Sharp asymptotics for the partition function of some continuous-time directed polymers

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Abstract

This paper is concerned with two related types of directed polymers in a random medium. The first one is a d-dimensional Brownian motion living in a random environment which is Brownian in time and homogeneous in space. The second is a continuous-time random walk on $\mathbb{Z}^d$, in a random environment with similar properties as in continuous space, albeit defined only on $\mathbb{R}_+ \times \mathbb{Z}^d$. The case of a space-time white noise environment can be acheived in this second setting. By means of some Gaussian tools, we estimate the free energy of these models at low temperature, and give some further information on the strong disorder regime of the objects under consideration.

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1 Introduction

1.1 Background, models, and motivation

Models for directed polymers in a random environment have been introduced in the physical literature [11, 14, 15, 19] for two main reasons. First, they provide a reasonably realistic model of a particle under the influence of a random medium, for which a number of natural questions can be posed, in terms of the asymptotic behavior for the path of the particle. The second point is that, in spite of the fact that polymers seem to be some more complicated objects than other disordered systems such as spin glasses, a lot more can be said about their behavior in the low temperature regime, as pointed out in [12, 14]. At a mathematical level, after two decades of efforts, a substantial amount of information about different models of polymer is now available, either in discrete or continuous space settings (see [9, 18, 20] and [4, 17] respectively).

The current article can be seen as a part of this global project consisting in describing precisely the polymer's asymptotic behavior, beyond the spin glass case. Except for some toy models such as the REM or GREM [2, 22], little is known about the low temperature behavior of the free energy for spin glasses systems, at least at a completely rigorous level. We shall see in this paper that polymer models are amenable to computations in this direction: we work to obtain some sharp estimates on the free energy of two different kind of polymers in continuous time, for which some scaling arguments seem to bring more information than in the discrete time setting. Here, in a strict polymer sense, time can also be interpreted as the length parameter of a directed polymer.

A word about random media appellations: we believe the term "random environment" normally implies that the underlying randomness is allowed to change over time; the appellation "random scenery" or "random landscape" is more specifically used for an environment that does not change over time; the models we consider herein fall under the time-varying "environment" umbrella. We now give some brief specifics about these models.

(1) We first consider a Brownian polymer in a Gaussian environment: the polymer itself is modeled by a Brownian motion $b = \{b_t; t \geq 0\}$, defined on a complete filtered probability space $(\mathcal{C}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (P_b^x)_{x\in\mathbb{R}^d})$, where P_b^x stands for the Wiener measure starting from the initial condition x. The corresponding expected value is denoted by E_b^x, or simply by E_b when $x = 0$.

The random environment is represented by a centered Gaussian random field W indexed by $\mathbb{R}_+ \times \mathbb{R}^d$, defined on another independent complete probability space $(\Omega, \mathcal{G}, \mathbf{P})$. Denoting by $\mathbf{E}$ the expected value with respect to $\mathbf{P}$, the covariance structure of W is given by

$$\mathbf{E}\left[W(t,x)W(s,y)\right] = (t \wedge s) \cdot Q(x-y), \tag{1}$$

for a given homogeneous covariance function $Q : \mathbb{R}^d \to \mathbb{R}$ satisfying some regularity conditions that will be specified later on. In particular, the function $t \mapsto [Q(0)]^{-1/2} W(t,x)$ will be a standard Brownian motion for any fixed $x \in \mathbb{R}^d$; for every fixed $t \in \mathbb{R}_+$, the process $x \mapsto t^{-1/2} W(t,x)$ is a homogeneous Gaussian field on $\mathbb{R}^d$ with covariance function Q. Notice that the homogeneity assumption is made here for sake of readability, but could be weakened for almost all the results we will show. The interested reader can consult [13] for the types of tools needed for such generalizations.

Once b and W are defined, the polymer measure itself can be described as follows: for any $t > 0$, the energy of a given path (or configuration) b on $[0,t]$ is given by the *Hamiltonian*

$$-H_t(b) = \int_0^t W(ds, b_s). \tag{2}$$

A completely rigorous meaning for this integral will be given in the next section, but for the moment, observe that for any fixed path b, $H_t(b)$ is a centered Gaussian random variable with variance $tQ(0)$. Based on this Hamiltonian, for any $x \in \mathbb{R}^d$, and a given constant β (interpreted as the inverse of the temperature of the system), we define our (random) polymer measure G_t^x (with $G_t := G_t^0$) as follows:

$$dG_t^x(b) = \frac{e^{-\beta H_t(b)}}{Z_t^x} dP_b^x(b), \quad \text{with} \quad Z_t^x = E_b^x \left[e^{-\beta H_t(b)} \right]. \tag{3}$$

(2) The second model we consider in this article is the continuous time random walk on $\mathbb{Z}^d$ in a white noise potential, which can be defined similarly to the Brownian polymer above: the polymer is modeled by a continuous time random walk $\hat{b} = \{\hat{b}_t; t \geq 0\}$ on $\mathbb{Z}^d$, defined on a complete filtered probability space $(\hat{\mathcal{C}}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0}, (\hat{P}_{\hat{b}}^x)_{x\in\mathbb{Z}^d})$. The corresponding expected value will be denoted by $\hat{E}_{\hat{b}}^x$, or simply by $\hat{E}_{\hat{b}}$ when $x = 0$. Notice that $\hat{b}$ can be represented in terms of its jump times $\{\tau_i; i \geq 0\}$ and its positions $\{x_i; i \geq 0\}$ between the jumps, as $\hat{b}_t = \sum_{i\geq 0} x_i \mathbf{1}_{[\tau_i, \tau_{i+1})}(t)$. Then, under $\hat{P}_{\hat{b}}$, $\tau_0 = x_0 = 0$, the sequence $\{\tau_{i+1} - \tau_i; i \geq 0\}$ is i.i.d with common exponential law $\mathcal{E}(2d)$, and the sequence $\{x_i; i \geq 0\}$ is a nearest neighbor symmetric random walk on $\mathbb{Z}^d$.

In this context, the random environment $\hat{W}$ will be defined as a sequence $\{\hat{W}(., z); z \in \mathbb{Z}^d\}$ of Brownian motions, defined on another independent complete probability space $(\hat{\Omega}, \hat{\mathcal{G}}, \hat{\mathbf{P}})$. Just like in the Brownian case described above, the covariance structure we assume on $\hat{W}$ is of the following type:

$$\hat{\mathbf{E}} \left[\hat{W}(t,x) \hat{W}(s,y) \right] = [t \wedge s] \, \hat{Q}(x-y), \tag{4}$$

for a covariance function $\hat{Q}$ defined on $\mathbb{Z}^d$. Note that the case where $\hat{Q}(z) = 0$ for all z except $\hat{Q}(0) > 0$, is the case where Brownian motions in the family $\{\hat{W}(\cdot, z); z \in \mathbb{Z}^d\}$ are independent, i.e. the case of space-time white noise. The Hamiltonian of our system can be defined formally similarly to the continuous case, as

$$-\hat{H}_t(\hat{b}) = \int_0^t \hat{W}(ds, \hat{b}_s).$$

Notice however that, since b is a piecewise constant function, the Hamiltonian $\hat{H}_t(\hat{b})$ can also be written as

$$-\hat{H}_t(\hat{b}) = \sum_{i=0}^{N_t} \hat{W}(\tau_{i+1}, x_i) - \hat{W}(\tau_i, x_i), \tag{5}$$

where N_t designates the number of jumps of $\hat{b}$ before time t, and $\tau_{N_t+1} = t$ by convention. Once the Hamiltonian $\hat{H}_t$ is defined, a Gibbs-type measure $\hat{G}_t$ can be introduced similarly to (3) in the Brownian case.

As mentioned before, our aim in this article is to give some sharp estimates on the free energies $p(\beta)$ and $\hat{p}(\beta)$ of the two systems described above, for large β. The quantities of interest are defined asymptotically as

$$p(\beta) = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\log(Z_t)\right], \quad \text{and} \quad \hat{p}(\beta) = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\log(\hat{Z}_t)\right];$$

it is well-known (see e.g. [20] for the Brownian case) that the limits above exist, are typically positive, and are both bounded from above by $Q(0)\beta^2/2$. It is then possible to separate a region of weak disorder from a region of strong disorder according to the value of $p(\beta)$: we will say that the polymer is in the weak disorder regime if $p(\beta) = Q(0)\beta^2/2$, while the strong disorder regime is defined by the strict inequality $p(\beta) < Q(0)\beta^2/2$. These two notions have some nice interpretations in terms of the behavior of the particle under the Gibbs measure (see e.g. [5, 10]), and it is expected, for any model of polymer in a random environment, that the strong disorder regime is attained whenever β is large enough. It is then natural to ask if one can obtain a sharper information than $p(\beta) < Q(0)\beta^2/2$ in the low temperature phase. Indeed, on the one hand, this may quantify in a sense how far we are from the weak disorder regime, and how much localization there is on our measures $G_t, \hat{G}_t$. On the other hand, the penalization method explained in [21] can be roughly summarized in the following way: if one can get a sharp equivalent for the quantity $E_b[e^{-\beta H_t(b)}]$, then this will also allow a detailed description of the limit $\lim_{t\to\infty} G_t$. This latter program is of course beyond the scope of the current article, but is a good motivation for getting some precise information about the function $p(\beta)$.

1.2 Summary of results

We now describe our main results. Our principal result in continuous space will be obtained in terms of the regularity of Q in a neighborhood of 0. In particular, we shall assume some upper and lower bounds on Q of the form

$$c_0|x|^{2H} \leq Q(0) - Q(x) \leq c_1|x|^{2H}, \quad \text{for all } x \text{ such that } |x| \in [0, r_0], \tag{6}$$

for a given exponent $H \in (0,1]$ and $r_0 > 0$. It should be noticed that condition (6) is equivalent to assuming that W has a specific almost-sure modulus of continuity in space, of order $|x|^H \log^{1/2}(1/|x|)$, i.e. barely failing to be H-Hölder continuous (see [23] for details). Then, under these conditions, we will get the following conclusions.

Theorem 1.1. *Assume that the function Q satisfies condition (6). Then the following hold true:*

1. *If $H \in [1/2, 1]$, we have for some constants $C_{0,d}$ and $C_{1,d}$ depending only on Q and d, for all $\beta \geq 1$,*
$$C_{0,d}\beta^{4/3} \leq p(\beta) \leq C_{1,d}\beta^{2-2H/(3H+1)}.$$

2. *If $H \in (0, 1/2]$, we have for some constants β_Q, $C'_{0,d}$, and $C'_{1,d}$ depending only on Q and d, for all $\beta \geq \beta_Q$,*
$$C'_{0,d}\beta^{2/(1+H)} \leq p(\beta) \leq C'_{1,d}\beta^{2-2H/(3H+1)}.$$

Corresponding almost sure results on $t^{-1}\mathbf{E}[\log(Z_t)]$ also hold, as seen in Corollary 1.3 and Proposition 2.1 below. Let us make a few elementary comments about the above theorem's bounds, which are also summarized in Figure 1.2. First of all, the exponent of β in those estimates is decreasing with H, which seems to indicate a stronger disorder when the Gaussian field W is smoother in space. Furthermore, in the case $H \in [1/2, 1]$, the gap between the two estimates decreases as H increases to 1; for $H = 1/2$, we get bounds with the powers of β equal to $4/3$ and $8/5$; and for $H = 1$, the bounds are $4/3$ and $3/2$. It should be noted that the case $H = 1/2$ is our least sharp result, while the case $H = 1$ yields the lowest power of β; one should not expect lower powers for any potential W even if W is so smooth that it is C^∞ in space: indeed, unless W is highly degenerate, the lower bound in (6) should hold with $H = 1$, while the upper bound will automatically be satisfied with $H = 1$. The case of small H is more interesting. Indeed, we can rewrite the lower and upper bounds above as
$$C'_{0,d}\beta^{2-2H+F(H)} \leq p(\beta) \leq C'_{1,d}\beta^{2-2H+G(H)}$$
where the functions F and G satisfy $F(x) = 2x^2 + O(x^3)$ and $G(x) = 6x^2 + O(x^3)$ for x near 0. We therefore see that the asymptotic β^{2-2H} is quite sharp for small H, but that the second order term in the expansion of the power of β for small H, while bounded, is always positive.

Using ideas introduced in [13] to deal with spatially non-homogeneous media, it is possible to extend Theorem 1.1. The reader will check that the first of the following two corollaries

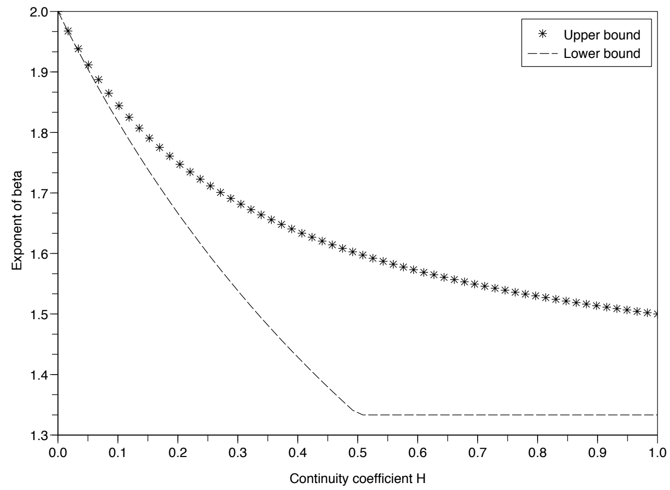

Figure 1: Exponent of β in function of H

is trivial to prove using the tools in this article. The second corollary requires techniques in [13], and can also be proved directly by using sub-Gaussian concentration results (see [27]). We do not give any details of its proof, for the sake of conciseness. Neither corollary assumes that W is spatially homogeneous. One will note that no assertion on the existence of $p(\beta)$ is made in these corollaries, but that the first corollary already implies strong disorder for large β in the sense that $\limsup t^{-1}\mathbf{E}\left[\log(Z_t)\right] < \beta^2 Q(0)/2$. [13] can be consulted for conditions under which $p(\beta)$ exists even if W is not spatially homogeneous.

Corollary 1.2. *In the non homogeneous case, the following bounds are satisfied:*

- [Upper bound] *Assume that for some $r_0, c_1 > 0$, for all $x, y \in \mathbb{R}^d$ such that $|x-y| \leq r_1$, the spatial canonical metric of W is bounded above as*

$$\delta^2(x,y) := \mathbf{E}\left[(W(1,x) - W(1,y))^2\right] \leq c_1 |x-y|^{2H}.$$

  Then, replacing $p(\beta)$ by $\limsup_{\beta\to\infty} t^{-1}\mathbf{E}\left[\log(Z_t)\right]$, the two upper bound results in Theorem 1.1 hold.

- [Lower bound] *Assume that for some $r_0, c_0 > 0$, for all $x, y \in \mathbb{R}^d$ such that $|x-y| \leq r_0$, we have*

$$\delta^2(x,y) := \mathbf{E}\left[(W(1,x) - W(1,y))^2\right] \geq c_0 |x-y|^{2H}.$$

  Then, replacing $p(\beta)$ by $\liminf_{\beta\to\infty} t^{-1}\mathbf{E}\left[\log(Z_t)\right]$, the two lower bound results in Theorem 1.1 hold.

Corollary 1.3. *Under the hypotheses of Corollary 1.2, its conclusions also hold* $\mathbf{P}$*-almost surely with* $\limsup_{\beta\to\infty} t^{-1}\mathbf{E}[\log(Z_t)]$ *replaced by* $\limsup_{\beta\to\infty} t^{-1}\log(Z_t)$*, and similarly for the* $\liminf$ *'s.*

Since our estimates become sharper as $H \to 0$, and also due to the fact that the behavior of $p(\beta)$ is nearly quadratic in β for small H (i.e. approaching the weak disorder regime), we decided to explore further the region of logarithmic spatial regularity for W, in order to determine whether one ever leaves the strong disorder regime. Namely, we also examine the situation of a covariance function Q for which there exist positive constants c_0, c_1, and r_1 such that for all x with $|x| \le r_1$,

$$c_0 \log^{-2\gamma}(1/|x|) \le Q(0) - Q(x) \le c_1 \log^{-2\gamma}(1/|x|), \tag{7}$$

where γ is a given positive exponent. Assumption (7) implies that W is not spatially Hölder-continuous for any exponent $H \in (0,1]$. Moreover, the theory of Gaussian regularity implies that, if $\gamma > 1/2$, W is almost-surely continuous in space, with modulus of continuity proportional to $\log^{-\gamma+1/2}(1/|x|)$, while if $\gamma \le 1/2$, W is almost-surely not uniformly continuous on any interval in space, and in fact is unbounded on any interval. We will then establish the following result, which is optimal, up to multiplicative constants.

Theorem 1.4. *Assume condition (7) where* $\gamma > 0$*. We have for some constants* $D_{0,d}$ *and* $D_{1,d}$ *depending only on* Q *and* d*, for all* β *large enough,*

$$D_{0,d}\beta^2 \log^{-2\gamma}(\beta) \le p(\beta) \le D_{1,d}\beta^2 \log^{-2\gamma}(\beta).$$

Besides giving a sharp result up to constants for the free energy $p(\beta)$, the last result will allow us to make a link between our Brownian model and the random walk polymer described by the Hamiltonian (5). Indeed, the following result will also be proved in the sequel.

Theorem 1.5. *Assume that* $\hat{Q}(0) - \hat{Q}(2) > 0$*, where* $\hat{Q}$ *has been defined at (4). Then the free energy* $\hat{p}(\beta)$ *of the random walk polymer* $\hat{b}$ *satisfies, for* β *large enough:*

$$D'_{0,d}\beta^2 \log^{-1}(\beta) \le \hat{p}(\beta) \le D'_{1,d}\beta^2 \log^{-1}(\beta), \tag{8}$$

for two constants $D_{0,d}$ *and* $D_{1,d}$ *depending only on* Q *and* d*.*

Relation (8) will be obtained here thanks to some simple arguments, which allow the extension to spatially inhomogeneous media. In the special homogeneous case of space-time white noise ($Q(x) = 0$ for all $x \ne 0$), more can be said: the exact value of the limit $\lim_{\beta\to\infty} \hat{p}(\beta)\log(\beta)/\beta^2$ can be computed in this situation; this result has been established by the authors of the work in preparation [3].

In relation with the continuous space model considered at Theorem 1.4, we see that to obtain the same behavior as with space-time white noise in discrete space, we need to use precisely the environment W in $\mathbb{R}^d$ with the logarithmic regularity corresponding to $\gamma = 1/2$ in (7). As mentioned before, this behavior of W happens to be exactly at the threshold in which W becomes almost-surely discontinuous and unbounded on every interval. Nevertheless such a W is still function-valued. Hence, for the purpose of understanding the polymer partition function, there is no need to study the space-time white noise in continuous space, for which $W(t,\cdot)$ is not a bonafide function (only a distribution), and for which the meaning of Z_t itself is difficult to even define. Another way to interpret the coincidence of behaviors for "*space-time white noise in* $\mathbb{R}_+ \times \mathbb{Z}^d$" and for "$\gamma = 1/2$" is to say that both models for W are function-valued and exhibit spatial discontinuity: indeed, in discrete space, one extends $W(t,\cdot)$ to $\mathbb{R}^d$ by making it piecewise constant, in order to preserve independence. The fact that the limit in Theorem 1.4 depends on γ does prove, however, that the continuous-space polymer model under logarithmic regularity is richer than the discrete-space one.

As in the Hölder-scale continuous space setting, we have the following corollaries, in which W is allowed to be spatially inhomogeneous. Again, we do not include proofs of these results for the sake of conciseness.

Corollary 1.6. *Assume the lower and upper bound hypotheses in Corollary 1.2 hold with* $|x-y|^{2H}$ *replaced by* $\log^{-2\gamma}(1/|x-y|)$*. Then the conclusions of Theorem 1.4 hold with* $p(\beta)$ *replaced by* $\liminf_{\beta\to\infty} t^{-1}\mathbf{E}[\log(Z_t)]$ *for the lower bound, and by* $\limsup_{\beta\to\infty} t^{-1}\mathbf{E}[\log(Z_t)]$ *for the upper bound. Almost-sure results as in Corollary 1.3 also hold.*

Corollary 1.7. *For the discrete space polymer in Theorem 1.5, assume, instead of* $\hat{Q}(0) > \hat{Q}(2)$*, that* $\mathbf{E}\left[(W(1,-1) - W(1,1))^2\right] > 0$*. Then the conclusions of Theorem 1.5 hold with* $\hat{p}(\beta)$ *replaced by* $\liminf_{\beta\to\infty} t^{-1}\hat{\mathbf{E}}[\log(\hat{Z}_t)]$ *for the lower bound, and with* $\hat{p}(\beta)$ *replaced by* $\limsup_{\beta\to\infty} t^{-1}\hat{\mathbf{E}}[\log(\hat{Z}_t)]$ *for the upper bound. Almost-sure results as in Corollary 1.3 also hold.*

Let us say a few words now about the methodology we have used in order to get our results. It is inspired by the literature on Lyapounov exponents for stochastic PDEs [6, 7, 8, 13, 25, 26]; our upper bound results rely heavily on the estimation of the supremum of some well-chosen Gaussian fields, using such results as Dudley's so-called entropy upper bound, and the Borell-Sudakov inequality (see [1] or [27]); our lower bound results are obtained more "by hand", by isolating very simple polymer configurations b or $\hat{b}$ which maximize the random medium's increments in the Hamiltonian $H_t(b)$ or $\hat{H}_t(\hat{b})$, and showing that these configurations contain enough weight to provide lower bounds. It turns out that these estimation procedures works better when the configuration b is simple enough, such as a piecewise constant or linear function. For the upper bound in the continuous case, a careful discretization of our Brownian path will thus have to be performed in order to get our main results; the resulting proof cannot exploit the discrete case itself because of the different nature of the discrete and continuous environments.

The structure of the article is as follows: Section 2 contains preliminary information on the partition function. Section 3 deals with the Brownian polymer. Section 4 covers the random walk polymer. In order to simplify the notation, throughout the paper we will use C to represent the constants, but acknowledge that the value it represents will change, even from line to line.

2 Preliminaries; the partition function

In this section, we will first recall some basic facts about the definition and the simplest properties of the partition functions Z_t and $\hat{Z}_t$ which have been already considered in the introduction. We will also give briefly some notions of Gaussian analysis which will be used later on.

We begin with basic information about the partition function of the Brownian polymer. Recall that W is a centered Gaussian field on $\mathbb{R}_+ \times \mathbb{R}^d$, defined by its covariance structure (1). The Hamiltonian $H_t(b)$ given by (2) can be defined more rigorously through a Fourier transform procedure: there exists (see e.g. [8] for further details) a centered Gaussian independently scattered $\mathbb{C}$-valued measure ν on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$W(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathbf{1}_{[0,t]}(s) e^{\imath ux} \nu(ds, du), \tag{9}$$

where the simple notation ux stands for the inner product $u \cdot x$ in $\mathbb{R}^d$. For every test function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$, set now

$$\nu(f) \equiv \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s,u) \nu(ds, du). \tag{10}$$

While the random variable $\nu(f)$ may be complex-valued, to ensure that it is real valued, it is sufficient to assume that f is of the form $f(s,u) = f_1(s) e^{\imath u f_2(s)}$ for real valued functions f_1 and f_2. Then the law of ν is defined by the following covariance structure: for any such test functions $f, g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{C}$, we have

$$\mathbf{E}[\nu(f)\nu(g)] = \int_{\mathbb{R}_+ \times \mathbb{R}^d} f(s,u) \overline{g(s,u)} \hat{Q}(du) ds, \tag{11}$$

where the finite positive measure $\hat{Q}$ is the Fourier transform of Q (see [24] for details).

From (9), we see that the Itô-stochastic differential of W in time can be understood as $W(ds,x) := \int_{u \in \mathbb{R}^d} e^{\imath ux} \nu(ds, du)$, or even, if the measure $\hat{Q}(du)$ has a density $f(u)$ with respect to the Lebesgue measure, which is typical, as

$$W(ds,x) := \int_{u \in \mathbb{R}^d} e^{\imath ux} \sqrt{f(u)} M(ds, du)$$

where M is a white-noise measure on $\mathbb{R}_+ \times \mathbb{R}^d$, i.e. a centered independently scattered Gaussian measure with covariance given by $\mathbf{E}[M(A)M(B)] = m_{Leb}(A \cap B)$ where m_{Leb} is Lebesgue's measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

We can go back now to the definition of $H_t(b)$: invoking the representation (9), we can write

$$-H_t(b) := \int_0^t W(ds, b_s) = \int_0^t \int_{\mathbb{R}^d} e^{\imath u b_s} \nu(ds, du), \tag{12}$$

taking this expression as a definition of $H_t(b)$ for each fixed path b; it can be shown (see [8]) that the right hand side of the above relation is well defined for any Hölder continuous path b, by a L^2-limit procedure. Such a limiting procedure can be adapted to the specific case of constructing $H_t(b)$, using the natural time evolution structure; we will not comment on this further. However, the reader will surmise that the following remark, given for the sake of illustration, can be useful: when $\hat{Q}$ has a density f, we obtain $-H_t(b) = \iint_{[0,t]\times\mathbb{R}^d} e^{\imath u b_s} \sqrt{f(u)} M(ds, du)$.

We use as the definition of the partition function Z_t^x, its expression in (3), and set its expectation under $\mathbf{P}$ as

$$p_t(\beta) := \frac{1}{t} \mathbf{E}[\log(Z_t^x)], \tag{13}$$

usually called the free energy of the system. It is easily seen that $p_t(\beta)$ is independent of the initial condition $x \in \mathbb{R}^d$, thanks to the spatial homogeneity of W. Thus, in the remainder of the paper, x will be understood as 0 when not specified, and E_b, Z_t will stand for E_b^0, Z_t^0, etc... We summarize some basic results on $p_t(\beta)$ and Z_t established in [20].

Proposition 2.1. *For all $\beta > 0$ there exists a constant $p(\beta) > 0$ such that*

$$p(\beta) := \lim_{t\to\infty} p_t(\beta) = \sup_{t\geq 0} p_t(\beta). \tag{14}$$

Furthermore, the function p satisfies:

1. *The map $\beta \mapsto p(\beta)$ is a convex nondecreasing function on $\mathbb{R}_+$.*
2. *The following upper bound holds true:*

$$p(\beta) \leq \frac{\beta^2}{2} Q(0). \tag{15}$$

3. $\mathbf{P}$*-almost surely, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t = p(\beta). \tag{16}$$

For the random walk polymer on $\mathbb{Z}^d$, the Hamiltonian $\hat{H}_t(\hat{b})$ is easier to define, and can be expressed in a simple way by (5). Recall then that $\hat{Z}_t, \hat{p}_t(\beta)$ are defined as:

$$\hat{Z}_t = \hat{E}_{\hat{b}}\left[e^{-\hat{H}_t(\hat{b})}\right], \qquad \text{and} \qquad \hat{p}_t(\beta) = \hat{\mathbf{E}}\left[\log(\hat{Z}_t)\right].$$

Then, using the same kind of arguments as in [20] (see also [3]), we get the following:

Proposition 2.2. *The same conclusions as in Proposition 2.1 hold true for the random walk polymer $\hat{b}$.*

3 Estimates of the free energy: continuous space

In this section, we will proceed to the proof of Theorems 1.1 and, 1.4, by means of some estimates for some well-chosen Gaussian random fields.

The hypothesis we use guarantees that there is some $H \in (0,1)$ such that W is no more than H-Hölder continuous in space. Accordingly, we define the homogeneous spatial *canonical metric* δ of W by

$$\delta^2(x-y) := \mathbf{E}\left[(W(1,x)-W(1,y))^2\right] = 2\left(Q(0)-Q(x-y)\right), \tag{17}$$

for all $x,y \in \mathbb{R}^d$. Our hypotheses on δ translate immediately into statements about Q via this formula.

In our results below, we have also tried to specify the dependence of our constants on the dimension of the space variable. An interesting point in that respect is given in the lower bound of Subsection 3.2 below, which has to do with weak versus strong disorder in very high-dimensional cases.

3.1 Upper bound in the Brownian case

The upper bound in Theorem 1.1 follows immediately from the following proposition, which proves in particular that strong disorder holds for all $H \in (0,1]$.

Proposition 3.1. *Assume that there exist a number $H \in (0,1]$ and numbers c_1, r_1 such that for all $x,y \in \mathbb{R}^d$ with $|x-y| \le r_1$ we have*

$$\delta(x-y) < c_1 \left|x-y\right|^H. \tag{18}$$

Then there exists a constant C depending only on Q and a constant β_0 depending only on r_1 and d, such that for all $\beta \ge \beta_0$,

$$p(\beta) \le C d^{\frac{7H}{1+3H}} \beta^{\frac{2+4H}{1+3H}}.$$

Proof. Let us divide the proof in several steps:

Step 1: Strategy. From relation (14), we have

$$p(\beta) \le \limsup_{t\to\infty} p_t(\beta).$$

Our strategy is then to give an estimation of $p_t(\beta)$ for a *discretized* path $\tilde{b} \in \varepsilon\mathbb{Z}^d$ that stays close to b and proceeds only by jumps. Thanks to this substitution, and using Hölder's and Jensen's inequalities, we shall obtain

$$\begin{aligned}
\mathbf{E}\left[\log(Z_t)\right] &= \mathbf{E}\left[\log E_b\left[\exp\left(-\beta\left[H_t(b)-H_t(\tilde{b})\right]\right)\exp -\beta H_t(\tilde{b})\right]\right] \qquad (19)\\
&\le \frac{1}{2}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta[H_t(b)-H_t(\tilde{b})]\right)\right]\right]+\frac{1}{2}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta H_t(\tilde{b})\right)\right]\right]\\
&\le \frac{1}{2}\log E_b\left[\exp 2\beta^2\int_0^t\left(\delta(b_s-\tilde{b}_s)\right)^2 ds\right]+\frac{1}{2}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta H_t(\tilde{b})\right)\right]\right].
\end{aligned}$$

Notice that the first term on the right-hand side represents the error made by considering the discretized path $\tilde{b}$ instead of b, but thanks to hypothesis (18) and the definition of $\tilde{b}$ we will easily control it.

Step 2: The discretized path. Let us describe now the discretized process we shall use in the sequel: we will approximate the Brownian path b with a path that stays in $\varepsilon\mathbb{Z}^d$, where ε is a small positive number. Let b^j be the j-th component of the d-dimensional path b. Let T_1^j be the first time that b^j exits the interval $(-\varepsilon,\varepsilon)$ and T_{i+1}^j be the first time after T_i^j that b^j exits $(b_{T_i^j}-\varepsilon, b_{T_i^j}+\varepsilon)$. So, for a fixed component j, the times $(T_{i+1}^j - T_i^j)_{i=0}^{\infty}$ are i.i.d. and the successive positions $x_m^j = b_{T_m^j}^j$, which are independent of the jump times, form a one-dimensional symmetric random walk on $\varepsilon\mathbb{Z}$ in discrete time.

Now let $(T_n)_{n=0}^{\infty}$ be the increasing sequence of all the $(T_m^j)_{j,m}$ and let $(x_n)_{n=0}^{\infty}$ be the nearest neighbor path in $\varepsilon\mathbb{Z}^d$ with $x_0=0$ whose j-th component takes the same step as x_m^j at time T_m^j. We define the *discretized path* $\tilde{b}$ as the path that jumps to site x_n at time T_n and it is constant between jumps.

Remark 3.2. *At any time s, each coordinate of $\tilde{b}_s$ is within ε of the corresponding one of b_s. So the distance separating the two paths is never more than $\varepsilon\sqrt{d}$. Thus we have, for all $s \ge 0$, $|b_s-\tilde{b}_s| \le \varepsilon d^{1/2}$.*

Remark 3.3. *Thanks to Remark 3.2 we can now control the error term we have defined at relation (19). In fact, owing to Hypothesis (18), we have*

$$\begin{aligned}
&\frac{1}{2t}\log E_b\left[\exp 2\beta^2\int_0^t\left(\delta(b_s-\tilde{b}_s)\right)^2 ds\right]\\
&\qquad\le \frac{1}{2t}\log E_b\left[\exp\left(2\beta^2C^2\int_0^t|b_s-\tilde{b}_s|^{2H}dt\right)\right]\le C\beta^2\varepsilon^{2H}d^H,
\end{aligned}$$

where we recall that C is a constant depending on Q that can change from line to line.

Plugging this last inequality into (19), and defining

$$p_t^{\varepsilon}(\beta)=\frac{1}{t}\mathbf{E}\left[\log E_b\left[\exp\left(-2\beta H_t(\tilde{b})\right)\right]\right],$$

we have thus obtained the following estimate for $p_t(\beta)$:

$$p_t(\beta) \leq C\beta^2\varepsilon^{2H}d^H + \frac{1}{2}p_t^\varepsilon(\beta). \tag{20}$$

We shall try now to get some suitable bounds on $p_t^\varepsilon(\beta)$.

Step 3: Study of $p_t^\varepsilon(\beta)$. Let N_t^j be the number of jumps of the j-th component of $\tilde{b}$ up to time t. For a multi-index $k = (k_1, \cdots, k_d)$ let $|k| = k_1 + \cdots + k_d$, so the total number of jumps of $\tilde{b}$ up to time t is $|N_t| = N_t^1 + \cdots + N_t^d$. Denote by $\mathcal{S}(t,n)$ the simplex of all possible sequences of n jump times up to time t, namely

$$\mathcal{S}(t,n) = \left\{ \mathbf{t} = (t_0, \cdots, t_n) : 0 = t_0 \leq \cdots \leq t_n \leq t \right\}. \tag{21}$$

The set of the first k_j jump times of the j-th component of $\tilde{b}$ is a point $(t_i^j)_{i=1}^{k_j}$ in $\mathcal{S}(t,k_j)$. Given the set of all jump times $\left\{t_i^j : j \in [1, \cdots, d]\, ; i \in [1, \cdots, k_j]\right\}$, let $\left\{\tilde{t}_l : l \in [0, |k|+1]\right\}$ be the same set but ordered and with the convention $\tilde{t}_0 = 0$, $\tilde{t}_{|k|+1} = t$. And finally let $\tilde{x}_l$ be the value of $\tilde{b}$ between the two jump times $\tilde{t}_l$ and $\tilde{t}_{l+1}$. Denote by $\mathcal{P}_n$ the set of all such $\tilde{x}$, i.e. the set of all nearest-neighbor random walk paths of length k starting at the origin.

Then if we fix $|N_t| = |k|$, we can write

$$H_t(\tilde{b}) = X\left(|k|, (\tilde{t}_l)_{l=1}^{|k|}, (\tilde{x}_l)_{l=1}^{|k|}\right),$$

where

$$X\left(|k|, (\tilde{t}_l)_{l=1}^{|k|}, (\tilde{x}_l)_{l=1}^{|k|}\right) = \sum_{i=0}^{|k|} \left[W(\tilde{t}_{i+1}, \tilde{x}_i) - W(\tilde{t}_i, \tilde{x}_i)\right].$$

Thanks to these notations, we have

$$\begin{aligned} tp_t^\varepsilon(\beta) &= \mathbf{E}\left[\log E_b\left[\exp(-2\beta H_t(\tilde{b}))\right]\right] \\ &= \mathbf{E}\left[\log E_b\left[\exp\left(-2\beta X\left(|N_t|, (\tilde{t}_l)_{l=1}^{|N_t|}, (\tilde{x}_l)_{l=1}^{|N_t|}\right)\right)\right]\right]. \end{aligned}$$

So we can write the expectation with respect to b as:

$$\begin{aligned} E_b\left[\exp(-2\beta H_t(\tilde{b}))\right] = \sum_{n\geq 1} E_b &\left[\exp(-2\beta H_t(\tilde{b}))\Big| |N_t| \in [t\alpha(n-1), t\alpha n]\right] \\ &\times P_b\left[|N_t| \in [t\alpha(n-1), t\alpha n]\right]. \end{aligned}$$

The number of jumps of the discretized path $\tilde{b}$ in a given interval $[0,t]$ will play a crucial role in our optimization procedure. For a parameter $\alpha > 0$ which will be fixed later on, let us thus define

$$T_{n\alpha} = \left\{ \left(k, \tilde{t}, \tilde{x}\right) : k \leq tn\alpha, \tilde{t} \in \mathcal{S}(t,k), \tilde{x} \in \mathcal{P}_k \right\}.$$

Then the following estimates will be essential for our future computations:

$$P_b\left[N_t^j > n\alpha t\right] \le \exp\left(-\frac{t}{2}(\alpha n\varepsilon)^2 + t\alpha n\right) \tag{22}$$

$$\mathbf{E}\left[\sup_{T_{n\alpha}} X(k,\tilde{t},\tilde{x})\right] \le Ktd\sqrt{n\alpha}, \tag{23}$$

where K is a constant that depends on the covariance of the environment Q. Inequality (22) can be found textually in [13]. Inequality (23) is established identically to equation (30) in [13], with the minor difference that the total number of paths in $\mathcal{P}_m$ is not 2^m but $(2d)^m$, which, in the inequality above (30) near the bottom of page 33 in [13], accounts for a factor $e^{1+\log(6d)} = 6ed$ instead of e^{c_1} therein, hence the factor d in (23).

Defining $Y_{n\alpha} = \sup_{T_{n\alpha}} X(k,\tilde{t},\tilde{x})$, we can now bound $p_t^\varepsilon(\beta)$ as follows:

$$tp_t^\varepsilon(\beta) \le \mathbf{E}\left[\log(A+B)\right] \le \mathbf{E}\left[(\log A)_+\right] + \mathbf{E}\left[(\log B)_+\right] + \log 2,$$

where

$$A = P_b\left[|N_t| \le \alpha t\right]\exp\left(2\beta Y_\alpha\right), \quad \text{and} \quad B = \sum_{n\ge 1} P_b\left[|N_t| \in [n\alpha t,(n+1)\alpha t]\right]\exp\left(2\beta Y_{\alpha(n+1)}\right).$$

We will now bound the terms A and B separately.

Step 4: The factor A. We can bound $P_b\left[|N_t| \le \alpha t\right]$ by 1 and we easily get, invoking (23),

$$\mathbf{E}\left[(\log A)_+\right] \le 2\beta\mathbf{E}\left[Y_\alpha\right] \le 2\beta Kdt\sqrt{\alpha}. \tag{24}$$

Step 5: The factor B. Let $\mu = \mathbf{E}\left[Y_{\alpha(n+1)}\right]$. Since X is a Gaussian field and since it is easy to show that

$$\sigma^2 := \sup_{(m,\tilde{t},\tilde{x})} \mathbf{Var}(X(k,\tilde{t},\tilde{x})) \le tQ(0),$$

the so called Borell-Sudakov inequality (see [1] or [27]) implies that, for a constant $a > 0$,

$$\mathbf{E}\left[\exp\left(a\left|Y_{\alpha n} - \mu\right|\right)\right] \le 2\exp\left(\frac{a^2\sigma^2}{2}\right) = 2\exp\left(\frac{a^2 tQ(0)}{2}\right). \tag{25}$$

Fix now a number $\gamma \in (1/2, 1)$ and let us denote $\log_+(B) = (\log B)_+$. We have

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] = \mathbf{E}\left[\log_+\left(\sum_{n\ge 1} P_b\left[|N_t| \in [nt\alpha,(n+1)t\alpha]\right]\exp\left(2\beta Y_{\alpha(n+1)}\right)\right)^{t^{-\gamma}}\right]$$

$$\le \mathbf{E}\left[\log_+\left(\sum_{n\ge 1} P_b\left[|N_t| > nt\alpha\right]\exp\left(2\beta(Y_{\alpha(n+1)} - \mu)\right)\exp\left(2\beta Ktd\sqrt{\alpha(n+1)}\right)\right)^{t^{-\gamma}}\right],$$

where we used that (23) implies $\mu \leq Kdt\sqrt{(n+1)\alpha}$. We also know that for any sequence of non-negative reals $(x_n)_n$ the following holds: $(\sum_n x_n)^{t^{-\gamma}} \leq \sum_n x_n^{t^{-\gamma}}$. Thus we have

$$\begin{aligned}
&\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] \\
&\leq \mathbf{E}\left[\log_+\left(\sum_{n\geq 1}\left(P_b\left[|N_t| > nt\alpha\right]\right)^{t^{-\gamma}} \exp\left(\frac{2\beta}{t^\gamma}(Y_{\alpha(n+1)} - \mu)\right)\exp\left(2t^{1-\gamma}\beta K d\sqrt{\alpha(n+1)}\right)\right)\right] \\
&\leq \mathbf{E}\left[\log_+\left[d^{t^{-\gamma}}\sum_{n\geq 1}\exp\left(\frac{2\beta}{t^\gamma}(Y_{\alpha(n+1)} - \mu)\right)\exp\left(-\frac{t^{1-\gamma}}{2}y_n\right)\right]\right],
\end{aligned}$$

where we used estimate (22) in the following way:

$$\begin{aligned}
P_b\left[|N_t| > nt\alpha\right] &\leq \sum_{j=1}^{d} P_b\left[N_t^j > \frac{nt\alpha}{d}\right] = dP_b\left[N_t^1 > \frac{nt\alpha}{d}\right] \\
&\leq d\exp\left(-\frac{t}{2}\left(\frac{\alpha n\varepsilon}{d}\right)^2 + \frac{t\alpha n}{d}\right),
\end{aligned}$$

and where we have obtained:

$$y_n = \left(\frac{\varepsilon\alpha n}{d}\right)^2 - \frac{2\alpha n}{d} - 4\beta K d\sqrt{\alpha(n+1)}.$$

Now, bounding $\log_+(x)$ from above by $\log(1+x)$, for $x \geq 1$, and using Jensen's inequality, we have:

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] \leq \log\left[1 + \sum_{n\geq 1}\mathbf{E}\left[\exp\left(\frac{2\beta}{t^\gamma}(Y_{\alpha(n+1)} - \mu)\right)\right]\exp\left(\frac{-t^{1-\gamma}}{2}y_n\right)\right],$$

so, using (25), it is readily checked that

$$\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] \leq \log\left[1 + 2\exp\left(\frac{2\beta^2 Q(0)}{t^{2\gamma-1}}\right)\sum_{n\geq 1}\exp\left(\frac{-t^{1-\gamma}}{2}y_n\right)\right].$$

In order for the series above to converge, we must choose α so as to compensate the negative terms in y_n. Specifically, we choose

$$\left(\frac{\alpha\varepsilon}{d}\right)^2 = 16\beta K d\sqrt{\alpha}, \quad \text{i.e.} \quad \alpha = (16\beta K d^3\varepsilon^{-2})^{2/3}. \tag{26}$$

With this choice, we end up with:

$$y_n = \left(\frac{\alpha\varepsilon}{d}\right)^2\left(n^2 - \frac{2dn}{\alpha\varepsilon^2} - \frac{1}{4}\sqrt{n+1}\right).$$

Now we note that:

$$\text{If we choose } \varepsilon, \beta \text{ such that } \beta\varepsilon \geq d^{-3/2} \quad \Rightarrow \quad \frac{\alpha\varepsilon^2}{d} = (16K\beta\varepsilon)^{2/3} d \geq 4, \tag{27}$$

so that

$$y_n \geq \left(\frac{\alpha\varepsilon}{d}\right)^2 \left(n^2 - \frac{n}{2} - \frac{1}{4}\sqrt{n+1}\right),$$

and since $n^2 - \frac{n}{2} - \frac{\sqrt{n+1}}{4} \geq \frac{n}{8}$, we get

$$\begin{aligned}
&\sum_{n\geq 1} \exp\left(-\frac{t^{1-\gamma}}{2}\left(\frac{\alpha\varepsilon}{d}\right)^2 \left(n^2 - \frac{2dn}{\alpha\varepsilon^2} - \frac{1}{4}\sqrt{n+1}\right)\right) \\
&\leq \sum_{n\geq 1} \exp\left(-\frac{t^{1-\gamma}}{2}\left(\frac{\alpha\varepsilon}{d}\right)^2 \frac{n}{8}\right) = \frac{1}{1-\exp\left(-\frac{1}{16}t^{1-\gamma}\left(\frac{\alpha\varepsilon}{d}\right)^2\right)} - 1.
\end{aligned}$$

Notice that this last term can be made smaller than 1 if t is large enough. Hence we can write a final estimate on $\mathbf{E}\left[\log_+ B\right]$ as follows: for large t we have

$$\begin{aligned}
\frac{1}{t^\gamma}\mathbf{E}\left[\log_+ B\right] &\leq \log\left[1 + 2d^{t^{-\gamma}} \exp\left(\frac{2\beta^2 Q(0)}{t^{2\gamma-1}}\right)\right] \\
&\leq \log(1 + 2d^{t^{-\gamma}}) + \frac{2\beta^2 Q(0)}{t^{2\gamma-1}}.
\end{aligned} \tag{28}$$

Final step. Using inequalities (24) and (28) and the value of α, we can estimate $p_t^\varepsilon(\beta)$ in the following way:

$$\begin{aligned}
p_t^\varepsilon(\beta) &\leq 2\beta K d\sqrt{\alpha} + \frac{\log 2}{t} + \frac{\log(1+2d^{t^{-\gamma}})}{t^{1-\gamma}} + \frac{2\beta^2 Q(0)}{t^\gamma} \\
&\leq 2\beta K d\sqrt{\alpha} + o(1).
\end{aligned}$$

So using the value of α given in (26) we have

$$p_t^\varepsilon(\beta) \leq C\frac{\beta^{4/3}d^2}{\varepsilon^{2/3}} + o(1), \tag{29}$$

where C is a constant that depends on Q and that can change from line to line. Putting this result in (20) and taking the limit for $t \to \infty$ we get

$$\limsup_{t\to\infty} p_t(\beta) \leq C\left(\beta^2 d^H \varepsilon^{2H} + d^2\beta^{4/3}\varepsilon^{-2/3}\right).$$

In order to make this upper bound as small as possible we can choose ε such that

$$\beta^2 d^H \varepsilon^{2H} = d^2\beta^{4/3}\varepsilon^{-2/3}, \quad \text{i.e.} \quad \varepsilon = d^{\frac{6-3H}{2+6H}}\beta^{-\frac{1}{1+3H}},$$

so that

$$\limsup_{t\to\infty} p_t(\beta) \le C\beta^{\frac{2+4H}{1+3H}} d^{\frac{7H}{1+3H}},$$

which is the announced result. We then only need to check for what values of β we are allowed to make this choice of ε. Condition (18) states that we must use $\varepsilon \le r_1$. This is equivalent to $\beta \ge \beta_0 =: (r_1)^{-1-3H} d^{3-3H/2}$. One can check in this case that the restriction on ε, β in (27) is trivially satisfied. □

3.2 Lower bound in the Brownian case

In the following proposition, which implies the lower bound in Theorem 1.1, we shall also try to specify the dependence of the constants with respect to the dimension d. Let us state an interesting feature of this dependence. The proof of the proposition below shows that the results it states hold only for $\beta \ge \beta_0 = cd^{1-H/2}$. One may ask the question of what happens to the behavior of the partition function when the dimension is linked to the inverse temperature via the relation $\beta = \beta_0$, and one allows the dimension to be very large. The lower bounds on the value $p(\beta)$ in the proposition below will then increase, and while they must still not exceed the global bound $\beta^2 Q(0)/2$, the behavior for large β turns out to be quadratic in many cases. The reader will check that, when $H > 1/2$, this translates as $p(\beta) \ge c\beta^{2/(2-H)}$ which is quadratic when $H = 1$, and $p(\beta) \ge c\beta^2$ for all $H \le 1/2$. This is an indication that for extremely high dimensions and inverse temperatures, for $H \le 1/2$ or $H = 1$, strong disorder may not hold. Strong disorder for Brownian polymers may break down for complex, infinite-dimensional polymers. This is only tangential to our presentation, however.

Proposition 3.4. *Recall that δ has been defined at (17) and assume that there exist a number $H \in (0,1]$ and some positive constants c_2, r_2 such that for all $x, y \in \mathbb{R}^d$ with $|x-y| \le r_2$, we have*

$$\delta(x-y) > c_2 |x-y|^H. \tag{30}$$

Then if $H \le 1/2$, there exists a constant C depending only on Q, and a constant β_0 depending only on Q and d, such that, for all $\beta > \beta_0$,

$$p(\beta) \ge Cd^{\frac{2H-1}{H+1}} \beta^{\frac{2}{H+1}}.$$

On the other hand if $H > 1/2$, there exists a constant C' depending only on Q, and a constant β_0' depending only on Q and d, such that for all $\beta > \beta_0'$

$$p(\beta) \ge C' d^{\frac{2H-1}{3}} \beta^{\frac{4}{3}}.$$

Proof. Here again, we divide the proof in several steps.

Step 1: Strategy. From relation (14), we have

$$p(\beta) = \sup_{t \geq 0} p_t(\beta),$$

where $p_t(\beta)$ is defined by equation (13). So a lower bound for $p(\beta)$ will be obtained by evaluating $p_t(\beta)$ for any fixed value t. Additionally, by the positivity of the exponential factor in the definition of Z_t, one may include as a factor inside the expectation E_b the sum of the indicator functions of any disjoint family of events of Ω_b. In fact, we will need only two events, which will give the main contribution to Z_t at a logarithmic scale.

Step 2: Setup. Let $A_+(b)$ and $A_-(b)$ be two disjoints events defined on the probability space Ω_b under P_b, which will be specified later on. Set

$$X_b = -\beta H_{2t} = \beta \int_0^{2t} W(ds, b_s).$$

Conditioning by the two events $A_+(b)$ and $A_-(b)$ and using Jensen's inequality we have

$$\mathbf{E}(\log Z_t) \geq \log\left(\min\left\{P_b(A_+); P_b(A_-)\right\}\right) + \mathbf{E}\left[\max\left\{\tilde{Z}_+; \tilde{Z}_-\right\}\right], \tag{31}$$

where

$$\tilde{Z}_+ := E_b\left[X_b \mid A_+\right] \quad \text{and} \quad \tilde{Z}_- := E_b\left[X_b \mid A_-\right].$$

These two random variables form a pair of centered jointly Gaussian random variables: indeed they are both limits of linear combinations of values of a single centered Gaussian field. Thus this implies

$$\mathbf{E}\left[\max\left\{\tilde{Z}_+; \tilde{Z}_-\right\}\right] = \frac{1}{\sqrt{2\pi}}\left(\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right]\right)^{1/2}.$$

Therefore we only have to choose sets A_+ and A_- not too small, but still decorrelated enough so that condition (30) guarantees a certain amount of positivity in the variance of $\tilde{Z}_+ - \tilde{Z}_-$.

Step 3: Choice of A_+ and A_-. Let f be a positive increasing function. We take

$$A_+ = \left\{f(t) \leq b_s^i \leq 2f(t), \forall i = 1, \ldots, d, \quad \forall s \in [t, 2t]\right\},$$

$$A_- = \left\{-2f(t) \leq b_s^i \leq -f(t), \forall i = 1, \ldots, d, \quad \forall s \in [t, 2t]\right\}.$$

In other words, we force each component of our trajectory b to be, during the entire time interval $[t, 2t]$, in one of two boxes of edge size $f(t)$ which are at a distance of $2f(t)$ from each other. Because these two boxes are symmetric about the starting point of b, the corresponding events have the same probability. While this probability can be calculated in an arguably explicit way, we give here a simple lower bound argument for it. Using time scaling, the Markov property of Brownian motion, the notation $a = f(t)/\sqrt{t}$, we have

$$\begin{aligned} P_b(A_+) &= \prod_{i=1}^{d} P_b\left(\forall s \in [1,2] : b_s^i \in [a, 2a]\right) \\ &= \prod_{i=1}^{d} \frac{1}{2\pi} \int_a^{2a} P_b\left(\forall s \in [0,1] : b_s^i + y \in [a, 2a]\right) e^{-y^2/2} dy \\ &\geq \prod_{i=1}^{d} \frac{1}{2\pi} \int_{5a/4}^{7a/4} P_b\left(\forall s \in [0,1] : b_s^i + y \in \left[y - \frac{a}{4}, y + \frac{a}{4}\right]\right) e^{-y^2/2} dy \\ &= \left[P_b\left(b_1^1 \in [5a/4, 7a/4]\right) P_b\left(\forall s \in [0,1] : |b_s^1| \leq a/4\right)\right]^d . \end{aligned} \tag{32}$$

Step 4: Estimation of $\tilde{Z}_+$ and $\tilde{Z}_-$. It was established in [13] that in dimension $d = 1$

$$\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right] \geq \beta^2 \int_t^{2t} \mathbf{E}\left[\left(\delta(x_{s,+}^* - x_{s,-}^*)\right)^2\right] ds$$

where the quantities $x_{s,+}^*$ and $x_{s,-}^*$ are random variables such that for all $s \in [t, 2t]$: $x_{s,+}^* \in [f(t), 2f(t)]$ and $x_{s,-}^* \in [-2f(t), -f(t)]$. In dimension $d \geq 1$ the result still holds. In fact in this case we have $x_{s,+}^*, x_{s,-}^* \in \mathbb{R}^d$, so it is sufficient to take each component of the $x_{s,+}^*$ in the interval $[f(t), 2f(t)]$ and each component of $x_{s,-}^*$ in $[-2f(t), -f(t)]$, so their distance is greater than $d^{1/2} f(t)$. Thus, using condition (30), we have

$$\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right] \geq \beta^2 \int_t^{2t} C \left|x_{s,+}^* - x_{s,-}^*\right|^{2H} ds \geq Ct\beta^2 d^H \left(f(t)\right)^{2H}, \tag{33}$$

where as usual C is a constant that can change from line to line. Hence, we obtain:

$$\mathbf{E}\left[\max\left\{\tilde{Z}_+; \tilde{Z}_-\right\}\right] = \frac{1}{\sqrt{2\pi}} \left(\mathbf{E}\left[\left(\tilde{Z}_+ - \tilde{Z}_-\right)^2\right]\right)^{1/2} \geq C\beta\sqrt{t}\left(f(t)\right)^H d^{H/2}, \tag{34}$$

Observe that in order to use condition (30) we have to impose $f(t) \leq r_2$.

Step 5: The case $H \leq 1/2$. It is possible to prove that in this case the optimal choice for f is $f(t) = \sqrt{t}$, which corresponds to $a = 1$, so that $P_b(A_+)$ is a universal constant that does not depend on t. Thus we have, from (31), (32) and (34), for any $t > 0$,

$$p_{2t}(\beta) = \frac{\mathbf{E}\left[\log Z_{2t}\right]}{2t} \geq \frac{d \log C}{2t} + C\beta d^{H/2} t^{\frac{H-1}{2}}. \tag{35}$$

Now we may maximize the above function over all possible values of $t > 0$. To make things simple, we choose t so that the second term equals twice the first, yielding t of the form $t = Cd^{\frac{2-H}{H+1}} \beta^{-\frac{2}{H+1}}$, and therefore

$$\sup_{t>0} p_{2t}(\beta) \geq Cd^{\frac{2H-1}{H+1}} \beta^{\frac{2}{H+1}}.$$

This result holds as long as the use of condition (30) can be justified, namely as long as $f(t) \leq r_2$. This is achieved as soon as $\beta > \beta_0$ where $\beta_0 = C r_2^{-H-1} d^{1-H/2}$, and since $H \leq 1/2$, $\beta_0 \geq C d^{3/4}$.

Step 6: The case $H > 1/2$. In this case we consider $f(t) = ct^{\alpha}$, for a given $\alpha \in [0, 1/2)$ and some constant c chosen below. Thus we have $a = ct^{\alpha-1/2}$. In this case, if a is larger than a universal constant K_u, the result (32) yields that, for some constant C, we have

$$P_b(A_+) \geq \prod_{i=1}^{d} \exp(-Ca^2) = \exp(-Cc^2 dt^{2\alpha-1}).$$

So, using again condition (30) and relation (34) we obtain

$$p_{2t}(\beta) \geq -Cdt^{2\alpha-2} + C\beta d^{H/2} t^{\alpha H - 1/2},$$

where the constant C may also include the factor c^2. Again, choosing t so that the second term equals twice the first, we have

$$t = Cd^{\frac{1-H/2}{\alpha(H-2)+3/2}} \beta^{-\frac{1}{\alpha(H-2)+3/2}}, \tag{36}$$

and so

$$\sup_{t>0} p_{2t}(\beta) \geq Cd^{\frac{H-1/2}{\alpha(H-2)+3/2}} \beta^{-\frac{2\alpha-2}{\alpha(H-2)+3/2}}.$$

In order to maximize the power of β in the lower bound for $\sup_{t>0} p_t(\beta)$ we should find the maximum of the function

$$g(\alpha) = \frac{2 - 2\alpha}{\alpha(H-2) + 3/2}$$

for $0 \leq \alpha < 1/2$. Since this function is monotone decreasing when $H > 1/2$, the maximum is reached for $\alpha = 0$, so $g(0) = 4/3$.

Recall once again that, in order to apply condition (30) in the computations above, we had to assume $f(t) \leq r_2$; since now $f(t)$ is the constant c, we only need to choose $c = r_2$. We also had to impose $a = r_2 t^{-1/2} > K_u$, which translates as $\beta > \beta_0' := (K_u/r_2)^{4/3} d^{1-H/2}$. □

3.3 Logarithmic regularity scale

As mentioned in the introduction, the special shape of our Figure 1.2 induces us to explore the regions of low spatial regularity for W, in order to investigate some new possible scaling in the strong disorder regime. In other words, we shall work in this section under the assumptions that there exist positive constants c_0, c_1, and r_1, and $\beta \in (0, \infty)$, such that for all x, y with $|x-y| \leq r_1$,

$$c_0 \log^{-\gamma}(1/|x-y|) \leq \delta(x-y) \leq c_1 \log^{-\gamma}(1/|x-y|), \tag{37}$$

where $\gamma > 0$. Assumption (37) implies that W is not spatially Hölder-continuous for any exponent $H \in (0,1]$. Moreover, the theory of Gaussian regularity implies that, if $\gamma > 1/2$, W is almost-surely continuous in space, with modulus of continuity proportional to $\log^{-\gamma+1/2}(1/|x-y|)$, while if $\gamma \leq 1/2$, W is almost-surely not uniformly continuous on any interval in space. The case $\gamma = 1/2$, which is the threshold between continuous and discontinuous W, is of special interest, since it can be related to the discrete space polymer which will be studied in the next section. The main result which will be proved here is the following:

Theorem 3.5. *Assume condition (37). We have for some constants C_0 and C_1 depending only on Q, for all β large enough,*

$$C_0 \frac{\beta^2}{d} \log^{-2\gamma}\left(\frac{\beta}{\sqrt{d}}\right) \leq p(\beta) \leq C_1 \beta^2 \log^{-2\gamma}\left(\frac{\beta}{\sqrt{d}}\right).$$

Proof. Step 1: Setup. Nearly all the calculations in the proof of Propositions 3.1 and 3.4 are still valid in our situation.

Step 2: Lower bound. For the lower bound, reworking the argument in Step 2 in the proof of Proposition 3.4, using the function $\log^{-\gamma}(x^{-1})$ instead of the function x^H, we obtain the following instead of (33):

$$\mathbf{E}\left[(Z_+ - Z_-)^2\right] \geq t(\beta c_0)^2 \left(\log\left(\frac{1}{\sqrt{d}f(t)}\right)\right)^{-2\gamma},$$

which implies, instead of (35) in Step 5 of that proof, the following:

$$p_{2t}(\beta) \geq \frac{d \log C}{2t} + C\beta t^{-1/2} \left(\log\left(\frac{1}{\sqrt{d}f(t)}\right)\right)^{-\gamma}.$$

In other words, now choosing $f(t) = t^{1/2}$ as we did in the case $H < 1/2$ (recall that we are in the case of small H, as stated in the introduction),

$$p_{2t}(\beta) \geq \frac{d \log C}{2t} + C\beta t^{-1/2} \left(\log\left(\frac{1}{\sqrt{dt}}\right)\right)^{-\gamma}.$$

Now choose t such that the second term in the right-hand side above equals twice the first, i.e.

$$t^{1/2} \log^{-\gamma}\left(\frac{1}{\sqrt{dt}}\right) = Cd\beta^{-1}.$$

For small t, the function on the left-hand side is increasing, so that the above t is uniquely defined when β is large. We see in particular that when β is large, t is small, and we have $t^{-1} \leq \beta^2$. This fact is then used to imply

$$\frac{1}{t} = \left(\frac{C\beta}{d}\right)^2 \left(\log\left(\frac{1}{\sqrt{dt}}\right)\right)^{-2\gamma} \geq 2(C\beta)^2 \log^{-2\gamma}(\beta).$$

Therefore, for some constants β_2 and c depending only on Q, for the t chosen above with $\beta \geq \beta_2$,

$$p_{2t}(\beta) \geq \frac{C\beta^2}{d}\left(\log\left(\frac{\beta}{\sqrt{d}}\right)\right)^{-2\gamma}.$$

Step 3: Upper bound. Here, returning to the proof of Proposition 3.1, the upper bound (29) in the final step of that proof holds regardless of δ, and therefore, using the result of Remark 3.3 with $\delta(r) = \log^{-\gamma}(1/r)$, we immediately get that there exists c depending only on Q such that for all $\varepsilon < r_1$ and all $\beta > \beta_3$,

$$\limsup_{t\to\infty} p_t(\beta) \leq C\beta^2\left(\log\left(\frac{1}{\varepsilon\sqrt{d}}\right)\right)^{-2\gamma} + Cd^2\beta^{4/3}\varepsilon^{-2/3},$$

as long as one is able to choose ε so that $\beta\varepsilon \geq 1$. By equating the two terms in the right-hand side of the last inequality above, we get

$$\varepsilon\left(\log\left(\frac{1}{\varepsilon\sqrt{d}}\right)\right)^{-3\gamma} = Cd^3\beta^{-1}.$$

Since the function $\varepsilon \mapsto \varepsilon\log^{-3\gamma}\left(1/(\varepsilon\sqrt{d})\right)$ is increasing for small ε, the above equation defines ε uniquely when β is large, and in that case ε is small. We also see that for any $\theta > 0$, for large β, $1/\varepsilon \geq \beta^{1-\theta}$. Therefore we can write, for $\beta \geq \beta_3$, almost surely,

$$\limsup_{t\to\infty} p_t(\beta) \leq C(1-\theta)^{-2\gamma}\beta^2\left(\log\left(\frac{\beta}{\sqrt{d}}\right)\right)^{-2\gamma}.$$

This finishes the proof of the theorem. □

4 Estimates of the free energy: discrete space

Recall that, up to now, we have obtained our bounds on the free energy in the following manner: the upper bound has been computed by evaluation of the supremum of a well-chosen random Gaussian field, while the lower bound has been obtained by introducing two different events, depending on the Brownian configuration, which capture most of the logarithmic weight of our polymer distribution. This strategy also works in the case of the random walk polymer whose Hamiltonian is described by (5), without many additional efforts, but a separate proof is still necessary. This section shows how this procedure works, resulting in the proof of Theorem 1.5.

Quantities referring to the random walk polymer have been denoted by $\hat{b}, \hat{W}, \hat{E}_{\hat{b}}, \hat{\mathbf{E}}$, etc... In this section, for notational sake, we will omit the hats in the expressions above, and write instead $b, W, E_b, \mathbf{E}$ like in the Brownian case. Recall our simple non-degeneracy condition on Q in this case:

$$c_Q := \sup_{1\leq i\leq d} \left(Q(0) - Q(2e_i)\right)^{1/2} > 0, \tag{38}$$

where e_i, $i = 1, \cdots, d$ are the unit vectors in $\mathbb{Z}^d$. Condition (38), which is used only in the lower bound result, is extremely weak. It essentially covers all possible homogeneous covariance functions, except the trivial one $Q(x) \equiv Q(0)$ for all x, which is the case where W does not depend on x, in which case the Hamiltonian has no effect. Indeed, assume that there exists an $x_0 \in \mathbb{Z}^d$ such that $W(t, 0)$ and $W(t, x_0)$ are not (a.s.) equal. Then $Q(x_0) < Q(0)$. Our lower bound proof below can then be adapted to use this condition instead of Condition (38). We do not comment on this point further.

4.1 Lower bound for the random walk polymer

The lower bound announced in Theorem 1.5 is contained in the following.

Proposition 4.1. *Assume condition (38) holds true. Then there exists a constant $\beta_0 > 0$, which depends on d and on c_Q and a constant $C > 0$, which depend only on c_Q, such that if $\beta > \beta_0$ then almost surely*

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t \geq C \frac{\beta^2}{\log \beta}.$$

Proof. Invoking Proposition 2.2, we have $p(\beta) = \lim_{t\to\infty} p_t(\beta) = \sup_{t\geq 0} p_t(\beta)$. Therefore, any chosen fixed value t yields $p_t(\beta)$ as a lower bound for $p(\beta)$.

We express Z_t by using the fact that each component of b is constant between its jump times, which are uniformly distributed on the simplex, given N_t the total number of jumps before time t, which is a Poisson r.v. with parameter $2dt$. Moreover the visited sites $(x_k)_{k=1}^{N_t}$ are uniformly distributed on the set of all nearest-neighbor paths of length N_t started at 0, given N_t. For a lower bound on $p_t(\beta)$, we throw out, in the expectation defining Z_t, all the paths b that do not jump exactly once before time t. We also throw out all jump positions that are not $\pm e_i$, where $c_Q = (Q(0) - Q(2e_i))^{1/2} > 0$. Therefore,

$$Z_t \geq P_b\left[N_t = 1\right] \frac{1}{2d} \int_0^t \frac{ds}{t} \left(e^{\beta W(s,0)+\beta W([s,t],e_i)} + e^{\beta W(s,0)+\beta W([s,t],-e_i)}\right),$$

where $W([s,t], x) := W(t, x) - W(s, x)$. Here, given $N_t = 1$, $1/(2d)$ is the weight of the path that jumps to $\pm e_i$, and $\mathbf{1}_{[0,t]}(s)\, ds/t$ is the law of the single jump time. Using this and Jensen's inequality, we get

$$Z_t \geq dte^{-2td} \int_0^t \frac{ds}{t} \left(e^{\beta W(s,0)+\beta W([s,t],e_i)} + e^{\beta W(s,0)+\beta W([s,t],-e_i)}\right),$$

$$\frac{1}{t}\mathbf{E}(\log Z_t) \geq \frac{\log t}{t} - 2d + \beta \int_0^t \frac{ds}{t^2} \mathbf{E}\left[\max\left(W([s,t], e_i); W([s,t], -e_i)\right)\right].$$

Now we evaluate the expected maximum above. The vector $(W([s,t],e_i), W([s,t],-e_i))$ is jointly Gaussian with common variances $\sqrt{t-s}Q(0)$ and covariance $\sqrt{t-s}Q(2)$. Therefore

$$\mathbf{E}\left[\max\left(W([s,t],e_i), W([s,t],-e_i)\right)\right] = \frac{1}{2}\mathbf{E}\left[\left|W([s,t],e_i) - W([s,t],-e_i)\right|\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\mathbf{Var}\left[W([s,t],e_i) - W([s,t],-e_i)\right]\right)^{1/2} = \frac{1}{\sqrt{\pi}}\sqrt{t-s}\sqrt{Q(0) - Q(2e_i)}. \tag{39}$$

Thus, recalling condition (38), and chosing $t = C\log^2\beta/\beta^2$, we obtain

$$\begin{aligned}\frac{1}{t}\mathbf{E}(\log Z_t) &\geq \frac{\log t}{t} - 2d + \frac{2\beta}{3\sqrt{\pi t}}c_Q \\ &\geq \frac{\beta^2}{\log\beta}\left(-\frac{2}{C} + \frac{2c_Q}{3\sqrt{c\pi}}\right) + \frac{\beta^2}{C\log^2\beta}\left(\log C + 2\log\log\beta\right) - 2d. \end{aligned}\tag{40}$$

The proof is completed by choosing C such that $-\frac{2}{C} + \frac{2c_Q}{3\sqrt{C\pi}} > 0$, i.e. $C > \frac{9\pi}{Q(0)-Q(2)}$, and β large enough so that the second and third terms in (40) contribute nonnegatively. □

4.2 Upper bound for the random walk polymer

The upper bound result in Theorem 1.5 can be summarized in the following proposition.

Proposition 4.2. *Under the assumption that $Q(0) < \infty$, there exists a constant $\beta_0' > 0$, which depends on Q and on d, and a constant $C > 0$, which depend only on Q, such that if $\beta > \beta_0'$ then almost surely*

$$\lim_{t\to\infty}\frac{1}{t}\log Z_t \leq Cd^3\frac{\beta^2}{\log\beta}.$$

Proof. Define $\mathcal{S}(t,n)$, k_j, t_i^j, $\tilde{t}_l$, N_t, $\mathcal{P}_n$ and $\tilde{x}_l$ like in Step 3 of the proof of Proposition 3.1. Then if we fix $N_t = m$, we can define

$$X\left(m,\tilde{t},\tilde{x}\right) := \sum_{i=0}^{m}\left\{W\left(\tilde{t}_{i+1},\tilde{x}_i\right) - W\left(\tilde{t}_i,\tilde{x}_i\right)\right\}.$$

Let α be a fixed positive number which will be chosen later. Let $I_\alpha = \cup_{m\leq\alpha t}J_m$, where $J_m := \{m\}\times\mathcal{S}_{m,t}\times\mathcal{P}_m$, and set also $Y_\alpha = \sup_{I_\alpha}X$. As in the Brownian case, we can bound $\mathbf{E}[\log Z_t]$ above as follows:

$$\mathbf{E}[\log Z_t] \leq \mathbf{E}[\log(A+B)] \leq \mathbf{E}\left[\log_+ A\right] + \mathbf{E}\left[\log_+ B\right] + \log 2, \tag{41}$$

where $\log_+ A = (\log A)_+ = \max(\log A, 0)$ and

$$\begin{aligned} A &:= P_b\left[N_t \leq \alpha t\right]\exp\left(\beta Y_\alpha\right) \\ B &:= \sum_{m>\alpha t} P_b\left[N_t = m\right]\, E_b\left[\exp\left(\beta X\left(m,\tilde{t},\tilde{x}\right)\right)\Big| N_t = m\right]. \end{aligned}\tag{42}$$

Step 1: The term A. As in the continuous case, we have that

$$\mathbf{E}\left[\sup_{T_{n\alpha}} X(k,\tilde{t},\tilde{x})\right] \leq Ktd\sqrt{n\alpha}, \tag{43}$$

where K depends only on Q. So, bounding $P_b\left[N_t \leq \alpha t\right]$ by 1, we have

$$\mathbf{E}\left[\log_+ A\right] \leq \beta\mathbf{E}\left[Y_\alpha\right] \leq \beta K dt\sqrt{\alpha}. \tag{44}$$

Step 2: The term B. The term B defined in (42) can be bounded as follows:

$$\begin{aligned}
&\mathbf{E}\left[\log B_+\right] \\
&= \mathbf{E}\left[\log_+ \sum_{m>\alpha t} P_b\left[N_t = m\right] \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{(2d)^m} \int_{\mathcal{S}_{m,t}} \exp\left(\beta X\left(m,\tilde{t},\tilde{x}\right)\right) d\tilde{t}\right] \\
&= \mathbf{E}\left[\log_+ \sum_{n\geq 1} \sum_{m\in[\alpha nt,\alpha(n+1)t]} P_b\left[N_t = m\right] \sum_{\tilde{x}\in\mathcal{P}_m} \frac{1}{(2d)^m} \int_{\mathcal{S}_{m,t}} \exp\left(\beta X\left(m,\tilde{t},\tilde{x}\right)\right) d\tilde{t}\right] \\
&\leq \mathbf{E}\left[\log_+ \sum_{n\geq 1} P_b\left[N_t > \alpha nt\right] \exp\left(\beta Y_{(n+1)\alpha}\right)\right].
\end{aligned}$$

So, using the fact that for $t > 1$, the power t^{-1} of a sum is less than the sum of the terms raised to the power t^{-1}, followed by Jensen's inequality, we have, similarly to what we did in the proof of Proposition 3.1,

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \leq \log\left(1 + \sum_{n\geq 1}\left(P_b\left[N_t > \alpha nt\right]\right)^{t^{-1}} \mathbf{E}\left[\exp\left(\frac{\beta Y_{(n+1)\alpha}}{t}\right)\right]\right).$$

Using once again Gaussian supremum analysis results (see [1] or [27]), for any $\alpha, x > 0$,

$$\begin{aligned}
\mathbf{E}\left[\exp\left(xY_\alpha\right)\right] &\leq \exp\left(x\mathbf{E}\left[Y_\alpha\right]\right)\exp\left(x^2 K_u \max_{(m,\tilde{t},\tilde{x})\in I_\alpha} \mathbf{Var}\left[X\left(m,\tilde{t},\tilde{x}\right)\right]\right) \\
&\leq \exp\left(xdKt\sqrt{\alpha}\right)\exp\left(x^2 K_u t Q(0)\right),
\end{aligned}$$

where K_u designate a universal constant, and where we used (43) and the trivial fact $\mathbf{E}[X\left(m,\tilde{t},\tilde{x}\right)^2] = Q\left(0\right)t$. Hence

$$\mathbf{E}\left[\exp\left(\frac{\beta Y_{(n+1)\alpha}}{t}\right)\right] \leq \exp\left(\beta dK\sqrt{\alpha(n+1)} + \frac{\beta^2 K_u Q(0)}{t}\right).$$

If we choose t such that $t > (2\beta K_u Q(0))/(dK\alpha^{1/2})$, the estimate on B becomes

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \leq \log\left\{1 + \sum_{n\geq 1}\left(P_b\left[N_t > \alpha nt\right]\right)^{t^{-1}} \exp\left(\beta dK_u\sqrt{\alpha}\left(\sqrt{n+1} + \frac{1}{2}\right)\right)\right\}. \tag{45}$$

Step 3: The tail of N_t. Using the presumably well-known tail estimate $P_b[N_t > \alpha t] \leq \exp\left(-\alpha t \log\left(\frac{\alpha}{2d}\right) - t(\alpha - 2d)\right)$, valid for all $\alpha \geq 1$ (see e.g. [16, pages 16-19]), if we set $\alpha' = \alpha/2d$ and we assume $\alpha' \geq \exp(1 - 1/2d)$ we have

$$P_b[N_t > \alpha t] \leq \exp(-t\alpha' \log \alpha') . \tag{46}$$

Step 4: Grouping our estimates and choosing α. From (45) and (46) we have

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \leq \log\left\{1 + \sum_{n\geq 1} \exp\left(-\alpha' n \log \alpha' n + d\beta K_u \sqrt{\alpha}\left(\sqrt{n+1} + \frac{1}{2}\right)\right)\right\}.$$

To exploit the negativity of the exponential term, we simply require

$$\alpha' \log \alpha' = 4dK_u\beta\sqrt{\alpha}. \tag{47}$$

Indeed, since $n \geq 1$, we then have that the term inside the exponential is

$$\begin{aligned}
&\alpha' n \log \alpha' n - \beta d K_u \sqrt{\alpha}\left(\sqrt{n+1} + \frac{1}{2}\right) = \alpha' n \log \alpha' n - \frac{1}{4}\alpha' \log \alpha' \left(\sqrt{n+1} + \frac{1}{2}\right) \\
&\geq \alpha' n \log \alpha' - \frac{1}{4}\alpha' \log \alpha' \left(\sqrt{n+1} + \frac{1}{2}\right) \\
&= \frac{1}{2}\alpha' n \log \alpha' n + \left(\frac{1}{2}n - \frac{1}{4}\left(\sqrt{n+1} + \frac{1}{2}\right)\right)\alpha' \log \alpha' \\
&\geq \frac{1}{2}\alpha' n \log \alpha',
\end{aligned}$$

which implies

$$\frac{1}{t}\mathbf{E}\left[\log_+ B\right] \leq \log\left\{1 + \sum_{n\geq 1} \exp\left(-\frac{1}{2}\alpha' n \log \alpha'\right)\right\} = \log\left\{1 + \frac{1}{\exp\left(\frac{1}{2}\alpha' \log \alpha'\right) - 1}\right\} := c_d.$$

The restriction $\alpha' \geq \exp(1 - 1/2d)$ implies that c_d is a constant that depends on the dimension d only. Combining this with (41) and (44), we get

$$\frac{1}{t}\mathbf{E}[\log Z_t] \leq \frac{\log 2}{t} + c_d + dK_u\beta\sqrt{\alpha}. \tag{48}$$

Step 5: Conclusion. It is easy enough to see that, with

$$x := \left(4d\sqrt{2d}\beta K_u\right)^2, \tag{49}$$

the equation (47), which is $\alpha' = x/\log^2 \alpha'$, has a unique solution α' when x exceeds e, and that α' also exceeds e in that case: indeed $\alpha' = e$ when $x = e$ and $d\alpha'/dx =$

$\left(\log^2 \alpha' + 2\log \alpha'\right)^{-1} > 0$ for all $\alpha' \geq e$. Therefore, since $\log^2 \alpha' > 1$, we can write $\alpha' \leq x$, and thus we also have:

$$\alpha' = \frac{x}{\log^2 \alpha'} \geq \frac{x}{\log^2 x}. \tag{50}$$

This lower bound on α' implies the following upper bound on α':

$$\alpha' = \frac{x}{\log^2 \alpha'} \leq \frac{x}{\log^2 \left(x/\log^2 x\right)} = \frac{x}{\left(\log x - 2\log(\log x)\right)^2}. \tag{51}$$

Since there exists x_0 such that, for any $x > x_0$, we have

$$\log x > 4\log(\log x), \tag{52}$$

and we can recast expression (51) into:

$$\alpha' \leq \frac{4x}{\log^2 x} = \frac{(4d\sqrt{2d}\beta K_u)^2}{(\log\beta + \log 4d\sqrt{2d}K_u)^2} \leq (4d\sqrt{2d}K_u)^2 \frac{\beta^2}{\log^2\beta},$$

from which we obtain $\alpha \leq (8d^2K_u)^2\beta^2\log^{-2}\beta$ (recall $\alpha =: 2d\alpha'$). Thus for t large enough:

$$\frac{\mathbf{E}[\log(Z_t)]}{t} \leq \frac{\log 2}{t} + c_d + 8d^3K_u^2\frac{\beta^2}{\log\beta}.$$

Taking limits as t tends to ∞ and choosing β so that

$$\beta^2 > \frac{c_d \log\beta}{8d^3K_u^2}, \tag{53}$$

the theorem is proved with $C = 16K_u^2$.

Finally, we show that the theorem holds for β large enough. Analyzing the conditions we used above, we only have to take $\beta \geq \beta_0' := \max(K_d, \beta^*, \beta_*)$,where K_d, β^*, β_* are now specified. This is due to the fact that we assumed $\alpha' \geq \exp(1 - 1/2d)$ and this implies, via (50), that $x \geq 2d\exp(1-1/2d)\left(\log 2d + 1 - 1/\left(2d\right)\right)$, and therefore, from (49), we have to take $\beta \geq K_d$, where K_d is a constant that depends only on the dimension d. In addition, according to (52) and (53), β_* and β^* are the solutions to the following equations:

$$\beta^2 = \frac{c_d\log\beta}{8d^3K_u^2} \quad \text{and} \quad \log(4d\sqrt{2d}K_u\beta) = 4\log(\log(4d\sqrt{2d}K_u\beta)).$$

□

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