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Uniform distribution on divisors and Behrend sequences

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1. Definitions and basic results

The purpose of this paper is twofold : to give a consistent, largely self contained account on the theory of uniform distribution on divisors, and to establish effective estimates with immediate applications to the construction of Behrend sequences.

We recall that a strictly increasing sequence $\mathcal{A}$ of integers exceeding 1 is called a Behrend sequence if its set of multiples

$$\mathcal{M}(\mathcal{A}) := \{ma : a \in \mathcal{A}, m \geqslant 1\}$$

has asymptotic density 1. As underlined by Erdős in [5], the problem of characterising Behrend sequences appears to be both very difficult and fundamental for describing the multiplicative structure of normal integers. Recent progress in the area of sets of multiples and Behrend sequences may be found in [6], [12], [15], [22], [24].

The definition of uniform distribution on divisors is due to Hall [9]. It may certainly be regarded as a concept of independent interest, which is worth being developed for its own sake. The idea is to give a rigourous content, given an arithmetic function f, to the assertion that, for almost all integers n, the numbers $f(d)$ are evenly distributed modulo 1 when d runs through the divisors of n. To this end, we define the *discrepancy* function

$$\Delta(n; f) := \sup_{0 \leqslant u \leqslant v \leqslant 1} \Big| \sum_{d|n,\, u < \langle f(d) \rangle \leqslant v} 1 - (v-u)\tau(n) \Big|,$$

where, here and throughout this paper, we let $\langle u \rangle$ denote the fractional part of the real number u. We then say that f *is uniformly distributed on divisors* (in short : erd, for the French *équirépartie sur les diviseurs*) if

$$\Delta(n; f) = o\big(\tau(n)\big) \qquad \text{pp,} \tag{1·1}$$

where $\tau(n)$ stands for the number of divisors of n. Here and in the sequel we use the notation pp (resp. ppℓ) to indicate that a relation holds on a sequence of asymptotic density 1 (resp. logarithmic density 1).

In 1978, Hall [10] introduced the closely connected notion of divisor density. An integer sequence $\mathcal{A}$ is said to have divisor density z, in which case we write $\mathbf{D}\mathcal{A} = z$, if

$$\tau(n, \mathcal{A}) := \sum_{d|n,\, d \in \mathcal{A}} 1 = \{z + o(1)\}\tau(n) \qquad \text{pp.}$$

The link with uniform distribution on divisors is as follows. Writing

$$\mathcal{A}(z; f) = \{d \geqslant 1 : \langle f(d) \rangle \leqslant z\}, \tag{1·2}$$

we obviously have

$$\mathbf{D}\mathcal{A}(z; f) = z \qquad (z \in [0,1]) \tag{1·3}$$

whenever f is erd. Moreover, as one might expect from classical results in the theory of uniform distribution modulo 1, it is not very difficult to prove that this last condition is also sufficient.

Theorem 1 (Hall [11]). *Let f be an arithmetic function. Then f is* erd *if, and only if, condition (1·3) holds.*

Proof. We only need to show that the condition is sufficient. Suppose that f is not erd. Then, for suitable $\varepsilon > 0$, we have $\Delta(n; f) > 4\varepsilon\tau(n)$ for all integers n in a sequence $\mathcal{B}$ with positive lower density. Hence for each $n \in \mathcal{B}$ there exists $z_n \in [0, 1]$ such that $|T_f(n, z_n) - z_n\tau(n)| > 2\varepsilon\tau(n)$, with

$$T_f(n, z) := \tau\big(n, \mathcal{A}(z; f)\big) = \big|\{d|n : \langle f(d)\rangle \leqslant z\}\big|. \tag{1·4}$$

Let q be any integer $> 1/\varepsilon$. By the monotonicity of the function $z \mapsto T_f(n, z)$, we can find an integer a, $0 \leqslant a \leqslant q$, such that $|T_f(n, a/q) - (a/q)\tau(n)| > \varepsilon\tau(n)$. Since $\mathcal{B}$ has positive lower density, this implies that (1·3) cannot hold for $z = a/q$.

Davenport & Erdős [2], [3], proved that a set of multiples necessarily has logarithmic density, equal to its lower asymptotic density. This implies that a (necessary and) sufficient condition for an integer sequence $\mathcal{A}$ to be Behrend is that $\boldsymbol{\delta}\mathcal{M}(A) = 1$. Here and in the remainder of the paper, we use the letter $\boldsymbol{\delta}$ to denote logarithmic density. The following result is a criterion for divisor density much in the same spirit.

Theorem 2 (Hall & Tenenbaum [13]). *Let $\mathcal{A}$ be an integer sequence. Then we have $\mathbf{D}\mathcal{A} = z$ if, and only if,*

$$\tau(n, \mathcal{A}) := \sum_{d|n,\, d\in\mathcal{A}} 1 = \{z + o(1)\}\tau(n) \qquad \text{pp}\ell.$$

The proof rests upon the Hardy-Littlewood-Karamata Tauberian theorem. An immediate corollary of Theorems 1 & 2 is the following slightly surprising statement.

Corollary 1. *Let f be an arithmetic function such that*

$$\Delta(n; f) = o\big(\tau(n)\big) \qquad \text{pp}\ell. \tag{1·5}$$

Then f is erd.

Proof. From (1·5), it is clear that $\tau\big(n, \mathcal{A}(z; f)\big) = \{z + o(1)\}\tau(n)$ ppℓ for all $z \in [0, 1]$. By Theorem 2, it follows that (1·3) holds, so Theorem 1 yields the required conclusion.

Corollary 1 opens new possibilities for constructing ‘thin’ Behrend sequences inasmuch as ppℓ upper bounds for the discrepancy are usually much easier to achieve than bounds valid on a set of asymptotic density 1. For convenience of further reference, we make a formal statement.

Theorem 3. *Let $\varepsilon(n)$ be a non-increasing function of n such that*

$$\varepsilon(n) = o(1), \qquad \varepsilon(n)\tau(n) \to \infty \qquad \text{pp}\ell,$$

and let f be an arithmetic function satisfying

$$\Delta(n; f) < \tfrac{1}{2}\varepsilon(n)\tau(n) \qquad \text{pp}\ell.$$

Then the integer sequence

$$\mathcal{A} = \{d \geqslant 1 : \langle f(d)\rangle \leqslant \varepsilon(d)\}$$

is a Behrend sequence.

Proof. We plainly have

$$\big|\{d|n : \langle f(d)\rangle \leqslant \varepsilon(n)\}\big| \geqslant \varepsilon(n)\tau(n) - \Delta(n;f) > \tfrac{1}{2}\varepsilon(n)\tau(n) \quad \text{pp}\ell.$$

Since $\varepsilon(d) \geqslant \varepsilon(n)$ whenever $d|n$, this implies $\boldsymbol{\delta}\mathcal{M}(\mathcal{A}) = 1$. By the Davenport-Erdős theorem, we deduce that $\mathcal{A}$ is a Behrend sequence.

Thus the problem of finding effective bounds on a set of logarithmic density 1 appears to be essential in both problems of obtaining erd-type results and of constructing Behrend sequences. The remainder of this paper is devoted to describing the various methods that have been devised to this end.

An obvious consequence of Theorem 1 is that any effective criterion for divisor density may be employed to decide whether a given function is erd. We now quote a result of this kind. The statement involves a function $R(x) \leqslant x$ which is increasing and has the property that, for all $y \in [0,1]$ (but actually $y = \frac{1}{4}$ suffices) there is a suitable Stieltjes measure $\mathrm{d}\lambda_y(t)$ on $[0, \frac{1}{2}]$ with $|\,\mathrm{d}\lambda_y(t)| \ll t^{-y}\,\mathrm{d}t$ and

$$\sum_{n\leqslant x} y^{\Omega(n)} = \int_0^{1/2} x^{1-t}\,\mathrm{d}\lambda_y(t) + O(x/R(x)), \tag{1·6}$$

where, here and in the sequel, we let $\Omega(n)$ (resp. $\omega(n)$) denote the total number of prime factors of n, counted with (resp. without) multiplicity. It is shown in [13] that

$$R(x) = \exp\left\{(\log x)^{3/5-\varepsilon}\right\}$$

is an admissible choice for all $\varepsilon > 0$, and an examination of the proof shows that $x/R(x)$ is essentially of the size of the error term in the prime number theorem — see also [25], chapters II.5, II.6 and notes on § II.5.4.

Theorem 4 (Hall & Tenenbaum [13]). *Let $\{u_j\}_{j=0}^\infty$ be a strictly increasing sequence of positive real numbers such that $|\{j : u_j \leqslant x\}| \leqslant R(x^{o(1)})$ and put $\mathcal{A} := \cup_{j=1}^\infty (u_{2j}, u_{2j+1}] \cap \mathbb{Z}^+$. Then $\boldsymbol{\delta}\mathcal{A} = z$ implies that $\mathbf{D}\mathcal{A} = z$.*

Theorem 4 provides a ready-to-use sufficient condition for smooth functions to be erd. For instance, it enables one to recover immediately the two following basic results. We let $\log_k$ denote the k-fold iterated logarithm.

Corollary 2 (Tenenbaum [23]). *The function $d \mapsto (\log d)^\alpha$ is* erd *if, and only if, $\alpha > 0$.*

This result was conjectured by Hall in [9].

Corollary 3 (Hall [9], [10]). *The function $d \mapsto (\log_2 d)^\beta$ is* erd *if, and only if, $\beta > 1$.*

It is straightforward to check that the sequences $\mathcal{A}(z;f)$ defined by (1·2) for $f = \log^\alpha$ and $f = \log_2^\beta$ satisfy the hypotheses of Theorem 4 whenever $\alpha > 0$, $\beta > 1$. On the other hand, as observed by Hall in [9], relation (1·1) does not hold for $f(d) = (\log_2 d)^\beta$ when $\beta \leqslant 1$. Indeed, let $0 < \delta < 2^{-1/\beta}$ and consider the set $\mathcal{S}(\delta)$ of integers of the form $n = mp$ with $p^\delta > m$, which has natural density $\log(1+\delta)$. For $n \in \mathcal{S}(\delta)$, there are at least $\frac{1}{2}\tau(n)$ divisors d of n which are divisible by p, and all of them verify $f(p) \leqslant f(d) \leqslant f(p^{1+\delta}) < f(p) + \delta^\beta$. Thus $\Delta(n;f) \geqslant \left(\frac{1}{2} - \delta^\beta\right)\tau(n)$.

The strongest limitation in Theorem 4 is the growth condition on the sequence $\{u_j\}_{j=0}^\infty$, which, in the present state of knowledge concerning the error term of the prime number theorem or the zero-free region for the Riemann zeta function, certainly implies that

$$|\{j : u_j \leqslant x\}| = \exp\{o\big((\log x)^{3/5}(\log_2 x)^{-1/5}\big)\}. \tag{1·7}$$

Thus, we can only obtain from Theorem 4 that

$$f(d) = \exp\{(\log d)^\alpha\}$$

is erd for $0 < \alpha < 3/5$, although it is natural to conjecture that this holds for all positive $\alpha \neq 1$. We shall see in section 3 that this can indeed be established for the range $0 < \alpha < 3/2$, $\alpha \neq 1$. To tackle functions f beyond the scope of Theorem 4, one possibility is to appeal, as already done in [13], to the criterion for uniform distribution on divisors established in [23]. In the spirit of the Weyl criterion for ordinary uniform distribution modulo 1, this is formulated in terms of exponential sums. We now provide an effective form of this criterion. Given an integer ν, we put

$$\varepsilon_\nu(x;f) := (\log x)^{-1/2} \sum_{k \leqslant x} \Bigg| \sum_{\substack{n \leqslant x \\ n \equiv 0 \,(\mathrm{mod}\, k)}} \frac{e\big(\nu f(n)\big)}{n 4^{\Omega(n)}} \Bigg|. \tag{1·8}$$

Here and throughout the paper we use the traditional notation $e(u) = e^{2\pi i u}$ $(u \in \mathbb{R})$.

Theorem 5. *Let f be an arithmetical function. Then f is* erd *if, and only if, we have, as $x \to \infty$,*

$$\varepsilon_\nu(x;f) = o(1) \qquad (\nu \neq 0). \tag{1·9}$$

Furthermore, if this is the case, then the upper bound

$$\Delta(n;f) < \xi(n)\tau(n) \min_{T \geqslant 1} \Big\{ \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{\varepsilon_\nu^+(n;f)}{\nu} \Big\}^{1/2} \qquad \mathrm{pp}\ell \tag{1·10}$$

holds for arbitrary $\xi(n) \to \infty$, where $\varepsilon_\nu^+(x;f)$ is, for each ν, any non-increasing function such that $x \mapsto \varepsilon_\nu^+(x;f)\sqrt{(\log x)}$ is non-decreasing and $\varepsilon_\nu(x;f) \leqslant \varepsilon_\nu^+(x;f)$ for large x.

Thus, the problem of finding effective ppℓ bounds for the discrepancy (with the by-product, which is essential here, of exhibiting new types of Behrend sequences) may be reduced to the study of appropriate exponential sums with multiplicative coefficients.

Proof of Theorem 5. First assume that f is erd. For $k \geqslant 1$, $x \geqslant 2$, $0 \leqslant z \leqslant 1$, put

$$\Phi_k(z;x) = (\log x)^{-1/2} \sum_{\substack{n \leqslant x,\, n \equiv 0 \,(\mathrm{mod}\, k) \\ \langle f(n) \rangle \leqslant z}} \frac{1}{n 4^{\Omega(n)}} \ll (\log x)^{-1/4} \frac{1}{k 4^{\Omega(k)}}.$$

By the author's criterion [23] for divisor density and Theorem 1, we have that

$$F_x(z) := \sum_{k \leqslant x} \big| \Phi_k(z;x) - z\Phi_k(1;x) \big| = o(1) \qquad (x \to \infty) \tag{1·11}$$

for all fixed $z \in [0,1]$. Now for any non-zero integer ν we have

$$\begin{aligned} \varepsilon_\nu(x;f) &= \sum_{k \leqslant x} \Big| \int_0^1 e(\nu z)\,\mathrm{d}\Phi_k(z;x) \Big| \\ &= \sum_{k \leqslant x} \Big| 2\pi\nu i \int_0^1 e(\nu z)\big\{\Phi_k(z;x) - z\Phi_k(1;x)\big\}\,\mathrm{d}z \Big| \\ &\leqslant 2\pi|\nu| \int_0^1 F_x(z)\,\mathrm{d}z. \end{aligned}$$

Since $F_x(z) = O(1)$ uniformly in x, z, the required conclusion (1·9) follows by Lebesgue's theorem of dominated convergence.

Conversely, we now assume that (1·9) holds and derive a ppℓ upper bound for $\Delta(n;f)$. By the Erdős-Turán inequality for the discrepancy (see e.g. Kuipers & Niederreiter [19], theorem 2.5) we have for all n and $T \geqslant 1$

$$\Delta(n;f) \ll \frac{\tau(n)}{T} + \sum_{1\leqslant\nu\leqslant T} \frac{|g_\nu(n)|}{\nu}, \tag{1·12}$$

with $g_\nu(n) := \sum_{d|n} e\big(\nu f(d)\big)$. By the Cauchy-Shwarz inequality, we infer that

$$\Delta(n;f)^2 \ll \frac{\tau(n)^2}{T^2} + \log T \sum_{1\leqslant\nu\leqslant T} \frac{|g_\nu(n)|^2}{\nu}. \tag{1·13}$$

We now estimate $|g_\nu(n)|^2$ on logarithmic average with weight $1/4^{\Omega(n)}$. The procedure is similar to the proof of theorem 1 of [23]. Writing $\sigma = 1/\log x$, we have

$$\begin{aligned} L_\nu(x) &:= \sum_{n\leqslant x} \frac{|g_\nu(n)|^2}{n4^{\Omega(n)}} \leqslant e \sum_{n=1}^\infty \frac{|g_\nu(n)|^2}{n^{1+\sigma}4^{\Omega(n)}} \\ &\ll \sum_{n=1}^\infty \frac{1}{n^{1+\sigma}4^{\Omega(n)}} \sum_{d|n,\, t|n} e\big(\nu f(d) - \nu f(t)\big) \\ &= \zeta(1+\sigma, \tfrac14) \sum_{d,t\geqslant 1} \frac{e\big(\nu f(d) - \nu f(t)\big)}{[d,t]^{1+\sigma}4^{\Omega([d,t])}}, \end{aligned} \tag{1·14}$$

where we used the notation

$$\zeta(s,y) = \sum_{n=1}^\infty \frac{y^{\Omega(n)}}{n^s} = \prod_p \big(1 - yp^{-s}\big)^{-1} \qquad (|y| < 2,\ \Re e\, s > 1). \tag{1·15}$$

We note that

$$\begin{aligned} \zeta(1+\sigma, \tfrac14) &= \zeta(1+\sigma)^{1/4} \prod_p \Big(1 - \frac{1}{4p^{1+\sigma}}\Big)^{-1}\Big(1 - \frac{1}{p^{1+\sigma}}\Big)^{1/4} \\ &\sim H(\tfrac14)(\log x)^{1/4} \qquad (x\to\infty), \end{aligned} \tag{1·16}$$

with

$$H(y) := \prod_p \Big(1 - \frac{y}{p}\Big)^{-1}\Big(1 - \frac{1}{p}\Big)^y \qquad (|y| < 2).$$

Using the identity

$$m^{1+\sigma}4^{\Omega(m)} = \sum_{k|m} \lambda(k,\sigma)k^{1+\sigma}4^{\Omega(k)}, \quad \text{with} \quad \lambda(k,\sigma) := \prod_{p|k} \big(1 - \frac{1}{4p^{1+\sigma}}\big) \leqslant 1,$$

we may rewrite the last double sum in (1·14) as

$$\begin{aligned} &\sum_{d,t\geqslant 1} \frac{e\big(\nu f(d) - \nu f(t)\big)}{(dt)^{1+\sigma}4^{\Omega(dt)}} \sum_{k|(d,t)} \lambda(k,\sigma)k^{1+\sigma}4^{\Omega(k)} \\ &\qquad = \sum_{k=1}^\infty \frac{\lambda(k,\sigma)}{k^{1+\sigma}4^{\Omega(k)}} \Big| \sum_{m=1}^\infty \frac{e\big(\nu f(km)\big)}{m^{1+\sigma}4^{\Omega(m)}} \Big|^2. \end{aligned} \tag{1·17}$$

Bounding $\lambda(k,\sigma)$ by 1, and noticing that the m-sum is at most $\zeta(1+\sigma,\frac14)$ in absolute value, we see that the quantity (1·17) does not exceed

$$\zeta(1+\sigma,\tfrac14)\sum_{k=1}^\infty \frac{1}{k^{1+\sigma}4^{\Omega(k)}}\Big|\sum_{m=1}^\infty \frac{e\big(\nu f(km)\big)}{m^{1+\sigma}4^{\Omega(m)}}\Big| = \zeta(1+\sigma,\tfrac14)\sum_{k=1}^\infty\Big|\sum_{\substack{n=1\\ k|n}}^\infty \frac{e\big(\nu f(n)\big)}{n^{1+\sigma}4^{\Omega(n)}}\Big|.$$

Inserting this upper bound into (1·14) and appealing to (1·16) we obtain

$$L_\nu(x) \ll (\log x)^{1/2}\sum_{k=1}^\infty\Big|\int_{0-}^\infty e^{-\sigma u}\,\mathrm{d}A_k(u)\Big|,$$

with $A_k(u) := \sum_{n\leqslant e^u,\,k|n} e\big(\nu f(n)\big)4^{-\Omega(n)}n^{-1}$. Integrating by parts, it follows that

$$\begin{aligned}L_\nu(x) &\ll \sigma(\log x)^{1/2}\sum_{k=1}^\infty\Big|\int_0^\infty e^{-\sigma u}A_k(u)\,\mathrm{d}u\Big| \\ &\ll (\log x)^{-1/2}\int_0^\infty A(u)e^{-\sigma u}\,\mathrm{d}u,\end{aligned}\tag{1·18}$$

with $A(u) := \sum_{k=1}^\infty |A_k(u)| \leqslant \varepsilon_\nu^+(e^u;f)\sqrt{u}$. The last integral may be easily estimated using the monotonicity properties of $\varepsilon_\nu^+(e^u;f)$. We have

$$\begin{aligned}\int_0^\infty A(u)e^{-\sigma u}\,\mathrm{d}u &\leqslant \int_0^{1/\sigma}\varepsilon_\nu^+(e^{1/\sigma};f)\sqrt{(1/\sigma)}\,\mathrm{d}u + \int_{1/\sigma}^\infty \varepsilon_\nu^+(e^{1/\sigma};f)e^{-\sigma u}\sqrt{u}\,\mathrm{d}u \\ &\ll (\log x)^{3/2}\varepsilon_\nu^+(x;f),\end{aligned}$$

and so

$$L_\nu(x) \ll (\log x)\varepsilon_\nu^+(x;f).$$

Inserting this into (1·13), we deduce that, for any $T\geqslant 1$,

$$(\log x)^{-1}\sum_{n\leqslant x}\frac{\Delta(n;f)^2}{n4^{\Omega(n)}} \ll \frac{1}{T^2} + \log T\sum_{1\leqslant\nu\leqslant T}\frac{\varepsilon_\nu^+(x;f)}{\nu}.\tag{1·19}$$

Using the inequality

$$\tau(n) \geqslant 2^{\Omega(n)}/\xi(n)^{1/3} \qquad \text{pp},$$

which follows from the fact that $\Omega(n)-\omega(n)$ is bounded on average, we infer from (1·19) that

$$\Delta(n;f) < \tau(n)\xi(n)\min_{T\geqslant 1}\Big\{\frac{1}{T^2}+\log T\sum_{1\leqslant\nu\leqslant T}\frac{\varepsilon_\nu^+(x;f)}{\nu}\Big\}^{1/2}\tag{1·20}$$

holds for all $n\leqslant x$ except those of a set $\mathcal{E}_x$ with $\sum_{n\in\mathcal{E}_x}(1/n) = o(\log x)$. The stated result follows since the quantity inside curly brackets in (1·20) is a non-increasing function of x for each fixed T.

It would be possible to obtain pp upper bounds for the discrepancy in Theorem 4 along the lines of the proof of Theorem 2, appealing to an effective form of the Hardy-Littlewood-Karamata Tauberian theorem. However, the resulting estimate would be much weaker than (1·10). Such an analysis might of course be pursued for its own sake, but is irrelevant in the present context, as we remarked earlier.

The ppℓ upper bound (1·10) is by no means a unique or optimal choice. We now summarise what we believe to be the three most important variations.

It is convenient to introduce the following definition.

Definition. *A function $F:\mathbb{R}^+\to\mathbb{R}^+$ is called slowly increasing (resp. slowly decreasing) if it satisfies for suitable $x_0>0$*

$$F(x)\ll_\varepsilon F(x^\varepsilon)\qquad \big(\text{resp. } F(x)\gg_\varepsilon F(x^\varepsilon)\big)\qquad (x>x_0)$$

for all $\varepsilon\in]0,1[$.

Recall the formula

$$g_\nu(n):=\sum_{d|n}e\big(\nu f(n)\big).$$

Then it is an immediate consequence of the Erdős-Turán inequality (1·12) that

$$\sum_{n\leqslant x}\frac{\Delta(n;f)}{n}\Big(\frac{y}{2}\Big)^{\Omega(n)}\ll\frac{(\log x)^y}{T}+\sum_{1\leqslant\nu\leqslant T}\frac{1}{\nu}\sum_{n\leqslant x}\frac{|g_\nu(n)|}{n}\Big(\frac{y}{2}\Big)^{\Omega(n)},\tag{1·21}$$

uniformly for $x\geqslant 2$, $T\geqslant 1$, $0\leqslant y\leqslant y_0<4$. Suppose $E_1(x,y)\log x$ is an upper bound for the right-hand side of (1·21), corresponding to some optimal or quasi-optimal choice $T=T(x,y)$, and has the property that $x\mapsto E_1(x,y)$ is slowly increasing. Then we deduce from (1·21) the following statement.

Theorem 6. *Let $\xi(n)\to\infty$ and $0<y_0<4$. Then, for any function $y=y(n)$ with values in $[0,y_0]$ such that $n\mapsto E_1\big(n,y(n)\big)$ is slowly increasing, we have*

$$\Delta(n;f)<\xi(n)\tau(n)E_1(n,y)y^{-\Omega(n)}\qquad \text{pp}\ell.\tag{1·22}$$

Of course, from our assumption on $E_1(x,y)$, any $y(n)=y$ independent of n will be an admissible choice. Since $\Omega(n)$ has normal order $\log_2 n$, the optimal function $y(n)$ in (1·22) will be close to $y=y_1(n)$ minimising the expression

$$(\log n)^{-\log y}E_1(n,y),\tag{1·23}$$

and indeed this choice always approximates the minimum of the right-hand side of (1·22) to within a factor $(\log n)^{o(1)}$.

Theorem 6 is only applicable when one disposes of non-trivial estimates for the right-hand side of (1·21). This is in particular the case when f is additive, for g_ν is then multiplicative. We shall study this situation in detail in section 5.

When individual bounds for $|g_\nu(n)|$ fail to yield non-trivial information on the weighted average appearing in (1·21), one can still perform a computation parallel to (1·13)–(1·17), but with $(\frac14)^{\Omega(n)}$ replaced by $(\frac14 y)^{\Omega(n)}$. This gives, uniformly for $x\geqslant 2$, $T\geqslant 1$, $0\leqslant y\leqslant y_0<8$,

$$\begin{aligned}(\log x)^{-1}&\sum_{n\leqslant x}\frac{\Delta(n;f)^2}{n}\Big(\frac{y}{4}\Big)^{\Omega(n)}\\ &\ll\frac{(\log x)^{y-1}}{T^2}+\log T\,(\log x)^{y/4-1}\sum_{1\leqslant\nu\leqslant T}\frac{1}{\nu}H_\nu(x,y),\end{aligned}\tag{1·24}$$

with

$$H_\nu(x,y) := \sum_{k=1}^\infty \Big(\frac{y}{4}\Big)^{\Omega(k)} \frac{1}{k^{1+\sigma}} \Big| \sum_{m=1}^\infty \Big(\frac{y}{4}\Big)^{\Omega(m)} \frac{e\big(\nu f(km)\big)}{m^{1+\sigma}}\Big|^2 \qquad (\sigma := 1/\log x). \tag{1·25}$$

At this stage, we may employ two distinct strategies. The first one corresponds to cases in which we can take advantage of the presence of the squared modulus in (1·25). If we then denote by $E_2(x,y)$ an upper bound for the right-hand side of (1·24) which is slowly increasing as a function of x, we obtain the following result.

Theorem 7. *Let $\xi(n) \to \infty$ and $0 < y_0 < 8$. Then, for any function $y = y(n)$ with values in $[0, y_0]$ and such that $n \mapsto E_2\big(n, y(n)\big)$ is slowly increasing, we have*

$$\Delta(n; f) < \xi(n)\tau(n)y^{-\Omega(n)/2}\sqrt{E_2(n,y)} \qquad \text{pp}\ell. \tag{1·26}$$

Here again the optimal y must be close to $y = y_2(n)$ minimising the expression

$$(\log n)^{-\log y}E_2(n,y).$$

The second strategy, which corresponds to cases when a 'linearized' upper bound is more convenient, consists in bounding trivially one of the two (identical) factors of the square in (1·25) by $\zeta(1+\sigma, y)$ and then repeating *mutatis mutandis* the procedure described in (1·18)–(1·20). For $0 < y < 8$, $\nu \geqslant 1$, let $\varepsilon_\nu^+(x,y;f)$ be a non-increasing function of x such that $x \mapsto \varepsilon_\nu^+(x,y;f)(\log x)^{y/2}$ is non-decreasing and, for all $x \geqslant 2$,

$$(\log x)^{-y/2} \sum_{k \leqslant x} \Big| \sum_{\substack{n \leqslant x \\ n \equiv 0 \,(\mathrm{mod}\, k)}} \Big(\frac{y}{4}\Big)^{\Omega(n)} \frac{e\big(\nu f(n)\big)}{n} \Big| \leqslant \varepsilon_\nu^+(x,y;f). \tag{1·27}$$

We arrive at the following estimate generalising (1·19) : the bound

$$\sum_{n \leqslant x} \Big(\frac{y}{4}\Big)^{\Omega(n)} \frac{\Delta(n;f)^2}{n} \ll (\log x)^y E_3(x,y) \tag{1·28}$$

holds uniformly for $x \geqslant 2$, $0 \leqslant y \leqslant y_0$, with

$$E_3(x,y) := \min_{T \geqslant 2} \Big\{ \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{\varepsilon_\nu^+(x,y;f)}{\nu} \Big\}.$$

The monotonicity hypotheses on the functions $\varepsilon_\nu^+(x,y;f)$ are slightly awkward in practical use, and, for convenience of further reference, we note right away that they may be slightly relaxed. Let us say that a positive function F is *weakly increasing* (resp. *weakly decreasing*) if it satisfies $F(t) \ll F(x)$ (resp. $F(t) \gg F(x)$) for $t \leqslant x$. Then it is enough for (1·28) to assume that $\varepsilon_\nu^+(x,y;f)$ and $(\log x)^{y/2}\varepsilon_\nu^+(x,y;f)$ are respectively weakly decreasing and weakly increasing functions of x.

The upper bound (1·28) immediately implies, in a straightforward way, our next theorem.

Theorem 8. *Let $\xi(n) \to \infty$, $0 < y_0 < 8$, $y = y(n) \in [0, y_0]$ and suppose that $E_3^*(n,y)$ is an upper bound for $E_3(n,y)$ which is slowly increasing as a function of n. Then we have*

$$\Delta(n;f) < \xi(n)\tau(n)(\log n)^{(y-1)/2}y^{-\Omega(n)/2}\sqrt{E_3^*(n,y)} \qquad \text{pp}\ell. \tag{1·29}$$

As before, we remark that the choice $y = y_3(n)$ minimising the expression

$$(\log n)^{y-1-\log y}E_3(n,y)$$

yields an approximation of the optimum to within a factor $(\log n)^{o(1)}$.

2. Functions of moderate growth

In this section, we investigate uniform distribution on divisors and effective ppℓ upper bounds for the discrepancy in the case of functions f for which the sets $\mathcal{A}(z;f)$ defined in (1·2) may be tackled by Theorem 4 or techniques of similar strength.

We say that a function $f:\mathbb{R}^+\to\mathbb{R}^+$ has *moderate growth* if it satisfies

$$f(t)\ll R\big(t^{o(1)}\big)\qquad(t\to\infty) \tag{2·1}$$

for some increasing function R satisfying (1·6) and having the property that

$$\exists\, b>0: R(\sqrt{t})\ll R(t)^{1-b}\quad(t\geqslant 1). \tag{2·2}$$

An easy calculation shows that this implies $R(x)\gg\exp\{(\log x)^c\}$ for some positive c.

Our first result establishes a connection between usual uniform distribution modulo 1 and uniform distribution on divisors. It was announced, with a sketched proof (and incidentally a slightly deficient statement), in [13].

Theorem 9 (Hall & Tenenbaum). *Let $F:\mathbb{R}^+\to\mathbb{R}^+$ be differentiable and satisfy*

(i) $F'(x)=o(1)\quad(x\to\infty)$,

(ii) $\{F(n)\}_{n=1}^{\infty}$ *is uniformly distributed modulo 1.*

Suppose that $\vartheta:\mathbb{R}^+\to\mathbb{R}^+$ has moderate growth and is ultimately of class C^1. Furthermore assume that, for large x,

$$x\mapsto x\vartheta'(x)\quad\text{is monotonic, and}\quad\vartheta(x)\ll x\vartheta'(x)\log x. \tag{2·3}$$

Then $f:=F\circ\vartheta$ is erd.

Proof. We observe that the assumptions on $\vartheta(x)$ imply that $\vartheta(x)\to\infty$ and in fact $\vartheta(x)\gg(\log x)^c$ for some positive c. Moreover, we may modify ϑ on any fixed, finite interval and hence assume without loss of generality that $\vartheta'(x)$ exists and is positive for all $x>0$, and that (2·3) holds for all $x>\frac{3}{2}$.

Let $z\in(0,1)$. We shall show that $\mathbf{D}\mathcal{A}(z;f)=z$, which implies the stated result in view of Theorem 1 : indeed, the cases $z=0$ or 1 then follow by a straightforward argument. For fixed $\varepsilon\in(0,\min(z,1-z))$, we set

$$\mathcal{A}^{\pm}(\varepsilon):=\Big\{d\geqslant 1:\big\langle F\big(\big[\vartheta(d)\big]\big)\big\rangle\leqslant z\pm\varepsilon\Big\}.$$

Our first aim is to prove that

$$\boldsymbol{\delta}\mathcal{A}^{\pm}(\varepsilon)=z\pm\varepsilon. \tag{2·4}$$

We only consider $\mathcal{A}^+(\varepsilon)$ since the other case is similar. Let x be large and put $N=\big[\vartheta(x)\big]$. Denoting by ψ the inverse function of ϑ, we have

$$\sum_{\substack{d\leqslant x\\ d\in\mathcal{A}^+(\varepsilon)}}\frac{1}{d}=\sum_{\substack{1\leqslant n<N\\ \langle F(n)\rangle\leqslant z+\varepsilon}}\ \sum_{\psi(n)<d\leqslant\psi(n+1)}\frac{1}{d}+O\big(1+\log\big(x/\psi(N)\big)\big), \tag{2·5}$$

where the error term corresponds to those d with $d\leqslant\psi(1)$ or $\psi(N)\leqslant d\leqslant x$.

The inner sum is

$$a(n) + O(1/\psi(n)), \qquad \text{with} \qquad a(n) := \log\big(\psi(n+1)/\psi(n)\big).$$

Since ϑ has moderate growth, we certainly have $\vartheta(x) \ll x^{o(1)} \ll \sqrt{x}$, whence $\psi(n) \gg n^2$. Therefore the double sum on the right-hand side of (2·5) is equal to

$$\sum_{\substack{1\leqslant n<N\\ \langle F(n)\rangle\leqslant z+\varepsilon}} a(n) + O(1) = (z+\varepsilon)\sum_{n<N} a(n) + \sum_{n<N}\chi(n)a(n) + O(1), \tag{2·6}$$

where

$$\chi(n) := \begin{cases} 1-(z+\varepsilon) & \text{if } \langle F(n)\rangle \leqslant z+\varepsilon,\\ -(z+\varepsilon) & \text{otherwise.}\end{cases}$$

We have $t\vartheta'(t) \geqslant t_0\vartheta'(t_0) \gg 1$ for $t \geqslant t_0$, so $\vartheta(x) - \vartheta(cx) > 1$ for sufficiently small c and large x. This implies $cx \leqslant \psi(N) \leqslant x$ and hence

$$\log\big(x/\psi(N)\big) \ll 1, \qquad \sum_{n<N} a(n) = \log\psi(N) + O(1) = \log x + O(1).$$

Inserting these estimates into (2·6), we see that proving (2·4) reduces to showing the asymptotic formula

$$\sum_{n<N}\chi(n)a(n) = o(\log x). \tag{2·7}$$

This will follow by partial summation, noting that the assumption that $\{F(n)\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 immediately implies

$$H(y) := \sum_{n\leqslant y}\chi(n) = o(y). \tag{2·8}$$

We first observe that we have for all $y \leqslant N-1$

$$\begin{aligned} a(y) &= \int_y^{y+1}\frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t = \int_y^{y+1}\frac{\mathrm{d}t}{\psi(t)\vartheta'(\psi(t))}\\ &\ll \int_y^{y+1}\frac{\log\psi(t)}{t}\,\mathrm{d}t \ll \frac{\log\psi(y+1)}{y} \ll \frac{\log x}{y}, \end{aligned} \tag{2·9}$$

where we have used (2·3) in the third stage. Now the left-hand side of (2·7) is

$$\int_1^{N-1} a(y)\,\mathrm{d}H(y) = a(N-1)H(N-1) - \int_1^{N-1} a'(y)H(y)\,\mathrm{d}y.$$

By (2·8) and (2·9), and since $a'(y) = 1/\psi(y+1)\vartheta'(\psi(y+1)) - 1/\psi(y)\vartheta'(\psi(y))$ has constant sign for large y by the monotonicity assumption on $x\vartheta'(x)$, this is

$$\ll \frac{\log x}{N}o(N) + o\Big(\int_1^{N-1} a'(y)y\,\mathrm{d}y\Big) = o(\log x) + o\Big(\int_1^{N-1} a(y)\,\mathrm{d}y\Big),$$

where we estimated the integral over $a'(y)$ by another partial summation. Now

$$\int_1^{N-1} a(y)\,\mathrm{d}y = \int_1^{N-1}\int_y^{y+1}\frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t\,\mathrm{d}y \leqslant \int_1^{N}\frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t = \log x + O(1).$$

This shows that (2·7) holds and hence establishes (2·4).

We may now apply Theorem 4 to the sequences $\mathcal{A}^{\pm}(\varepsilon)$: indeed they are composed of at most $\big[\vartheta(x)\big]+1$ blocks, and this has the required order of magnitude since ϑ is of moderate growth. Thus we obtain

$$\mathbf{D}\mathcal{A}^{\pm}(\varepsilon)=z\pm\varepsilon. \tag{2·10}$$

From the facts that $\vartheta(d)\to\infty$ and $F'(x)=o(1)$, we deduce that, for each $\varepsilon>0$, there exists a $d_0(\varepsilon)$ such that

$$\big|F\big(\big[\vartheta(d)\big]\big)-F\big(\vartheta(d)\big)\big|<\varepsilon \qquad (d>d_0(\varepsilon)).$$

This implies that for all n

$$\tau\big(n,\mathcal{A}^{-}(\varepsilon)\big)-d_0(\varepsilon)\leqslant\tau(n,\mathcal{A})\leqslant\tau\big(n,\mathcal{A}^{+}(\varepsilon)\big)+d_0(\varepsilon),$$

whence, in view of (2·10),

$$\{z-\varepsilon+o(1)\}\tau(n)\leqslant\tau(n,\mathcal{A})\leqslant\{z+\varepsilon+o(1)\}\tau(n) \qquad \text{pp.}$$

Since ε is arbitrary, a routine argument yields $\tau(n,\mathcal{A})=\{z+o(1)\}\tau(n)$ pp, as required. This completes the proof of Theorem 9.

The following corollary was also stated (with a slight oversight in the monotonicity assumption) in [13].

Corollary 4 (Hall & Tenenbaum). *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be differentiable and such that, for some function $\vartheta(x)$ satisfying the conditions of Theorem 9,*

(i) $\vartheta'(x)/f'(x)+1/xf'(x)$ *is ultimately monotonic,*

(ii) $|\vartheta'(x)/f'(x)|+|xf'(x)|=o\big(\vartheta(x)\big) \quad (x\to\infty)$.

Then f is erd.

Proof. Set $\vartheta_1(x):=\vartheta(x)\log x$. Then ϑ_1 also satisfies the assumptions of Theorem 9. The only condition which is non-trivial to check is that (2·1) holds for $f=\vartheta_1$; however we have by (2·2), for some function $\eta(x)\to0$ sufficiently slowly,

$$\vartheta_1(x)\ll R\big(x^{\eta(x)}\big)\log x\ll R\big(x^{2\eta(x)}\big)^{1-b}\log x\ll R\big(x^{2\eta(x)}\big).$$

We also observe that $\vartheta_1'(x)/\vartheta_1(x)\asymp\vartheta'(x)/\vartheta(x)$ by (2·3).

It is clear that ϑ_1 is ultimately strictly increasing, and hence ultimately one-to-one. Let ψ_1 denote the inverse of ϑ_1, and put $F(x)=f\circ\psi_1(x)$ for sufficiently large x, so that $f(x)=F\big(\vartheta_1(x)\big)$. We want to apply Theorem 9 to F and hence must check that $F'(x)=o(1)$ and that $\{F(n)\}_{n=1}^{\infty}$ is uniformly distributed modulo 1.

By a well-known criterion of Fejér (see e.g. Rauzy [20], Corollary II.1.2) we only need to prove, in addition to $F'(x)=o(1)$, that F' is ultimately monotonic and that $xF'(x)\to\infty$. Since, for large x,

$$F'\big(\vartheta_1(x)\big)=f'(x)/\vartheta_1'(x)\asymp f'(x)\vartheta(x)/\vartheta'(x)\vartheta_1(x)\ll xf'(x)/\vartheta(x),$$

$$\vartheta_1(x)F'\big(\vartheta_1(x)\big)\asymp f'(x)\vartheta(x)/\vartheta'(x),$$

our assumption (ii) implies that $F'(x)\to0,\ xF'(x)\to\infty$. Moreover, replacing $\vartheta_1'(x)$ by $\vartheta'(x)+1/x$ in the first equality above yields, by assumption (i), that F' is monotonic. This completes the proof of Corollary 4.

Our next result provides effective ppℓ bounds for the discrepancy under slightly stronger assumptions. This is a refinement of theorem 5 of [13] and is obtained by the same technique. The proof has been (re)written jointly with R.R. Hall.

Theorem 10 (Hall & Tenenbaum). *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be continuously differentiable, such that $xf'(x)$ is ultimately monotonic. Assume that, for some increasing function R satisfying (1·6) and (2·2), there is a further non-decreasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\varphi_1(x):=(\log x)/\varphi(x)$ is ultimately non-decreasing and*

(2·11) $$\varphi(x)\gg(\log_2 x)^2,\qquad \varphi_1(x)\to\infty,$$

(2·12) $$1/\varphi(x)\ll x|f'(x)|\ll R\big(e^{\varphi(x)}\big).$$

Then f is erd *and we have for any $\xi(n)\to\infty$*

(2·13) $$\Delta(n;f)<\tau(n)\xi(n)\Big(1-\frac{\log\varphi_1(n)}{2\log_2 n}\Big)^{\Omega(n)/2}\log_2\varphi_1(n)\qquad \text{pp}\ell.$$

We note that the upper bound (2·13) is always non trivial under the conditions of the theorem. It yields in fact

(2·14) $$\Delta(n;f)\ll\tau(n)\varphi_1(n)^{-1/4+o(1)}\qquad \text{pp}\ell,$$

since the normal order of $\Omega(n)$ is $\log_2 n$. It is clear that the theorem only applies to functions of moderate growth, and one can get a fairly precise idea of the quality of the quantitative result by considering the functions $f(x)=(\log x)^\alpha$ with $\alpha>0$ and $f(x)=(\log_2 x)^\beta$ with $\beta>1$. In the first instance we may choose $\varphi(x)=(\log x)^{1-\alpha}+(\log_2 x)^2$, and hence obtain

(2·15) $$\Delta(n;\log^\alpha)<\tau(n)^{1-\kappa(\alpha)+o(1)}\qquad \text{pp}\ell,$$

with $\kappa(\alpha)=-\log\big(1-\frac12\min(1,\alpha)\big)/\log 4>0$. In the second instance, we select $\varphi(x)=(\log x)/(\log_2 x)^{\beta-1}$ and get similarly

(2·16) $$\Delta(n;\log_2^\beta)<\tau(n)(\log_2 n)^{-(\beta-1)/4+o(1)}\qquad \text{pp}\ell.$$

From the point of view of constructing Behrend sequences, the uniform distribution approach is usually weaker than the block sequences technique developed in the author's recent paper [24], which rests upon a probabilistic argument. This is to be expected since, in the former case, one derives the conclusion from a very strong hypothesis (namely that $f(d)$ is occasionally small modulo 1 because the corresponding frequency is asymptotically equal to the expectation), whereas, in the latter case, the density of the set of multiples is tackled by an *ad hoc* method. Thus, from (2·15) one can only infer, via Theorem 3, that

(2·17) $$\mathcal{A}(\alpha,t):=\{d>1:\big\langle(\log d)^\alpha\big\rangle\leqslant(\log d)^{-t}\}$$

is Behrend for $t<\kappa(\alpha)\log 2=-\frac12\log\big(1-\frac12\min(1,\alpha)\big)$, whereas Theorem 1 of [24] provides, after a straightforward calculation, the larger range

(2·18) $$t<t_0(\alpha):=(\log 2)\min\{1,\alpha/(1-\log 2)\},$$

which is sharp except for the possibility of taking $t=t_0(\alpha)$. However, some upper bounds methods for exponential sums are so powerful that the discrepancy approach enables one to deal with block sequences composed of intervals which are far too short for the induction technique of [24] to be applicable. We shall discuss some examples of this situation in the next two sections.

At this stage, it is worthwhile to note that one can deduce *lower bounds* for the discrepancy from theorem 1 of Hall & Tenenbaum [15], which provides a necessary condition for block sequences to be Behrend. Indeed, if a block sequence $\mathcal{A}$ is defined, for some function $\varepsilon(n)$ which fulfils the assumptions of Theorem 3, by a formula of the type

$$\mathcal{A} = \{d \geqslant 1 : \langle f(d) \rangle \leqslant \varepsilon(d)\}$$

and yet does not satisfy the corresponding necessary condition of [15], we may deduce that

$$\Delta(n; f) \geqslant \tfrac{1}{2}\varepsilon(n)\tau(n)$$

on a set of positive logarithmic density. Actually the necessary condition of [15] and the sufficient condition of [24] are "adjacent" (in a sense precisely described in [24]), and it follows in particular that the sequence $\mathcal{A}(\alpha, t)$ of (2·17) is not Behrend when $t > t_0(\alpha)$. As a consequence, we obtain that, for all $\alpha < 1 - \log 2$, the lower bound

$$\Delta(n; \log^\alpha) > (\log n)^{\log 2 - t_0(\alpha) + o(1)} = (\log n)^{(\log 2)(1-\alpha/(1-\log 2)) + o(1)} \tag{2·19}$$

holds on a set of positive logarithmic density. It is not very difficult to show by a direct argument (using theorem 07 of [14] or exercise III.5.6 of [25]) that (2·19) in fact holds pp.

The true order of magnitude of $\Delta(n, \log^\alpha)$ ppℓ is an interesting open problem, especially in the case $\alpha = 1$. From (2·15) we have

$$\Delta(n; \log) < \tau(n)^{1/2 + o(1)} \qquad \text{pp}\ell,$$

and, as shown in section 5, the exponent $\frac{1}{2}$ can be further reduced to $\log(4/\pi)/\log 2 \approx 0{\cdot}34850$ by exploiting the additivity of $\log d$. However, in view of the fact explained above that $\mathcal{A}(1, t)$ is Behrend for all $t < \log 2$, it seems not unreasonable to conjecture that

$$\Delta(n; \log) = \tau(n)^{o(1)} \qquad \text{pp}\ell.$$

For the sake of further reference, we make the following formal and more general statement.

Conjecture. *Let $\ell(\alpha)$ be the infimum of the set of those real numbers ξ such that $\Delta(n; \log^\alpha) < \tau(n)^\xi$ ppℓ. Then for all positive α we have*

$$\ell(\alpha) = 1 - t_0(\alpha)/\log 2 = \max\{0, 1 - \alpha/(1 - \log 2)\}.$$

It follows from (2·19) that $\ell(\alpha) \geqslant \max\{0, 1 - \alpha/(1 - \log 2)\}$.

Proof of Theorem 10. We use Theorem 7 with $0 < y_0 < 4$, and set out to find an upper bound for

$$S_\nu(k) := \sum_{m=1}^{\infty} \Big(\frac{y}{4}\Big)^{\Omega(m)} \frac{e\big(\nu f(km)\big)}{m^{1+\sigma}}$$

where $0 \leqslant y \leqslant y_0$, $\sigma = 1/\log x$ and ν, k are positive integers. Let x_0 be so large that $xf'(x)$ is monotonic, and $\varphi_1(x)$ is decreasing, for $x > x_0$. It will be convenient to introduce a parameter $M = M(k)$ such that

$$M > x_0, \qquad \varphi(kM) \leqslant \tfrac{1}{2}\log M \qquad (k \geqslant 1). \tag{2·20}$$

Such an M does exist since, for each fixed k, $\varphi(kM)/\log M \sim 1/\varphi_1(kM) \to 0$ as $M \to \infty$. We note that for $u \geqslant M(k)$ we have

$$\varphi(ku) = \frac{\log(ku)}{\varphi_1(ku)} \leqslant \frac{\log(ku)}{\varphi_1(kM)} = \frac{\varphi(kM)\log(ku)}{\log(kM)} \leqslant \tfrac{1}{2}\frac{\log M \log(ku)}{\log(kM)} \leqslant \tfrac{1}{2}\log u. \tag{2·21}$$

For given $k, \nu \geqslant 1$, put $h(u) := e\big(\nu f(ku)\big),\ A(u) := \sum_{m\leqslant u}(y/4)^{\Omega(m)}$, so that

$$S_\nu(k) = \int_{1-}^{\infty} \frac{h(u)}{u^{1+\sigma}}\,\mathrm{d}A(u), \tag{2·22}$$

and by (1·6), since $0 \leqslant y \leqslant y_0 < 4$,

$$A(u) = \int_0^{1/2} u^{1-t}\,\mathrm{d}\lambda_{y/4}(t) + O\big(u/R(u)\big).$$

We insert this into (2·22), make the trivial estimate $|h(u)| \leqslant 1$ for $u \leqslant M$ and integrate by parts on $[M, \infty)$ the contribution of the remainder term. We obtain

$$\begin{aligned} S_\nu(k) = {}& O\big((\log M)^{y/4}\big) + \int_M^{\infty} \frac{h(u)}{u^{1+\sigma}} \int_0^{1/2} (1-t)u^{-t}\,\mathrm{d}\lambda_{y/4}(t)\,\mathrm{d}u \\ &+ O\Big(\frac{1}{R(M)}\Big) - \int_M^{\infty} \frac{\mathrm{d}}{\mathrm{d}u}\Big(\frac{h(u)}{u^{1+\sigma}}\Big) O\Big(\frac{u}{R(u)}\Big)\,\mathrm{d}u. \end{aligned}$$

The last term is

$$\begin{aligned} &\ll \int_M^{\infty} \frac{u|h'(u)| + |h(u)|}{u^{1+\sigma}R(u)}\,\mathrm{d}u \ll \nu \int_M^{\infty} \frac{R\big(e^{\varphi(ku)}\big)+1}{u^{1+\sigma}R(u)}\,\mathrm{d}u \\ &\ll \nu \int_M^{\infty} \frac{R(\sqrt{u})}{u^{1+\sigma}R(u)}\,\mathrm{d}u \ll \nu \int_M^{\infty} \frac{\mathrm{d}u}{uR(u)^b} \ll \frac{\nu}{R(M)^{b/2}}\,, \end{aligned} \tag{2·23}$$

where we have used (2·21) in the third step, and (2·2) in the fourth. Next, we consider the main term in $S_\nu(k)$. This is

$$\int_0^{1/2} (1-t) \int_M^{\infty} \frac{h(u)}{u^{1+\sigma+t}}\,\mathrm{d}u\,\mathrm{d}\lambda_{y/4}(t) \ll \int_0^{1/2} \Big| \int_M^{\infty} \frac{h(u)}{u^{1+\sigma+t}}\,\mathrm{d}u \Big| t^{-y/4}\,\mathrm{d}t. \tag{2·24}$$

We substitute $u = Me^v$ in the inner integral which becomes

$$\int_0^{\infty} \frac{e\big(\nu f(kMe^v)\big)}{M^{\sigma+t}e^{(\sigma+t)v}}\,\mathrm{d}v = \int_0^{\infty} \frac{H'(v)}{M^{\sigma+t}e^{(\sigma+t)v}}\,\mathrm{d}v = \int_0^{\infty} \frac{(\sigma+t)H(v)}{M^{\sigma+t}e^{(\sigma+t)v}}\,\mathrm{d}v,$$

with $H(v) := \displaystyle\int_0^{v} e\big(\nu f(kMe^w)\big)\,\mathrm{d}w$. The function

$$\frac{\mathrm{d}}{\mathrm{d}w}\big\{\nu f(kMe^w)\big\} = \nu kMe^w f'(kMe^w)$$

is monotonic on the whole half-line $w \geqslant 0$ by the choice of M, and, by (2·12), it is $\gg \nu/\varphi\big(kMe^w\big) \gg \nu/\varphi\big(kMe^v\big)$ for $w \leqslant v$. By a well-known lemma on exponential integrals (see e.g. Titchmarsh [26], lemma 4.2), we obtain that

$$H(v) \ll \nu^{-1}\varphi\big(kMe^v\big),$$

so the upper bound in (2·24) is

$$\ll \frac{1}{\nu} \int_0^{1/2} \int_0^{\infty} \frac{(\sigma+t)\varphi\big(kMe^v\big)}{M^{\sigma+t}e^{(\sigma+t)v}t^{y/4}}\,\mathrm{d}v\,\mathrm{d}t.$$

At this stage, we note that $\varphi_1(x) \ll \varphi_1(x')$ for $1 \leqslant x \leqslant x'$. This readily follows from the facts that φ_1 is non-decreasing for $x > x_0$ and that $\varphi(x) \asymp 1$ for $1 \leqslant x \leqslant x_0$, so we omit the details. Therefore, we have for $\xi \geqslant 1, \eta \geqslant 1$,

$$\varphi(\xi\eta) = \frac{\log(\xi\eta)}{\varphi_1(\xi\eta)} \ll \frac{\log \xi}{\varphi_1(\xi)} + \frac{\log \eta}{\varphi_1(\eta)} = \varphi(\xi) + \varphi(\eta). \tag{2·25}$$

Thus, the last double integral is

$$\ll \int_0^{1/2} \frac{\varphi(kM)}{M^t t^{y/4}} \,\mathrm{d}t + \int_0^{1/2} \int_0^{\infty} \frac{(\sigma + t)\varphi\big(e^v\big)}{M^t e^{(\sigma+t)v} t^{y/4}} \,\mathrm{d}v \,\mathrm{d}t.$$

The first term can be computed explicitly. In the inner v-integral of the second term, we substitute $v = w/(\sigma + t)$ and split the range at $w = 1$. We obtain altogether

$$\begin{aligned}
&\ll \varphi(kM)(\log M)^{y/4-1} + \int_0^{1/2} \int_0^{\infty} \frac{\varphi\big(e^{w/(\sigma+t)}\big)}{M^t e^w t^{y/4}} \,\mathrm{d}w \,\mathrm{d}t \\
&\ll \varphi(kM)(\log M)^{y/4-1} + \int_0^{1/2} \frac{\varphi\big(e^{1/(\sigma+t)}\big)}{M^t t^{y/4}} \,\mathrm{d}t + \int_0^{1/2} \int_1^{\infty} \frac{\varphi\big(e^{w/(\sigma+t)}\big)}{M^t e^w t^{y/4}} \,\mathrm{d}w \,\mathrm{d}t \\
&\ll \{\varphi(kM) + \varphi\big(e^{1/\sigma}\big)\}(\log M)^{y/4-1} + \int_0^{1/2} \int_1^{\infty} \frac{w\varphi\big(e^{1/(\sigma+t)}\big)}{M^t e^w t^{y/4}} \,\mathrm{d}w \,\mathrm{d}t \\
&\ll \{\varphi(kM) + \varphi\big(e^{1/\sigma}\big)\}(\log M)^{y/4-1},
\end{aligned}$$

where we have used in the penultimate stage the upper bound

$$\varphi\big(e^{w/(\sigma+t)}\big) = \frac{w}{(\sigma + t)\varphi_1\big(e^{w/(\sigma+t)}\big)} \ll \frac{w}{(\sigma + t)\varphi_1\big(e^{1/(\sigma+t)}\big)} = w\varphi\big(e^{1/(\sigma+t)}\big).$$

Collecting our estimates so far and inserting them into (2·23), we obtain, since $e^{1/\sigma} = x$,

$$\begin{aligned}
S_\nu(k) &\ll (\log M)^{y/4} + \nu R(M)^{-b/2} + \nu^{-1}\{\varphi(kM) + \varphi(x)\}(\log M)^{y/4-1} \\
&\ll (\log M)^{y/4} + \nu R(M)^{-b/2} + \nu^{-1}\varphi(x),
\end{aligned} \tag{2·26}$$

as $\varphi(kM) \leqslant \frac{1}{2}\log M$ by (2·20). Let C be an absolute constant which is at least three times as large as the implicit constant in (2·25). We select

$$M := e^{C\varphi(k)+\varphi(x)},$$

so that, when $x \geqslant x_1(C)$, we have $\varphi(M) \leqslant (1/2C)\log M$ (because M is large) and hence

$$\varphi(kM) \leqslant \tfrac{1}{3}C\{\varphi(M) + \varphi(k)\} \leqslant \tfrac{1}{6}\log M + \tfrac{1}{3}\log M = \tfrac{1}{2}\log M.$$

Thus (2·20) is satisfied with this choice of M. Moreover, we also have $\varphi(x)/\nu \ll \log M$, so we finally obtain from (2·26) that

$$\begin{aligned}
S_\nu(k) &\ll (\log M)^{y/4} + \nu R(M)^{-b/2} \ll \varphi(k)^{y/4} + \varphi(x)^{y/4} + \nu R\big(e^{\varphi(x)}\big)^{-b/2} \\
&\ll \varphi(x)^{y/4}\Big\{1 + \Big(\frac{\log k}{\log x}\Big)^{y/4}\Big\} + \nu R\big(e^{\varphi(x)}\big)^{-b/2},
\end{aligned}$$

where we have used in the last stage the inequality $\varphi_1(x) \leqslant \varphi_1(k)$ for $k \geqslant x$.

We are now in a position to embark on the final part of the proof. Inserting the above estimate for $S_\nu(k)$ into (1·25), we find that

$$H_\nu(x,y)=\sum_{k=1}^{\infty}\Big(\frac{y}{4}\Big)^{\Omega(n)}\frac{1}{k^{1+\sigma}}\big|S_\nu(k)\big|^2\ll\varphi(x)^{y/2}(\log x)^{y/4}+\frac{\nu^2(\log x)^{y/4}}{R\big(e^{\varphi(x)}\big)^b}.$$

Hence, for $T\geqslant 2$,

$$\sum_{1\leqslant\nu\leqslant T}\frac{1}{\nu}H_\nu(x)\ll\varphi(x)^{y/2}(\log x)^{y/4}\log T+\frac{T^2(\log x)^{y/4}}{R\big(e^{\varphi(x)}\big)^b}.$$

We therefore deduce from (1·24) that

$$\begin{aligned}(\log x)^{-1}&\sum_{n\leqslant x}\frac{\Delta(n;f)^2}{n}\Big(\frac{y}{4}\Big)^{\Omega(n)}\\&\ll\frac{(\log x)^{y-1}}{T^2}+\varphi(x)^{y/2}(\log x)^{y/2-1}(\log T)^2+\frac{T^2(\log T)(\log x)^{y/2-1}}{R\big(e^{\varphi(x)}\big)^b}.\end{aligned}$$

We choose $T=\varphi_1(x)^{y/4}=\big(\log x/\varphi(x)\big)^{y/4}$. The upper bound above becomes

$$\ll\varphi(x)^{y/2}(\log x)^{y/2-1}\big(\log\varphi_1(x)\big)^2. \tag{2·27}$$

Indeed the last term is easily seen to be negligible by the lower bound (2·11) imposed on $\varphi(x)$, and because $R(x)=\exp\{(\log x)^{4/7}\}$ is an admissible choice for R. Thus we may define $E_2(x,y)$ as being equal to a suitable constant multiple of the right-hand side of (2·27), and apply Theorem 7 to obtain that

$$\Delta(n;f)\ll\xi(n)\tau(n)y^{\Omega(n)/2}\sqrt{E_2(n,y)}\qquad pp\ell, \tag{2·28}$$

provided $0\leqslant y\leqslant y_0<4$ and $y=y(n)$ is such that $E_2(n,y)$ is slowly increasing as a function of n. We choose

$$y=2\Big/\Big(1+\frac{\log\varphi(n)}{\log_2 n}\Big)=1\Big/\Big(1-\frac{\log\varphi_1(n)}{2\log_2 n}\Big),$$

which minimises $(\log n)^{-\log y}E_2(n,y)$ up to a power of $\log_2\varphi_1(n)$. This value of y is always in the range $[1,2]$. Inserting into (2·27) yields

$$E_2(n,y)\asymp\big(\log\varphi_1(n)\big)^2,$$

which implies that this function is slowly decreasing. The required estimate (2·13) hence follows from (2·28). This completes the proof of Theorem 10.

3. Functions of excessive growth : the case $f(d) = d^{\alpha}$

Here, we address the problem of bounding the discrepancy ppℓ for functions which increase too fast for the techniques of the previous section to be applicable. More precisely, let us recall the quantity

$$H_\nu(x,y) := \sum_{k=1}^{\infty} \Big(\frac{y}{4}\Big)^{\Omega(k)} \frac{1}{k^{1+\sigma}} \Big| \sum_{m=1}^{\infty} \Big(\frac{y}{4}\Big)^{\Omega(m)} \frac{e\big(\nu f(km)\big)}{m^{1+\sigma}} \Big|^2, \tag{3·1}$$

with $\sigma := 1/\log x$, which appears implicitly in the upper bound (1·26) of Theorem 7 for the discrepancy $\Delta(n; f)$. This was primarily defined for $y < 8$, but we restrict it here to values of $y \leqslant 4$. The functions of moderate growth are essentially those for which the inner m-sum can be estimated by partial summation, using the available results on the mean value of $m \mapsto (y/4)^{\Omega(m)}$. When the rate of growth of f prohibits such a treatment, we may consider $H_\nu(x,y)$ as a 'type II sum', according to the poetic terminology of Vinogradov. For all intents and purposes, this means making the trivial estimate $\big|(y/4)^{\Omega(k)}\big| \leqslant 1$, expanding the square and, after permuting summations, estimating the inner k-sum by an *ad hoc* exponential sum method.

This programme may be carried out, in principle, for any smooth function f of, say, at most polynomial growth, and indeed one could even aim at a general theorem established along these lines and providing, under suitable sufficient conditions, explicit upper bounds for the discrepancy. Due to the considerable amount of calculations that this would involve, we have preferred to treat only examples which reflect all the difficulties of the general case, but avoid tedious technicalities that would hide the main stream of the argument. In this context, we believe that the functions

$$d \mapsto d^{\alpha} \quad (\alpha \in \mathbb{R}^+ \smallsetminus \mathbb{Z}^+), \qquad d \mapsto \vartheta d \quad (\vartheta \in \mathbb{R} \smallsetminus \mathbb{Q})$$

are of special interest. We treat the first of these immediately and the second one in the next section. The following theorem provides an effective version of the corresponding qualitative result obtained by Hall & Tenenbaum in [13].

Theorem 11. *Let $\alpha > 0$ be a given real number, not an integer. Then the function $g_\alpha(d) := d^{\alpha}$ is* erd. *More precisely, we have*

$$\Delta(n; g_\alpha) < \tau(n)(\log n)^{-\delta} \qquad \text{pp}\ell \tag{3·2}$$

for all $\delta < \delta_0 := \frac{1}{2}\log\frac{12}{11}$.

We note that $\delta_0 > \frac{1}{23}$. We have not attempted here to find the best exponent available from latest developments in exponential sums theory and have confined ourselves to using a result of Karatsuba [16] on Vinogradov-type bounds which is expressed in an easily applicable form. We remark that van der Corput-type estimates would in general yield weaker bounds for the discrepancy and would actually only save a power of $\log_2 n$ in (3·2).

It is also worthwhile to note at this stage that the method of proof of Theorem 11 will readily yield that the function

$$f(d) = \exp\{(\log d)^{\alpha}\}$$

is erd for $0 < \alpha < 1$, and indeed it will provide a bound comparable with (3·2) for the discrepancy. In particular, this shows that the limitation $\alpha < 3/5$ which arises from mere application of Theorems 4 or 10 is purely technical. The range $1 < \alpha < 3/2$ may also be handled by the same technique, but with a weaker effective result — see Theorem 2 of Karatsuba [16].

By Theorem 3, we immediately derive from the above result the following corollary.

Corollary 5. *Let α, δ be as in the statement of Theorem 11. Then the sequence*

$$\{n \geqslant 2 : \langle n^\alpha \rangle \leqslant (\log n)^{-\delta}\} \tag{3·3}$$

is a Behrend sequence.

Of course we can rewrite the condition in (3·3) introducing $j := [n^\alpha]$. This yields the following reformulation in terms of block sequences.

Corollary 6. *Let $\beta > 0$, $1/\beta \notin \mathbb{Z}$, $\delta < \delta_0 = \frac{1}{2} \log \frac{12}{11}$. Then the sequence*

$$\mathcal{B}(\delta) := \bigcup_{j=2}^{\infty} \Big] j^\beta, j^\beta \Big(1 + \frac{1}{j(\log j)^\delta}\Big)\Big] \cap \mathbb{Z}^+ \tag{3·4}$$

is a Behrend sequence.

As far as block sequences are concerned, this is only significant when $\beta > 1$: otherwise the 'blocks' have lengths smaller than 1 and looking at $\mathcal{B}(\delta)$ as a block sequence is meaningless. As we remarked in the previous section, the above result is unreachable, in the present state of knowledge, by the technique applied in [24]. The natural conjecture in accord with the results of [15] and [24] would be that $\mathcal{B}(\delta)$ is Behrend for all $\delta < \log 2$, this exponent then being optimal. This is also out of reach of the present technique, which implies a systematic loss due, among other causes, to the trivial estimate for $(y/4)^{\Omega(k)}$ in (3·1).

We now embark on the proof of Theorem 11. We give ourselves two parameters x_1, x_2 satisfying

$$e^{\sqrt{\log x}} \leqslant x_1 \leqslant x, \qquad x_2 := x^{\log_2 x},$$

and introduce the following further notation

$$J = J(x) := \frac{\log(x_2/x_1)}{\log 2}, \quad K_j := 2^j x_1 \quad (0 \leqslant j \leqslant J),$$
$$B_j(m, n; \nu, f) := \sum_{K_j < k \leqslant K_{j+1}} e\big(\nu f(kn) - \nu f(km)\big) \qquad (0 \leqslant j \leqslant J).$$

Lemma 1. *Let $\alpha > 0$. Then we have uniformly for $x \geqslant 3$, $\nu \geqslant 1$, $0 \leqslant y \leqslant 4$, and a real valued arithmetical function f*

$$\begin{aligned} H_\nu(x, y) \ll &(\log x)^{y/2} (\log x_1)^{y/4} \\ &+ \sum_{\substack{x_1 < m < n \leqslant x_2 \\ n^\alpha - m^\alpha \geqslant 1}} \frac{(\frac{1}{4}y)^{\Omega(mn)}}{mn} \sum_{0 \leqslant j \leqslant J} \frac{\Re e\, B_j(m, n; \nu, f)}{K_j}. \end{aligned} \tag{3·5}$$

Proof. To lighten the presentation, we temporarily set $z := y/4$. We first split the k-sum in (3·1) according to whether $k \leqslant x_1$, $x_1 < k < x_2$ or $k > x_2$, so as to write correspondingly

$$H_\nu(x, y) = H_\nu^{(1)}(x, y) + H_\nu^{(2)}(x, y) + H_\nu^{(3)}(x, y).$$

Using the bounds

$$\sum_{k \leqslant x_1} z^{\Omega(k)} k^{-1} \ll (\log x_1)^z, \tag{3·6}$$

and

$$\sum_{k>x_2} z^{\Omega(k)}k^{-1-\sigma} \leqslant x_2^{-\sigma/2}\sum_{k\geqslant 1} z^{\Omega(k)}k^{-1-\sigma/2} \ll (\log x)^{z-1/2}, \tag{3·7}$$

we readily obtain

$$\begin{aligned} H_\nu^{(1)}(x,y) + H_\nu^{(3)}(x,y) &\ll (\log x_1)^z(\log x)^{2z} + (\log x)^{3z-1/2} \\ &\ll (\log x)^{y/2}(\log x_1)^{y/4}. \end{aligned} \tag{3·8}$$

Next, we split the inner m-sum in $H_\nu^{(2)}(x,y)$ at x_1 and x_2 and use the inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$ to obtain

$$H_\nu^{(2)}(x,y) \leqslant 3H_\nu^{(21)}(x,y) + 3H_\nu^{(22)}(x,y) + 3H_\nu^{(23)}(x,y)$$

with

$$\begin{aligned} H_\nu^{(21)}(x,y) &\leqslant \sum_{x_1<k\leqslant x_2} z^{\Omega(k)}k^{-1}\Big(\sum_{m\leqslant x_1} z^{\Omega(m)}m^{-1}\Big)^2 \\ &\ll (\log x)^z(\log x_1)^{2z} \ll (\log x)^{y/2}(\log x_1)^{y/4}, \end{aligned} \tag{3·9}$$

by (3·6), and

$$\begin{aligned} H_\nu^{(23)}(x,y) &\leqslant \sum_{x_1<k\leqslant x_2} z^{\Omega(k)}k^{-1}\Big(\sum_{m>x_2} z^{\Omega(m)}m^{-1-\sigma}\Big)^2 \\ &\ll (\log x)^{3z-1} = (\log x)^{3y/4-1} \ll (\log x)^{y/2}, \end{aligned} \tag{3·10}$$

by (3·7).

It remains to estimate $H_\nu^{(22)}(x,y)$. We have

$$H_\nu^{(22)}(x,y) \leqslant \sum_{0\leqslant j\leqslant J} K_j^{-1} \sum_{K_j<k\leqslant K_{j+1}} \Big| \sum_{x_1<m\leqslant x_2} \frac{z^{\Omega(m)}}{m^{1+\sigma}} e\big(\nu f(km)\big)\Big|^2, \tag{3·11}$$

where we have made the trivial estimate $z^{\Omega(k)} \leqslant 1$. Expanding the square, we find that it does not exceed

$$2\Re e \sum_{\substack{m<n\leqslant x_2\\ n^\alpha-m^\alpha\geqslant 1}} \frac{z^{\Omega(mn)}}{(mn)^{1+\sigma}} e\big(\nu f(kn) - \nu f(km)\big) + 2\sum_{\substack{x_1<m\leqslant n\leqslant x_2\\ n^\alpha-m^\alpha<1}} \frac{1}{mn}. \tag{3·12}$$

We claim that the second sum on the right is $\ll x_1^{-\min(1,\alpha)}$. This plainly holds if $\alpha \geqslant 1$ since the summation conditions then imply that $m = n$. When $0 < \alpha < 1$, we note that $n^\alpha - m^\alpha \geqslant \alpha(n-m)n^{\alpha-1}$, so for fixed n the m-sum is $\ll n^{1-\alpha}/n = n^{-\alpha}$ and the conclusion is still valid. Inserting this estimate into (3·12) and (3·11) and using the fact that $J \ll x_1^{\min(1,\alpha)}$, we obtain

$$\begin{aligned} &H_\nu^{(22)}(x,y) \\ &\quad\ll \Re e \sum_{0\leqslant j\leqslant J} \frac{1}{K_j} \sum_{K_j<k\leqslant K_{j+1}} \sum_{\substack{x_1<m<n\leqslant x_2\\ n^\alpha-m^\alpha\geqslant 1}} \frac{z^{\Omega(mn)}}{(mn)^{1+\sigma}} e\big(\nu f(kn) - \nu f(km)\big) + 1. \end{aligned}$$

We permute summations on k and m,n and see that the new, inner k-sum equals $B_j(m,n;\nu,f)$. Together with (3·8), (3·9) and (3·10), this completes the proof of our lemma.

We now apply Karatsuba's estimate to bound the exponential sum

$$B(K;v) := \sum_{K<k\leqslant 2K} e\big(vk^\alpha\big) \tag{3·13}$$

for relevant values of v, K.

Lemma 2. *Let $\alpha\in\mathbb{R}^+\smallsetminus\mathbb{Z}^+$. There exists a constant $c=c(\alpha)>0$ such that the estimate*

$$B(K;v)\ll K^{1-c}+K\exp\{-c\frac{(\log K)^3}{(1+\log v)^2}\} \tag{3·14}$$

holds uniformly for $K\geqslant 1$, $v\geqslant 1$.

Proof. If $v\leqslant K^{1-\alpha}$, and so $0<\alpha<1$, we apply a classical estimate of van der Corput (see e.g.Titchmarsh [26], lemma 4.7) to get

$$B(K;v)=\int_K^{2K} e\big(vt^\alpha\big)\,\mathrm{d}t+O(1)\ll K^{1-\alpha}.$$

(The same estimate also follows from theorem 2.1 of Graham & Kolesnik [7].) Thus (3·14) holds in this case.

If $v>K^{1-\alpha}$, set $n:=\big[3\alpha+3(\log v)/\log K\big]$, so that $n\geqslant 3$ and

$$K^{n/3}\leqslant vK^\alpha<K^{(n+1)/3}.$$

Put $g(t):=vt^\alpha$. We have for all non-negative integers s

$$\frac{g^{(s)}(t)}{s!}=v\binom{\alpha}{s}t^{\alpha-s}.$$

Writing $\binom{\alpha}{s}=(-1)^s\prod_{1\leqslant j\leqslant s}\big\{1-(\alpha+1)/j\big\}$, we see that we have for suitable positive constants $c_1=c_1(\alpha)$, $c_2=c_2(\alpha)$,

$$c_1s^{-\alpha-1}\leqslant\left|\binom{\alpha}{s}\right|\leqslant c_2s^{-\alpha-1}\qquad(s\geqslant 0). \tag{3·15}$$

Hence for large K and $K\leqslant t\leqslant 2K$ we have

$$\Big|\frac{g^{(n+1)}(t)}{(n+1)!}\Big|\leqslant\frac{2^\alpha c_2}{(n+1)^{\alpha+1}}vK^{\alpha-n-1}\leqslant 2^\alpha c_2K^{-2(n+1)/3}\leqslant K^{-(n+1)/2}.$$

Similarly, a straightforward computation enables us to deduce from (3·15) that for all s in the range $3n/4\leqslant s\leqslant n$ (so $s\geqslant 3$) and large K we have

$$K^{-3s/4}\leqslant c_1s^{-\alpha-1}2^{-s}K^{-2/3}\leqslant\Big|\frac{g^{(s)}(t)}{s!}\Big|\leqslant 2^\alpha K^{-5s/9+1/3}\leqslant K^{-s/3}.$$

By Theorem 1 of Karatsuba [16], it follows that, for suitable positive absolute constants c_3 and c_4 and $K>K_0(\alpha)$ we have

$$|B(K;v)|\leqslant c_3K^{1-c_4/n^2}.$$

This implies the required bound.

We are now in a position to complete the proof of Theorem 11. We want to apply Theorem 7 and use Lemmas 1 and 2 to obtain an upper bound for the quantity $E_2(n, y)$. We select, in Lemma 1, $f = g_\alpha$, as defined in the statement of the theorem, and

$$x_1 = \exp\left\{c(\alpha)(\log x)^{2/3}\log_2 x\right\},$$

with $c(\alpha) > 0$ at our disposal. Then, with the notation of Lemma 2, $B_j(m, n; \nu, g_a) = B(K_j; v)$ where $v := \nu(n^\alpha - m^\alpha)$, so $v \leqslant x_2^{1+\alpha}$ provided $\nu \leqslant x$. By Lemma 2 there is a positive constant $c_5 = c_5(\alpha)$ such that, for all $m, n \leqslant x$ with $n^\alpha - m^\alpha \geqslant 1$,

$$B_j(m, n; \nu, g_\alpha)/K_j \ll \exp\left\{-c_5\frac{(\log K_j)^3}{(\log x_2)^2}\right\} \ll 1/(\log x)^4$$

provided $c(\alpha)$ is large enough. By (3·5), we infer that we have uniformly for $1 \leqslant \nu \leqslant x$, $0 \leqslant y \leqslant 4$,

$$H_\nu(x, y) \ll (\log x)^{2y/3}(\log_2 x)^{y/4}. \tag{3·16}$$

Inserting the above estimate into (1·24) with, say, $T := \log x$, we see that we may choose

$$E_2(x, y) \asymp (\log x)^{11y/12-1}(\log_2 x)^{2+y/4}, \tag{3·17}$$

which is hence slowly increasing. Therefore we get by Theorem 7 that

$$\Delta(n, g_\alpha) < \tau(n)(\log n)^{11y/24-1/2-(1/2)\log y+o(1)} \qquad \text{pp}\ell. \tag{3·18}$$

The required estimate (3·2) now follows on taking optimally $y = \frac{12}{11}$.

4. Functions of excessive growth : the case $f(d) = \vartheta d$

We now investigate, in a quantitative form, the uniform distribution on divisors of the function

$$h_\vartheta(d) := \vartheta d$$

when ϑ is a given irrational real number. This study is similar in principle to that of the previous section, but more complicated inasmuch as the effective bounds for $\Delta(n; h_\vartheta)$ will depend on the arithmetic nature of ϑ. On the other hand we shall not need, as might be expected, any involved tool for the estimation of the relevant exponential sums.

More explicitly, let us define $Q(x) := x/(\log x)^{10}$, and

$$q(x; \vartheta) := \inf\{q : 1 \leqslant q \leqslant Q(x),\ \|q\vartheta\| \leqslant 1/Q(x)\} \tag{4·1}$$

where $\|u\|$ denotes the distance of u to the set of integers. Our results depend on a free parameter y, $0 < y \leqslant 4$, and may be expressed conveniently in terms of any increasing lower bound for $q(x; \vartheta)$, say $q^*(x; y, \vartheta)$, with the property that $q^*(x; y, \vartheta)/(\log x)^{y/4}$ is decreasing. A possible choice is

$$q^*(x; y, \vartheta) := 4(\log x)^{y/4}\Big/\int_1^x \frac{y(\log t)^{y/4-1}}{t\inf_{u\geqslant t} q(u; \vartheta)}\,\mathrm{d}t. \tag{4·2}$$

Unless ϑ has abnormally good rational approximations, we have

$$q^*(x; y, \vartheta) \asymp (\log x)^{y/4}. \tag{4·3}$$

Indeed, let us define, for real positive γ, the set

$$E(\gamma) := \{\vartheta \in \mathbb{R} \smallsetminus \mathbb{Q} : \liminf_{x\to\infty} q(x; \vartheta)/(\log x)^\gamma > 0\}.$$

Then $E(\gamma)$ contains almost all real numbers, and in particular, by Liouville's theorem, all algebraic numbers. Moreover, it is not difficult to show that $\mathbb{R} \smallsetminus E(\gamma)$ has zero Hausdorff dimension. We readily see from (4·2) that (4·3) holds for all $\vartheta \in E(\gamma)$ whenever $\gamma > \frac{1}{4}y$.

We shall establish the following result.

Theorem 12. *Let $\vartheta \in \mathbb{R} \smallsetminus \mathbb{Q}$, $0 < \delta < 1$. Uniformly for $x \geqslant 2$, $0 < y \leqslant 4$, we have*

$$\sum_{n \leqslant x} \Big(\frac{y}{4}\Big)^{\Omega(n)} \frac{\Delta(n; h_\vartheta)^2}{n} \ll (\log x)^y / q^*(x; y, \vartheta)^\delta. \tag{4·4}$$

Taking $y = 1$, we immediately obtain an effective uniform distribution result which is valid without any restriction on ϑ. The corresponding qualitative result had been established by Dupain, Hall & Tenenbaum [4].

Corollary 7. *The function $h_\vartheta(d) = \vartheta d$ is* erd *for each irrational number ϑ. Moreover, if $0 < \delta < 1$, then we have*

$$\Delta(n; h_\vartheta) < \tau(n)/q^*(n; 1, \vartheta)^{\delta/2} \qquad \text{pp}\ell.$$

The above bound is always $o(\tau(n))$ and $\ll \tau(n)/(\log n)^{-\delta/8}$ for $\vartheta \in E(\frac14)$. However, if we are prepared to exclude a set of ϑ of Hausdorff dimension zero, we may achieve a better ppℓ estimate by taking $y = \frac43$ in (4·4). Indeed, the following statement stems from Theorem 12 by optimising the parameter y under the assumption that (4·3) holds.

Corollary 8. *Let $\gamma > \frac13$. Then we have for all $\vartheta \in E(\gamma)$*

$$\Delta(n; h_\vartheta) < \tau(n)^{(\log 3)/\log 4 + o(1)} \qquad \text{pp}\ell.$$

It is very likely that the estimate $\Delta(n, h_\vartheta) < \tau(n)^{1/2+o(1)}$ ppℓ holds outside a set of ϑ with Hausdorff dimension 0, but this is beyond the scope of the method employed here. If we only require that the set of exceptional ϑ have Lebesgue measure zero, this last bound does actually hold and can be easily established by the variance argument used for the proof of Theorem 14 below. Moreover, with this level of generality, the exponent 1/2 is sharp.

Of course, Corollaries 7 & 8 may be used to exhibit Behrend sequences. An immediate application of these results and Theorem 3 yields the following proposition.

Corollary 9. *Let $\vartheta \in \mathbb{R} \smallsetminus \mathbb{Q}$, $0 < \delta < 1$. Then the sequence*

$$\mathcal{U}(\vartheta, \delta) := \Big\{ n \geqslant 2 : \langle \vartheta n \rangle \leqslant q^*(n; 1, \vartheta)^{-\delta/2} \Big\}$$

is a Behrend sequence. Furthermore, if $\vartheta \in E(\gamma)$ for some $\gamma > \frac13$, and in particular if ϑ is algebraic, then the sequence

$$\mathcal{W}(\vartheta, \varrho) := \Big\{ n \geqslant 2 : \langle \vartheta n \rangle \leqslant (\log n)^{-\varrho} \Big\}$$

is a Behrend sequence for all $\varrho < \frac12 \log \frac43$.

Let $\varrho^*(\gamma)$ denote the supremum of those exponents ϱ such that $\mathcal{W}(\vartheta, \varrho)$ is Behrend for all $\vartheta \in E(\gamma)$. The above result implies that $\varrho^*(\gamma) \geqslant \frac12 \log \frac43 \approx 0{\cdot}14384$ for $\gamma > \frac13$, and it is natural to conjecture that there exists a γ_0 such that $\varrho^*(\gamma) = \log 2$ for $\gamma > \gamma_0$. By techniques similar to those presented below for the proof of Theorem 12, it can be shown that the distributions of $\Omega(n)$ and $\langle \vartheta n \rangle$ are largely independent. Moreover, using Vaughan's bound for exponential sums over primes (see e.g.Davenport [1], chapter 25), this statement can be put in an effective form which is sufficiently strong to yield

$$\sum_{n \in \mathcal{W}(\vartheta, \varrho)} \frac{(\log n)^{\log 2 - 1/2}}{n 2^{\Omega(n)}} < \infty$$

for all $\varrho > \log 2$ and, e.g., $\vartheta \in E(2)$. By a result of Hall ([12], theorem 1), this implies that, when $\vartheta \in E(2)$ and $\varrho > \log 2$, the sequence $\mathcal{W}(\vartheta, \varrho)$ is not Behrend. A weak consequence of this is that $\varrho^*(\gamma) \leqslant \log 2$ for all $\gamma \geqslant 2$.

For the proof of Theorem 12, we have chosen to avoid some technical complications by applying Theorem 8 rather than Theorem 7, although the latter could in principle lead to better quantitative estimates. In connection with the general upper bound (1·28) for weighted logarithmic averages of $\Delta(n; f)^2$, we introduce the expressions

$$T(x; z, \vartheta) := \sum_{n\leqslant x} z^{\Omega(n)} \frac{e(\vartheta n)}{n}, \quad S(x, z, \vartheta) := \sum_{k\leqslant x} \frac{z^{\Omega(k)}}{k} \big|T(x/k; z, k\vartheta)\big|, \tag{4·5}$$

so that (1·28) reads, for $f = h_\vartheta$,

$$\sum_{n\leqslant x} \Big(\frac{y}{4}\Big)^{\Omega(n)} \frac{\Delta(n; h_\vartheta)^2}{n} \ll (\log x)^y \Big\{\frac{1}{T^2} + \log T \sum_{1\leqslant \nu\leqslant T} \frac{1}{\nu} \varepsilon_\nu^+(x, y; h_\vartheta)\Big\}, \tag{4·6}$$

uniformly for $x \geqslant 2$, $0 \leqslant y \leqslant y_0 < 8$, $T \geqslant 2$, where $\varepsilon_\nu^+(x, y; h_\vartheta)$ is any non-increasing function of x such that $x \mapsto \varepsilon_\nu^+(x, y; h_\vartheta)(\log x)^{y/2}$ is non-decreasing and satisfies

$$S(x; \tfrac{1}{4}y, \nu\vartheta) \ll (\log x)^{y/2} \varepsilon_\nu^+(x, y; h_\vartheta).$$

For technical reasons, it will be more convenient to use at certain stages Cesàro-type averages, so we set

$$T^*(x; z, \vartheta) := \sum_{n\leqslant x} z^{\Omega(n)} e(\vartheta n), \quad S^*(x; z, \vartheta) := \sum_{k\leqslant x} z^{\Omega(k)} \big|T^*(x/k; z, k\vartheta)\big|, \tag{4·7}$$

from which we shall derive information on the quantities in (4·5) by partial summation.

We need several preliminary estimates which we state as independent lemmata.

Lemma 3. *For $0 \leqslant z \leqslant 1$, $1 \leqslant a \leqslant q \leqslant x$, $(a, q) = 1$, $|\vartheta - a/q| \leqslant 1/q^2$, we have*

$$S^*(x; z, \vartheta) \ll x(\log x)^{z-1} (\log_2 x)^z + x(\log x)^2 \Big\{\frac{\sqrt{q}}{\sqrt{x}} + \frac{1}{\sqrt{q}}\Big\}. \tag{4·8}$$

Proof. This is a variant of a familiar lemma in Vinogradov's method. We first note the trivial estimate

$$|T^*(w; z, \vartheta)| \leqslant \sum_{n\leqslant w} z^{\Omega(n)} \ll w(\log w)^{z-1} \qquad (w \geqslant 2), \tag{4·9}$$

which stems from (6) or e.g.theorem III.3.5 of [25]. Then, assuming, as we may, that x is large, we put $y = (\log x)^6 \leqslant \sqrt{x}$ and we split the outer k-sum in $S^*(x; z, \vartheta)$, applying (4·9) with $w = x/k$ for the ranges $k \leqslant y$ and $x/y < k \leqslant x$. Using (4·9) again with partial summation for the corresponding resulting summation over k, and bounding $z^{\Omega(k)}$ by 1 in the complementary sum, we arrive at

$$S^*(x; z, \vartheta) \ll x(\log x)^{z-1} (\log_2 x)^z + \sum_{0\leqslant j<J} W(2^j y), \tag{4·10}$$

with $J := \dfrac{\log(x/y)}{\log y}$ and

$$W(K) := \sum_{K<k\leqslant 2K} \big|T^*(x/k; z, k\vartheta)\big|. \tag{4·11}$$

Now we have for $y \leqslant K \leqslant x/y$, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
W(K)^2 &\leqslant K \sum_{K<k\leqslant 2K} \sum_{1\leqslant m,n\leqslant x/k} z^{\Omega(mn)} e\big(k\vartheta(n-m)\big) \\
&= K \sum_{1\leqslant m,n\leqslant x/K} z^{\Omega(mn)} \sum_{K<k\leqslant \min(2K,\, x/m,\, x/n)} e\big(k\vartheta(n-m)\big) \\
&\ll K \sum_{1\leqslant m,n\leqslant x/K} \min\big(K, 1/\|\vartheta(n-m)\|\big) \\
&\ll K\frac{x}{K} \sum_{0\leqslant h\leqslant x/K} \min\big(K, 1/\|\vartheta h\|\big).
\end{aligned}
\tag{4·12}
$$

To estimate the h-sum, we write $\vartheta = a/q+\beta$ with $|\beta| \leqslant 1/q^2$ and $h = tq + r$ with $0 \leqslant r < q$. Then, for each given t, we have $\|\vartheta h\| = \|\alpha_r\|$ with $\alpha_r := ra/q + r\beta + tq\beta$. For $0 \leqslant r \neq s < q$, and if $\langle\alpha_r\rangle - \frac{1}{2}$ and $\langle\alpha_s\rangle - \frac{1}{2}$ have the same sign, we may write $\big|\|\alpha_s\| - \|\alpha_r\|\big| = \|\alpha_s - \alpha_r\| > \|(s-r)a/q\| - 1/q$. Hence there are at most 6 values of r, $0 \leqslant r < q$, such that α_r belongs to any given interval $(v/q, (v+1)/q]$ modulo 1. This implies that

$$
\begin{aligned}
\sum_{0\leqslant h\leqslant x/K} \min\big(K, 1/\|\vartheta h\|\big) &\ll \sum_{0\leqslant t\leqslant x/Kq} \sum_{0\leqslant r<q} \min\big(K, 1/\|ar/q\|\big) \\
&\ll (1 + x/Kq)(K + q\log q) \\
&\ll x\log x\Big(\frac{K}{x} + \frac{q}{x} + \frac{1}{q} + \frac{1}{K}\Big) \ll x\log x\Big(\frac{1}{y} + \frac{q}{x} + \frac{1}{q}\Big).
\end{aligned}
$$

By (4·12), we infer that

$$
W(K) \ll x\log x\Big\{\frac{\sqrt{q}}{\sqrt{x}} + \frac{1}{\sqrt{q}}\Big\} + \frac{x}{(\log x)^2} \qquad (y \leqslant K \leqslant x/y).
\tag{4·13}
$$

Inserting this into (4·10), we readily get the required estimate.

Lemma 4. *Let $\vartheta \in \mathbb{R}\smallsetminus\mathbb{Q}$. For all $x \geqslant 3$ such that $q(x;\vartheta) > (\log x)^{10}$ and uniformly for $0 \leqslant z \leqslant 1$, $1 \leqslant \nu \leqslant \log x$, we have*

$$
S^*(x; z, \nu\vartheta) \ll x(\log x)^{z-1}(\log_2 x)^z.
\tag{4·14}
$$

Proof. Let $q = q(x;\vartheta)$. Then, by Dirichlet's theorem, $q \leqslant Q(x) = x/(\log x)^{10}$ and, for suitable integer a, we have $|\vartheta - a/q| \leqslant 1/qQ \leqslant 1/q^2$. Moreover the minimality assumption on q implies that $(a,q) = 1$. Thus for each ν with $1 \leqslant \nu \leqslant \log x$ we have $|\nu\vartheta - a_\nu/q_\nu| \leqslant 1/q_\nu^2$ for some integers a_ν, q_ν with $(a_\nu, q_\nu) = 1$, $(\log x)^9 < q_\nu \leqslant q$. Applying Lemma 3, we obtain

$$
S^*(x; z, \nu\vartheta) \ll x(\log x)^{z-1}(\log_2 x)^z + x/(\log x)^{5/2}.
$$

The result follows.

Our next lemma concerns the distribution of the numbers $z^{\Omega(n)}$ on arithmetic progressions. We put

$$
H(x; z; q, a) := \sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\, q)}} z^{\Omega(n)}, \qquad H(x; z; q) := \sum_{\substack{n\leqslant x\\ (n,q)=1}} z^{\Omega(n)}.
\tag{4·15}
$$

Lemma 5. *Let $A > 0$. Then there is a positive constant c such that, for $0 \leqslant z \leqslant 1$, $x \geqslant 2$, $1 \leqslant q \leqslant (\log x)^A$, $(a,q)=1$, we have*

$$H(x;z;q,a) = \frac{1}{\varphi(q)} H(x;z;q) + O\big(x\exp\{-c(\log x)^{1/3}\}\big). \tag{4·16}$$

Proof. It would be possible, as in Rieger [21], to obtain an exponent $\frac{1}{2}$ instead of $\frac{1}{3}$ in the remainder term using contour integration and standard analytic information on powers $L(s,\chi)^z$ of Dirichlet L-functions. The result stated will be more than sufficient for our actual purpose. It may be given a short proof which we include for the convenience of the reader. We introduce the Dirichlet characters to the modulus q and write

$$H(x;z;q,a) = \frac{1}{\varphi(q)} H(x;z;q) + O\Big(\max_{\chi\neq\chi_0}\Big|\sum_{n\leqslant x}\chi(n)z^{\Omega(n)}\Big|\Big), \tag{4·17}$$

where the maximum is taken over all non-principal characters χ modulo q. This remainder may be bounded above by appealing to the prime number theorem for arithmetic progressions in the form

$$\sum_{p\leqslant t}\chi(p) \ll te^{-c(b)\sqrt{(\log t)}} \qquad (q\leqslant(\log t)^b), \tag{4·18}$$

valid for any non-principal character χ to the modulus q. Here b is any fixed parameter and $c(b)>0$. This estimate may be found e.g. in Davenport [1], p. 132.

We also introduce the largest prime factor function $P^+(n)$ and recall from [25] (theorem III.5.1) the estimate

$$\Psi(x,y) := \sum_{\substack{n\leqslant x\\ P^+(n)\leqslant y}} 1 \ll x^{1-1/(2\log y)} \quad (x\geqslant 2,\ y\geqslant 2). \tag{4·19}$$

For any non-principal character χ to the modulus q, we have

$$\begin{aligned}
\sum_{n\leqslant x}\chi(n)z^{\Omega(n)} &= \sum_{\substack{n\leqslant x\\ P^+(n)>\exp\{\sqrt{(\log x)}\}}} \chi(n)z^{\Omega(n)} + O\big(xe^{-\frac{1}{2}\sqrt{(\log x)}}\big)\\
&= \sum_{\substack{mr\leqslant x\\ P^+(mr)=r\\ r>\exp\{\sqrt{(\log x)}\}}} \chi(m)\chi(r)z^{\Omega(m)+1} + O\big(xe^{-\frac{1}{2}\sqrt{(\log x)}}\big)\\
&= \sum_{mP^+(m)\leqslant x}\chi(m)z^{\Omega(m)+1}\sum_{\substack{P^+(m)\leqslant r\leqslant x/m\\ r\text{ prime}\\ r>\exp\{\sqrt{(\log x)}\}}}\chi(r) + O\big(xe^{-\frac{1}{2}\sqrt{(\log x)}}\big)\\
&\ll \sum_{mP^+(m)\leqslant x}\frac{x}{m}e^{-c(2A)\sqrt{(\log(x/m))}} + O\big(xe^{-\frac{1}{2}\sqrt{(\log x)}}\big),
\end{aligned}$$

by (4·18). Write $c(2A)=c_1$ for brevity and set $M_j := x/e^j$ $(j=0,1,\ldots)$. The above m-sum does not exceed

$$\sum_{0\leqslant j\leqslant\log x}\frac{x}{M_j}e^{-c_1\sqrt{j}}\sum_{\substack{M_{j+1}<m\leqslant M_j\\ P^+(m)\leqslant e^{j+1}}}1 \ll \sum_{0\leqslant j\leqslant\log x} xe^{-c_1\sqrt{j}-(\log x)/(2j)},$$

by (4·19). Since $c_1\sqrt{j}+(\log x)/(2j) \gg (\log x)^{1/3}$, we obtain that the estimate

$$\sum_{n\leqslant x}\chi(n)z^{\Omega(n)} \ll x\exp\{-c(\log x)^{1/3}\}$$

holds, under the prescribed conditions, for a suitable positive constant c. In view of (4·17) this readily yields the required result.

Lemma 6. *Let $A > 0$. There exists a positive constant c_0 such that, uniformly for $0 \leqslant z \leqslant 1$, $x \geqslant 2$, $1 \leqslant q \leqslant (\log x)^A$, $(a,q) = 1$, we have*

$$T^*(x; z, a/q) = \sum_{t|q} \frac{\mu(t) z^{\Omega(q/t)}}{\varphi(t)} H(tx/q; z; t) + O\big(x \exp\{-c_0(\log x)^{1/3}\}\big).$$

Proof. We have

$$\begin{aligned} T^*(x; z, a/q) &= \sum_{0 \leqslant b < q} e(ab/q) \sum_{\substack{n \leqslant x \\ n \equiv b \,(\mathrm{mod}\, q)}} z^{\Omega(n)} \\ &= \sum_{0 \leqslant b < q} e(ab/q) z^{\Omega((b,q))} \sum_{\substack{m \leqslant x/(b,q) \\ m \equiv b/(b,q) \,(\mathrm{mod}\, q/(b,q))}} z^{\Omega(m)} \\ &= \sum_{t|q} \sum_{\substack{0 \leqslant h < t \\ (h,t)=1}} e(ah/t) z^{\Omega(q/t)} H(tx/q; z; t, h), \end{aligned}$$

where we have put $t = q/(b,q)$, $b = hq/t = h(b,q)$ with $(h,t) = 1$. Using Lemma 5 to evaluate $H(tx/q; z; t, h)$, and appealing to Ramanujan's formula

$$\sum_{\substack{0 \leqslant h < t \\ (h,t)=1}} e(ah/t) = \mu(t),$$

we get

$$T^*(x; z, a/q) = \sum_{t|q} \frac{\mu(t) z^{\Omega(q/t)}}{\varphi(t)} H(tx/q; z; t) + O\Big(qx \exp\{-c(\log x)^{1/3}\}\Big).$$

Hence the required estimate holds for all $c_0 < c$.

We are now in a position to evaluate $S^*(x; z, \nu\vartheta)$ when $q(x;\vartheta)$ is small.

Lemma 7. *Let $\vartheta \in \mathbb{R} \smallsetminus \mathbb{Q}$, and $0 < \delta < 1$. For all real numbers $x \geqslant 3$ such that $q(x;\vartheta) \leqslant (\log x)^{10}$ and uniformly for $0 \leqslant z \leqslant 1$, $1 \leqslant \nu \leqslant \log 2q(x;\vartheta)$, we have*

$$S^*(x; z, \nu\vartheta) \ll x(\log x)^{2z-1}\Big\{(\log x)^{-\delta z} + q(x;\vartheta)^{-\delta}\Big\}. \tag{4·20}$$

Proof. As previously, we start with Dirichlet's theorem which implies that $q = q(x,\vartheta) \leqslant Q(x) = x/(\log x)^{10}$. Hence we have $|\vartheta - a/q| \leqslant 1/qQ(x)$ with $(a,q) = 1$. For $1 \leqslant \nu \leqslant \log 2q$, we write $a_\nu := a\nu/(q,\nu)$, $q_\nu := q/(q,\nu)$. Putting $Q := x/(\log x)^{11}$, we obtain that

$$0 < |\nu\vartheta - a_\nu/q_\nu| \leqslant 1/q_\nu Q, \quad (a_\nu, q_\nu) = 1, \quad q/\log 2q \leqslant q_\nu \leqslant (\log x)^{10}. \tag{4·21}$$

In the sequel, we write $\beta_\nu := \nu\vartheta - a_\nu/q_\nu$.

Let $\eta := \frac{1}{2}(1-\delta)$, $x_2 := \exp\,(\log x)^\eta$. We plainly have

$$\begin{aligned} S^*(x; z, \nu\vartheta) = \sum_{k \leqslant x/x_2} z^{\Omega(k)} \Big| \int_{x_2}^{x/k} \mathrm{d}T^*(u; z, k\nu\vartheta) + T^*(x_2; z, k\nu\vartheta) \Big| \\ + \sum_{x/x_2 < k \leqslant x} z^{\Omega(k)} \big| T^*(x/k; z, k\nu\vartheta) \big|. \end{aligned}$$

Using the trivial bounds

$$T^*(x_2;z,k\nu\vartheta)\ll x_2(\log x_2)^{z-1}\quad\text{and}\quad T^*(x/k;z,k\nu\vartheta)\ll (x/k)(\log(x/k))^{z-1}$$

for $x/x_2<k\leqslant x$, and noting that $\mathrm{d}T^*(u;z,k\nu\vartheta)=e(k\beta_\nu u)\,\mathrm{d}T^*(u;z,ka_\nu/q_\nu)$, we arrive at

$$\begin{aligned}S^*(x;z,\nu\vartheta)\ll & x(\log x)^{(1+\eta)z-1}\\ &+\sum_{k\leqslant x/x_2} z^{\Omega(k)}\Big|\int_{x_2}^{x/k} e(k\beta_\nu u)\,\mathrm{d}T^*(u;z,ka_\nu/q_\nu)\Big|.\end{aligned}\tag{4·22}$$

For each $k\leqslant x/x_2$, we put $q_\nu(k):=q_\nu/(q_\nu,k)$, $a_\nu(k):=ka_\nu/(q_\nu,k)$. Then $T^*(u;z,ka_\nu/q_\nu)=T^*(u;z,;a_\nu(k)/q_\nu(k))$ and we may apply Lemma 6 with $A=10/\eta$ to write, whenever $x_2\leqslant u\leqslant x$,

$$T^*(u;z,ka_\nu/q_\nu)=M(u)+R(u)$$

with

$$M(u):=\sum_{t|q_\nu(k)}\frac{\mu(t)z^{\Omega(q_\nu(k)/t)}}{\varphi(t)}H(tu/q_\nu(k);z;t),\ \ R(u)\ll u\exp\{-c_0(\log x)^{\eta/3}\}.$$

The contribution of R to the integral in (4·22) may be estimated by partial summation. We have

$$\int_{x_2}^{x/k} e(k\beta_\nu u)\,\mathrm{d}R(u)\ll\frac{x}{k}\exp\{-c_0(\log x)^{\eta/3}\}(1+|\beta_\nu x|)\ll\frac{x}{k}\exp\{-c_1(\log x)^{\eta/3}\}$$

for a suitable positive constant c_1, since $|\beta_\nu|x\leqslant(\log x)^{11}$ by (4·21). Thus the total contribution of the remainder term R to the right-hand side of (4·22) is

$$\ll x\exp\{-c_1(\log x)^{\eta/3}\}\sum_{k\leqslant x/x_2} z^{\Omega(k)}/k\ll x/\log x.$$

We estimate the contribution of the main term $M(u)$ to the integral of (4·22) by considering $M(u)$ as a double summation and bounding all the summands in absolute value. Moreover, we may also delete in this process the coprimality conditions appearing in the H-functions. In other words, we use the inequality between Stieltjes measures

$$|dM(u)|\leqslant\sum_{t|q_\nu(k)}\frac{1}{\varphi(t)}\,\mathrm{d}H(tu/q_\nu(k);z,1).\tag{4·23}$$

Therefore we obtain

$$\begin{aligned}\Big|\int_{x_2}^{x/k} e(k\beta_\nu u)\,\mathrm{d}M(u)\Big|&\leqslant\sum_{t|q_\nu(k)}\frac{1}{\varphi(t)}\int_0^{x/k}\mathrm{d}H(tu/q_\nu(k);z,1)\\&=\sum_{t|q_\nu(k)}\frac{1}{\varphi(t)}\sum_{n\leqslant tx/kq_\nu(k)} z^{\Omega(n)}\\&\ll\sum_{t|q_\nu(k)}\frac{1}{\varphi(t)}\frac{tx}{kq_\nu(k)}\Big(\log\frac{x}{k}\Big)^{z-1}\\&\ll\frac{\tau(q)(\log 2q)^2}{q}(k,q)\frac{x}{k}\Big(\log\frac{x}{k}\Big)^{z-1},\end{aligned}$$

by (4·9) and (4·21). In the last stage, we have used the bound $t/\varphi(t) \leqslant q/\varphi(q) \ll \log 2q$ for all $t|q_\nu(k)$. Since $\tau(q)(\log 2q)^2 \ll q^\eta$, we see that the total contribution of the main term $M(u)$ to the right-hand side of (4·22) is

$$\begin{aligned}
&\ll xq^{-1+\eta} \sum_{k\leqslant x/x_2} (k,q)\frac{z^{\Omega(k)}}{k}\Big(\log\frac{x}{k}\Big)^{z-1}\\
&\ll xq^{-1+\eta} \sum_{d|q} z^{\Omega(d)} \sum_{\ell\leqslant x/x_2 d} \frac{z^{\Omega(\ell)}}{\ell}\Big(\log\frac{x}{\ell d}\Big)^{z-1}\\
&\ll xq^{-1+\eta}\tau(q)(\log x)^{2z-1} \ll q^{-\delta}x(\log x)^{2z-1}.
\end{aligned}$$

Inserting this into (4·22), we obtain the required estimate and this finishes the proof of the lemma.

Completion of the proof of Theorem 12. We want to apply (4·6) and hence need an upper bound for $S(x;\tfrac14 y,\nu\vartheta)$. We select $T=\log 2q^*(x;y,\vartheta)$. Since $q^*(x;y,\vartheta)\leqslant q(x;\vartheta)$ and $q^*(x;y,\vartheta)\ll(\log x)^{y/4}$, we infer from Lemmas 4 and 7 that we have uniformly for $1\leqslant\nu\leqslant T$, $1\leqslant y\leqslant 4$,

$$S^*(x;\tfrac14 y,\nu\vartheta) \ll x(\log x)^{y/2-1}q^*(x;y,\vartheta)^{-\delta}. \tag{4·24}$$

Now

$$\begin{aligned}
S(x;\tfrac14 y,\nu\vartheta) &= \sum_{k\leqslant x}\frac{(\frac14 y)^{\Omega(k)}}{k}\Big|\int_{1-}^{x/k}\frac1u\,\mathrm{d}T^*(u;\tfrac14 y,k\nu\vartheta)\Big|\\
&= \sum_{k\leqslant x}\frac{(\frac14 y)^{\Omega(k)}}{k}\Big|\frac{k}{x}T^*(x/k;\tfrac14 y,k\nu\vartheta)+\int_1^{x/k}\frac1{u^2}T^*(u;\tfrac14 y,k\nu\vartheta)\,\mathrm{d}u\Big|\\
&= \sum_{k\leqslant x}(\tfrac14 y)^{\Omega(k)}\Big|\frac1x T^*(x/k;\tfrac14 y,k\nu\vartheta)+\int_1^{x}\frac1{u^2}T^*(u/k;\tfrac14 y,k\nu\vartheta)\,\mathrm{d}u\Big|\\
&\leqslant \frac1x S^*(x;\tfrac14 y,\nu\vartheta)+\int_1^x\frac1{u^2}S^*(u;\tfrac14 y,\nu\vartheta)\,\mathrm{d}u.
\end{aligned}$$

By our monotonicity assumptions on $q^*(x;y,\vartheta)$ and (4·24), we have for $1\leqslant u\leqslant x$

$$S^*(u;\tfrac14 y,\nu\vartheta)\ll u(\log 2u)^{(2-\delta)y/4-1}\big\{(\log x)^{y/4}/q^*(x;y,\vartheta)\big\}^\delta.$$

Inserting this in the previous bound, we obtain

$$S(x;\tfrac14 y,\nu\vartheta)\ll(\log x)^{y/2}/q^*(x;y,\vartheta)^\delta. \tag{4·25}$$

It follows that, with the value of T given above, we may take in (4·6), for all ν with $1\leqslant\nu\leqslant T$,

$$\varepsilon_\nu^+(x,y;h_\vartheta):=q^*(x;y,\vartheta)^{-\delta}. \tag{4·26}$$

This is clearly a non-increasing function of x and $\varepsilon_\nu^+(x,y;h_\vartheta)(\log x)^{y/2}$ is plainly non-decreasing. Therefore we obtain

$$\sum_{n\leqslant x}\Big(\frac y4\Big)^{\Omega(n)}\frac{\Delta(n;h_\vartheta)^2}{n}\ll(\log x)^y q^*(x;y,\vartheta)^{-\delta}\{\log 2q^*(x;y,\vartheta)\}^2.$$

Altering the value of δ, the factor $\{\log 2q^*(x;y,\vartheta)\}^2$ may be deleted. This yields (4·4) and finishes the proof of Theorem 12.

5. Additive functions

As we noted in the first section, the study of uniform distribution on divisors of additive functions is made much easier by the fact that the Weyl sums are multiplicative. Indeed this is the only case when we are able to achieve estimates for the discrepancy which go beyond the statistical bound $\sqrt{\tau(n)}$.

We shall prove the following theorem, which provides a simple, but nevertheless effective criterion. We write for integer ν

(5·1) $$L_\nu(x;f) := \sum_{p\leqslant x} \frac{\|\nu f(p)\|^2}{p}, \qquad C_\nu(x;f) = \sum_{p\leqslant x} \frac{1-|\cos \nu\pi f(p)|}{p},$$

and note that we have for all x

(5·2) $$4L_\nu(x;f) \leqslant C_\nu(x;f) \leqslant \tfrac{1}{2}\pi^2 L_\nu(x;f).$$

Theorem 13. *Let f be an additive function. Then f is* erd *if, and only if,*

(5·3) $$\sum_p \frac{\|\nu f(p)\|^2}{p} = \infty \qquad (\nu \neq 0).$$

Moreover, if this is the case then we have

(5·4) $$\sum_{n\leqslant x} \frac{\Delta(n;f)}{n}\Big(\frac{y}{2}\Big)^{\Omega(n)} \ll (\log x)^y \Big\{\frac{1}{T} + \sum_{1\leqslant\nu\leqslant T} \frac{e^{-yC_\nu(x;f)}}{\nu}\Big\}$$

uniformly for $x \geqslant 2$, $T \geqslant 1$, $0 \leqslant y \leqslant y_0 < 4$.

The qualitative result corresponding to this statement (i.e. the criterion (5·3)) was first established by Kátai [18].

By (5·3) and the lower bound of (5·2), the expression between curly brackets tends to zero for suitable $T = T(x,y)$. Hence (5·4) provides, as described in Theorem 6, a family of effective ppℓ estimates for $\Delta(n;f)$ which may of course be further optimised with respect to y. This process must lead to a non trivial result since $y = 1$ is admissible.

We give below two applications of Theorem 13, respectively devoted to the functions $f(d) = \vartheta\Omega(d)$ for irrational ϑ, and $f(d) = \log d$. Recovering a result first obtained independently by Hall and Kátai, the latter case furnishes the best known ppℓ upper bound for the discrepancy for any function we are aware of, although this falls short of the current conjecture stated in section 2. The former case provides an example of a function which is erd but has the same rate of growth than, and indeed is asymptotically equal to, a function which is not, namely $\vartheta \log_2 n$ — see Corollary 3. Actually, if we define $\Omega(n;E)$ as the number of those prime factors of n which belong to some set of primes E, then Theorem 13 implies that $\vartheta\Omega(n;E)$ is erd for all $\vartheta \in \mathbb{R} \smallsetminus \mathbb{Q}$ if, and only if, $\sum_{p\in E} 1/p = \infty$. We thus exhibit erd additive functions with an arbitrary slow growth.

The effective bounds for $\Delta(n;\vartheta\Omega)$ naturally depend on the rational approximations to ϑ. We set $Q_1(x) := \sqrt{(\log_2 x)}/\sqrt{(\log_3 x)}$, and define

(5·5) $$q_1(x;\vartheta) := \inf_{t\geqslant x} \max\{q \leqslant Q_1(t) : \|\vartheta q\| \leqslant 1/Q_1(t)\}.$$

Then $q_1(x;\vartheta)$ increases to ∞ for all irrational ϑ and $q_1(x;\vartheta) = (\log_2 x)^{1/2+o(1)}$ for almost all ϑ.

Corollary 10. *Let $\vartheta\in\mathbb{R}\smallsetminus\mathbb{Q}$. Then the function $\vartheta\Omega(n)$ is* erd *and we have*

$$\Delta(n;\vartheta\Omega)<\tau(n)q_1(n;\vartheta)^{-1+o(1)}\qquad \text{pp}\ell. \tag{5·6}$$

Corollary 11 (Hall [8] ; Kátai [17]). *Let $\alpha>\big(\log(4/\pi)\big)/\log 2\approx 0{\cdot}34850$. We have*

$$\Delta(n;\log)<\tau(n)^\alpha\qquad \text{pp}\ell. \tag{5·7}$$

We now embark on the proof of Theorem 13. That the condition is necessary is a straightforward consequence of the definition of uniform distribution on divisors in the form (1·1). Indeed, suppose that (5·3) fails to hold for $\nu\neq 0$. Writing $F(z;n):=\sum_{d\mid n,\,\langle f(d)\rangle\leqslant z}1$, we have

$$\begin{aligned} g_\nu(n) &:=\sum_{d\mid n}e\big(\nu f(n)\big)=\int_0^1 e(\nu z)\,\mathrm{d}F(z;n)=\int_0^1 e(\nu z)\,\mathrm{d}\big(F(z;n)-z\tau(n)\big)\\ &=2\pi i\nu\int_0^1 e(\nu z)\big(F(z;n)-z\tau(n)\big)\,\mathrm{d}z, \end{aligned}$$

hence

$$2\pi|\nu|\Delta(n;f)\geqslant|g_\nu(n)|. \tag{5·8}$$

Now for all $\varepsilon\in]0,1[$, we have

$$\begin{aligned} \sum_{n\leqslant x}\frac{\mu(n)^2|g_\nu(n)|}{n\tau(n)} &\geqslant \sum_{P^+(n)\leqslant x^\varepsilon}\frac{\mu(n)^2|g_\nu(n)|}{n\tau(n)}-\sum_{\substack{n>x\\ P^+(n)\leqslant x^\varepsilon}}\frac1n\\ &=\prod_{p\leqslant x^\varepsilon}\Big(1+\frac{|\cos\pi\nu f(p)|}{p}\Big)-O\big(e^{-1/2\varepsilon}\log x\big), \end{aligned} \tag{5·9}$$

where the O-estimate follows from (4·19) by partial summation. The product is

$$\gg\exp\Big\{\sum_{p\leqslant x^\varepsilon}\frac{|\cos\pi\nu f(p)|}{p}\Big\}\geqslant\exp\Big\{\sum_{p\leqslant x^\varepsilon}\frac{1-\frac12\pi^2\|\nu f(p)\|^2}{p}\Big\}\gg\varepsilon\log x,$$

by our assumption that the series (5·3) converges. Inserting this into (5·9) and choosing ε small enough but fixed, we obtain that the left-hand side is $\gg\log x$, whence by (5·8)

$$\sum_{n\leqslant x}\frac{\Delta(n;f)}{n\tau(n)}\gg\log x.$$

This cannot hold if $\Delta(n;f)=o\big(\tau(n)\big)$ pp (or even ppℓ), and we obtain the required necessity assertion.

The sufficiency part of the theorem readily follows from the upper bound (1·21) which we recall for convenience : we have uniformly for $x\geqslant 2$, $T\geqslant 1$, $0\leqslant y\leqslant y_0<4$,

$$\sum_{n\leqslant x}\frac{\Delta(n;f)}{n}\Big(\frac y2\Big)^{\Omega(n)}\ll\frac{(\log x)^y}{T}+\sum_{1\leqslant\nu\leqslant T}\frac1\nu\sum_{n\leqslant x}\frac{|g_\nu(n)|}{n}\Big(\frac y2\Big)^{\Omega(n)}. \tag{5·10}$$

The inner n-sum on the right does not exceed

$$\sum_{P^+(n)\leqslant x}\frac{|g_\nu(n)|}{n}\Big(\frac{y}{2}\Big)^{\Omega(n)}\ll\exp\Big\{\sum_{p\leqslant x}\frac{y|\cos\pi\nu f(p)|}{p}\Big\}\ll(\log x)^y e^{-yC_\nu(x;f)}.$$

Inserting this into (5·10) yields (5·4). Taking $y=1$ implies $\Delta(n;f)=o\big(2^{\Omega(n)}\big)$ ppℓ, from which we deduce in turn that $\Delta(n;f)=o\big(\tau(n)\big)$ ppℓ by a familiar argument. Corollary 1 hence implies that f is erd, and this finishes the proof of Theorem 13.

Proof of Corollary 10. We apply the upper bound (5·4) of Theorem 13 with $y=1$. We have

$$C_\nu(x;\vartheta\Omega)=(1-|\cos\pi\nu\vartheta|)\log_2 x+O(1)$$

and need a lower bound for this. For given x, there exist integers a, q with $(a,q)=1$, $q\leqslant Q_1(x)=\surd(\log_2 x)/\surd(\log_3 x)$, and $|\vartheta-a/q|\leqslant 1/qQ_1(x)$. Furthermore, we have

$$q\geqslant q_1(x;\vartheta), \tag{5·11}$$

where the right-hand side is defined by (5·5). We select $T:=q/\log q$, and note that for $\nu\leqslant T$ we have $|\vartheta\nu-a_\nu/q_\nu|\leqslant T/qQ_1(x)\leqslant 1/q\log q$, with $a_\nu=a\nu/(q,\nu)$, $q_\nu=q/(q,\nu)$. This implies $\|\vartheta\nu\|\geqslant(1/q)-1/q\log q$, hence, for large x,

$$1-|\cos\pi\nu\vartheta|\geqslant(\tfrac12\pi^2+o(1))/q^2>(\log_3 x)/\log_2 x.$$

Inserting this into (5·4), we arrive at

$$(\log x)^{-1}\sum_{n\leqslant x}\frac{\Delta(n;\vartheta\Omega)}{n2^{\Omega(n)}}\ll\frac{1}{T}+\frac{\log q}{\log_2 x}\ll q_1(x;\vartheta)^{-1+o(1)}.$$

Since $q_1(x;\vartheta)$ is a non-decreasing function of x, this implies (5·6) and the proof is thereby completed.

Proof of Corollary 11. Put $\tau(n,\vartheta):=\sum_{d|n}d^{i\vartheta}$. When $f=\log$, we have that $g_\nu(n)=\tau(n,2\pi\nu)$. By lemma 30.2 of [14] we infer that, uniformly for $1\leqslant|\vartheta|\leqslant\exp\sqrt{\log x}$,

$$\sum_{p\leqslant x}\frac{|\tau(p,\vartheta)|}{p}=\sum_{p\leqslant x}\frac{|\cos(\frac12\vartheta\log p)|}{p}=\frac{2}{\pi}\log x+O(1).$$

This is proved by partial summation from a strong form of the prime number theorem. Thus we obtain that we have uniformly for $1\leqslant\nu\leqslant\log x$

$$C_\nu(x;\log)=(1-2/\pi)\log_2 x+O(1).$$

Inserting this into (5·4) with $T=\log x$ and choosing optimally $y=\pi/2$, we obtain

$$\Delta(n;f)<\xi(n)(\log_2 n)\Big(\frac{4}{\pi}\Big)^{\Omega(n)}\qquad\text{pp}\ell$$

for all $\xi(n)\to\infty$. This implies the required result by a now standard device.

6. Metric results

In this last section, we investigate the problem of uniform distribution on divisors from a further statistical point of view, regarding as random not only the integers n but also the function f. Thus, we define a measure $\boldsymbol{\mu}$ on the set $\mathbb{A}$ of all real valued arithmetical function as the inverse image of the Haar measure on the compact group $(\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ by the canonical mapping $f \mapsto \langle f\rangle$. In other words, $\boldsymbol{\mu}$ is characterised by the property that for all finite families $\{E_j : 1 \leqslant j \leqslant k\}$ of measurable subsets of the torus $\mathbb{R}/\mathbb{Z}$ and for all integers $n_1, n_2, \ldots, n_k$, we have

$$\boldsymbol{\mu}\{f \in \mathbb{A} : \langle f(n_j)\rangle \in E_j \ (1 \leqslant j \leqslant k)\} = \prod_{j=1}^{k} \lambda(E_j),$$

where λ stands for the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. The only basic property of $\boldsymbol{\mu}$ that we shall use is the orthogonality relation

$$\int_{\mathbb{A}} e\big(\nu f(n) - \nu f(m)\big)\,\mathrm{d}\boldsymbol{\mu}(f) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{6·1}$$

If an arithmetic function behaves statistically, then one expects that the Weyl sums

$$g_\nu(n) := \sum_{d|n} e\big(\nu f(d)\big),$$

and hence the discrepancy $\Delta(n; f)$, will normally have size roughly $\sqrt{\tau(n)}$. The purpose of the next theorem is to establish that this is indeed the case.

Theorem 14. *Let $\xi(n) \to \infty$. For $\boldsymbol{\mu}$-almost all arithmetic functions f, we have*

$$\Delta(n; f) < \xi(n)(\log_2 n)^3\big(\tau(n)\big)^{1/2} \qquad \mathrm{pp}\ell. \tag{6·2}$$

Moreover, the exponent $\frac{1}{2}$ is sharp in this statement.
Proof. The upper bound follows from (1·24) with $y = 2$, $T = (\log x)^2$, namely

$$\begin{aligned} S(x; f) &:= \frac{1}{\log x} \sum_{n \leqslant x} \frac{\Delta(n; f)^2}{2^{\Omega(n)} n} \\ &\ll \frac{1}{(\log x)^3} + \frac{\log_2 x}{\sqrt{(\log x)}} \sum_{1 \leqslant \nu \leqslant T} \frac{1}{\nu} H_\nu(x; f), \end{aligned} \tag{6·3}$$

with

$$H_\nu(x; f) := \sum_{k=1}^{\infty} \Big(\frac{1}{2}\Big)^{\Omega(k)} \frac{1}{k^{1+\sigma}} \bigg| \sum_{m=1}^{\infty} \frac{e\big(\nu f(km)\big)}{m^{1+\sigma} 2^{\Omega(m)}} \bigg|^2 \quad (\sigma := 1/\log x).$$

We have $H_\nu(x; f) \leqslant 2H_\nu^\dagger(x; f) + 2H_\nu^\ddagger(x; f)$ where the m-sum is restricted to $m \leqslant (\log x)^3$ in $H_\nu^\dagger$ and to $m > (\log x)^3$ in $H_\nu^\ddagger$. We note right away the trivial estimate

$$H_\nu^\dagger(x; f) \ll \sqrt{(\log x)} \log_2 x,$$

which follows from (1·6) by partial summation. We deduce from this and (6·3) that

$$S(x; f) \ll (\log_2 x)^3 + R(x; f), \tag{6·4}$$

with

$$R(x;f) := \frac{\log_2 x}{\surd(\log x)} \sum_{1\leqslant \nu\leqslant T} \frac{1}{\nu} H_\nu^\ddagger(x;f).$$

Expanding the square and integrating over f with respect to $\boldsymbol{\mu}$ we get from (6·1) that

$$\begin{aligned}\int_{\mathbb{A}} H_\nu^\ddagger(x;f)\,\mathrm{d}\boldsymbol{\mu}(f) &= \sum_{k=1}^{\infty} \frac{1}{2^{\Omega(k)}k^{1+\sigma}} \sum_{m>(\log x)^3} \frac{1}{4^{\Omega(m)}m^{2+2\sigma}} \\ &\ll 1/(\log x)^2.\end{aligned}$$

Hence

$$\int_{\mathbb{A}} R(x;f)\,\mathrm{d}\boldsymbol{\mu}(f) \ll \frac{(\log_2 x)^2}{(\log x)^{5/2}}$$

Markov's inequality thus implies that

$$\boldsymbol{\mu}\{f\in\mathbb{A} : R(2^\ell;f)\geqslant 1\} \ll 1/\ell^2 \qquad (\ell=1,2,\ldots),$$

so it follows by the Borel-Cantelli theorem, that for $\boldsymbol{\mu}$-almost all f we have

$$R(2^\ell;f)\ll 1 \qquad (\ell=1,2,\ldots).$$

In view of (6·4), we see that the estimate $S(2^\ell;f)\ll(\log 2\ell)^3$ holds $\boldsymbol{\mu}$-almost surely in f and uniformly for $\ell\geqslant 1$. However, using the trivial bound $\Delta(n;f)\leqslant\tau(n)$, we readily see that

$$S(x;f)-S(2^\ell;f)\ll 1 \qquad (2^\ell\leqslant x<2^{\ell+1}).$$

This yields that for $\boldsymbol{\mu}$-almost all f and uniformly in $x\geqslant 3$, we have

$$S(x;f)\ll(\log_2 x)^3,$$

which in turn implies (6·2).

To show that the exponent $\frac{1}{2}$ is sharp, we simply use (5·8) with $\nu=1$ in the form

$$4\pi^2\int_{\mathbb{A}}\Delta(n;f)^2\,\mathrm{d}\boldsymbol{\mu}(f)\geqslant\int_{\mathbb{A}}|g_1(n)|^2\,\mathrm{d}\boldsymbol{\mu}(f)=\tau(n),$$

where the equality follows from (6·1). This plainly implies that there is no $\alpha<\frac{1}{2}$ such that $\Delta(n;f)\ll\tau(n)^\alpha$ ppℓ for $\boldsymbol{\mu}$-almost all f : such a bound is actually false as soon as $\tau(n)$ is large enough.

The same quadratic mean approach that we used for Theorem 14 yields metric results for more restricted classes of arithmetic functions. We quote without proof the following theorem.

Theorem 15. *The function $d\mapsto\vartheta^d$ is* erd *for almost all $\vartheta>1$ and the function $d\mapsto\lambda\vartheta^d$ is* erd *for all $\vartheta>1$ and almost all λ. More precisely, the corresponding discrepancies satisfy*

$$\Delta(n;f)\ll\tau(n)^{1/2+o(1)} \qquad \text{pp}\ell, \tag{6·5}$$

under the indicated hypotheses, and the exponent $1/2$ is sharp.

Theorems 14 & 15 together provide an optimal strengthening of theorem 5 of Dupain, Hall & Tenenbaum [4].

References

[1] Davenport, H., *Multiplicative number theory*, second edition, Springer, New York, Heidelberg, Berlin, 1980.
[2] Davenport, H., and P. Erdős, On sequences of positive integers, *Acta Arith.* **2** (1937), 147–151.
[3] Davenport, H., and P. Erdős, On sequences of positive integers, *J. Indian Math. Soc.* **15** (1951), 19–24.
[4] Dupain, Y., R.R. Hall and G. Tenenbaum, Sur l'équirépartition modulo 1 de certaines fonctions de diviseurs, *J. London Math. Soc.* (2) **26**, (1982), 397–411.
[5] Erdős, P., Some unconventional problems in number theory, *Astérisque* **61** (1979), 73–82.
[6] Erdős, P., R.R Hall and G. Tenenbaum, On the densities of sets of multiples, *J. reine angew. Math.* **454** (1994), 119–141.
[7] Graham, S. W., and G. Kolesnik, *Van der Corput's method of exponential sums*, London Math. Soc. Lecture Notes 126, Cambridge University Press, 1991.
[8] Hall, R.R., Sums of imaginary powers of the divisors of integers, *J. London math. Soc.* (2) **9** (1975), 571–580.
[9] Hall, R.R., The distribution of $f(d)$ (mod 1), *Acta Arith.* **31** (1976), 91–97.
[10] Hall, R.R., A new definition of the density of an integer sequence, *J. Austral. Math. Soc.* Ser. A **26** (1978), 487–500.
[11] Hall, R.R., The divisor density of integer sequences, *J. London Math. Soc.* (2) **24** (1981), 41–53.
[12] Hall, R.R., Sets of multiples and Behrend sequences, in : *A tribute to Paul Erdős* (editors A. Baker, B. Bollobás, A. Hajnal), Cambridge University Press, 1990, pp. 249–258.
[13] Hall, R.R., and G. Tenenbaum, Les ensembles de multiples et la densité divisorielle, *J. Number Theory* **22** (1986), 308–333.
[14] Hall, R.R., and G. Tenenbaum, *Divisors*, Cambridge University Press, 1988.
[15] Hall, R.R., and G. Tenenbaum, On Behrend sequences, *Math. Proc. Camb. Phil. Soc.* **112** (1992), 467–482.
[16] Karatsuba, A.A., Estimates for trigonometric sums by Vinogradov's method and some applications, *Proc. Steklov Inst. Math.* **112** (1971), 251–265.
[17] Kátai, I., The distribution of divisors (mod 1), *Acta Math. Acad. Sci. Hungar.* **27** (1976), 149–152.
[18] Kátai, I., Distribution mod 1 of additive functions on the set of divisors, *Acta Arith.* **30** (1976), 209–212.
[19] Kuipers, L., and H. Niederreiter, *Uniform distribution of sequences*, Wiley, New York, 1976.
[20] Rauzy, G., *Propriétés statistiques de suites arithmétiques,* Presses universitaires de France, 1976.
[21] Rieger, G.J., Zum Teilerproblem von Atle Selberg, *Math. Nachr.* **30** (1965), 181–192.
[22] Ruzsa, I., and G. Tenenbaum, A note on Behrend sequences, *Acta Math. Hung.*, to appear.
[23] Tenenbaum, G., Sur la densité divisorielle d'une suite d'entiers, *J. Number Theory* **15**, no. 3 (1982), 331–346.
[24] Tenenbaum, G., On block Behrend sequences, *Math. Proc. Camb. Phil. Soc.*, to appear.
[25] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge studies in advanced mathematics, no. 46, Cambridge University Press, 1995.
[26] E.C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford University Press, 1986, second edition, revised by D.R. Heath-Brown.

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