligned} \quad (20)$$

where in the last step we have used Jensen's inequality. Suppose now that $B - \bar{B} = \tilde{g}$, where $\tilde{g}(t) = \int_0^t g(s) ds$ and g is the inverse Fourier transform of an element of \mathcal{H} . Notice that g admits the representation

$$g(s, x) = \int_{\mathbb{R}^d} e^{i\lambda x} h(s, \lambda) \hat{Q}(d\lambda), \quad \text{with } |h|_{\mathcal{H}}^2 = \int_0^\infty \int_{\mathbb{R}^d} |h(s, \lambda)|^2 \hat{Q}(d\lambda) ds < \infty.$$

Furthermore,

$$\begin{aligned} & \left| \left\langle \int_0^t B(ds, \omega_s) - \bar{B}(ds, \omega_s) \right\rangle_t^{\bar{B}} \right| = \left| \int_0^t \int_{\mathbb{R}^d} \langle e^{\iota \lambda \omega_s} \rangle_t^{\bar{B}} h(s, \lambda) \hat{Q}(d\lambda) ds \right| \\ & \leq |h|_{\mathcal{H}} \left(\int_0^t \int_{\mathbb{R}^d} \langle e^{\iota \lambda (\omega_s^1 - \omega_s^2)} \rangle_t^{\bar{B}} \hat{Q}(d\lambda) ds \right)^{1/2} = |h|_{\mathcal{H}} \left(\left\langle I_t^{(2)}(Q) \right\rangle_t^{\bar{B}} \right)^{1/2}. \end{aligned} \quad (21)$$

Thus, putting together (20) and (21), we get, if $\bar{B} \in A$ and $B - \bar{B} = \tilde{g}$,

$$\begin{aligned} \log(Z_t) & \geq \log(Z_t(\bar{B})) + \beta \left\langle \int_0^t B(ds, \omega_s) - \bar{B}(ds, \omega_s) \right\rangle_t^{\bar{B}} \\ & \geq \log(\mathbf{E}[Z_t]) - \log(2) - \beta |h|_{\mathcal{H}} K_3^{\frac{1}{2}} \\ & = \frac{\beta^2 t Q(0)}{2} - \log(2) - \beta |h|_{\mathcal{H}} K_3^{\frac{1}{2}}. \end{aligned}$$

Obviously, one can choose, in the above inequality, the norm $|h|_{\mathcal{H}}$ as close as desired to q_A . Thus, we get

$$\log(Z_t) \geq \frac{\beta^2 t Q(0)}{2} - \log(2) - \beta q_A K_3^{\frac{1}{2}},$$

and using (19) we have that, for all $u > 0$, the event

$$\log(Z_t) \geq \frac{\beta^2 t Q(0)}{2} - \log(2) - \beta K_3^{\frac{1}{2}}(u + K_4)$$

holds with probability larger than $1 - \exp(-\frac{u^2}{2})$. The proof is now easily completed. \square

4 The strong disorder regime

In this section, we will give some examples of Gaussian polymers in the strong disorder regime. We will begin with a general sufficient condition. Recall that Q is the covariance of our noise B .

Theorem 4.1 *Let $p > 1$ be a constant, $\{\Lambda_s; s \in \mathbb{R}^+\}$ a family of subsets of \mathbb{R}^d and*

$$\kappa = \frac{1}{2} \beta^2 Q(0) (1 - 4q)^2 q^{-1},$$

where q is the conjugate exponent of p . Based on these notations, set

$$\begin{aligned} v(s) &= \inf_{x \in \Lambda_s} Q(x) \\ w(s) &= \left(\inf_{x \in \Lambda_s} Q(x) \right) \hat{P}^{\frac{1}{p}}(\omega_s^1 - \omega_s^2 \in \Lambda_s^c) e^{\kappa s}, \end{aligned}$$

and assume that

$$\mathbf{(H1)} \quad \int_0^\infty v(s) = \infty \text{ and } \int_0^\infty w(s) < \infty.$$

Then

$$\mathbf{P}(W_\infty = 0) = 1.$$

Proof: Since $W_\infty \geq 0$, we have, for any $\theta > 0$,

$$\mathbf{E}[W_\infty^\theta] = \mathbf{E}[\liminf_{t \rightarrow \infty} W_t^\theta] \leq \liminf_{t \rightarrow \infty} \mathbf{E}[W_t^\theta].$$

Thus, it is enough to check that

$$\liminf_{t \rightarrow \infty} \mathbf{E}[W_t^\theta] = 0.$$

Recall now the martingale decomposition we got for W at (15): setting

$$X_s = \exp\left(N_s - \frac{\beta^2 Q(0)s}{2}\right),$$

one can write, for $t \geq 0$,

$$W_t = 1 + \beta \int_0^t \int_{\mathbb{R}^d} E_\omega [e^{\iota \lambda \omega_s} X_s] M(ds, d\lambda).$$

Pick now $0 < \theta < 1$. An application of Itô's formula gives

$$\begin{aligned} W_t^\theta &= 1 + \beta\theta \int_0^t \int_{\mathbb{R}^d} W_s^{\theta-1} E_\omega [X_s e^{\iota \lambda \omega_s}] M(ds, d\lambda) \\ &\quad - \frac{\beta^2}{2} \theta(1-\theta) \int_0^t \int_{\mathbb{R}^d} W_s^{\theta-2} (E_\omega [X_s e^{\iota \lambda \omega_s}])^2 \hat{Q}(d\lambda) ds. \end{aligned}$$

Then, taking expectations, we obtain

$$\begin{aligned}\mathbf{E} [W_t^\theta] &= 1 - \frac{\beta^2}{2}\theta(1-\theta)\mathbf{E} \left[\int_0^t \int_{\mathbb{R}^d} W_s^{\theta-2} (E_\omega [X_s e^{\lambda\omega_s}])^2 \hat{Q}(d\lambda) ds \right] \\ &= 1 - \frac{\beta^2}{2}\theta(1-\theta)\mathbf{E} \left[\int_0^t W_s^{\theta-2} E_\omega [X_s^1 X_s^2 Q(\omega_s^1 - \omega_s^2)] ds \right].\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{E} [W_s^{\theta-2} E_\omega [X_s^1 X_s^2 Q(\omega_s^1 - \omega_s^2)]] \\ &\geq \left(\inf_{\Lambda_s} Q \right) \mathbf{E} [W_s^{\theta-2} E_\omega [X_s^1 X_s^2 \mathbf{1}_{\{\omega_s^1 - \omega_s^2 \in \Lambda_s\}}]] \\ &= \left(\inf_{\Lambda_s} Q \right) \mathbf{E} [W_s^\theta] - \left(\inf_{\Lambda_s} Q \right) \mathbf{E} [W_s^{\theta-2} E_\omega [X_s^1 X_s^2 \mathbf{1}_{\{\omega_s^1 - \omega_s^2 \in \Lambda_s^c\}}]].\end{aligned}\quad (22)$$

On the other hand, Hölder's inequality yields, for any conjugate exponents p, q ,

$$\begin{aligned}&\mathbf{E} [W_s^{\theta-2} E_\omega [X_s^1 X_s^2 \mathbf{1}_{\{\omega_s^1 - \omega_s^2 \in \Lambda_s^c\}}]] \\ &= E_\omega [\mathbf{1}_{\{\omega_s^1 - \omega_s^2 \in \Lambda_s^c\}} \mathbf{E} [W_s^{\theta-2} X_s^1 X_s^2]] \\ &\leq \hat{P}_p^{\frac{1}{p}} (\omega_s^1 - \omega_s^2 \in \Lambda_s^c) E_\omega^{\frac{1}{q}} [\mathbf{E}^q [W_s^{\theta-2} X_s^1 X_s^2]].\end{aligned}\quad (23)$$

In particular, if $q = \theta^{-1}$, invoking the fact that $E [X^\rho] \leq E^\rho [X]$ for $\rho \leq 1$ and $X \geq 0$, we get that

$$\begin{aligned}&E_\omega [\mathbf{E}^q [W_s^{\theta-2} X_s^1 X_s^2]] \\ &= e^{-\frac{\beta^2}{2}sQ(0)} E_\omega \left[\mathbf{E}^q \left[E_\omega^{\theta-2} \left[e^{\beta \int_0^s B(du, \omega_u)} \right] e^{\beta \int_0^s B(du, \omega_u^1)} e^{\beta \int_0^s B(du, \omega_u^2)} \right] \right] \\ &\leq e^{-\frac{\beta^2}{2}sQ(0)} E_\omega \left[\mathbf{E} \left[E_\omega^{q(\theta-2)} \left[e^{\beta \int_0^s B(du, \omega_u)} \right] e^{q\beta \int_0^s B(du, \omega_u^1) + B(du, \omega_u^2)} \right] \right] \\ &\leq e^{-\frac{\beta^2}{2}sQ(0)} \mathbf{E} \left[E_\omega \left[e^{q\beta(\int_0^s B(du, \omega_u^1) + B(du, \omega_u^2)) + (1-2q)\beta \int_0^s B(du, \omega_u^3)} \right] \right] \leq e^{q\kappa s}.\end{aligned}\quad (24)$$

Then, putting together (22), (23) and (24), we obtain

$$\mathbf{E} [W_s^{\theta-2} E_\omega [X_s^1 X_s^2 Q(\omega_s^1 - \omega_s^2)]] \geq v(s) \mathbf{E} [W_s^\theta] - w(s),$$

Consequently,

$$\begin{aligned}\mathbf{E} [W_t^\theta] &\leq 1 - \frac{\beta^2}{2}\theta(1-\theta) \int_0^t v(s) \mathbf{E} [W_s^\theta] ds \\ &\quad + \frac{\beta^2}{2}\theta(1-\theta) \int_0^t w(s) ds.\end{aligned}$$

However, using our assumptions **(H1)** and setting

$$\gamma = \frac{\beta^2}{2}\theta(1 - \theta), \quad \text{and} \quad \delta = 1 + \gamma \int_0^\infty w(s)ds,$$

we get

$$\mathbf{E} [W_t^\theta] \leq \delta - \gamma \int_0^t v(s)\mathbf{E} [W_s^\theta] ds,$$

and by a standard comparison argument for ordinary differential equations, this yields

$$\mathbf{E} [W_t^\theta] \leq \delta e^{-\gamma \int_0^t v(s)ds},$$

and hence, invoking again Hypotesis **(H1)**,

$$\lim_{t \rightarrow \infty} \mathbf{E} [W_t^\theta] = 0,$$

which proves our claim. □

Example 4.2 Consider $d \geq 1$, and assume that the covariance function Q satisfies

$$c_1(1 + |x|^2)^{-\lambda} \leq Q(x) \leq c_2(1 + |x|^2)^{-\hat{\lambda}}, \quad (25)$$

for some constants $c_1 > 0, c_2 > 0$ and $0 < \hat{\lambda} \leq \lambda < \frac{1}{2}$. Then the polymer will be in the strong disorder regime for any value of $\beta > 0$.

Proof: Observe that there exist some positive definite functions Q satisfying (25), since a function of the type $c_1(1 + |x|^2)^{-\lambda}$ is the Fourier transform of a tempered measure (see [14, page 288]).

Now, Theorem 4.1 can be applied with an arbitrary constant $p > 1$, by choosing the set Λ_s as the centered ball of radius s^α in \mathbb{R}^d , with $\alpha > 1$ such that $\alpha\lambda < \frac{1}{2}$. Indeed, it is easily seen in this case that

$$v(s) \geq c_1(s^{2\alpha} + 1)^{-\lambda},$$

and

$$w(s) \leq c_2(s^{2\alpha} + 1)^{-\hat{\lambda}} \exp\left(-\frac{s^{2\alpha-1}}{8p} + \kappa s\right),$$

which proves that the assumption **(H1)** is verified. □

References

- [1] Albeverio, S; Zhou, X. (1996). A martingale approach to directed polymers in a random environment. *J. Theoret. Probab.* **9** , no. 1, 171–189.
- [2] Bolthausen, E. (1989). A note on the diffusion of directed polymers in a random environment. *Comm. Math. Phys.* **123** (1989), no. 4, 529–534.
- [3] Carmona, P; Hu, Y. (2002). On the partition function of a directed polymer in a Gaussian random environment. *Probab. Theory Relat. Fields*, **124**, 431-457.
- [4] Carmona, R; Molchanov, S. A. (1994). Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108**.
- [5] Carmona, R; Viens, F. (1998). Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter. *Stochastics and Stochastic Reports*, **62**, 251-273.
- [6] Comets, F; Yoshida, N. (2003). Brownian Directed Polymers in Random Environment. *Preprint*.
- [7] Conlon, J; Olsen, P. (1996). A Brownian motion version of the directed polymer problem. *J. Statist. Phys.* **84** , no. 3-4, 415–454.
- [8] Coyle, L. (1996). A continuous time version of random walks in a random potential. *Stochastic Process. Appl.* **64** , no. 2, 209–235.
- [9] Cranston, M; Mountford, T; Shiga, T. (2002). Lyapunov exponents for the parabolic Anderson model. *Acta Math. Univ. Comenian. (N.S.)* **71**, no. 2, 163–188.
- [10] Dawson, D.A.; Salehi, H. (1980). Spatially homogeneous random evolutions. *J. Multivariate Anal.* **10**, no. 2, 141-180.
- [11] Derrida, B; Spohn, H. (1988). Polymers on disordered trees, spin glasses, and traveling waves. New directions in statistical mechanics (Santa Barbara, CA, 1987). *J. Statist. Phys.* **51** , no. 5-6, 817–840. Derrida Spohn
- [12] Fernique, X. (1997). Fonctions aléatoires gaussiennes, vecteurs aléatoires gaussiens. Centre de Recherches Mathématiques, Montreal.

- [13] Feyel, D; Üstünel, A.S. (2004). Monge-Kantorovich Measure Transportation and Monge-Ampère Equation on Wiener Space. *Probab. Theory Relat. Fields*, **128**, 347-385.
- [14] Gel'fand, I.M.; Vilenkin, N. Ya.(1964). *Generalized Functions. Volume I*. Academic Press, New York.
- [15] Imbrie, J; Spencer, T. (1988). Diffusion of directed polymers in a random environment. *J. Statist. Phys.* **52**, no. 3-4, 609–626.
- [16] Major, P. (1981). *Multiple Wiener-It integrals. With applications to limit theorems*. Lecture Notes in Mathematics, **849**. Springer, Berlin.
- [17] Malliavin, P. (1997) *Stochastic Analysis*. Springer, Berlin.
- [18] Nualart, D. (1995) *The Malliavin Calculus and Related Topics*. Springer, Berlin.
- [19] Pitman, J; Yor, M. (1982). A decomposition of Bessel bridges. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **59**, 425-457.
- [20] Revuz,D; Yor, M. (1991). *Continuous martingales and Brownian motion*. Springer, Berlin.
- [21] Salminen, P; Yor, M. (2003). Properties of perpetual integral functionals of Brownian motion with drift. *Preprint*.
- [22] Sznitman, A. (1998). *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin.
- [23] Sinai, Yakov G. (1995). A remark concerning random walks with random potentials. *Fund. Math.* **147**, no. 2, 173–180.
- [24] Talagrand, M. (2003). *Spin Glasses: A challenge for Mathematicians*. Springer, Berlin.
- [25] Tindel, S; Viens, F. (2002). Almost sure exponential behaviour for a parabolic SPDE on a manifold. *Stochastic Process. Appl.* **100** , 53–74.
- [26] Tindel, S; Viens, F. (2001). Relating the almost-sure Lyapunov exponent of a parabolic SPDE and its coefficients' spatial regularity. To appear at *Potential Anal.*

- [27] Üstünel, A.S. (1995). *An introduction to analysis on Wiener space*. Lecture Notes in Mathematics **1610**. Springer-Verlag, Berlin.
- [28] Üstünel, A.S. *Private Communication*.
- [29] Wüthrich, M. (1998). Superdiffusive behavior of two-dimensional Brownian motion in a Poissonian potential. *Ann. Probab.* **26** , no. 3, 1000–1015.
- [30] Yor, M. (1992). *Some aspects of Brownian motion. Part I. Some special functionals*. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel.