

SOME THEOREMS ON FOURIER COEFFICIENTS

WALTER RUDIN¹

I. Trigonometric polynomials with coefficients ± 1 . Consider the trigonometric polynomial

$$(1.1) \quad P(e^{i\theta}) = \sum_{n=1}^N \epsilon_n e^{in\theta}$$

where $\epsilon_n = \pm 1$. If we set $\|P\|_\infty = \max_\theta |P(e^{i\theta})|$, the Parseval theorem shows that $\|P\|_\infty \geq N^{1/2}$, and the following problem arises: *does there exist an absolute constant A with the property that for each N one can find $\epsilon_1, \dots, \epsilon_N$, equal to ± 1 , so that*

$$(1.2) \quad \|P\|_\infty \leq AN^{1/2},$$

where P is given by (1.1)?

If one allows the coefficients ϵ_n to be complex numbers of absolute value 1, an affirmative answer to the question is furnished by the partial sums of the series $\sum_1^\infty e^{in \log n} e^{in\theta}$; this example is due to Hardy and Littlewood [4, pp. 116–118]. A theorem of Salem and Zygmund [2, pp. 270, 278] shows, roughly speaking, that $(N \log N)^{1/2}$ is the "most probable" order of magnitude for $\|P\|_\infty$ if $\epsilon_n = \pm 1$.

During the summer of 1958, Salem drew my attention to the problem stated in the first paragraph. It turns out that an affirmative answer can be given by a construction which uses nothing more sophisticated than the parallelogram law

$$(1.3) \quad |\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2.$$

After I found this construction I learned that the problem had been solved earlier, by essentially the same method, in the 1951 Master's Thesis of H. S. Shapiro [3]. Since the result is needed in Part II of this paper, I am publishing the proof here, with Shapiro's consent. As in the Hardy-Littlewood example, the polynomials (1.1) may actually be taken as the partial sums of a fixed series $\sum_1^\infty \epsilon_n e^{in\theta}$:

THEOREM I. *There exists a sequence $\{\epsilon_n\}$ ($n=1, 2, 3, \dots$), with $\epsilon_n = 1$ or -1 , such that*

$$(1.4) \quad \left| \sum_{n=1}^N \epsilon_n e^{in\theta} \right| < 5N^{1/2} \quad (0 \leq \theta < 2\pi; N = 1, 2, 3, \dots).$$

Received by the editors March 26, 1959.

¹ The author is a Research Fellow of the Alfred P. Sloan Foundation. This work was supported in part by Air Force Contract SAR/AF 49(638)153.

PROOF. Set $P_0(x) = Q_0(x) = x$, and define polynomials P_k and Q_k inductively by

$$(1.5) \quad \begin{cases} P_{k+1}(x) = P_k(x) + x^{2^k}Q_k(x), \\ Q_{k+1}(x) = P_k(x) - x^{2^k}Q_k(x), \end{cases} \quad (k = 0, 1, 2, \dots).$$

Then $P_k(e^{i\theta})$ is of the form (1.1), with $N = 2^k$, and P_k is a partial sum of P_{k+1} . Hence we can define a sequence $\{\epsilon_n\}$ by setting ϵ_n equal to the n th coefficient of P_k , where $2^k > n$; this sequence will be shown to have the desired properties.

For $|x| = 1$, (1.3) and (1.5) imply

$$\begin{aligned} |P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 &= |P_k(x) + x^{2^k}Q_k(x)|^2 + |P_k(x) - x^{2^k}Q_k(x)|^2 \\ &= 2|P_k(x)|^2 + 2|Q_k(x)|^2, \end{aligned}$$

and since $|P_0(x)|^2 + |Q_0(x)|^2 = 2$, we conclude that

$$(1.6) \quad |P_k(e^{i\theta})|^2 + |Q_k(e^{i\theta})|^2 = 2^{k+1}.$$

Hence

$$(1.7) \quad |P_k(e^{i\theta})| \leq 2^{1/2} \cdot 2^{k/2},$$

which proves (1.4) for $N = 2^k$.

If now $s_n(P_k)$ and $s_n(Q_k)$ denote the n th partial sums of P_k and Q_k respectively, where $1 \leq n \leq 2^k$, then

$$(1.8) \quad \left. \begin{array}{l} |s_n(P_k)(e^{i\theta})| \\ |s_n(Q_k)(e^{i\theta})| \end{array} \right\} \leq (2 + 2^{1/2})2^{k/2} \quad (k = 0, 1, 2, \dots).$$

This is obviously true if $k=0$. Suppose (1.8) holds for some k , and consider $s_n(P_{k+1})$ and $s_n(Q_{k+1})$, with $1 \leq n \leq 2^{k+1}$. If $n \leq 2^k$, (1.5) shows that

$$|s_n(P_{k+1})| = |s_n(Q_{k+1})| = |s_n(P_k)| < (2 + 2^{1/2})2^{(k+1)/2}.$$

If $2^k < n \leq 2^{k+1}$, (1.5) and (1.7) show that

$$\begin{aligned} |s_n(P_{k+1})| &\leq |P_k| + |s_{n-2^k}^k(Q_k)| \\ &\leq 2^{(k+1)/2} + (2 + 2^{1/2})2^{k/2} = (2 + 2^{1/2})2^{(k+1)/2}. \end{aligned}$$

The same estimate holds for $|s_n(Q_{k+1})|$, and (1.8) is proved by induction.

To complete the proof of (1.4), suppose $2^{k-1} \leq N \leq 2^k$. By (1.8), we have

$$|s_N(P_k)(e^{i\theta})| \leq (2 + 2^{1/2})2^{k/2} \leq 2(1 + 2^{1/2})N^{1/2} < 5N^{1/2}.$$

II. Transformations of Fourier coefficients. In this section, p and

q will always denote conjugate exponents, i.e., $1/p + 1/q = 1$. For $1 \leq p < \infty$, L^p denotes the usual Lebesgue space of complex functions on the unit circle, normed by

$$(2.1) \quad \|f\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta \right\}^{1/p}.$$

L^∞ is the space of all essentially bounded measurable functions on the circle. The Fourier coefficients of any $f \in L^1$ will be denoted by

$$(2.2) \quad \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n = 0, \pm 1, \pm 2, \dots).$$

If F is a complex function defined in the plane, we say that F maps A into B , where A and B are function spaces on the circle, if to every $f \in A$ there corresponds a $g \in B$ (we shall write: $g = F \circ f$) such that $\hat{g} = F(\hat{f})$. In other words, it is required that the series $\sum F(c_n) e^{in\theta}$ should be the Fourier series of a function in B whenever $\sum c_n e^{in\theta}$ is the Fourier series of a function in A .

The functions F which map L^1 into L^1 have recently been determined [1]; they are precisely those which are real-analytic near the origin (i.e., in some neighborhood of the origin); of course we must also have $F(0) = 0$. For the other Lebesgue spaces, the situation is quite different. We first state some sufficient conditions:

THEOREM II. *Suppose $1 < p \leq 2$, and suppose there is a constant A such that $|F(z)| \leq A|z|^{q/2}$ near the origin. Then F maps L^p into L^2 .*

PROOF. If $f \in L^p$, the Hausdorff-Young theorem [4, p. 190] shows that $\sum |\hat{f}(n)|^q < \infty$, so that $\sum |F(\hat{f}(n))|^2 < \infty$.

THEOREM III. *Suppose $1 \leq p \leq 2$. If $|F(z)| \leq A|z|^{2/p}$ near the origin, then F maps L^q into L^q .*

PROOF. If $f \in L^q$, then $\sum |\hat{f}(n)|^2 < \infty$, so that $\sum |F(\hat{f}(n))|^p < \infty$, and the Hausdorff-Young theorem implies that $F \circ f \in L^q$.

REMARKS. 1. For $q = 2$, this condition is necessary as well as sufficient.

2. For $q = \infty$, the hypothesis of Theorem III is: $|F(z)| \leq A|z|^2$. It follows that F maps L^∞ (even L^2) into the class of functions which are sums of absolutely convergent trigonometric series.

3. If F is of the form

$$(2.3) \quad F(z) = a_1 z + a_2 \bar{z} + |z|^{2/p} b(z),$$

where b is a function which is bounded near the origin, then F also maps L^q into L^q . Note that no smoothness conditions are imposed on

b (not even measurability is needed), in strong contrast to the results in [1].

I do not know whether (2.3) holds whenever F maps L^q into L^q . However, if we restrict ourselves to *even* functions F , Theorem I can be used to show that *Theorem III states a condition which is necessary as well as sufficient*. In fact, the following stronger assertion holds:

THEOREM IV. *Suppose $1 \leq p < \infty$, F is an even function, and $|z|^{-2/p} |F(z)|$ is not bounded near the origin. Then there is a continuous function f on the circle to which corresponds no $g \in L^q$ with $\hat{g} = F(\hat{f})$.*

In other words, F does not map the space of all continuous functions into L^q , hence it does not map L^q into L^q .

PROOF. The hypothesis implies the existence of numbers $z_m \neq 0$ ($m = 1, 2, 3, \dots$), such that $m^2 z_m \rightarrow 0$ and $|F(z_m)| > m^5 |z_m|^{2/p}$. Define $N_m = [m^{-4} z_m^{-2}]$. These choices produce the relations

$$(2.4) \quad \sum_{m=1}^{\infty} |z_m| N_m^{1/2} < \infty$$

and

$$(2.5) \quad |F(z_m)| N_m^{1/p} \rightarrow \infty \text{ as } m \rightarrow \infty.$$

Now choose integers n_m so that

$$(2.6) \quad n_m + N_m < n_{m+1} - N_{m+1}$$

and define

$$(2.7) \quad T_m(e^{i\theta}) = z_m e^{in_m \theta} (\epsilon_1 e^{i\theta} + \dots + \epsilon_{N_m} e^{iN_m \theta}),$$

where $\{\epsilon_n\}$ is the sequence of Theorem I. The series

$$(2.8) \quad f(e^{i\theta}) = \sum_{m=1}^{\infty} T_m(e^{i\theta})$$

converges uniformly, by (2.4) and Theorem I, so that f is continuous.

Define the kernels K_m by

$$(2.9) \quad K_m(e^{i\theta}) = e^{i(n_m + N_m)\theta} \sum_{n=-2N_m}^{2N_m} \min\left(1, 2 - \frac{|n|}{N_m}\right) e^{in\theta}.$$

Suppose there is a function $g \in L^q$ such that $\hat{g} = F(\hat{f})$, i.e., $g = F \circ f$. Our choice of $\{n_m\}$ implies that $g * K_m = F \circ T_m$, where

$$(2.10) \quad (g * K_m)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i(\theta-\phi)}) K_m(e^{i\phi}) d\phi.$$

Since $\|K_m\|_1 < 3$, we see that

$$(2.11) \quad \|F \circ T_m\|_q < 3\|g\|_q \quad (m = 1, 2, 3, \dots).$$

On the other hand, the assumption that $F(-z_m) = F(z_m)$ shows that

$$(2.12) \quad (F \circ T_m)(e^{i\theta}) = F(z_m)e^{in_m\theta}(e^{i\theta} + \dots + e^{iN_m\theta}),$$

so that

$$(2.13) \quad |(F \circ T_m)(e^{i\theta})| = |F(z_m)| \cdot \left| \frac{\sin(N_m\theta/2)}{\sin(\theta/2)} \right|.$$

An easy computation now yields

$$(2.14) \quad \|F \circ T_m\|_q > C_q |F(z_m)| N_m^{1/p},$$

where C_q is a positive constant, depending only on q . By (2.5), (2.14) implies that $\|F \circ T_m\|_q \rightarrow \infty$ as $m \rightarrow \infty$, and this contradicts (2.11).

The theorem follows.

REFERENCES

1. H. Helson, P. J. Kahane, Y. Katznelson and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. vol. 102 (1959) pp. 135-157.
2. Raphael Salem and Antoni Zygmund, *Some properties of trigonometrical series whose terms have random signs*, Acta Math. vol. 91 (1954) pp. 245-301.
3. H. S. Shapiro, *Extremal problems for polynomials and power series*, Thesis for S. M. Degree, Massachusetts Institute of Technology, 1951.
4. Antoni Zygmund, *Trigonometrical series*, Warsaw, 1935.

YALE UNIVERSITY