

# Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation

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## 1 Introduction

This article has two purposes: to present a sharp *regularity theory* for Gaussian fields on the unit circle  $S^1$ , and to apply this theory to formulate a spatial regularity theory for the stochastic heat equation (SHE). In this introduction, we describe the results that we aim at, and the road that we take to achieve them.

We have in mind the equation

$$\frac{\partial X}{\partial t}(t, x) = \Delta_x X(t, x) + \frac{\partial B}{\partial t}(t, x) : \quad x \in S^1; \quad t \in [0, 1]; \quad u(0, x) = 0, \quad (1)$$

where  $B$  is a Gaussian field on  $[0, 1] \times S^1$  whose behavior in time is that of fractional Brownian motion (fBm) with any parameter  $H \in (0, 1)$ , and whose behavior in space is homogeneous, and can be completely arbitrary within that restriction.

By “regularity theory” for a Gaussian field  $Y$  we mean a *characterization* of almost-sure modulus of continuity for  $Y$  that can be written using information about  $Y$ ’s covariance. We seek necessary *and* sufficient conditions whenever possible, hence the use of the word “characterization”. By “spatial regularity theory” for the stochastic heat equation, we mean a characterization of the almost-sure modulus of continuity for the equation’s solution in its space parameter  $x \in S^1$ , that can be written using information about the spatial covariance of the equation’s data (additive Gaussian fractional noise  $\partial B/\partial t$ ), or that can be formulated in exact relation to the data’s almost sure modulus of continuity in  $x$ .

Let us be more specific about the distinction between the various characterizations. Let  $Y(x) := (I - \Delta_x)^{-H} B(1, x)$ . This defines a homogeneous Gaussian field on  $S^1$ . We can also abusively use the notation  $Y$  for the random field  $Y := (I - \Delta)^{-H} B$  on  $[0, 1] \times S^1$ , which can be called the “ $2H$ -fractional spatial antiderivative” of  $B$ . It is well understood (for the Brownian case, see [19], or more recently [16], [17], [12]) that in our one-dimensional situation,  $B$  does not need to be a bonafide function in  $x$  for the SHE (1) to have a solution. In fact only  $Y$  needs to be a bonafide function; in [15] it is shown that this is a necessary and sufficient condition even in the fractional Brownian case.

Once a condition for existence is given, it is natural to seek conditions for regularity. We consider two types of conditions for guaranteeing/characterizing the fact that the solution  $X$  of the SHE (1) admits a given fixed function  $f$  as an almost-sure uniform modulus of continuity:

- Type I (an *intrinsic* or *pathwise* condition): the fact that the same almost-sure continuity holds for  $Y := (I - \Delta)^{-H} B$ ;
- Type II (a condition *on the distribution*): a condition that can be written using the covariance function of  $Y$ .

From the applied physical point of view, the necessary Type I condition may be quite useful. Indeed, the solution of a stochastic PDE can be a model for a turbulent physical observation, such as the velocity of a fluid flow; the necessary Type I condition asserts that the level of regularity (turbulence) that is observed is precisely related to the regularity of the equation’s forcing term; but in many physical situations, this forcing term is unobserved; our result allows to draw almost-sure conclusions about the turbulence – and even the distribution – of the unobserved forcing term from one sample observation, without any statistical (Type II) analysis. This is of course an ideal situation, which has a lot to do with the Gaussian framework we have chosen.

This article goes fairly far in proving both Type I and Type II necessary and sufficient conditions for  $X$ ’s almost sure spatial continuity. This is achieved in Section 3. The results are entirely new, even in the case of standard white noise ( $H = 1/2$ ). In the small number of recent works on stochastic PDEs driven by fBm ([7], [8], [11]), we found no effort to analyze almost-sure spatial regularity. [IN WHAT SENSE IS THIS TRUE? CHECK TUBARO ET BONACCORSI TO MAKE SURE]. Our tools are inspired by work done by two of the three authors of this article in [16] and [17]. However, these articles had led the two authors to formulate conjectures on the nature of a necessary and sufficient Type II condition; we prove herein that this conjecture was erroneous (see Remark 4 below). Moreover, still in [16], a necessary and sufficient Type I condition had been proved, but only the Hölder scale  $f(r) = r^\alpha$  had been considered, and the was

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the Type I condition was stated left gaps within the Hölder scale, which could not be filled using the techniques in that article. In particular, it was believed that Type II conditions were a first step in proving Type I conditions. We prove herein that such a strategy is unnecessary (see Remark 5), that the two types of conditions can be proved essentially disjointly, and we believe that this more efficient method of proof is the reason we were able to fill the gaps left in the Holder scale in [16] and [17].

The proofs of our Type I and Type II theorems rely heavily on a more basic set of underlying results for a single Gaussian field  $Y = Y(x)$ , proved in Section 2, which constitute our general Gaussian regularity theory on the circle. In particular we prove

- *regularity* theorem (the basis for our Type I theorem):  $Y$  admits a fixed function  $f$  as an almost-sure modulus of continuity if and only if  $f$  exceeds, up to a non-random constant, a canonical (entropy-based) modulus of continuity;
- *summability* theorem (the basis for our Type II theorem): any bound on the canonical metric  $\delta(x) = \left[ \mathbf{E} (Y(x) - Y(e^{i0}))^2 \right]^{1/2}$  is equivalent to a summability condition on the Fourier coefficients of  $Y$ ’s covariance.

The “if” part in our regularity theorem is well-known as part of the Fernique-Talagrand theory (see [1]), while the “only if” part seems entirely new. Our proof relies on revisiting the original work of Talagrand in [14], sharpening the proofs by relying on the particularly simple structure of the circle. Our summability theorem seems entirely new; it can be considered as an elementary – albeit non-trivial – result in the harmonic analysis of the circle.

The theorems of Section 3 also depend in a crucial way on sharp estimates developed in [15] in proving that the solution  $X$  to (1) exists as a member of  $L^2(\Omega \times [0, 1] \times S^1)$  if and only if the same holds for  $Y$ . These estimates, and the theory of the SHE (1), are recalled in Section 3.1. The basic result is of the following form.

- Let  $t \in [0, 1]$  be fixed and let  $r_n$  and  $s_n$  be the Fourier coefficients of the covariance functions of  $X(t, \cdot)$  and  $Y(t, \cdot)$  respectively. Then up to a constant  $K$  that may depend on  $t$ , we have  $K^{-1}r_n \leq s_n \leq Kr_n$ .

In our effort to give results that are as sharp as possible, we have specialized to the case of the Laplacian on the one-dimensional circle  $S^1$ . It is easy to extend all our sufficient conditions for continuity of  $X$  to higher-dimensional spaces, and/or much more general operators; the difficulty is in extending the necessary conditions. We will tackle the issue of sharp necessary conditions (“lower bounds”) for more general operators and spaces in a subsequent publication. The extension to a uniformly hypoelliptic operator on a smooth compact one-dimensional manifold should be trivial, and we will not comment on this further. The reader may notice that our “lower bounds” proofs (necessary conditions) below make extensive use of a property of spatial isotropy for  $B$ . Since we always assume that  $B$  is spatially homogeneous, in the case of the circle  $S^1$  isotropy is always satisfied. In higher-dimensional problems, we believe there is hope of extending our one-dimensional lower bound results only in the isotropic case.

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To establish the “necessary”, or “lower bound” portion of the Type I condition, we will need a technical assumption (Condition B, see below) which amounts to requiring that  $X$  is not Hölder-continuous. Without this assumption, in the Hölder scale, we will show a slightly weaker result. The “necessary” or “lower bound” Type II condition, and the “sufficient”, or “upper bound” Type II condition require separate mild technical assumptions which do not limit the regularity scales one may wish to consider.

Extending the range of validity of the necessary condition of Type I to include Hölder-continuous moduli will be the subject of future work. However, in Corollary 2 below (also see Corollary 3), it can be seen that a Type I condition is necessary without any restriction on the regularity, provided one is satisfied with knowing that a certain canonical function  $f_Y$  is the common modulus of continuity for  $Y$  and  $X$ . This function  $f_Y$  is known only if one has precise statistical (Type II) information on  $B$ . In other words, the result is useful only if one knows the distribution of  $B$  with some accuracy. In this sense, one is dealing with a hybrid Type I – Type II condition. An arbitrary measurement of the regularity of  $X$  cannot alone be guaranteed to transfer sharply to  $Y$  without the “non-Hölder” regularity condition B. This is why we have spent the extra effort to exploit the full sharpness of Gaussian regularity theory, despite the mathematical price we have to pay in assuming Condition B.

Summarizing our results on the stochastic heat equation,

- the Type I condition is always sufficient;
- Type II conditions are necessary and sufficient, modulo mild technical assumptions;
- the Type I condition is necessary if the modulus is not too regular, i.e. not Hölder;
- in the Hölder scale, the Type I condition is nearly necessary;
- a Type I condition is necessary and sufficient with no regularity restriction if one is satisfied with having an a.s. modulus of continuity that is defined in terms of a formula of Type II.

## 2 Sharp Gaussian regularity theory on the circle

This section presents the basic regularity theory for homogeneous Gaussian fields on the circle. It is not difficult to modify the arguments to fit many one-dimensional situations; as alluded to above, many isotropic higher-dimensional settings as well.

### 2.1 Definitions and statements of the regularity results

We start with some definitions.

**Definition 1** *Let  $f$  be a continuous increasing function on  $\mathbb{R}_+$  such that  $\lim_{0^+} f = 0$ . Let  $\{Y(x) : x \in S^1\}$  be a bonafide random field on  $S^1$  (meaning  $Y(\cdot)$  is almost-surely a bonafide function on  $S^1$ ).*

- *We say that  $f$  is an almost-sure spatial uniform modulus of continuity for  $Y$  if there exists an almost-surely positive (non-zero) random variable  $\alpha_0$  such that*

$$\alpha < \alpha_0 \implies \sup_{x,y \in S^1; |x-y| < \alpha} \{|Y(x) - Y(y)| \leq f(\alpha)\}.$$

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- *The canonical metric  $\delta$  of  $Y$  is defined for all  $x, y \in S^1$  by*

$$\delta(x, y) = \left( E \left[ (Y(x) - Y(y))^2 \right] \right)^{1/2}.$$

- *$Y$  is said to be homogeneous if for any  $x$  in  $S^1$ ,  $Y$  and  $Y(\cdot + x)$  have the same distribution.*

**Remark 1** *If  $Y$  is a homogeneous Gaussian field on  $S^1$  then  $\delta(x, y) = \delta(|x - y|)$  where  $\mathbb{R}_+ \ni r \mapsto \delta(r)$  is some function on a neighborhood of 0. Indeed, by homogeneity there exists some function  $\delta$  such that  $\delta(x, y) = \delta(x - y)$ , and by symmetry this also equals  $\delta(y - x)$ , i.e.  $\delta(r) = \delta(|r|)$ . If  $Y$  is a separable random field, then  $\delta$  will be continuous on  $\mathbb{R}_+$  and increasing in a neighborhood of 0. We leave the proof of this last sentence to the reader.*

**Definition 2** *We call the function  $\delta(\cdot)$  on  $\mathbb{R}_+$  the canonical metric function of  $Y$ .*

**Notation** We use the notation  $\asymp$  for commensurate quantities: for positive functions  $A$  and  $B$  of any variable  $\xi$ ,  $A(\xi) \asymp B(\xi)$  means that the ratio  $A(\xi)/B(\xi)$  is bounded away from 0 and  $\infty$ .

Note for example that for scalar fBm  $\{B^H(t) : t \in [0, 1]\}$  we have  $\delta(r) = r^H$ . One can construct a random field on  $S^1$  that is homogeneous and that has the same regularity properties than fBm; in particular we can construct it so that its canonical metric function satisfies  $\delta(r) \asymp r^H$ ; we leave it to the reader to check this. [JE VAIS PEUT-ETRE ECRIRE LA DEMO EXPLICITEMENT, CAR J'AI ESSAYE DE LE FAIRE A LA MAIN, ET C'EST NON TRIVIAL, ET JE N'Y SUIS MEME PAS ARRIVE, ALORS CA VAUT PEUT-ETRE LE COUP DE S'Y ACHARNER UN PEU...].

**Je crois qu'on peut prendre comme exemple les solutions stationnaires a base de ornstein uhlenbeck fractionnaire de maslo et schmalfluss**

One should note however that there exists a major difference between this process and the fBm on  $[0, 1]$ : the latter is adapted to a Brownian filtration, while even the concept of being adapted is unclear for a random field on  $S^1$ .

**Assumption A** We will assume throughout that  $\delta$  is differentiable except perhaps at 0, and that  $\lim_{0+} \delta' = +\infty$ . Without loss of generality this implies that  $\delta$  is concave in a neighborhood of 0.

In terms of regularity properties of  $Y$ , differentiability of  $\delta$  except at 0 introduces no loss of generality. The condition that  $\delta'$  at zero is infinite introduces no loss of generality outside of the very narrow class of processes  $Y$  that are a.s.  $\beta$ -Hölder-continuous for all  $\beta < 1$  but that are not a.s. of class  $C^1$ . For this class of processes, similar results to those we prove herein hold, but the methods of proof are substantially different. We do not comment on these processes further.

**Remark 2** *Assumption A implies that  $\delta$  is strictly increasing in a neighborhood of 0.*

**Remark 3 (Random Fourier series representation)** *For any spatially homogeneous Gaussian field  $Y$  on  $S^1$  with canonical metric function  $\delta$ , there exists a sequence  $\{r_n\}_{n=0}^{\infty}$  of non-negative terms such that*

$$\delta(r)^2 = \sum_{n=1}^{\infty} r_n (1 - \cos(nr)). \quad (2)$$

*Indeed, any such  $Y$  can be written as a random Fourier series*

$$Y(x) = Y_0 \sqrt{q_0} + 2 \sum_{n=1}^{\infty} \sqrt{r_n} \{Y_n \cos(nx) + Z_n \sin(nx)\} \quad (3)$$

*where all  $Y_n$ 's and  $Z_n$ 's are IID  $N(0, 1)$  r.v.'s. Then just calculating  $\delta^2$  yields the former statement.*

We now introduce a condition needed for the lower bound proof of the next theorem which characterizes the regularity of homogeneous Gaussian processes. This condition is satisfied for the class of functions  $\delta$  defined by  $\delta(r) = (\log(1/r))^{-p}$  for any  $p > 0$  no matter how large, and for all functions that are more irregular than this class, but is not satisfied in the power scale defined by  $\delta(r) = r^\alpha$  for any  $\alpha \in (0, 1)$ .

**Condition B** There exists a constant  $c > 0$  such that in a neighborhood of 0 we have

$$\int_0^\alpha \delta(r) \frac{dr}{r \sqrt{\log(r^{-1})}} > c \delta(\alpha) \sqrt{\log(\alpha^{-1})}$$

Our first result is the main elementary *regularity* theorem, which is key to our Type I characterization.

**Theorem 1** *Let  $Y$  be a Gaussian random field on  $S^1$  with canonical metric function  $\delta$ . Let*

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$$f_\delta(\alpha) = \int_0^{\delta(\alpha)} \sqrt{\log \frac{1}{\delta^{-1}(\varepsilon)}} d\varepsilon \quad (4)$$

$$= \int_0^\alpha \delta(r) \frac{dr}{2r \sqrt{\log(r^{-1})}} + \delta(\alpha) \sqrt{\log(\alpha^{-1})} \quad (5)$$

$$= \int_0^1 \frac{\delta(\min(r, \alpha)) dr}{2r \sqrt{\log(r^{-1})}} \quad (6)$$

$$= \int_0^\infty \delta(\min(e^{-x^2}, \alpha)) dx. \quad (7)$$

There is a non-random constant  $K$  depending only on the law of  $Y$  such that the following hold.

**(a) Lower bound / Necessary condition.** *If Assumption A and Condition B hold, if  $f$  is an almost-sure uniform modulus of continuity for  $Y$  on  $S^1$ , then for all  $\alpha$  small enough,*

$$Kf(\alpha) \geq f_\delta(\alpha). \quad (8)$$

**(b) Upper bound / Sufficient condition.** *If  $\lim_{\delta \rightarrow 0} f_\delta = 0$  then  $Kf_\delta$  is an almost-sure modulus of continuity for  $Y$  on  $S^1$ .*

This theorem, whose upper bound is well-known, shows that the function  $f_\delta$  is, up to a constant, an exact uniform modulus of continuity for  $Y$ , as long as  $Y$  is more irregular than Hölder. The next theorem shows that one can do nearly as well in the Hölder scale.

**Corollary 1** *Assume  $\delta(r) \asymp r^\alpha$  for some  $\alpha > 0$  (the “fractional Brownian” scale). Then the Lower Bound (a) in Theorem 1 holds even though Condition B is not satisfied, if one replaces  $f(\alpha)$  by  $f(\alpha) \log(1/\alpha)$  in line (8).*

## 2.2 Proof of Theorem 1

### 2.2.1 Proof of the lower bound of Theorem 1.

Let  $X = \{X(t) : t \in I\}$  be a bounded Gaussian field on an index set  $I$ . Let  $\delta$  be its canonical metric, and  $B(x, \varepsilon)$  be the ball of radius  $\varepsilon$  centered at  $x$  in this metric. In this general situation, we introduce some notation. For a fixed measure  $m$  on  $I$ , let

$$\begin{aligned} \gamma_m(\eta) &= \sup_{x \in I} \int_0^\eta \sqrt{\log(1/m(B(x, \varepsilon)))} d\varepsilon, \\ \theta_m(\eta) &= \sup_{x \in I} \int_0^\eta \sqrt{\log(1/\sup\{m(\{u\}) : u \in B(x, \varepsilon)\})} d\varepsilon, \\ \phi_\delta(\eta) &= E[\sup\{X(x) - X(y) : x, y \in I; \delta(x, y) < \eta\}] \\ \beta_\delta(\eta) &= \sup_{x \in I} E[\sup\{|X(x) - X(y)| : y \in I; \delta(x, y) < \eta\}]. \end{aligned}$$

We note that  $\theta_m \geq \gamma_m$ . Since  $X$  is centered, we have  $\beta_\delta \leq 2\phi_\delta$ . Also we introduce the metric entropy of  $\delta$ :  $N(\varepsilon)$  is the smallest number of balls of radius  $\varepsilon$  in the metric  $\delta$  that are needed to cover  $I$ . Let  $D$  be the diameter of  $I$  in the metric  $\delta$ . Recall the following result from Fernique’s general theory of suprema for Gaussian processes.

**Proposition 1** *[Theorem 17, part (a) in [14].] There exists a universal constant  $K$  (not dependent on  $X$ ) such that with the notation as above, for any probability measure  $m$  on  $I$ ,*

$$\phi_\delta(\eta) \leq K\gamma_m(\eta).$$

Theorem 17 in [14] also establishes the following lower bound which is original to Talagrand’s paper: for some probability measure  $m$  on  $I$ ,

$$\theta_m(\eta) \leq K\beta_\delta(\eta) + \eta(\log(2N(\eta)) + 2\log(2D/\eta) / \log 2)^{1/2}.$$

Yet this estimate is not sufficient for our purposes. In our specific situation however, we are able to bring a slight improvement to Talagrand’s original lower bound proof, by assuming condition B.

**Step 1.** *A Talagrand-type lower bound.* We begin with a lemma inspired by Talagrand’s lower bound [proof of Theorem 17 part (b) in [14]].

**Lemma 1** *With the notation as above, with  $K$  denoting a universal constant (not dependent on  $X$ ), let  $\eta > 0$  be fixed. Let  $\{B_j : j = 1, \dots, N(\eta)\}$  be a covering of  $I$  with balls of radius no greater than  $\eta$ . There exists a probability measure  $m_j$  on  $B_j$  such that for all  $x \in B_j$*

$$\int_0^{\text{diam}(B_j)} \left( \log \left[ (\sup \{m_j(\{u\}) : \delta(x, u) \leq \varepsilon\})^{-1} \right] \right)^{1/2} d\varepsilon \leq K\beta_\delta(\eta), \quad (9)$$

and moreover, defining the probability measure  $m = N(\eta)^{-1} \sum_{i=1}^{N(\eta)} m_j$ ,

$$\theta_m(\eta) \leq \sup_{j \in \{1, \dots, N(\eta)\}} \sup_{x \in B_j} \int_0^\eta \left( \log N(\eta) + \log \left[ (\sup \{m_j(\{u\}) : \delta(x, u) \leq \varepsilon\})^{-1} \right] \right)^{1/2} d\varepsilon \quad (10)$$

**Proof.** Let  $u_j$  be the center of the ball  $B_j$ . On each ball  $B_j : j = 1, \dots, N(\eta)$  we apply Theorem 14 in [14] to obtain the existence of a probability measure  $m_j$  on  $B_j$  such that

$$\begin{aligned} \int_0^{\text{diam}(B_j)} \left( \log \left[ (\sup \{m_j(\{u\}) : \delta(x, u) \leq \varepsilon\})^{-1} \right] \right)^{1/2} d\varepsilon &\leq K\mathbf{E} \left[ \sup_{x \in B_j} X(x) \right] \\ &= K\mathbf{E} \left[ \sup_{x \in B_j} X(x) - X(u_j) \right] \leq K\mathbf{E} \left[ \left| \sup_{x \in B_j} X(x) - X(u_j) \right| \right] \leq K\beta_\delta(\eta), \end{aligned}$$

proving the lemma's first assertion. For the second, fix  $x \in I$  and let  $j$  be such that  $x \in B_j$ ; then for any  $\varepsilon \leq \eta$ ,

$$\sup \{m(\{u\}) : \delta(x, u) \leq \varepsilon\} \geq \frac{1}{N(\eta)} \sup \{m_j(\{u\}) : \delta(x, u) \leq \varepsilon\},$$

so that

$$\begin{aligned} &\int_0^\eta (\log(1/\sup \{m(\{u\}) : u \in B(x, \varepsilon)\}))^{1/2} d\varepsilon \\ &\leq \int_0^\eta (\log N(\eta) + \log(1/\sup \{m_j(\{u\}) : u \in B(x, \varepsilon)\}))^{1/2} d\varepsilon, \end{aligned}$$

and the result follows.  $\square$ .

**Step 2.** *First majorizing measure integral estimation.* Assume  $I$  is a compact Lie group, let  $|\cdot|$  and  $dx$  denote the Haar measure on  $I$ , and its differential, and assume  $X$  is homogeneous on  $I$ . Fix  $x_0$  in  $I$ . Let  $B(x_0, \varepsilon)$  be the ball in the metric  $\delta$  centered at  $x_0$  with radius  $\varepsilon$ . Let

$$\tau(\eta) = \int_0^\eta \sqrt{\log(1/|B(x_0, \varepsilon)|)} d\varepsilon.$$

Since  $X$  is homogeneous,  $\tau$  does not depend on  $x_0$ , and we have the following equalities for any fixed probability measure  $m$  on  $I$ :

$$\begin{aligned} \int_I m(B(x, \varepsilon)) dx &= \int_I m(x - x_0 + B(x_0, \varepsilon)) dx \\ &= \int_I m(x' + B(x_0, \varepsilon)) dx' \\ &= \int_I |y + B(x_0, \varepsilon)| m(dy) \\ &= |B(x_0, \varepsilon)|. \end{aligned}$$

**je propose une legere modification de ce bout de preuve qui me turlupine depuis l'an dernier:**

Since  $X$  is homogeneous, we have  $|B(y, \varepsilon)| = |B(x_0, \varepsilon)|$  for any  $x_0, y \in I$ . In particular,  $\tau$  does not depend on  $x_0$ , and we have the following equalities for any fixed probability measure  $m$  on  $I$ :

$$\begin{aligned} \int_I m(B(x, \varepsilon)) dx &= \int_I dx \int_I \mathbf{1}_{\{|x-y| \leq \varepsilon\}} m(dy) \\ &= \int_I m(dy) \int_I \mathbf{1}_{\{|x-y| \leq \varepsilon\}} dx \\ &= \int_I m(dy) |B(y, \varepsilon)| = |B(x_0, \varepsilon)|. \end{aligned}$$

Since the function  $u(z) = \sqrt{\log(1/z)}$  is convex for  $0 < z < 1/2$ , for small  $\eta$ , we can use **ci-dessus: small  $\varepsilon$**

$$u\left(\int_I m(B(x, \varepsilon)) dx\right) \leq \int_I u(m(B(x, \varepsilon))) dx.$$

Therefore

$$u(|B(x_0, \varepsilon)|) \leq \int_I u(m(B(x, \varepsilon))) dx,$$

which implies

$$\begin{aligned} \tau(\eta) &\leq \int_I dx \int_0^\eta u(m(B(x, \varepsilon))) d\varepsilon \\ &\leq \sup_{x \in I} \int_0^\eta u(m(B(x, \varepsilon))) d\varepsilon \\ &= \gamma_m(\eta). \end{aligned}$$

**Step 3.** *Using the Talagrand-type lower bound.* The last inequality, together with inequality (10), proves, with the measures  $m$  and  $m_j : j = 1, \dots, N(\eta)$ , identified in Lemma 1,

$$\begin{aligned} \tau(\eta) &\leq \gamma_m(\eta) \leq \theta_m(\eta) \\ &\leq \sup_{j \in \{1, \dots, N(\eta)\}} \sup_{x \in B_j} \int_0^\eta \left( \log N(\eta) + \log \left[ (\sup \{m_j(\{u\}) : \delta(x, u) \leq \varepsilon\})^{-1} \right] \right)^{1/2} d\varepsilon. \end{aligned} \tag{11}$$

Since we have no control over the term involving  $m_j$  in the above expression, in comparison to  $N(\eta)$ , we have no choice, as did Talagrand himself, but to use the estimate  $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$ . Then, with inequality (9), we obtain

$$\begin{aligned} \tau(\eta) &\leq \eta (\log N(\eta))^{1/2} + K\beta_\delta(\eta) \\ &\leq \eta (\log N(\eta))^{1/2} + K\phi_\delta(\eta). \end{aligned} \tag{12}$$

**Step 4.** *Calculation of the majorizing measure integral.* The next step in the proof is to calculate  $\tau$ . Here we specialize to the case of  $\delta$  on the circle  $I = S^1$ . We denote by  $\check{\delta}$  the inverse function of  $\delta$ . We have

$$|B(x_0, \varepsilon)| = |\{x : \delta(|x - x_0|) < \varepsilon\}| = |\{x : |x - x_0| < \check{\delta}(\varepsilon)\}| = 2\check{\delta}(\varepsilon).$$

Therefore  $\tau$  becomes

$$\begin{aligned}\tau(\eta) &:= \int_0^\eta \sqrt{\log(1/|B(x_0, \varepsilon)|)} d\varepsilon \\ &= \int_0^\eta \sqrt{\log(1/(2\check{\delta}(\varepsilon)))} d\varepsilon.\end{aligned}\tag{13}$$

We also note that because we are working on a one-dimensional index set, we have

$$N(\eta) = \frac{1}{2\check{\delta}(\eta)}\tag{14}$$

for any value of  $\eta$  that yields an integer in this formula.

**Step 5.** *Using the hypothesis of a.s. modulus of continuity with a zero-one law of Fernique.* To complete the proof of the lower bound, we need to estimate  $\phi_\delta$ . To this end, we use a zero-one-type result due to Fernique. Let  $C^f(I)$  be the space of continuous functions on  $I$  that have  $f$  as an almost sure uniform modulus of continuity, up to a multiplicative constant. For any  $\alpha \leq 1$ , for any function  $g$  defined on  $I$ , set

$$A_\alpha(g) = \sup_{|x-y| \leq \alpha} |g(x) - g(y)|, \quad N_f(g) = \sup_{\alpha \leq 1} \frac{A_\alpha(g)}{f(\alpha)}.$$

Then, following Fernique's definitions [13, Definition 1.2.1],  $N_\theta$  is a gauge on  $C^f(G)$ . Indeed, it suffices to see that  $N_f$  is lower-semi-continuous, that is, for every  $M > 0$ , the set  $K_M = \{\Phi; N_f(\Phi) \leq M\}$  is closed. Let  $\Phi_n$  be a sequence in  $K_M$  converging uniformly to  $\Phi$ . Then, for every  $n$  and for every  $\alpha \leq 1$  and for every  $x, y$  such that  $|x - y| \leq \alpha$ , we have

$$|\Phi_n(x) - \Phi_n(y)| \leq Mf(\alpha).$$

We obtain that  $\Phi \in K_M$  when we let  $n \rightarrow \infty$ .

**ci-dessous: je rajouterais qu'on considere maintenant un module de continuite  $f$  pour  $Y$**

By assumption, we have the existence of an almost surely positive random variable  $\alpha_0$  such that, if  $\alpha < \alpha_0$  then

$$A_\alpha(Y) \leq f(\alpha).$$

Since  $Y$  is almost surely continuous, it is also almost surely bounded. This, together with the last inequality, implies  $N_f(Y)$  is almost surely finite. A theorem of Fernique [13, Lemma 1.2.3] implies  $E[N_f(Y)] := c < \infty$  where  $c = c(f, Y)$  is a constant depending only on  $f$  and the law of  $B$ . Therefore

$$\begin{aligned}\phi_\delta(\eta) &\leq E \sup_{\delta(|x-y|) \leq \eta} |Y(x) - Y(y)| \\ &= E \sup_{|x-y| \leq \check{\delta}(\eta)} |Y(x) - Y(y)| \\ &\leq cf(\check{\delta}(\eta)).\end{aligned}\tag{15}$$

**Step 6.** *Conclusion.* Combining (12), (13), and (14), we obtain

$$\int_0^\eta \sqrt{\log(1/(2\check{\delta}(\varepsilon)))} d\varepsilon \leq K\phi_\delta(\eta) + \eta \left( \log \left( \frac{1}{2\check{\delta}(\eta)} \right) \right)^{1/2}$$

Now let  $\alpha$  be defined by  $\alpha = \check{\delta}$ . Then for small  $\alpha$ , with (15) and formula (5),

$$\int_0^\alpha \delta(r) \frac{dr}{2r\sqrt{\log(r^{-1})}} + \delta(\alpha) \sqrt{\log(\alpha^{-1}/2)} \leq \delta(\alpha) \sqrt{\log(\alpha^{-1}/2)} + cKf(\alpha),$$

or in other words

$$\int_0^\alpha \delta(r) \frac{dr}{2r\sqrt{\log(r^{-1})}} \leq cKf(\alpha). \quad (16)$$

Condition B and the formula for  $f_\delta$  in (5) finish the proof of (a).  $\square$

### 2.2.2 Proof of the upper bound of Theorem 1.

Part (b) is a consequence of a well-known property of homogeneous Gaussian processes and the general theory of Gaussian regularity. Indeed, one only needs to apply Theorem 4.4 and Corollary 4.7 in [1]. The details are left to the reader.  $\square$

### 2.2.3 Proof of Corollary 1.

In the proof of Theorem 1, in the Hölder case  $\delta(r) = r^\alpha$ , all lower bound calculations are valid up to inequality (16). The conclusion of the corollary follows by calculating the left-hand side of (16) and comparing it to  $f(\alpha) \log(1/\alpha)$ .  $\square$

## 2.3 Summability result

Our basic *summability* theorem translates the magnitude of  $\delta$  – and thus, by Theorems 1 and Corollary 1, the regularity of  $Y$  – into a condition on the summability of the  $q_n$ 's. This explains why it is the basis for our Type II characterization. The “sufficient condition” requires the Condition C below, which is stated relative to an upper bound  $g$  on the canonical metric (see statement (c) in the theorem). One can think of Condition C as a condition on  $g = \delta^2$ .

**Condition C** There exists constants  $c, y_0 > 0$  such that for all  $0 < x < y < y_0$

$$g(x)/x^2 - g(y)/y^2 \geq c(g(y) - g(x))/y^2.$$

Condition C essentially places no restriction on the regularity of the canonical metrics that can be used in Type II characterizations. Indeed, Condition C is satisfied for all the following basic examples:

- “Hölder” scale:  $\delta(r)^2 = r^{2\alpha}$  for any  $\alpha \in (0, 1)$ ; up to logarithmic corrections, this scale corresponds to the Hölder scale of almost-sure uniform moduli of continuity  $f(r) = r^\alpha$ ;
- logarithmic scale:  $\delta(r)^2 = (\log(1/r))^{-1-2\varepsilon}$  for any  $\varepsilon > 0$ ; this scale corresponds to the scale of moduli of continuity given by  $f(r) = (\log(1/r))^{-\varepsilon}$ ;
- iterated logarithmic scale:  $\delta(r)^2 = (\log(1/r))^{-1} \left( \log \log(1/r) \cdots \log_{(n-1)}(1/r) \right)^{-2} \left( \log_{(n)}(1/r) \right)^{-2-2\varepsilon}$  for any  $n \in \{2, 3, \dots\}$  and any  $\varepsilon > 0$ ; here  $\log_{(n)}$  denotes the  $n$ -fold iterated logarithm; this scale corresponds to the scale of moduli of continuity given by  $f(r) = \left( \log_{(n)}(1/r) \right)^{-\varepsilon}$ .

Condition C even work in a scale which yields a.s. discontinuous  $Y$ , although this scale cannot be used for our purposes:

- logarithmic scale for discontinuous processes:  $\delta(r)^2 = (\log(1/r))^{-\varepsilon}$  for any  $\varepsilon \in (0, 1]$ .

EN PRINCIPE IL FAUDRAIT VERIFIER QUE TOUS CES EXEMPLES SONT REELLEMENT DES “CANONICAL METRIC FUNCTIONS”, C’EST A DIRE QU’ILS DEFINISSENT VRAIMENT DES COVARIANCES. MAIS ON PEUT LAISSER LE FLOU ARTISTIQUE POUR L’INSTANT.

**je crois que c’est pas tout a fait exact: la condition C porte sur  $g$ , qui ne fait que dominer  $\delta$**

Our basic *summability* theorem is the following.

**Theorem 2** *Let  $Y$  be a homogeneous Gaussian random field on  $S^1$  with canonical metric function  $\delta$ . Let  $\{r_n\}_n$  be the sequence defined by the random Fourier series representation (3) for  $Y$ . Let  $g$  be a strictly increasing continuous function on  $\mathbb{R}_+$ , continuously differentiable on  $(0, \infty)$ , with  $\lim_{0+} g = 0$ . Consider the following statements:*

(c) *There exist a constant  $K > 0$  such that for all  $r \geq 0$ ,  $\delta(r) \leq K\sqrt{g(r)}$ .*

(d) *For any strictly decreasing, positive function  $h$  on a neighborhood of 0 with  $\int_0^\infty h(x) dx < \infty$ :*

$$\sum_n r_n h\left(g\left(\frac{1}{n}\right)\right) < \infty.$$

*The implication (c)  $\implies$  (d) always holds. The converse (d)  $\implies$  (c) holds if we assume Condition C.*

## 2.4 Proof of Theorem 2

### 2.4.1 Proof of (c) $\implies$ (d)

To exploit the hypothesis on  $\delta$  in condition (c) note that we have for all  $n$ ,

$$K^2 g\left(\frac{1}{n}\right) \geq \sum_{j=1}^{\infty} r_j (1 - \cos(j/n)) \geq \frac{1}{3} \sum_{j=n}^{5n} r_j \quad (17)$$

since for  $x \in [1, 5]$ ,  $1 - \cos x > 1/3$ . Fix an  $h$  as in condition (d). Fix an integer  $k > 0$ . Let

$$\begin{aligned} I_k &= \sum_{n=1}^{5^{k+1}} r_n h\left(g\left(\frac{1}{n}\right)\right) \\ &= \sum_{l=0}^k \sum_{n=5^l}^{5^{l+1}-1} r_n h\left(g\left(\frac{1}{n}\right)\right). \end{aligned}$$

We only need to show that  $I_k$  is bounded. Since  $h$  is decreasing and  $g$  is increasing, we have  $h\left(g\left(\frac{1}{n}\right)\right) \leq h\left(g\left(\frac{1}{5^{l+1}}\right)\right)$  for any  $n \in [5^l, 5^{l+1}]$ . This, coupled with inequality (17), and the notation  $F(x) = 1/(xh(x))$ , yields

$$\begin{aligned} I_k &\leq \sum_{l=0}^k \left( \sum_{n=5^l}^{5^{l+1}-1} r_n \right) h\left(g\left(\frac{1}{5^{l+1}}\right)\right) \\ &\leq 3K^2 \sum_{l=0}^k g\left(\frac{1}{5^l}\right) h\left(g\left(\frac{1}{5^{l+1}}\right)\right) \\ &= 3K^2 \sum_{l=0}^k \frac{g(5^{-l})}{g(5^{-(l+1)})} \frac{1}{F(g(5^{-(l+1)}))} \end{aligned}$$

By Assumption A, without loss of generality, we can assume that  $g(0) = 0$  and that  $\tilde{g} = g^{1/2}$  is concave (near 0), and also that  $g(x) \geq x^2$  (near 0). This implies that (near 0)

$$\frac{g(x)}{g(x/5)} \leq 25; \quad g(x) \geq x^2.$$

Also, without loss of generality, we can assume that  $1/F(x) = xh(x)$  is non-decreasing near 0.

LA JE SUIS PAS SUR QUE C'EST TOUT A FAIT CACHER DE FAIRE CETTE HYPOTHESE DE CROISSANCE DE  $xh(x)$ , MAIS LAISSONS LE FLOU ARTISTIQUE POUR L'INSTANT. C'EST EVIDEMMENT MORALEMENT CLAIR PUISQUE  $h(x) = o(1/(x \log x))$ .

**j'ai l'impression que ca cafouille un peu au niveau des croissants-décroissants. Dans ce bout de demo, je noterais plutôt  $U(x) = xh(x)$ , et on a besoin de  $U$  décroissant, si je me trompe pas. Je tente d'inclure le lemme suivant, vous me direz ce que vous en pensez.**

**Lemma 2** *Let  $h$  be a function satisfying the assumptions of condition (d). Then one can find a function  $\phi$  satisfying the same assumptions, such that  $\phi \geq h$  and  $x\phi(x)$  is a decreasing function in a neighborhood of 0.*

**Proof:** Without loss of generality, one can assume that  $h$  is  $C^1((0,1))$  and that  $\lim_0 h = \infty$ . Hence, in a neighborhood of 0,  $h$  is a decreasing convex function. Now, if there exists a constant  $c_1$  such that, for  $\beta > 1$ ,

$$h(x) \leq \frac{c_1}{x|\log(x)|^\beta} \equiv \psi_1(x),$$

just take  $\phi = \psi_1$ . Otherwise, if there exists a  $x_0$  and a strictly positive constant  $c_2$  such that, for  $\beta > 1$ ,

$$h(x_0) \geq \frac{c_2}{x_0|\log(x_0)|^\beta} \equiv \psi_2(x_0),$$

then, by convexity of  $h$ , we have  $h \geq \psi_2$  in a neighborhood of 0, and

$$\lim_{x \rightarrow 0} xh'(x) = -\infty. \tag{18}$$

On the other hand, if  $\int_0 h < \infty$  and  $h$  is a decreasing convex function, then

$$xh(x) \leq c_3 \tag{19}$$

in a neighborhood of 0. Putting together (18) and (19), we get

$$h(x) + xh'(x) < 0$$

for  $x$  small enough. Thus,  $xh(x)$  is a decreasing function in a neighborhood of 0. □

The estimate of  $I_k$  now yields

$$\begin{aligned} I_k &\leq 75K^2 \sum_{l=0}^k \frac{1}{F(g(5^{-(l+1)}))} \\ &\leq 75K^2 \sum_{l=0}^k \frac{1}{F(25^{-(l+1)})}. \end{aligned}$$

The conclusion the proof of (c) $\implies$ (d).is now immediate from the following trivial calculation,

$$\infty > \int_0^{x_0} h = \int_0^{x_0} \frac{dx}{xF(x)} = \ln(25) \int_{y_0}^{\infty} \frac{dy}{F(25^{-y})} \geq \ln(25) \sum_{n=n_0}^{\infty} \frac{1}{F(25^{-(l+1)})},$$

where the values  $x_0$ ,  $y_0$ , and  $n_0$  are obvious notation signifying that we ignore the first terms in  $I_k$  to ensure the validity of the inequalities used above when  $x$  is close to 0.  $\square$

### 2.4.2 Proof of (d) $\implies$ (c)

The following lemma on summation by parts will be useful.

**Lemma 3** *Let  $(A_n)_n$  and  $(B_n)_n$  be sequences of real numbers. Let  $a_1 = A_1$ ,  $b_1 = B_1$ , and for all  $j \geq 2$ ,  $a_j = A_j - A_{j-1}$  and  $b_j = B_j - B_{j-1}$ . Then*

$$A_n B_n = \sum_{j=1}^n A_j b_j + \sum_{j=2}^n B_{j-1} a_j.$$

Proof: iterate the relation:

$$A_n B_n = A_n (B_n - B_{n-1}) + B_{n-1} (A_n - A_{n-1}) + A_{n-1} B_{n-1}.$$

from  $n$  to 1.  $\square$

**Step 1. Space discretization.** We first show that it is sufficient to show the conclusion of (c) for the  $x$ 's of the form  $x = 1/n$  where  $n$  is an integer. Indeed assume that there exist  $K > 0$  and  $n_{\min}$  an integer such that for all  $n \geq n_{\min}$ :

$$\delta(1/n)^2 \leq K g(1/n).$$

For an arbitrary  $x \in (0; 1/n_{\min}]$ , let  $n$  be such that  $x \in (1/(n+1); 1/n]$ . We have

$$\begin{aligned} \delta^2(x) &\leq \delta^2(1/n) \leq K g(1/n) \\ &= K g(x) \left( 1 + \frac{g(1/n) - g(x)}{g(x)} \right). \end{aligned} \tag{20}$$

By Assumption A, without loss of generality, we can assume that  $g(0) = 0$  and that  $\tilde{g} = g^{1/2}$  is concave near 0. This implies that if  $b > x > 0$ ,  $[\tilde{g}(b) - \tilde{g}(x)] / [b - x] \leq g(x) / x$ . Using this fact and the fact that  $1/(n+1) < x$  implies  $1/n < 2x$  as long as  $x < 1/2$ , we obtain:

$$\begin{aligned} \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{1/n - x} \cdot \frac{1/n - x}{g(x)} \\ &\leq \frac{g(x)}{x} \cdot \frac{1/n - x}{g(x)} \\ &= \frac{1/n - x}{x} < 1. \end{aligned}$$

Then we can estimate

$$\begin{aligned} \frac{g(1/n) - g(x)}{g(x)} &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} \cdot \frac{\tilde{g}(1/n) + \tilde{g}(x)}{\tilde{g}(x)} \\ &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} \cdot \left[ \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} + 2 \right] \\ &< 3. \end{aligned}$$

Returning to (20) we get

$$\delta^2(x) \leq 3Kg(x).$$

**Step 2.** *Separating the head and the tail of  $\delta$ .* Let  $n_0$  be a fixed integer larger than  $n_{\min}$ . We have

$$\begin{aligned} \delta^2(1/n_0) &= \sum_{n=1}^{n_0-1} r_n (1 - \cos(n/n_0)) + \sum_{n=n_0}^{\infty} r_n (1 - \cos(n/n_0)) \\ &\leq \sum_{n=1}^{n_0-1} r_n (n/n_0)^2 + \sum_{n=n_0}^{\infty} r_n. \end{aligned}$$

We only need to show that there exists  $K > 0$  such that for all  $n_0 > n_{\min}$ , the following two inequalities hold:

$$\sum_{n=1}^{n_0} r_n (n/n_0)^2 \leq Kg(1/n_0), \quad (21)$$

$$\sum_{n=n_0}^{\infty} r_n \leq Kg(1/n_0). \quad (22)$$

We will assume (d) holds and will assume successively that each of these two inequalities does not hold; we will obtain a contradiction in each case.

**Step 3.** *Controlling the tail.*

**Step 3.1.** *Assuming the tail is unbounded.* The negation of inequality (22) is equivalent to the existence of a sequence of integers  $(N_m)_m$  that increases to  $+\infty$ , and a sequence of positive reals  $(K_m)_m$  that increases to  $+\infty$ , satisfying for all  $m \in \mathbf{N}$ ,

$$\sum_{n=N_m}^{\infty} r_n \geq g(1/N_m) K_m. \quad (23)$$

It will be convenient below to use the fact that without loss of generality, we can choose  $K_m$  to increase to infinity as slowly as we want, without effecting the sequence  $(N_m)_m$ . Let  $h$  be a function as in (d). Recall that  $n \mapsto h(g(1/n))$  is strictly increasing. We introduce the following notation:

$$\begin{aligned} B^{(m)} &:= \sum_{n=N_m}^{\infty} r_n, \\ -b^{(m+1)} &:= B^{(m)} - B^{(m+1)} = r_{N_m} + \cdots + r_{N_{m+1}-1}, \\ A^{(m)} &:= h(g(1/N_m)) \\ a^{(m+1)} &:= A^{(m+1)} - A^{(m)} > 0. \end{aligned}$$

By hypothesis (d) the tail  $\sum_{n=N_m}^{\infty} r_n h(g(1/n))$  converges to 0 as  $m \rightarrow \infty$ . We will calculate this tail using the summation-by-parts lemma 3 with the  $A$ 's and  $B$ 's as above. This will enable us to use the hypothesis (23) on this tail, and another application of Lemma 3 will

yield a contradiction thanks to an appropriately chosen  $h$ . Let  $m_0$  be fixed. We have

$$\begin{aligned} \sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &= \sum_{m=m_0}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} r_n h(g(1/n)) \\ &\geq \sum_{m=m_0}^{\infty} A^{(m)} \left( -b^{(m+1)} \right) \\ &= \sum_{m=m_0-1}^{\infty} a^{(m+1)} B^{(m+1)} - \lim_{m \rightarrow \infty} A^{(m)} B^{(m)}, \end{aligned}$$

where the last equality is by Lemma 3. We can prove that the last limit does in fact exist and is equal to 0. Indeed

$$\begin{aligned} A^{(m)} B^{(m)} &= h(g(1/N_m)) \sum_{n=N_m}^{\infty} r_n, \\ &\leq \sum_{n=N_m}^{\infty} r_n h(g(1/n)) \end{aligned}$$

and the last term converges to 0 as  $m \rightarrow \infty$  by hypothesis (d). Now we use (23), which yields

$$\begin{aligned} \sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &\geq \sum_{m=m_0}^{\infty} a^{(m)} g(1/N_m) K_m \\ &\geq K_{m_0} \sum_{m=m_0}^{\infty} a^{(m)} g(1/N_m). \end{aligned}$$

We calculate this last series by Lemma 3 again, using the notation  $C^{(m)} := g(1/N_m)$  and  $-c^{(m)} := g(1/N_{m-1}) - g(1/N_m)$ :

$$\begin{aligned} \sum_{m=m_0}^{\infty} a^{(m)} g(1/N_m) &= \sum_{m=m_0+1}^{\infty} A^{(m-1)} \left( -c^{(m)} \right) + \lim_{m \rightarrow \infty} A^{(m)} C^{(m)} \\ &= \sum_{m=m_0+1}^{\infty} h(g(1/N_{m-1})) [g(1/N_{m-1}) - g(1/N_m)] \end{aligned}$$

where the fact that the last limit is zero is a trivial consequence of the integrability of  $h$  at 0. To summarize we have proved:

$$\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geq K_{m_0} \sum_{m=m_0+1}^{\infty} h(g(1/N_{m-1})) [g(1/N_{m-1}) - g(1/N_m)]. \quad (24)$$

It is now sufficient to show that  $h$  can be chosen integrable at 0 and strictly decreasing, and such that for all  $m_0$  large enough,

$$\sum_{m=m_0}^{\infty} h(g(1/N_{m-1})) [g(1/N_{m-1}) - g(1/N_m)] \geq \frac{1}{\sqrt{K_{m_0}}}. \quad (25)$$

**Step 3.2. Choosing  $h$ .** We let  $g_m = g(1/N_{m-1})$  and introduce a arbitrary sequence  $(k_m)_m$  such that  $\lim k_m = 0$  and  $k_m \geq (K_m)^{-1/2}$ . First we show that we can reduce the

problem of finding  $h$  as above to the problem of finding a strictly increasing sequence of positive numbers  $(h_m)_m$  such that

$$\sum_{m=m_0}^{\infty} h_m [g_m - g_{m+1}] \geq k_{m_0}, \quad (26)$$

and such that the series on the left converges. Indeed define a function  $h$  as follows: for each fixed  $m$ , define  $h$  to be linear on the interval  $(g_m; g_{m-1}]$ , with endpoints set to

$$\begin{aligned} h(g_{m-1}) &= h_{m-1}, \\ \lim_{x \downarrow g_m} h(x) &= \min(h_m; 2h_{m-1}). \end{aligned}$$

Since  $(h_m)_m$  is strictly increasing, this  $h$  is strictly decreasing. Moreover it is clear that

$$\begin{aligned} & \sum_{m=m_0}^{\infty} h_m [g_m - g_{m+1}] \\ &= \sum_{m=m_0}^{\infty} h(g(1/N_{m-1})) [g(1/N_{m-1}) - g(1/N_m)] \\ &\leq \int_0^{g_{m_0}} h(x) dx \leq 2 \sum_{m=m_0}^{\infty} h_m [g_m - g_{m+1}]. \end{aligned}$$

This implies that (25) holds and also that  $h$  is integrable at 0, which is what we want. Thus we only need to find  $(h_m)_m$  as in (26). Because of the flexibility we have on the choice of  $(K_m)_m$  (being able to decrease all the values of  $K_m$  as long as the resulting sequence still converges to  $+\infty$ ), there is no actual loss of generality in fixing the values of  $h_m$  and searching for new values of  $K_m$  such that (26) holds and  $k_m \geq (K_m)^{-1/2}$ . More precisely we choose  $h_m = f(g_m)$  where  $f$  is any positive strictly decreasing integrable function (e.g.  $f(x) = x^{-1/2}$ ). Then we can simply define  $(k_m)_m$  by imposing that (26) hold as an equality. Note that we have

$$\int_0^{g_1} f(x) dx \geq \sum_{m=1}^{\infty} f(g_m) [g_m - g_{m+1}].$$

Therefore,  $k_m$  is the tail of this convergent series, and so it tends to zero. Therefore there is no loss of generality in reassigning the values of  $K_m$  to satisfy for all  $m \geq 1$

$$K_m \leq \frac{1}{(\sup_{l \geq m} k_l)^2}.$$

**Step 4. Controlling the head.** This step follows a similar structure to Step 3.

**Step 4.1. Negating the head bound.** The negation of inequality (21) is equivalent to the existence of a sequence of integers  $(N_m)_m$  that increases to  $+\infty$ , and a sequence of positive reals  $(K_m)_m$  that increases to  $+\infty$ , satisfying for all  $m \in \mathbf{N}$ ,

$$\sum_{n=1}^{N_m} n^2 r_n \geq (N_m)^2 g(1/N_m) K_m. \quad (27)$$

Let  $h$  be a function as in (d). We introduce the following notation:

$$\begin{aligned}
B^{(m)} &:= \sum_{n=1}^{N_m} n^2 r_n, \\
b^{(m)} &:= B^{(m)} - B^{(m-1)} = \sum_{n=N_{m-1}+1}^{N_m} n^2 r_n, \\
A^{(m)} &:= \frac{1}{(N_m)^2} h\left(g\left(\frac{1}{N_m}\right)\right), \\
a^{(m)} &:= A^{(m)} - A^{(m-1)}.
\end{aligned}$$

We will show in Step 4.3 that  $h$  can be chosen in such a way that with  $A_n := n^{-2}h(g(1/n))$ , the sequence  $(A_n)_n$  is decreasing. This yields

$$\begin{aligned}
\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &= \sum_{n=N_{m_0}}^{\infty} n^2 r_n \frac{1}{n^2} h(g(1/n)) \\
&\geq \sum_{n=N_{m_0}}^{\infty} \frac{1}{(N_m)^2} h\left(g\left(\frac{1}{N_m}\right)\right) \sum_{n=N_{m-1}+1}^{N_m} n^2 r_n \\
&= \sum_{m=m_0}^{\infty} A^{(m)} b^{(m)} \\
&= \sum_{m=m_0+1}^{\infty} \left(-a^{(m)}\right) B^{(m-1)} + \lim_{m \rightarrow \infty} A^{(m)} B^{(m)} \\
&\geq \sum_{m=m_0+1}^{\infty} \left(-a^{(m)}\right) K_{m-1} g\left(\frac{1}{N_{m-1}}\right) (N_{m-1})^2 \\
&\geq K_{m_0+1} \sum_{m=m_0+1}^{\infty} \left(-a^{(m)}\right) g\left(\frac{1}{N_{m-1}}\right) (N_{m-1})^2 \\
&= K_{m_0+1} \sum_{m=m_0}^{\infty} A^{(m)} c^{(m)} - \lim_{m \rightarrow \infty} A^{(m)} C^{(m)},
\end{aligned}$$

where  $C^{(m)} = g(1/N_m)(N_m)^2$ . We note that  $A^{(m)}C^{(m)} = g(1/N_m)h(g(1/N_m))$ . However, since we assumed that  $h$  is decreasing and  $\int_0 h < \infty$ , we get immediately that  $\lim_{x \rightarrow 0} xh(x) = 0$ , so that the last limit above is zero.

**Step 4.2.** *Applying the method of Step 3.* Thanks to the previous step we have:

$$\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geq K_{m_0} \sum_{m=m_0}^{\infty} h(g(1/N_m)) \left[ g(1/N_m) - \frac{(N_{m-1})^2}{(N_m)^2} g(1/N_{m-1}) \right]$$

In fact this implies that inequality (24) holds. Indeed by Condition C, we get

$$\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geq cK_{m_0} \sum_{m=m_0}^{\infty} h(g(1/N_m)) [g(1/N_{m-1}) - g(1/N_m)]$$

which is stronger than (24) since  $h(g(1/N_m)) > h(g(1/N_{m-1}))$ . We now apply the strategy of Step 3 by saying that it is sufficient to find a function  $h$  integrable at 0 and strictly decreasing, and such that for all  $m_0$  large enough, (25) holds.

**Step 4.3. Choosing  $h$ .** We choose  $h(y) = 1 - y$ , defined in a neighborhood of 0. Clearly this  $h$  is integrable at 0 and strictly decreasing. Now for each integer  $m_0$  we define  $k_{m_0}$  by

$$\sum_{m=m_0}^{\infty} h(g(1/N_m)) [g(1/N_{m-1}) - g(1/N_m)] = k_{m_0}.$$

Since  $h$  is bounded by 1, the left-hand side of this equality is the tail of a converging series. Therefore  $k_m$  decreases to 0, and (25) holds by invoking the reassignment of the values of  $(K_m)_m$  described at the end of Step 3.2.

The only thing left to prove is that this  $h$  is consistent with the condition, announced at the beginning of Step 4.1, that  $A_n = n^{-2}h(g(1/n))$  is a decreasing sequence for  $n$  large enough. That is, we want to show that for all  $n$  large enough,

$$\left(\frac{1}{n+1}\right)^2 (1 - g(1/(n+1))) \leq \left(\frac{1}{n}\right)^2 (1 - g(1/n))$$

which is equivalent to:

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} < 1 + 2/n + 1/n^2.$$

To see this we can assume without loss of generality that near 0, either  $g$  is concave or  $g(x) \leq x$ . In the first case, we have

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} = 1 + [1 + g(1/n) + o(g(1/n))] g'(\xi_n) \frac{1}{n(n+1)}$$

where  $\xi \in (1/(n+1); 1/n)$ . Since  $g$  is concave we have  $g'(\xi_n) \leq g'(1/(n+1))$ , and also  $g'(x) \leq g(x)/x < 1/x$  for small  $x$ . Thus we get:

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} < 1 + 2/n.$$

For the other case,  $g(x) \leq x$ , we can also assume without loss of generality that  $g(x) \geq x^2$  because of Assumption A. Then we get for large  $n$ :

$$\frac{1 - g(1/(n+1))}{1 - g(1/n)} \leq \frac{1 - (n+1)^{-2}}{1 - n^{-2}} = 1 + \frac{1}{n} + o\left(\frac{1}{n}\right) < 1 + \frac{2}{n}.$$

□.

### 3 Sharp regularity for the stochastic heat equation

#### 3.1 The SHE with infinite-dimensional fractional Brownian noise

The metric  $|\cdot|$  on the circle  $S^1$  identified to  $[0, 2\pi)$  coincides with the usual Euclidean distance on any subinterval of length  $\pi$ , normalized by the factor  $2\pi$ , with an obvious extension to  $M$  because of the identification of 0 and  $2\pi$ . Let  $\{e_n\}$  be the orthonormal basis of  $L^2 = L^2(S^1)$  made of trigonometric functions, namely the set of eigenfunctions of

$\Delta$  on  $S^1$ . As in [15], and as is often done in the theory of stochastic PDEs, let us write the SHE (1) in its weaker *evolution* form:

$$X(t, x) = \int_0^t P_{t-s} [B_n(ds, \cdot)](x) : t \in [0, 1]; \quad x \in S^1. \quad (28)$$

Clearly, because the original equation had only an additive noise term, the evolution form of the equation gives the solution explicitly. Here  $(P_t)_{t \geq 0}$  is the semigroup of operators generated by the Laplacian on  $S^1$ . The action of this semigroup on  $L^2(S^1)$  is characterized by

$$P_t [e_n] = \exp(-tn) e_n$$

for all the functions  $e_n$ . In accordance with our announcement in the introduction, we assume, as was done in [15], and as is often done in the theory of stochastic PDEs with any sort of infinite-dimensional noise, that the random field  $B$  is an infinite-dimensional fBm, with values in  $L^2(S^1)$ , given by a random Fourier series: for  $\{q_n\}_n$  a sequence of nonnegative terms, we let

$$B(t, x) = B^H(t, x) = \sum_n \sqrt{q_n} e_n(x) \beta_n^H(t), \quad (29)$$

where  $\beta_n^H$  are IID fBm's with constant Hurst parameter  $H \in (0, 1)$ . The reader can refer to any recent article on fractional Brownian motion for a detailed treatment of how each process  $\beta_n^H$  can be defined. We suggest our own [15], or for a more detailed treatment, [6], [2], or [3].

The random field  $B$  can be *generalized* in the  $x$  parameter – meaning that  $X(t, \cdot)$  can be almost-surely a (Schwartz) distribution (a generalized function) – while still allowing for the solution  $X$  in (28) to be defined as a bonafide function of both  $t$  and  $x$ . In [15], the following (easy) random Fourier representation is given for  $X$  when it exists:

$$X(t, x) = \sum_n \sqrt{q_n} e_n(x) \int_0^t e^{-n^2(t-s)} \beta_n^H(ds); \quad (30)$$

moreover it is proved that a necessary and sufficient condition for existence of  $X$  is

$$\sum_n \frac{q_n}{n^{4H}} < \infty. \quad (31)$$

Only under the stronger condition that  $q_n$  is summable can we guarantee that  $B$  is a true function in  $L^2(S^1)$ , but this is somewhat irrelevant to the purpose of studying  $X$ .

Note that both  $B$  and  $X$  are spatially homogeneous centered Gaussian random fields. In particular, for fixed  $t$ , they can be represented as

$$\begin{aligned} B(t, \cdot) &= \sum_n \sqrt{q_n(t)} e_n W_n, \\ X(t, \cdot) &= \sum_n \sqrt{s_n(t)} e_n G_n, \end{aligned}$$

where clearly  $q_n(t) = q_n t^{2H}$  and where  $s_n$  is another function of  $t$ , and where  $(W_n)_n$  and  $(G_n)$  are I.I.D. sequences of standard normal r.v.'s.  $W$  and  $G$  are not independent, but their correlation will be of no consequence for the sequel. Also, for fixed  $t$ ,  $X(t, \cdot)$  is almost-surely in  $L^2$  if and only if for each fixed  $x$ ,  $E[X(t, x)^2] < \infty$ . The above facts are where condition (31) come from. More specifically, in the proof of Theorems 2 and 3 in [15], it is established that the variance of the centered Gaussian r.v.  $\int_0^t e^{-n^2(t-s)} B_n^H(ds)$  is commensurate with  $n^{-4H}$ , which means we have the following single fundamental estimate:

**Lemma 4** For fixed  $t \in [0, 1, ]$  we have  $s_n(t) \asymp q_n(t) / n^{4H}$ , where the commensurability constants depend only on  $t, H$ , and on the original sequence  $(q_n)_n$ .

Our purpose now is to seek a stronger condition on  $(q_n)_n$  than (31) which characterizes the fact that the solution has a specified almost-sure spatial modulus of continuity.

## 3.2 Type I (pathwise) characterization

### 3.2.1 General result and examples

We now describe how to use the first theorem to compare the almost-sure regularities of  $B$  and  $X$ . In what follows  $t$  is a fixed positive value, and with Lemma 4 in mind, we abusively use the notation  $q_n$  and  $s_n$  for  $q_n(t)$  and  $s_n(t)$ , omitting the  $t$ . It is reassuring to note that from the proof of Theorems 2 and 3 in [15],  $q_n$  and  $s_n$  (and similar time-dependent constant used below) are bounded away from 0 and  $\infty$  as soon as the same holds for  $t$ .

Let  $Y = (I - \Delta)_x^{-H} B$ . The operator  $(I - \Delta)_x^{-H}$  on  $L^2$  is defined by saying that

$$(I - \Delta)_x^{-H} e_n(x) = e_n(x) / (1 + n^2)^H.$$

$Y$  can be interpreted as an ‘‘antiderivative of order  $2H$ ’’ for  $B$ . For fixed  $t > 0$ , the expansion of  $Y$  is a random Fourier series

$$Y(t, \cdot) = \sum \sqrt{r_n} e_n(\cdot) W'_n$$

where, by Lemma 4,  $r_n$  is commensurate with  $s_n$ :

$$s_n \asymp r_n. \tag{32}$$

Now assume that  $Y$  has for fixed  $t$ , an almost-sure uniform spatial modulus of continuity  $f$ . Let  $\delta_Y$  be the canonical metric function for  $Y(t, \cdot)$ . Assume  $\delta_Y$  satisfies Assumption A and Condition B. Then by Theorem 1 part (a), for some  $K > 0$ , for all  $\alpha$  small enough,

$$Kf(\alpha) \geq \int_0^\infty \delta_Y \left( \min \left( e^{-x^2}, \alpha \right) \right) dx.$$

Because of formula (2) and the fact that  $s_n \asymp r_n$  we get that for some (possibly different) constant  $K$ , for all small  $r$ ,

$$\begin{aligned} \delta_Y(r) &= \sum_{n=1}^\infty \sqrt{r_n} (1 - \cos(nr)) \\ &\geq K \sum_{n=1}^\infty \sqrt{s_n} (1 - \cos(nr)) \\ &= K \delta_X(r), \end{aligned}$$

where  $\delta_X$  is the canonical metric function for  $X(t, \cdot)$ . Thus for some constant  $K$ ,

$$Kf(\alpha) \geq \int_0^\infty \delta_X \left( \min \left( e^{-x^2}, \alpha \right) \right) dx = f_{\delta_X}(\alpha),$$

Now use part (b) of Theorem 1: since  $\lim_{0+} f = 0$ , the same holds for  $f_{\delta_X}$ , and we get that  $f_{\delta_X}$  is an almost-sure uniform modulus of continuity for  $X(t, \cdot)$  up to a constant. Since  $Kf \geq f_{\delta_X}$  we get that  $f$  itself is an almost-sure uniform modulus of continuity for  $X(t, \cdot)$ . Since  $s_n \asymp r_n$ , the roles of  $X$  and  $Y$  can be swapped, which proves the following theorem, modulo the statements in the Hölder case, which are clear given Corollary 1.

**Theorem 3** *Let  $X, Y$  be as above, relative to  $B$ . Let the function  $\delta_Y$  be defined by*

$$\delta_Y(r) = \sum_{n=1}^{\infty} q_n \frac{1}{n^{4H}} (1 - \cos(nr))$$

*We assume Condition B hold for  $\delta_Y$ . Let  $f$  be an increasing continuous function on  $\mathbb{R}_+$  with  $\lim_{0^+} f = 0$ . For any fixed  $t > 0$ ,  $f$  is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for  $Y(t, \cdot)$  if and only if  $f$  is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for  $X(t, \cdot)$ . Also,  $\delta_Y$  is the canonical metric function of  $Y(1, \cdot)$ , and the function  $f_Y$  defined by*

$$f_Y(\alpha) = \int_0^{\infty} \delta_Y\left(\min\left(e^{-x^2}, \alpha\right)\right) dx$$

*is also an almost-sure uniform modulus of continuity for both  $Y(t, \cdot)$  and  $X(t, \cdot)$ , and is bounded above by a constant multiple of  $f$ .*

*In the Hölder case  $\delta_Y(r) = r^\alpha$  for some  $\alpha \in (0, 1)$ , Condition B is not satisfied. However, we can assert that if  $f$  is an almost-sure uniform modulus of continuity for  $Y(t, \cdot)$ , then  $\bar{f}(r) = f(r) \log(1/r)$  is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for  $X(t, \cdot)$ , and  $f_Y$  is bounded above by a constant multiple of  $f \log(1/\cdot)$ ; the same statements hold if one exchanges the roles of  $X$  and  $Y$ .*

We now consider some illustrative examples. In this development, we omit the appellation “almost-sure, uniform” when talking about spatial moduli of continuity.

- Assume that  $Z$  is a Gaussian field on  $R_+ \times S^1$  that is fBm in time with parameter  $H$  and has fBm behavior in space with parameter  $H'$ , and that  $B^H = (I - \Delta)^{1/2} Z$ . In this way,  $q_n \asymp n^{1-2H}$ , and we have a generalized random field  $\partial B / \partial t$  in the SHE (1) that is the first derivative of fBm both in time and in space (with different Hurst parameters), and can thus be considered as space-time fractional noise. Then we have  $Y = (I - \Delta)^{1/2-H} Z$ . We see that there is existence of  $X$  if and only if  $H' > 1 - 2H$ . Certainly if  $H' < 1 - 2H$ , we do not have existence. But if we cannot guarantee that  $H'$  actually exceeds  $1 - 2H$  – that is, if in the Hölder scale it looks like  $H' = 1 - 2H$  – we can still have a solution with a certain amount of regularity. Indeed Theorem 3 asserts that only a spatial “derivative” of  $Z$  of order  $1 - 2H$  needs to exist; for example, the spatial modulus of continuity of  $X$  is commensurate with  $f_\alpha(r) = (\log(1/r))^{-\alpha}$  for some fixed  $\alpha > 0$  if and only if the same holds for the spatial derivative or order  $1 - 2H$  of  $Z$ . In this situation,  $Z$  is spatially more regular than fBm of parameter  $1 - 2H$ , but is not spatially fBm for any parameter  $H' > 1 - 2H$ .
- Consider now the case where  $Z$  is spatially fBm with parameter  $H' > 1 - 2H$ . One can check that a sharp spatial modulus of continuity for  $Y$  is  $f(r) = r^{H'-1+2H} \log^{1/2}(1/r)$ . Theorem 3 then asserts the following.

- For the equation (1) driven by space-time fractional noise with Hurst parameters  $H$  and  $H'$  respectively, the evolution solution  $X$  admits

$$f(r) = r^{H'-1+2H} \log^{1/2}(1/r)$$

as a modulus of continuity. Note here that the full force of the characterization is being used because we start with a bound on the canonical metric of the potential, and can reprove Theorem 3 without needing to invoke the “lower bound” portion (a) of Theorem 1 and Corollary 1 (see Corollary 2).

- If the evolution solution of equation (1) has fractional Brownian regularity in space, in the sense that for some  $H'' \in (0, 1)$ , it admits  $f(r) = r^{H''} \log^{1/2}(1/r)$  as a spatial modulus of continuity, then the equation's potential is the spatial derivative of a Gaussian field which admits  $\bar{f}(r) = r^{H''+1-2H} \log^{3/2}(1/r)$  as a spatial modulus of continuity
- In the previous “necessary condition implication”, we do not know if the logarithmic corrections can be disposed of, because we do not know whether Corollary 1 is sharp. However, in the Hölder scale, these corrections can be viewed as irrelevant.
- When  $H = H' = 1 - 2H = 1/3$ , since then  $Y$  cannot be Hölder continuous, we can try to invoke Theorem 1 without needing Corollary 1. We get the following.
  - For the equation (1) driven by space-time fractional noise with common Hurst parameters  $1/3$  in time and space, the evolution solution  $X$  does not exist. This can be established using the results of [15] only. It can be considered as the limit of non-existence of  $X$ ...
  - However, in the case  $H = 1/3$ , assume  $Y$  admits  $f(r) = r^{1/3} \tilde{f}(r)$  as a spatial modulus of continuity where  $\lim_0 \tilde{f} = 0$  and  $\tilde{f}(r) \gg r^\alpha$  for all  $\alpha > 0$ . Then  $\tilde{f}(r)$  is a spatial modulus of continuity for  $X$ , and the converse holds, still assuming  $H = 1/3$ .

### 3.2.2 Condition B is morally not necessary for regularity

A slightly weaker version of Theorem 3 can be formulated without Condition B if one is willing to change from a *pathwise* to a *distributional* hypothesis. The distributional hypothesis we make here is that the function  $f_Y$ , which can be calculate directly from the law of  $B$ , is continuous at 0. The final conclusion of the corollary seems to be a pathwise statement, but we consider it a hybrid Type I – Type II characterization because  $f_Y$  is characterized by the distribution of  $B$ . In other words, an empirical measurement of the almost-sure modulus of continuity of  $X$  cannot be guaranteed to apply also to  $B$  without Condition B; the only almost sure modulus of continuity they are guaranteed to share is  $f_Y$ , which requires a precise knowledge of  $B$ 's distribution.

**Corollary 2** *Let  $B, X, Y, \delta_Y, f_Y$  be as in Theorem 3, and let  $\delta_X$  and  $f_X$  be defined similarly relative to  $X$ . We have*

$$\lim_{r \downarrow 0} f_Y(r) = 0 \iff \lim_{r \downarrow 0} f_X(r) = 0.$$

*In that situation  $X(1, \cdot)$  and  $Y(1, \cdot)$  share both  $f_Y$  and  $f_X$  as a.s. uniform moduli of continuity. Consequently  $f_Y$  is an a.s. uniform spatial modulus of continuity of continuity for  $X$  if and only if the same holds for  $Y$ . This last statement holds as soon as one knows that either  $X$  or  $Y$  is a.s. continuous, or even merely a.s. bounded.*

**Proof:** The proof follows from Theorem 3 part (b) and the fact, made trivial by relations (2), (6), and (32), that  $f_Y \asymp f_X$ . Note that the last statement is a well-known fact from the general theory of homogeneous Gaussian processes (see [1]).  $\square$

### 3.3 Type II characterization: summability interpretation

#### 3.3.1 Statement of the summability results

The next lemma shows how to invert the formula that gives the almost-sure modulus of continuity from the canonical metric function. In the notation of Theorem 1, it is interesting to note that this lemma implies that  $\delta \mapsto f_\delta$  is a bijective linear map.

**Lemma 5** *Let  $\delta$  be an increasing continuous function on  $\mathbb{R}_+$  with  $\lim_{0+} \delta = 0$ . Let*

$$f(\alpha) = f_\delta(\alpha) := \int_0^\infty \delta\left(\min\left(e^{-x^2}, \alpha\right)\right) dx = \int_0^{\delta(\alpha)} \sqrt{\log 1/\delta^{-1}(\varepsilon)} d\varepsilon. \quad (33)$$

Then

$$\begin{aligned} \delta(\alpha) &= \delta_f(\alpha) = \int_0^{f(\alpha)} (\log 1/f^{-1}(\varepsilon))^{-1/2} d\varepsilon \\ &= \int_0^\alpha f'(r) (\log(1/r))^{-1/2} dr \end{aligned} \quad (34)$$

$$= f(\alpha) (\log(1/\alpha))^{-1/2} - \int_0^\alpha f(r) (\log(1/r))^{-3/2} (2r)^{-1} dr \quad (35)$$

**Proof:** trivial. □

This lemma poses a difficulty in terms of monotonicity of  $\delta$  with respect to  $f$ , because of the negative term in expression (35). However, by virtue of (34), some knowledge of  $f'$  may help exploit this expression in a monotone way.

**Theorem 4** *Let  $f$  be an increasing continuous function on a neighborhood of 0 in  $\mathbb{R}_+$ , continuously differentiable everywhere except at 0, with  $\lim_{0+} f = 0$ . Let  $\delta$  be given by (35). Let  $B$  be defined by (29), and  $X, Y$  be as above relative to  $B$ . Let  $\delta_Y$  be the canonical metric function of  $Y(1, \cdot)$ . Recall the following three statements, which are equivalent if Condition B holds for  $\delta_Y$ :*

- (e) *for some fixed  $t > 0$ ,  $X(t, \cdot)$  has a constant multiple of  $f$  as an almost-sure uniform modulus of continuity;*
- (e') *for some fixed  $t > 0$ ,  $Y(t, \cdot)$  has a constant multiple of  $f$  as an almost-sure uniform modulus of continuity;*
- (f) *for all  $t > 0$ ,  $X(t, \cdot)$  and  $Y(t, \cdot)$  both have a constant multiple of  $f$  as an almost-sure modulus of continuity;*

*If  $\delta$  satisfies Condition C, then (e), (e') and (f) are implied by the following:*

- (g) *for any continuous, decreasing, differentiable function  $h$  on  $[0, 1]$  with  $\int_0^1 h(x) dx < \infty$ ,*

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2\right) < \infty. \quad (36)$$

*For the converse, assume that  $F(y) = f_Y \circ f^{-1}$  is continuously differentiable at 0. Then (g) is implied by (e'). If moreover  $\delta_Y$  satisfies Condition B then (g) is implied by (e). If Condition B is not satisfied, (g) is still implied by (e) provided one replaces, in (36), the expression  $\delta(1/n)$  by  $\delta(1/n) \log(n)$ .*

**Remark 4** In [17], a conjecture in the direction of Theorem 4 was formulated. The authors believed the above result would hold with  $h(r) = r^{-1}$  in condition (g). This Theorem shows that such a condition (g) would be too strong. In fact, one can say that the gap in regularity that is introduced by the stronger version of (g) translates into a factor of order  $(\log(\delta^{-2}(r)))^{1/2}$ ; this factor is not visible in the Hölder scale, which explains why in [16], in which only the Hölder scale is considered, it had been possible to formulate necessary and sufficient conditions whose naive generalization would lead the authors to the slightly erroneous conjecture of [17].

**Remark 5** In [16] and [17], the authors had formulated results similar to Theorems 3 and 4 in the belief that a Type II characterization was a necessary intermediate step in the proof of a Type I characterization. The proofs we propose here show that the two types of characterizations can be established independently of each other.

**Remark 6** The differentiability of  $F$  at 0 essentially brings no restriction. Indeed, if the measured empirical modulus of continuity  $f$  is such that  $f \gg f_Y$ , then we can easily see that  $F'(0) = 0$  with continuity at 0. On the other hand, if it turns out that  $f \asymp f_Y$ , then since our moduli of continuity are only defined up to a multiplicative constant, one may as well assume that  $f_Y = f$ , in which case, as seen in Corollary 2 above and Corollary 3 below, Condition B and the condition on  $F'$  can be dispensed with. A situation in which neither  $f \gg f_Y$  nor  $f \asymp f_Y$  hold can perhaps be considered as pathological, with a discontinuous  $F'$  as an even more of a stretch.

### 3.3.2 Proof of Theorem 4

**Step 1 (converse).** We begin by proving the “Converse” part.

**Step 1.a: (e')  $\implies$  (g) and [Condition B and (e)]  $\implies$  (g).** Assume first that Condition B holds for  $\delta_Y$ . Then since the equivalence of (e) and (e') follows from Theorem 3, we only need to prove (e') implies (g). First note by Theorem 3, under condition B, with  $\delta_Y$  the canonical metric function of  $Y(1, \cdot)$ , (e') implies

$$f_Y(\alpha) := \int_0^\infty \delta_Y(\min(e^{-x^2}, \alpha)) dx \leq Kf(\alpha) \quad (37)$$

for some constant  $K > 0$ . We wish for a similar statement relating  $\delta_Y$  and  $\delta$ .

C'EST ICI QUE J'AI REUSSI A ME DEBARASSER DE LA CONDITION D.

Consider  $F(y) = f_Y(f^{-1}(y))$ . By hypothesis, this function is continuously differentiable everywhere in a neighborhood of 0 in  $\mathbb{R}_+$ . Also note by (37) that  $F(y) \leq Ky$ . The following argument shows that we have, in a neighborhood of 0 in  $\mathbb{R}_+$ ,

$$f'_Y(y) \leq 2Kf'(y). \quad (38)$$

Indeed, if this were not the case, there would exist a sequence  $\{y_n\}_n$  decreasing to 0 such that

$$f'_Y(y_n) > 2Kf'(y_n).$$

Since we have  $F'(y) = f'_Y(f^{-1}(y)) / f'(f^{-1}(y))$ , for all  $y$  in a neighborhood of 0 in  $\mathbb{R}_+$ , this means that with  $z_n = f(y_n)$ , we get  $F'(z_n) > 2K$  for all  $n$ . Now by the mean value theorem on each interval  $[0, y_n]$ , there exists a value  $\xi_n \in (0, y_n)$  such that  $F(y_n) = y_n F'(\xi_n)$ . Therefore  $F'(\xi_n) \leq K$  for all  $n$ . We thus have two sequences  $\{z_n\}$  and  $\{\xi_n\}$  that tend to 0 along which  $F'$  cannot tend to a common value, contradicting the hypothesis that  $F$  is continuously differentiable at 0.

Therefore, using (34) and (38), in a neighborhood of 0, we have,

$$\delta_Y(\alpha) \leq 2K\delta(\alpha). \quad (39)$$

Recall that  $r_n = q_n n^{-4H}$  are the coefficients of the expansion of  $\delta_Y$  in the form of (2). Inequality (39) allows us to use the implication “(c) implies (d)” from Theorem 2 with  $g = \delta^2$  to conclude that for any decreasing, differentiable function  $h$  on  $[0, 1]$  with  $\int_0^1 h(x) dx < \infty$ ,

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2\right) < \infty.$$

We have thus proved (e') implies (g) using Condition B.

**Step 1.b: (e)  $\implies$  (g) with log correction factor.** If (e) is assumed without Condition B, the Hölder-case portion of Theorem 3 proves that  $\bar{f}(r) := f(r) \log(1/r)$  exceeds  $f_Y$  up to a multiplicative constant. Then applying the implication (e')  $\implies$  (g), which we have just established, we obtain inequality (36) with  $\delta$  replaced by  $\delta_{\bar{f}}$ :

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta_{\bar{f}}\left(\frac{1}{n}\right)^2\right) < \infty. \quad (40)$$

However we can write

$$\begin{aligned} \delta_{\bar{f}}(\alpha) &:= \bar{f}(\alpha) (\log(1/\alpha))^{-1/2} - \int_0^\alpha \bar{f}(r) (\log(1/r))^{-3/2} (2r)^{-1} dr \\ &= \log(1/\alpha) f(\alpha) (\log(1/\alpha))^{-1/2} - \int_0^\alpha \log(1/r) f(r) (\log(1/r))^{-3/2} (2r)^{-1} dr \\ &\leq \log(1/\alpha) f(\alpha) (\log(1/\alpha))^{-1/2} - \log(1/\alpha) \int_0^\alpha f(r) (\log(1/r))^{-3/2} (2r)^{-1} dr \\ &= \delta(\alpha) \log(1/\alpha). \end{aligned}$$

Since  $h$  in (40) is decreasing, we obtain

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2 \log^2(n)\right) < \infty,$$

which completes Step 1.b.

**Step 2: (g)  $\implies$  (f):** For the first statement of the theorem, assuming (g), and since  $r_n = q_n n^{-4H}$  are still the coefficients of the expansion of  $\delta_Y$ , assuming Condition C, the implication “(d) implies (c)” in Theorem 2 proves that there exists  $K > 0$  such that the inequality  $\delta_Y(\alpha) < K\delta(\alpha)$  holds for small  $\alpha$ . Applying the transformation  $\delta \mapsto f_\delta$  to this inequality yields for small  $\alpha$ ,

$$f_Y(\alpha) \leq Kf_\delta(\alpha) = Kf(\alpha).$$

where the last equality is by the definition of  $\delta$ . Theorem 3 could now be used directly to conclude on the moduli of continuity of  $X$  and  $Y$ . However, our claim is that condition B is not needed. To see this, note that by hypothesis  $\lim_{0+} f = 0$ , so that the previous inequality justifies invoking Corollary 2, which does not require any conditions, and implies here that both  $X$  and  $Y$  share both  $f$  and  $f_Y$  as a.s. uniform spatial moduli of continuity. The theorem is proved.  $\square$

### 3.3.3 Condition B is morally not necessary for summability

The presence of Condition B and the differentiability condition on  $F$  in Theorem 4 masks the fact that the summability condition (g) is a necessary condition for continuity even when  $F$  is not differentiable at 0, and even when Condition B is not satisfied. We can state this by rephrasing Theorem 4 in a slightly weaker form, assuming only condition C: The resulting corollary operates in the same way as our hybrid theorem Corollary 2: the usage of the corollary below assumes that we can estimate  $f_Y$  sharply, i.e. that we have a handle on the distribution of  $B$ .

**Corollary 3** *Let  $B$  be as in (29), and assume that  $Y = (I - \Delta)^{-H} X$  has canonical metric function  $\delta$  satisfying Condition C. Define  $f = f_\delta$  as in (33). Then Conditions (e') and (g) are equivalent. Moreover, they are equivalent to each of the following:*

- (i)  $Y(1, \cdot)$  is almost-surely bounded;
- (ii)  $Y(1, \cdot)$  is almost-surely continuous;
- (iii)  $\lim_{0+} f_\delta = 0$

*All these conditions are also equivalent to (e) and to (f).*

**Proof.** That (e') is equivalent to (i), (ii) and (iii) is a well-known fact from the general theory of homogeneous Gaussian processes (see [1]). The only part that has not already been established is (e') $\implies$ (g). This follows by the proof of the same implication in the proof of Theorem 4 because here since  $\delta = \delta_Y$ , inequality (39) holds automatically, so there is no need to invoke the differentiability of  $F$ , and for the same reason there is no need to use part (a) of Theorem 1, which makes Condition B superfluous. The last statement is obvious by Corollary 2.  $\square$ .

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