

A note on Behrend sequences

I.Z. Ruzsa* & G. Tenenbaum

1. Introduction

The concept of a Behrend sequence is one of the most fundamental and challenging in the theory of sets of multiples. A sequence \mathcal{A} of integers exceeding 1 is called a Behrend sequence if almost all integers n have at least one divisor in \mathcal{A} , or, in other words, if its set of multiples $\mathcal{M}(\mathcal{A}) = \{ma : m \geq 1, a \in \mathcal{A}\}$ has natural density 1. This was recently defined formally by Hall (1990), but the idea has been constantly used by Erdős in the last half-century. Recent progress on this topic may be found in Hall-Tenenbaum (1992), Erdős-Hall-Tenenbaum (1994), Tenenbaum (1994).

By the Davenport-Erdős theorem (1937, 1951), any set of multiples $\mathcal{M}(\mathcal{A})$ has a logarithmic density $\delta\mathcal{M}(\mathcal{A})$, equal to its lower asymptotic density, moreover

$$(1.1) \quad \delta\mathcal{M}(\mathcal{A}) = \lim_{z \rightarrow \infty} \mathbf{d}\mathcal{M}(\mathcal{A} \cap [1, z]).$$

Here and in the sequel we let $\mathbf{d}\mathcal{S}$ (resp. $\delta\mathcal{S}$) denote the asymptotic (resp. logarithmic) density of an integer sequence \mathcal{S} . Thus logarithmic density appears to be best suited to the study of sets of multiples. With the notation

$$(1.2) \quad \mu\mathcal{A} = \delta\mathcal{M}(\mathcal{A}),$$

we can state Behrend's fundamental inequality (1948) as

$$(1.3) \quad 1 - \mu(\mathcal{A} \cup \mathcal{B}) \geq (1 - \mu\mathcal{A})(1 - \mu\mathcal{B}),$$

valid for all sequences \mathcal{A} , \mathcal{B} . This has been given several proofs in the literature, and we present in Section 3 a completely arithmetical version of a probabilistic proof of Ruzsa (1976), which also has the advantage of yielding easily the cases of equality for finite primitive sequences.

By (1.1) and (1.3), we have for any sequence \mathcal{A}

$$(1.4) \quad \mu\mathcal{A} \leq 1 - \prod_{a \in \mathcal{A}} (1 - 1/a)$$

with equality if the elements of \mathcal{A} are pairwise coprime. Thus, under this last assumption, the condition

$$(1.5) \quad \sum_{a \in \mathcal{A}} 1/a = \infty$$

* Supported by CEE fund no. CIPA-CT92-4022 and Hungarian National Science Fund, Grant no. 1901.

is necessary and sufficient for \mathcal{A} to be Behrend. It is very difficult, if not hopeless, to give general criteria to decide whether a given sequence is or not Behrend, but the situation may become somewhat simpler when the study is restricted to sequences with a special structure. Some examples are given in Hall-Tenenbaum (1992), and the case of the so-called ‘block sequences’ (i.e. sequences that are composed of sufficiently long blocks of consecutive integers) is essentially solved by the results of Hall-Tenenbaum (1992) and Tenenbaum (1994).

Here we address the following problem, initially posed by Hall (private communication) and motivated by the fact that the case of a subsequence of the primes is trivially covered by (1.5). Let \mathcal{A} comprise solely integers $n = pq$ with two prime factors, distinct or not. By Theorem 3 of Erdős-Hall-Tenenbaum (1994), we know that \mathcal{A} is Besicovitch, that is $\mathcal{M}(\mathcal{A})$ has asymptotic density, and we ask for a criterion which decides when it is a Behrend sequence. Moreover, we would naturally like this criterion to apply successfully in Erdős’ situation (1959)

$$(1.6) \quad \mathcal{A} = \{pq : p < q \leq p^{1+\varepsilon_p}\},$$

when the condition

$$(1.7) \quad \sum_p \min(1, \varepsilon_p)/p = \infty$$

turns out to be a simple necessary and sufficient condition for \mathcal{A} to be Behrend.

From (1.7), one might be led to the straightforward conjecture that the desired criterion is

$$(1.8) \quad \sum_p \min(1, \lambda_p)/p = \infty$$

with $\lambda_p := \sum_{pq \in \mathcal{A}, p < q} 1/q \in \mathbb{R}^+ \cup \{\infty\}$, and indeed, it is easy to show, using Behrend’s inequality, that this is necessary. However, (1.8) is not sufficient, as shown by the following counterexample, which illustrates the underlying difficulties. Let $\{I_k\}_{k=1}^\infty, \{J_k\}_{k=1}^\infty$ denote two sequences of pairwise disjoint intervals with

$$(1.9) \quad \min J_k > \max I_k, \quad \sum_{p \in I_k} \frac{1}{p} \sum_{q \in J_k} \frac{1}{q} \asymp 1 \quad (k = 1, 2, \dots),$$

and

$$(1.10) \quad \sum_{k=1}^\infty \sum_{q \in J_k} \frac{1}{q} < \infty.$$

A possible choice is $I_k :=] \exp e^{k^3}, \exp e^{k^3+k^2}]$, $J_k :=] \exp e^{(k+1)^3}, \exp e^{(k+1)^3+1/k^2}]$. For each p in $\cup_{k=1}^\infty I_k$, Let \mathcal{A} comprise all pq with $q \in J_k$ if and only if $p \in I_k$. We plainly have that \mathcal{A} is not Behrend since Behrend’s inequality yields, in view of (1.10), that the non-multiples of the q have positive lower density. However, the second requirement in (1.9) implies (1.8).

Our main result stated below is a criterion for sequences composed of integers with two prime factors to be Behrend. We state it in a form which is easily applicable to recover Erdős’ theorem.

Theorem 1. Let $\mathcal{A} \subset \{pq : p, q \text{ primes}\}$. Define

$$\alpha_p := \sum_{pq \in \mathcal{A}} \frac{1}{q}, \quad \alpha_p^*(z) := \sum_{\substack{pq \in \mathcal{A} \\ p < q, \alpha_q \leq z}} \frac{1}{q} \quad (z > 0).$$

Then \mathcal{A} is Behrend if, and only if, we have for all z

$$(1.11) \quad \sum_{\alpha_p > z} \frac{1}{p} + \sum_{\alpha_p \leq z} \frac{\alpha_p^*(z)}{p} = \infty.$$

Corollary (Erdős, 1959). Let $\mathcal{A} = \{pq : p < q \leq p^{1+\varepsilon_p}\}$. Then \mathcal{A} is a Behrend sequence if, and only if, condition (1.7) is fulfilled.

Let $\tau(n, \mathcal{A})$ denote the number of divisors of n belonging to \mathcal{A} , so that \mathcal{A} is a Behrend sequence if and only if

$$(1.12) \quad \tau(n, \mathcal{A}) \geq 1 \quad \text{pp},$$

where, here and in the sequel, we use the notation pp (for the French *presque partout*) to indicate that a relation holds on a sequence of integers with density one. It was shown in Hall-Tenenbaum (1992) that (1.12) is equivalent to

$$(1.13) \quad \tau(n, \mathcal{A}) \rightarrow \infty \quad \text{pp}.$$

We prove slightly more in the following result.

Theorem 2. Let \mathcal{A} be a Behrend sequence. Then \mathcal{A} may be decomposed as an infinite disjoint union $\mathcal{A} = \cup_{k=1}^{\infty} \mathcal{A}_k$ of Behrend sequences.

Remark. This indeed implies (1.13) since, for any fixed $K \geq 1$,

$$\tau(n, \mathcal{A}) = \sum_{k=1}^{\infty} \tau(n, \mathcal{A}_k) \geq \sum_{1 \leq k \leq K} \tau(n, \mathcal{A}_k) \geq K \quad \text{pp}$$

if the \mathcal{A}_k are disjoint and each \mathcal{A}_k is Behrend.

Proof. Let us establish the announced decomposition property. By (1.1), one can choose z_1 so large that $1 - \mu \mathcal{A}_1^* \leq 1/2$, with $\mathcal{A}_1^* = \mathcal{A} \cap (1, z_1]$. However, \mathcal{A}_1^* is finite and hence not Behrend. So (1.3) implies that $\mathcal{A} \setminus \mathcal{A}_1^*$ is also Behrend, and one can choose z_2 such that $\mathcal{A}_2^* = \mathcal{A} \cap (z_1, z_2]$ satisfies $1 - \mu \mathcal{A}_2^* \leq 1/3$. Repeating this process, we may write \mathcal{A} as a disjoint union of finite sequences, viz $\mathcal{A} = \cup_{j=1}^{\infty} \mathcal{A}_j^*$, with $1 - \mu \mathcal{A}_j^* \leq 1/(j+1)$. Now we simply write \mathbb{Z}^+ as an infinite disjoint union of infinite sequences $\mathbb{Z}^+ = \cup_{k=1}^{\infty} \mathcal{Z}_k$, e.g. by defining \mathcal{Z}_k as the sequence of those integers having exactly k distinct prime factors. Then $\mathcal{A}_k = \cup_{j \in \mathcal{Z}_k} \mathcal{A}_j^*$ is Behrend because $1 - \mu \mathcal{A}_k \leq \inf_{j \in \mathcal{Z}_k} (1 - \mu \mathcal{A}_j^*) = 0$.

2. Proof of Theorem 1 and Corollary

First, let us assume Theorem 1 and show how Erdős' theorem may be derived from it.

We observe that, for the sequence \mathcal{A} of the Corollary, we have

$$(2.1) \quad \alpha_p = \sum_{p < q \leq p^{1+\varepsilon_p}} \frac{1}{q} + \sum_{q < p \leq q^{1+\varepsilon_q}} \frac{1}{q} = \log(1 + \varepsilon_p) + O\left(\frac{1}{\log p}\right) + \eta_p \quad (\text{say}),$$

and also, writing $\alpha_p^{**}(z) := \sum_{p < q \leq p^{1+\varepsilon_p}, \alpha_q > z} 1/q$,

$$(2.2) \quad \alpha_p^*(z) + \alpha_p^{**}(z) = \sum_{p < q \leq p^{1+\varepsilon_p}} \frac{1}{q} = \log(1 + \varepsilon_p) + O\left(\frac{1}{\log p}\right).$$

Suppose (1.7) does not hold. Then

$$\eta_p \leq \sum_{\varepsilon_q > 1} \frac{1}{q} + \sum_{\sqrt{p} \leq q < p} \frac{1}{q} \ll 1,$$

and hence, by (2.1),

$$\alpha_p \ll \varepsilon_p + 1 \ll \alpha_p + 1.$$

Using this and (2.2) in the form $\alpha_p^*(z) \ll \varepsilon_p + 1/\log p$, we obtain that, for large z and suitable $C > 0$, we have

$$\begin{aligned} \sum_{\alpha_p > z} \frac{1}{p} + \sum_{\alpha_p \leq z} \frac{\alpha_p^*(z)}{p} &\ll \sum_{\varepsilon_p > 1} \frac{1}{p} + \sum_{\varepsilon_p \leq C(z+1)} \frac{\varepsilon_p + 1/\log p}{p} \\ &\ll_z \sum_p \frac{\min(\varepsilon_p, C(z+1))}{p} + 1 \\ &\ll_z \sum_p \frac{\min(\varepsilon_p, 1)}{p} + 1 \ll 1. \end{aligned}$$

Hence (1.11) does not hold and \mathcal{A} is not Behrend. Thus, (1.7) is necessary.

Conversely, assume (1.7) holds and let z be such that

$$(2.3) \quad \sum_{\alpha_p > z} \frac{1}{p} < \infty.$$

If there is no such z , then (1.11) holds for all z , hence \mathcal{A} is Behrend and there is nothing to prove. Now

$$(2.4) \quad \sum_{\alpha_p \leq z} \frac{\alpha_p^{**}(z)}{p} = \sum_{\alpha_p \leq z} \frac{1}{p} \sum_{\substack{p < q \leq p^{1+\varepsilon_p} \\ \alpha_q > z}} \frac{1}{q} = \sum_{\alpha_q > z} \frac{1}{q} \sum_{\substack{p < q \leq p^{1+\varepsilon_p} \\ \alpha_p \leq z}} \frac{1}{p}.$$

By (2.1), we have $\varepsilon_p \leq e^z - 1 + O(1/\log p)$ in the inner sum, and this is hence $\ll_z 1$, ignoring the condition on α_p . Inserting this estimate in (2.4) yields

$$(2.5) \quad \sum_{\alpha_p \leq z} \frac{\alpha_p^{**}(z)}{p} \ll \sum_{\alpha_q > z} \frac{1}{q} \ll 1,$$

in view of (2.3). Now we see that (1.7), (2.2) and (2.5) imply

$$\sum_{\alpha_p \leq z} \min(1, \alpha_p^*(z))/p = \infty,$$

which in turn gives

$$\sum_{\alpha_p \leq z} \alpha_p^*(z)/p = \infty.$$

This shows that (1.11) holds, and hence that \mathcal{A} is Behrend. Thus, condition (1.7) is sufficient, and this completes the proof of the Corollary.

We now prove Theorem 1. We first show that (1.11) is necessary. Let \mathcal{A} be Behrend and let $z > 0$ be such that

$$(2.6) \quad \sum_{\alpha_p > z} \frac{1}{p} < \infty.$$

We write $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, with

$$\begin{aligned} \mathcal{A}_0 &:= \{p^2 \in \mathcal{A}\}, \\ \mathcal{A}_1 &:= \{pq \in \mathcal{A} : p < q, \alpha_p > z \text{ or } \alpha_q > z\}, \\ \mathcal{A}_2 &:= \{pq \in \mathcal{A} : p < q, \alpha_p \leq z \text{ and } \alpha_q \leq z\}. \end{aligned}$$

\mathcal{A}_0 is clearly not Behrend. Furthermore, Behrend's inequality and (2.6) yield

$$1 - \mu \mathcal{A}_1 \geq \prod_{\alpha_p > z} (1 - 1/p) > 0,$$

and hence \mathcal{A}_1 is also non-Behrend. Therefore \mathcal{A}_2 must be Behrend and, by (1.1),

$$(2.7) \quad \sum_{pq \in \mathcal{A}_2} \frac{1}{pq} = \sum_{\alpha_p \leq z} \frac{\alpha_p^*(z)}{p} = \infty.$$

This proves that the condition is necessary.

Next, we establish the sufficiency. First assume that

$$(2.8) \quad \sum_{\alpha_p > z} \frac{1}{p} = \infty$$

holds for all z . Then, for large z , we can find a finite subsequence $\mathcal{P}(z)$ of $\{p : \alpha_p > z\}$ such that

$$z < \sum_{p \in \mathcal{P}(z)} \frac{1}{p} \leq 2z.$$

Moreover, there is a finite number $Q(z)$ such that, for all $p \in \mathcal{P}(z)$,

$$\sum_{pq \in \mathcal{A}, q \leq Q(z)} \frac{1}{q} \geq \frac{1}{2}\alpha_p > \frac{1}{2}z.$$

We are going to show that the counting function $T(x)$ of the set of non-multiples $\mathcal{T}(\mathcal{A}) = \mathbb{Z}^+ \setminus \mathcal{M}(\mathcal{A})$ satisfies, for $x > x_0(z)$,

$$(2.9) \quad T(x) \ll x(1+z)e^{-z/2}.$$

Since z may be taken arbitrarily large, this shows that the upper natural density $\overline{\mathfrak{d}}\mathcal{T}(\mathcal{A})$ is zero, and hence that \mathcal{A} is Behrend, as required. For the proof of (2.9), we use the simple estimate

$$(2.10) \quad \begin{aligned} \sum_{n \leq x, (n, M)=1} 1 &= \sum_{n \leq x} \sum_{d|(n, M)} \mu(d) = \sum_{d|M} \mu(d) \left[\frac{x}{d} \right] \\ &= x \frac{\varphi(M)}{M} + O(\tau(M)) \asymp x \prod_{p|M} (1 - 1/p) \quad (x > x_0(M)), \end{aligned}$$

valid for any integer M . Indeed, this gives, for x sufficiently large in terms of z ,

$$\begin{aligned} T(x) &\leq \sum_{\substack{n \leq x \\ p \in \mathcal{P}(z) \Rightarrow p \nmid n}} 1 + \sum_{\substack{mp \leq x \\ p \in \mathcal{P}(z) \\ pq \in \mathcal{A}, q \leq Q(z) \Rightarrow q \nmid m}} 1 \\ &\ll x \left\{ \prod_{p \in \mathcal{P}(z)} \left(1 - \frac{1}{p}\right) + \sum_{p \in \mathcal{P}(z)} \frac{1}{p} \prod_{\substack{pq \in \mathcal{A} \\ q \leq Q(z)}} \left(1 - \frac{1}{q}\right) \right\} \\ &\ll x(e^{-z} + ze^{-z/2}). \end{aligned}$$

Thus (2.9) holds and \mathcal{A} is Behrend.

It remains to study the case when (2.6) holds for some z . Then (1.11) implies (2.7). With \mathcal{A}_2 as above, we fix some large number Z and consider $\mathcal{A}_2(Z) := \mathcal{A}_2 \cap (1, Z]$. We set out to prove, by Turán's variance method, that $\mathbf{dM}(\mathcal{A}_2(Z))$ tends to 1 as $Z \rightarrow +\infty$. Put

$$L(Z) := \sum_{a \in \mathcal{A}_2(Z)} \frac{1}{a},$$

so that (2.7) implies that $L(Z) \rightarrow +\infty$ as $Z \rightarrow +\infty$. Now, for fixed Z and $x \rightarrow +\infty$, we have

$$(2.11) \quad x^{-1} \sum_{n \leq x} \tau(n, \mathcal{A}_2(Z)) = L(Z) + o(1).$$

Furthermore

$$(2.12) \quad x^{-1} \sum_{n \leq x} \tau(n, \mathcal{A}_2(Z))^2 \leq L(Z)^2 + S(Z) + o(1),$$

with

$$S(Z) := \sum_{\substack{a, a' \in \mathcal{A}_2(Z) \\ (a, a') > 1}} \frac{1}{[a, a']}.$$

In this double sum, we have $a = pq$, $a' = p'q'$, with $p < q$, $p' < q'$, so one of the following possibilities must hold : $p = p'$, $q = q'$, $p = q'$, or $p' = q$. Let $S_1(Z), \dots, S_4(Z)$ denote the corresponding subsums. We have

$$\begin{aligned} S_1(Z) &\leq \sum_{\alpha_p \leq z} \frac{1}{p} \sum_{\substack{pq \in \mathcal{A}_2(Z) \\ p < q, \alpha_q \leq z}} \frac{1}{q} \sum_{pq' \in \mathcal{A}} \frac{1}{q'} \leq zL(Z), \\ S_2(Z) &\leq \sum_{\alpha_p \leq z} \frac{1}{p} \sum_{\substack{pq \in \mathcal{A}_2(Z) \\ p < q, \alpha_q \leq z}} \frac{1}{q} \sum_{p'q \in \mathcal{A}} \frac{1}{p'} \leq zL(Z), \\ S_3(Z) = S_4(Z) &\leq \sum_{\alpha_p \leq z} \frac{1}{p} \sum_{\substack{pq \in \mathcal{A}_2(Z) \\ p < q, \alpha_q \leq z}} \frac{1}{q} \sum_{p'p \in \mathcal{A}} \frac{1}{p'} \leq zL(Z). \end{aligned}$$

Inserting in (2.12), we obtain

$$x^{-1} \sum_{n \leq x} \tau(n, \mathcal{A}_2(Z))^2 \leq L(Z)^2 + O(L(Z)),$$

whence, in view of (2.11),

$$x^{-1} \sum_{n \leq x} \left\{ \tau(n, \mathcal{A}_2(Z)) - L(Z) \right\}^2 \ll L(Z).$$

By Chebyshev's inequality, it follows that the number of integers $n \leq x$ which do not belong to $\mathcal{M}(\mathcal{A}_2(Z))$ is $\ll x/L(Z)$. Letting x and then Z tend to infinity, we obtain that \mathcal{A} is Behrend. This finishes the proof of Theorem 1.

3. Behrend's inequality : the probabilistic proof in an arithmetical setting

We now establish (1.3). This is a mere translation of the probabilistic proof given by Ruzsa (1976), however it seems interesting to us to write up the details as a completely self-contained arithmetical proof which might lend itself to specific extensions. For the probabilistic background and earlier versions of this argument, see Esary, Proschan & Walkup (1967), and Fortuin, Kasteleyn & Ginibre (1971).

By (1.1), we can clearly restrict to finite sequences \mathcal{A} , \mathcal{B} . If M is any integer with

$$(3.1) \quad (\forall a \in \mathcal{A}) \quad M \equiv 0 \pmod{a},$$

then the membership of an integer n to $\mathcal{M}(\mathcal{A})$ only depends on (n, M) . For a given divisor d of M , there are exactly $\varphi(M/d)$ classes \bar{n} modulo M such that $(n, M) = d$ for $n \in \bar{n}$. Thus, if we denote by $\vartheta_{\mathcal{A}}$ the characteristic function of the set $\mathcal{T}(\mathcal{A})$ of non-multiples, and put, for any arithmetic function f ,

$$E(f; M) := \frac{1}{M} \sum_{d|M} \varphi(M/d) f(d),$$

then we have

$$(3.2) \quad 1 - \mu_{\mathcal{A}} = E(\vartheta_{\mathcal{A}}; M)$$

provided M satisfies (3.1). The fundamental property of the functions $\vartheta_{\mathcal{A}}$ that we shall use is that they are *multiplicatively non-increasing* in the following sense

$$d' | d \Rightarrow \vartheta_{\mathcal{A}}(d) \leq \vartheta_{\mathcal{A}}(d').$$

Since we obviously have $\vartheta_{\mathcal{A} \cup \mathcal{B}} = \vartheta_{\mathcal{A}} \vartheta_{\mathcal{B}}$ for all sequences \mathcal{A} , \mathcal{B} , it is clear from (3.2) that, for finite sequences, Behrend's inequality is a special case of the following lemma, which has an intrinsic interest.

Lemma. *Let f, g be multiplicatively non-increasing arithmetic functions. Then, for all $M \geq 1$, we have*

$$(3.3) \quad E(fg; M) \geq E(f; M)E(g; M).$$

Proof. We proceed by induction on $k = \omega(M)$. If $k = 1$, then $M = p^\nu$ for some prime p , and the property that f and g are non-increasing implies

$$(3.4) \quad (f(p^j) - f(p^h))(g(p^j) - g(p^h)) \geq 0 \quad (0 \leq j, h \leq \nu).$$

We multiply by $\varphi(p^{\nu-h})\varphi(p^{\nu-j})$ and sum over all j, h to obtain

$$2E(fg; M) - 2E(f; M)E(g; M) \geq 0.$$

Let $\omega(M) = k > 1$, and $p^\nu \parallel M$. We put $M = M_1 p^\nu$ and, for each $j = 1, \dots, \nu$, define the functions f_j, g_j by $f_j(d) = f(p^j d)$, $g_j(d) = g(p^j d)$. These are obviously multiplicatively non-increasing. Now

$$\begin{aligned}
 E(fg; M) &= \frac{1}{M_1 p^\nu} \sum_{j=0}^{\nu} \sum_{d|M_1} \varphi(M_1 p^{\nu-j}/d) f(p^j d) g(p^j d) \\
 (3.5) \qquad &= \frac{1}{p^\nu} \sum_{j=0}^{\nu} \varphi(p^{\nu-j}) E(f_j g_j; M_1) \\
 &\geq \frac{1}{p^\nu} \sum_{j=0}^{\nu} \varphi(p^{\nu-j}) E(f_j; M_1) E(g_j; M_1),
 \end{aligned}$$

where we have used, in the second step, the multiplicativity of the Euler function and the induction hypothesis in the third step.

Put $\tilde{f}(p^j) = E(f_j; M_1)$ and $\tilde{g}(p^j) = E(g_j; M_1)$ ($0 \leq j \leq \nu$). We have shown that

$$(3.6) \qquad E(fg; M) \geq E(\tilde{f}\tilde{g}; p^\nu).$$

The functions \tilde{f}, \tilde{g} are multiplicatively non-increasing and we may apply (3.3) with $k = 1$ to obtain

$$(3.7) \qquad E(\tilde{f}\tilde{g}; p^\nu) \geq E(\tilde{f}; p^\nu) E(\tilde{g}; p^\nu).$$

Since

$$E(\tilde{f}; p^\nu) = p^{-\nu} \sum_{0 \leq j \leq \nu} \varphi(p^{\nu-j}) E(f_j; M_1) = E(f; M)$$

(with a similar equation involving g), we may assemble (3.6) and (3.7) to obtain (3.3). This completes the proof of the induction step and hence of the lemma.

As in the case of Behrend's original proof, it is not difficult to determine the cases of equality of (1.3) when \mathcal{A} and \mathcal{B} are finite. We leave to the reader the easy task of checking that equality holds if $(a, b) = 1$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. Conversely, assume that \mathcal{A} and \mathcal{B} are both primitive, that is no element of each sequence divides any other. Then the coprimality condition is also necessary. Indeed, assume that $(a, b) > 1$ for some $a \in \mathcal{A}, b \in \mathcal{B}$ so that for some prime factor p of M we may write $M = p^\nu M_1, a = p^\alpha a_1, b = p^\beta b_1$ with $p \nmid a_1 b_1 M_1$ and ν, α, β are positive integers. We claim that equality cannot hold in (3.3) for $f = \vartheta_{\mathcal{A}}, g = \vartheta_{\mathcal{B}}$. Indeed, since \mathcal{A} and \mathcal{B} are primitive, $a_1 \notin \mathcal{A}, b_1 \notin \mathcal{B}$, that is $\vartheta_{\mathcal{A}}(a) \neq \vartheta_{\mathcal{A}}(a_1), \vartheta_{\mathcal{B}}(b) \neq \vartheta_{\mathcal{B}}(b_1)$. Thus at least one of the inequalities (3.4) is strict with $f = f_j, g = g_j$ ($0 \leq j \leq \nu$). This implies in turn that, for some j , we have $E(f_j g_j; M_1) > E(f_j; M_1) E(g_j; M_1)$, hence equality does not hold in (3.5), which is all we need.

References

- F.A. Behrend, Generalizations of an inequality of Heilbron and Rohrbach, *Bull. Amer. Math. Soc.* **54** (1948), 681–684.
- H. Davenport & P. Erdős, On sequences of positive integers, *Acta Arith.* **2** (1937), 147–151.
- H. Davenport & P. Erdős, On sequences of positive integers, *J. Indian Math. Soc.* **15** (1951), 19–24.
- P. Erdős, Some remarks on prime factors of integers, *Canadian J. Math.* **11** (1959), 161–167.
- P. Erdős, R.R. Hall & G. Tenenbaum, On the densities of sets of multiples, *J. reine angew. Math.* **454** (1994), 119–141.
- J.D. Esary, F. Proschan & D.-W. Walkup, Associations of random variables, with applications, *Ann. Math. Stat.* **38** (1967), 1466–1474.
- C.-M. Fortuin, P.W. Kasteleyn & J. Ginibre, Correlations inequalities on some partially ordered sets, *Comm. Math. Phys.* **22** (1971), 89–103.
- R.R. Hall, Sets of multiples and Behrend sequences, in : A. Baker, B. Bollobás, A. Hajnal (Eds) *A tribute to Paul Erdős*, pp. 240-258, Cambridge University Press 1990.
- R.R. Hall & G. Tenenbaum, On Behrend sequences, *Math. Proc. Camb. Phil. Soc.* **112** (1992), 467-482.
- I.Z. Ruzsa, Probabilistic generalization of a a number theoretic inequality *Amer. Math. Soc. Monthly* **83**, no 9 (1976), 723-725.
- G. Tenenbaum, On block Behrend sequences, preprint.

Mathematical Institute
of the Hungarian Academy of Sciences
PO Box 127
Budapest
H-1364 Hungary

Département de Mathématiques
Université de Nancy I
BP 239
54506 Vandœuvre Cedex
France