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Short sums of certain arithmetic functions

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1. Introduction

Let $\mathcal{M}$ denote the class of all non-negative multiplicative functions g with the property that

(i) $\exists\, A : \ g(p^\nu) \leqslant A^\nu \qquad (\nu \in \mathbb{N},\, p \text{ prime}),$

(ii) $\forall \varepsilon > 0 \ \exists\, B = B(\varepsilon) > 0 : \ g(n) \leqslant Bn^\varepsilon \qquad (n \in \mathbb{N}).$

In 1980, Shiu [7] obtained a general upper bound for short sums of functions $g \in \mathcal{M}$: *Let $\alpha, \beta \in]0,1[$, and let x, y satisfy $x \geqslant y \geqslant x^\alpha$. Then for positive integers a, q with $(a,q) = 1$ we have*

$$\sum_{\substack{x<n\leqslant x+y \\ n\equiv a \,(\mathrm{mod}\, q)}} g(n) \ll \frac{y}{\varphi(q)\log x} \exp\Big\{ \sum_{p\leqslant x,\, p\nmid q} \frac{g(p)}{p} \Big\}$$

uniformly for $1 \leqslant q \leqslant y^\beta$.

This result has turned out to be very useful in a wide range of applications. A closer inspection of its proof reveals, in the case $q = 1$, that :

(a) g needs only be sub-multiplicative, i.e. $g(mn) \leqslant g(m)g(n)$ for $(m,n) = 1$ with $g(1) = 1$;

(b) the constant implicit in the $\ll$ sign depends only on A, B and α ;

(c) given α, condition (ii) above need only hold for a particular $\varepsilon = \varepsilon(\alpha)$.

Shiu's result has been generalised by Nair [5] to sub-multiplicative functions of polynomial values in a short interval.

In this paper, we weaken the property of sub-multiplicativity significantly to appreciably widen the range of application of such a result. Consider, for any fixed $k \in \mathbb{N}$, the class $\mathcal{M}_k(A,B,\varepsilon)$ of non-negative arithmetic functions $F(n_1,\ldots,n_k)$ such that

$$(1) \qquad F(m_1n_1,\ldots,m_kn_k) \leqslant \min(A^{\Omega(m)}, Bm^\varepsilon)F(n_1,\ldots,n_k) \qquad (m := m_1\cdots m_k)$$

for all k-tuples $(m_1,\ldots,m_k)$, $(n_1,\ldots,n_k)$ with $(m_j,n_j) = 1$ $(1 \leqslant j \leqslant k)$. Here and in the sequel, $\Omega(m)$ denotes the total number of prime factors of m, counted with multiplicity.

Such functions F need not be multiplicative or even sub-multiplicative. For instance, the Hooley Δ-function (see [3], and chapters 4, 6 and 7 of [2]) defined by

$$\Delta(n) := \max_{u\in\mathbb{R}} \sum_{d|n,\, \mathrm{e}^u<d\leqslant \mathrm{e}^{u+1}} 1$$

satisfies $\Delta(mn) \leqslant \tau(m)\Delta(n)$ for $(m,n) = 1$, where $\tau(m)$ is the total number of divisors of m. Hence $\Delta \in \mathcal{M}_1(2,B,\varepsilon)$ for any $\varepsilon > 0$ and suitable $B = B(\varepsilon)$.

Let $Q_j \in \mathbb{Z}[X]$ $(1 \leqslant j \leqslant k)$ be polynomials such that $Q = \prod_{j=1}^k Q_j$ has no fixed prime divisor. Our main result (Theorem 1 below) is an upper bound of the form

$$\sum_{x<n\leqslant x+y} F\big(|Q_1(n)|,\ldots,|Q_k(n)|\big) \ll y \prod_{p\leqslant x} \Big(1 - \frac{\varrho(p)}{p}\Big) \sum_{n\leqslant x} v(n;F,\boldsymbol{\varrho})$$

uniformly for $x^\alpha \leqslant y \leqslant x$ with x sufficiently large and where ε and α can be arbitrary small positive real numbers satisfying certain conditions. Here $\varrho(m) = \varrho_Q(m)$ denotes the number of roots of Q in $\mathbb{Z}/m\mathbb{Z}$. The function $v(n; F, \boldsymbol{\varrho})$ will be precisely defined in the next section—see (10) and (16)— and is linked to the decomposition of Q into irreducible factors in $\mathbb{Z}[X]$.

In the very special case $k = 1$ and Q irreducible, our bound reads

$$\sum_{x<n\leqslant x+y} F(|Q(n)|) \ll y \prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{n\leqslant x}\frac{F(n)\varrho(n)}{n}. \tag{2}$$

The essential novelty of such an estimate is immediately evident even in this simple situation. For instance, we may apply it with $F(n) = \Delta(n)^t$ $(t > 0)$, the result being that

$$\sum_{x<n\leqslant x+y} \Delta(|Q(n)|)^t \ll_{\varepsilon,t,Q} \frac{y}{\log x}\sum_{n\leqslant x}\frac{\Delta(n)^t\varrho(n)}{n} \qquad (x^\varepsilon \leqslant y \leqslant x).$$

When combined with existing bounds for the sum on the right-hand side [9], this yields

$$\sum_{x<n\leqslant x+y} \Delta(|Q(n)|)^t \ll_{\varepsilon,t,Q} y(\log y)^{2^t-t-1}\mathrm{e}^{\sqrt{\{2t+o(1)\}\log_2 x\log_3 x}} \qquad (x\to\infty). \tag{3}$$

Here and in the remainder of this paper we let $\log_k$ denote the k-fold iterated logarithm.

Let $P^+(n)$ denote the largest prime factor of the integer n, with the convention that $P^+(1) = 1$. By a modification of the argument described in [9], we can further show that (3) leads to the lower bound

$$P^+\Big(\prod_{x<n\leqslant x+y}|Q(n)|\Big) > y\exp\{(\log x)^\alpha\}$$

for any irreducible $Q \in \mathbb{Z}[X]$ of degree exceeding 1, any $\alpha \in]0, 2-\log 4[$ and any $y = x^{1/k}$, with arbitrary fixed $k \in \mathbb{N}$.(1) This seems to be the first result of this kind and it also mirrors the corresponding current best estimate over the long interval $[1, x]$.

A seemingly more trivial application of (3) with $t = 1$ and $Q(X) = X$ is the estimate

$$\max_{D\in\mathbb{R}}\sum_{D<d\leqslant 2D}\Big(\Big[\frac{x+y}{d}\Big]-\Big[\frac{x}{d}\Big]\Big) \ll y\mathrm{e}^{\sqrt{\{2+o(1)\}\log_2 x\log_3 x}} = y(\log x)^{o(1)} \qquad (x^\varepsilon \leqslant y \leqslant x)$$

which is obtained by bounding the expression on the left by $\sum_{x<n\leqslant x+y}\Delta(n)$.

The uniformity with respect to the polynomial Q which we obtain in Theorem 1 enables us to generalise the result to the variable n restricted to an arithmetic progression : this is Corollary 1. This derivation closely follows the corresponding argument in [5].

More involved applications of our main theorem are obtained by considering functions F in many variables. By way of example, let us take $F_1, F_2 \in \mathcal{M}_1(A, B, \varepsilon)$, so that $F(n_1, n_2) = F_1(n_1)F_2(n_2)$ lies in $\mathcal{M}_2(A, B^2, \varepsilon)$. Our theorem yields that

$$\sum_{x<n\leqslant x+y} F_1\big(|Q_1(n)|\big)F_2\big(|Q_2(n)|\big) \ll y\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{n\leqslant x} v(n; F_1F_2, \boldsymbol{\varrho}) \tag{4}$$

1. A slightly more precise statement is given in Theorem 2 below.

where $\varrho = \varrho_Q$ with $Q = Q_1Q_2$. Here the quantity $v(n; F_1F_2, \boldsymbol{\varrho})$ can be made explicit by introducing the decomposition $Q = \prod_{h=1}^{r} R_h^{\gamma_h}$ into products of irreducible factors. Writing $\varrho_h := \varrho_{R_h}$ $(1 \leqslant h \leqslant r)$ and $Q_j = \prod_{h=1}^{r} R_h^{\gamma_{jh}}$ $(j = 1, 2)$, then

$$v(n; F_1F_2, \boldsymbol{\varrho}) = \sum_{\substack{n_1,\dots,n_r \geqslant 1\\ n_1^{\gamma_1}\cdots n_r^{\gamma_r} = n}} F_1\Big(\prod_{h=1}^{r} n_h^{\gamma_{1h}}\Big) F_2\Big(\prod_{h=1}^{r} n_h^{\gamma_{2h}}\Big) \prod_{h=1}^{r} \frac{\varrho_h(n_h)}{n_h}. \tag{5}$$

If, for instance, we choose $F_2(n) = 1$ when $P^+(n) \leqslant z$ and 0 otherwise, we obtain upon simplification and a further application of our Lemma 2 below that, for any $\kappa > 0$,

$$\sum_{\substack{x<n\leqslant x+y\\ P^+(|Q_2(n)|)\leqslant z}} F_1\big(|Q_1(n)|\big) \ll \frac{\mathrm{e}^{-\kappa u} y}{(\log x)^{\widehat{\omega}(Q_1)}} \sum_{n\leqslant x} v(n; F_1, \boldsymbol{\varrho}) \tag{6}$$

where $\widehat{\omega}(Q_1)$ is the number of irreducible factors of Q_1, and $u = (\log x)/\log z$. The special case of (6) with $Q_1(X) = Q_2(X) = X$, $F_1 \equiv 1$ is only slightly weaker than the current best available estimate of Hildebrand [4] for the sum $\sum_{\substack{x<n\leqslant x+y\\ P^+(n)\leqslant z}} 1$ —see the remark in the end of section 7.

It may also be observed from (5) that for $(Q_1, Q_2) = 1$ the bound (4) simplifies to

$$\begin{aligned}&\sum_{x<n\leqslant x+y} F_1\big(|Q_1(n)|\big)F_2\big(|Q_2(n)|\big)\\ &\quad\ll y \prod_{p\leqslant x}\Big(1 - \frac{\varrho_{Q_1}(p)}{p}\Big)\Big(1 - \frac{\varrho_{Q_2}(p)}{p}\Big)\bigg\{\sum_{n\leqslant x}\frac{F_1(n)\varrho_{Q_1}(n)}{n}\sum_{n\leqslant x}\frac{F_2(n)\varrho_{Q_2}(n)}{n}\bigg\}.\end{aligned} \tag{7}$$

Choosing Q_1 with $Q_1(0) \neq 0$, $Q_2(X) = X$ and $F_2(n) = 1$ if $P^-(n) > x$, $F_2(n) = 0$ otherwise, we obtain from (7) that

$$\sum_{x<p\leqslant x+y} F_1\big(|Q_1(p)|\big) \ll \frac{|Q_1(0)|}{\varphi\big(|Q_1(0)|\big)} \frac{y}{\log x} \prod_{p\leqslant x}\Big(1 - \frac{\varrho_{Q_1}(p)}{p}\Big) \sum_{n\leqslant x} \frac{F_1(n)\varrho_{Q_1}(n)}{n}.$$

When applied with $F_1 = \Delta$ and $Q_1(n) = n + a$, $a \neq 0$, this yields in turn

$$\sum_{x<p\leqslant x+y} \Delta(p + a) \ll \frac{|a|}{\varphi(|a|)} \frac{y}{(\log x)^2} \sum_{n\leqslant x} \frac{\Delta(n)}{n}, \tag{8}$$

uniformly for $1 \leqslant |a| \leqslant x$. Let $\pi(x; q, a)$ denote, as usual, the number of prime numbers $p \leqslant x$ with $p \equiv a \,(\mathrm{mod}\, q)$. Since

$$\sum_{K<q\leqslant 2K} \big\{\pi(x + y; q, a) - \pi(x; q, a)\big\} = \sum_{K<q\leqslant 2K} \sum_{\substack{x<p\leqslant x+y\\ q\mid(p-a)}} 1 \leqslant \sum_{x<p\leqslant x+y} \Delta(p - a),$$

we thus derive from (8) the striking bound

$$\max_{K\in\mathbb{R}} \sum_{K<q\leqslant 2K} \big\{\pi(x + y; q, a) - \pi(x; q, a)\big\} \ll \frac{y}{\log x} \mathrm{e}^{\sqrt{\{2+o(1)\}\log_2 x \log_3 x}},$$

valid uniformly in $1 \leqslant |a| \leqslant x$ and $x \geqslant 16$.

Throughout this introduction, we have sacrificed precision in the statement of our results in order to gain a clearer and more immediate presentation of the wide range of applicability of our main theorem. We should however emphasise that every estimate cited in this section is described in complete detail in section 3, with all possible dependencies explicitly mentioned.

2. Notation and definitions

On polynomials. We consider a finite number of polynomials $Q_j \in \mathbb{Z}[X]$ $(1 \leqslant j \leqslant k)$, with $\deg Q_j = g_j$ and put

$$Q = \prod_{j=1}^{k} Q_j = \prod_{h=1}^{r} R_h^{\gamma_h}, \qquad g := \deg Q, \qquad r_h := \deg R_h \quad (1 \leqslant h \leqslant r),$$

where r, k and γ_h $(1 \leqslant h \leqslant r)$ are positive integers and the $R_h \in \mathbb{Z}[X]$ are irreducible over $\mathbb{Q}$. We then write canonically

$$Q_j = \prod_{h=1}^{r} R_h^{\gamma_{jh}} \qquad (1 \leqslant j \leqslant k) \tag{9}$$

where $\gamma_{jh} \geqslant 0$ and note that $g = \sum_{j=1}^{k} g_j = \sum_{h=1}^{r} \gamma_h r_h$. Clearly, we have $\gamma_h = \sum_{j=1}^{k} \gamma_{jh}$ for all j. We also introduce the squarefree kernel of Q

$$Q^* := \prod_{h=1}^{r} R_h.$$

For any polynomial $T \in \mathbb{Z}[X]$, let $\varrho_T(n)$ denote the number of solutions of the congruence $T(m) \equiv 0 \pmod{n}$. We define

$$\varrho := \varrho_Q, \qquad \varrho^* := \varrho_{Q^*}, \qquad \varrho_h := \varrho_{R_h} \quad (1 \leqslant h \leqslant r). \tag{10}$$

Clearly, $\varrho(p) = \varrho^*(p)$ for any prime p. To preserve the uniformity of our results, we shall make the assumption that Q has no fixed prime factor, i.e. that

$$\varrho(p) < p \quad \text{for all primes } p.$$

We shall assume, implicitly, several familiar properties of the ϱ-function. See e.g. [5], p. 258 for a list of such properties.

We let D^* denote the (non-zero) discriminant of Q^*, and put $\overline{D} := \prod_{p^\nu \| D^*,\, \varrho(p)\neq 0} p^\nu$.

Finally, we write $\|T\| := \max_i |c_i|$ for any $T \in \mathbb{Z}[X]$ with $T(x) = \sum_{i \geqslant 0} c_i x^i$.

On arithmetic functions. $P^+(n)$, $P^-(n)$ denote respectively the greatest and the least prime factor of an integer n, with the convention that $P^+(1) = 1$, $P^-(1) = \infty$.

$\Omega(n)$, $\omega(n)$ denote the number of prime factors of n, counted respectively with or without multiplicity, and we write $\varphi(n)$ for Euler's function.

By $a|b^\infty$ $(a, b \in \mathbb{Z}^+)$ we mean that all prime factors of a divide b.

For any A, $B \geqslant 1$, $\varepsilon > 0$ and $k \in \mathbb{Z}^+$ we let $\mathcal{M}_k(A, B, \varepsilon)$ denote the class of non-negative arithmetic functions $F(n_1, \dots, n_k)$ in k variables satisfying (1) whenever $(m_j, n_j) = 1$ $(1 \leqslant j \leqslant k)$. For such F, we may define a minimal function $G = G_F$ by

$$G(n_1, \dots, n_k) := \max_{\substack{m_1 \geqslant 1, \dots, m_k \geqslant 1 \\ (m_j, n_j) = 1\ (1 \leqslant j \leqslant k) \\ F(m_1, \dots, m_k) \neq 0}} F(m_1 n_1, \dots, m_k n_k)/F(m_1, \dots, m_k). \tag{11}$$

We of course have the obvious properties that

$$(m_j, n_j) = 1\ (1 \leqslant j \leqslant k) \Rightarrow F(m_1 n_1, \dots, m_k n_k) \leqslant G(m_1, \dots, m_k) F(n_1, \dots, n_k) \tag{12}$$

and that

$$G(n_1, \ldots, n_k) \leqslant \min(A^{\Omega(n)}, Bn^\varepsilon) \qquad (n_1 \geqslant 1, \ldots, n_k \geqslant 1,\ n = n_1 \cdots n_k). \tag{13}$$

But we also note that G is sub-multiplicative with respect to each variable, that $G \in \mathcal{M}_k(A, B, \varepsilon)$ and that

$$G(n_1, \ldots, n_k) \leqslant \prod_{p^\nu \| n_1 \cdots n_k} \min(A^\nu, Bp^{\nu\varepsilon}). \tag{14}$$

When $k = 1$ we simply write $\mathcal{M}(A, B, \varepsilon)$ for $\mathcal{M}_1(A, B, \varepsilon)$. We also denote by $\mathcal{M}$ the class of functions f which belong to $\mathcal{M}(A, B, \varepsilon)$ for some $A \geqslant 1$ and every $\varepsilon > 0$ with corresponding $B = B(\varepsilon) \geqslant 1$.

Special notation. Because of the frequent interplay between algebraic properties of the polynomial Q and arithmetical properties of the functions F and G, it is convenient and natural to introduce the following notation. Let k, r and γ_{jh} $(1 \leqslant j \leqslant k,\ 1 \leqslant h \leqslant r)$ be defined as in the above sub-section on polynomials. Given r natural numbers $n_1, \ldots, n_r$, we put

$$n'_j := \prod_{h=1}^r n_h^{\gamma_{jh}} \quad (1 \leqslant j \leqslant k), \qquad n'' := \prod_{j=1}^k n'_j = \prod_{h=1}^r n_h^{\gamma_h}.$$

This arises from the fact that if, for some integer n, we have $n_h = R_h(n)$ $(1 \leqslant h \leqslant r)$, then $n'_j = Q_j(n)$ for all j and $n'' = Q(n)$. Observe that if $a_h = b_h c_h$ $(1 \leqslant h \leqslant r)$, then $a'_j = b'_j c'_j$ $(1 \leqslant j \leqslant k)$ and $a'' = b'' c''$. Given any function H of k integral variables, we define an associated function $\widetilde{H}$ of r variables by the formula

$$\widetilde{H}(n_1, \ldots, n_r) := H(n'_1, \ldots, n'_k). \tag{15}$$

Given a k-tuple $\mathbf{Q} := (Q_1, \ldots, Q_k)$ of polynomials satisfying the assumptions described above, an arithmetic function in k variables $F \in \mathcal{M}_k(A, B, \varepsilon)$ and an r-dimensional vector

$$\boldsymbol{\vartheta} := (\vartheta_1, \ldots, \vartheta_r)$$

whose components ϑ_h $(1 \leqslant h \leqslant r)$ are arithmetic functions in one variable, we put

$$v(n; F, \mathbf{Q}, \boldsymbol{\vartheta}) = v(n; F, \boldsymbol{\vartheta}) := \sideset{}{^\dagger}\sum_{n_1^{\gamma_1} \cdots n_r^{\gamma_r} = n} \widetilde{F}(n_1, \ldots, n_r) \frac{\vartheta_1(n_1) \cdots \vartheta_r(n_r)}{n_1 \cdots n_r}, \tag{16}$$

where, here and in the sequel, we let the dagger indicate that an r-fold sum is restricted to pairwise coprime variables which are in turn coprime to D^*.

Note that $v(n; F, \boldsymbol{\vartheta})$ only depends on $\mathbf{Q}$ via D^* and the exponents γ_{jh} occurring in the canonical decompositions of the Q_j as products of irreducible factors.

A useful role is also played by the multiplicative functions $f_h(n) = f_h(n, \varepsilon)$ defined for each $h = 1, \ldots, r$ by

$$f_h(p^\nu) := \begin{cases} A^{\nu\gamma_h} & \text{if } p > A^{1/\varepsilon}, \\ Bp^{\nu\gamma_h\varepsilon} & \text{if } p \leqslant A^{1/\varepsilon}. \end{cases}$$

A property that we shall make use of on more than one occasion is that

$$\widetilde{G}(n_1, \ldots, n_r) \leqslant \prod_{h=1}^r f_h(n_h) \qquad (n_1 \geqslant 1, \ldots, n_r \geqslant 1) \tag{17}$$

where $G = G_F$ is the function defined by (11). This follows immediately from (14) on observing that

$$\min(A^\nu, Bp^{\nu\varepsilon}) \leqslant \prod_{h=1}^{r} f_h(p^{\nu_h}) \qquad (\nu \geqslant 1)$$

for all r-tuples $(\nu_1, \ldots, \nu_r)$ such that $\sum_{h=1}^r \gamma_h \nu_h = \nu$.

We finally observe that if $k = r$ and $\gamma_{jh} = \delta_{jh}$ (with Kronecker's notation), then for any $(n_1, \ldots, n_r) \in \left(\mathbb{Z}^+\right)^k$, we have

$$n'_j = n_j \quad (1 \leqslant j \leqslant k) \qquad \text{and} \qquad n'' = n_1 \cdots n_k.$$

This corresponds to the situation where the Q_j are irreducible over $\mathbb{Q}$ and pairwise coprime.

3. Results

We now state our main theorem, from which all other results in this paper follow in a relatively simple way.

Theorem 1. *Let k be an arbitrary positive integer and let $Q_j \in \mathbb{Z}[X]$ $(1 \leqslant j \leqslant k)$ be such that $Q = \prod_{j=1}^k Q_j$ has no fixed prime divisor. Denote by g the degree of Q, by r the number of irreducible factors of Q and put $\varrho = \varrho_Q$. Then for any $A \geqslant 1$, $B \geqslant 1$, $0 < \varepsilon < 1/8g^2$, $0 < \delta \leqslant 1$, and $F \in \mathcal{M}_k(A, B, \varepsilon\delta/3)$ we have*

$$\sum_{x<n\leqslant x+y} F\big(|Q_1(n)|, \ldots, |Q_k(n)|\big) \ll y \prod_{p\leqslant x} \left(1 - \frac{\varrho(p)}{p}\right) \sum_{n\leqslant x} v(n; F, \boldsymbol{\varrho}) \tag{18}$$

uniformly for $x \geqslant c_0\|Q\|^\delta$ and $x^{4g^2\varepsilon} \leqslant y \leqslant x$. The implicit constant in the $\ll$ sign depends at most on A, B, ε, δ, k, r, g, $\overline{D}$ and the constant c_0 depends at most on A, B, ε, δ, k, r, and g. The terms $\boldsymbol{\varrho} := (\varrho_1, \ldots, \varrho_r)$, $v(n; F, \boldsymbol{\varrho})$ and $\overline{D}$ are as described earlier.

As we remarked in the introduction, the uniformity with respect to the coefficients of the polynomials in Theorem 1 furnishes *ipso facto* the generalisation to arithmetic progressions.

Corollary 1. *Let $A \geqslant 1$, $B \geqslant 1$, $0 < \varepsilon < 1/8g^2$, $0 < \beta < 1$, $0 < \delta \leqslant 1/2g$, and r, k be arbitrary positive integers. Let $F \in \mathcal{M}_k(A, B, \varepsilon\beta\delta/6)$ and $Q_j \in \mathbb{Z}[X]$ $(1 \leqslant j \leqslant k)$ be such that $Q = \prod_{j=1}^k Q_j$ has no fixed prime divisor. Let a, $q \in \mathbb{Z}^+$, with $a \leqslant q$, $(q, Q(a)) = 1$. Then we have*

$$\sum_{\substack{x<n\leqslant x+y \\ n\equiv a \,(\mathrm{mod}\, q)}} F\big(|Q_1(n)|, \ldots, |Q_k(n)|\big) \ll \frac{y}{q} \prod_{\substack{p\leqslant x \\ p\nmid q}} \left(1 - \frac{\varrho(p)}{p}\right) \sum_{\substack{n\leqslant x \\ (n,q)=1}} v(n; F, \boldsymbol{\varrho}) \tag{19}$$

uniformly for $x \geqslant c_1\|Q\|^{2\delta}$, $x^{4g^2\varepsilon} \leqslant y \leqslant x$, $1 \leqslant q \leqslant y^{1-\beta}$ with $\boldsymbol{\varrho} := (\varrho_1, \ldots, \varrho_r)$. The implicit constant in the $\ll$ sign depends at most on A, B, ε, β, δ, k, r, g, $\overline{D}$. The constant c_1 depends at most on A, B, ε, δ, k, r, g.

The two following corollaries provide simplified versions of the upper bounds in Theorem 1 or Corollary 1 when the polynomials Q_j are pairwise coprime.

Corollary 2. *Let the hypotheses of Corollary 1 hold and assume furthermore that the polynomials* Q_j $(1 \leqslant j \leqslant k)$ *are irreducible and pairwise coprime. Then we have*

$$\begin{aligned} &\sum_{\substack{x<n\leqslant x+y\\ n\equiv a\,(\mathrm{mod}\,q)}} F\big(|Q_1(n)|,\dots,|Q_k(n)|\big) \\ &\qquad \ll \frac{y}{q}\prod_{\substack{p\leqslant x\\ p\nmid q}}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{\substack{n_1\cdots n_k\leqslant x\\ (n_i,D^*qn_j)=1\ (1\leqslant i<j\leqslant k)}} F(n_1,\dots,n_k)\prod_{j=1}^k\frac{\varrho_j(n_j)}{n_j} \end{aligned} \tag{20}$$

in the same ranges and under the same uniformity conditions.

Corollary 3. *Let* k *be an arbitrary positive integer and let* $Q_j \in \mathbb{Z}[X]$ $(1 \leqslant j \leqslant k)$ *be pairwise coprime polynomials. Assume that* $Q = \prod_{j=1}^k Q_j$ *has no fixed prime divisor. Let* $g := \deg Q$*. Then for any* $A \geqslant 1$, $B \geqslant 1$, $0 < \varepsilon < 1/8g^2$, $0 < \delta \leqslant 1$, *and* $F_j \in \mathcal{M}(A, B, \varepsilon\delta/3)$ $(1 \leqslant j \leqslant k)$, *we have*

$$\sum_{x<n\leqslant x+y}\prod_{j=1}^k F_j\big(|Q_j(n)|\big) \ll y\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\prod_{j=1}^k\sum_{n\leqslant x}\frac{F_j(n)\varrho_{Q_j}(n)}{n}$$

uniformly for $x \geqslant c_0\|Q\|^\delta$ *and* $x^{4g^2\varepsilon} \leqslant y \leqslant x$*. The dependencies of the various constants are as described in the statement of Theorem 1.*

We next combine Theorem 1 in dimension 1 for $F(n) = \Delta(n)^t$ with the best current estimates for long weighted averages of powers of the Hooley function [9]. Similar results could of course be derived for the generalised Hooley functions $\Delta_r(n)$—see [2], chapters 6 & 7.

Corollary 4. *Let* $Q \in \mathbb{Z}[X]$ *be irreducible with no fixed prime divisor. Then, for any* $t \geqslant 1$ *and* $\varepsilon > 0$, *we have*

$$\sum_{x<n\leqslant x+y}\Delta(|Q(n)|)^t \ll y(\log x)^{\beta(t)-1}\mathcal{L}(\log x)^{\sqrt{2t}+o(1)} \qquad (x\to\infty)$$

provided that $x^\varepsilon \leqslant y \leqslant x$*. Here* $\beta(t) := 2^t - 1$ *and* $\mathcal{L}(z) := \mathrm{e}^{\sqrt{\log z\log_2 z}}$ $(z \geqslant 3)$*.*

The condition that Q has no fixed prime divisor is actually redundant here since we are indifferent in this corollary to the precise nature of the implicit constant in the $\ll$ notation. Also, using Corollary 2 instead of Corollary 3 as well as a messy but straightforward generalisation of Lemma 2.2 of [9], we could derive a corresponding result for any Q, not necessarily irreducible, namely

$$\sum_{x<n\leqslant x+y}\Delta(|Q(n)|)^t \ll y(\log x)^{\gamma_Q(t)}\mathcal{L}(\log x)^{B(t)+o(1)} \qquad (x\to\infty),$$

with $\gamma_Q(t) := \sum_{h=1}^r\{(\gamma_h+1)^t-1\}$ and some suitable constant $B(t)$—of course a variant for arithmetic progressions is also available. We have refrained from proving the more general result since we only need Corollary 4 as stated in our proof of Theorem 2.

Already in the very special case $Q(X) = X$, Corollary 4 implies a curious result which appears to be well beyond the reach of any exponential sums method we are aware of.

Corollary 5. *Let $\varepsilon \in]0,1[$. Then we have, uniformly for $D \geqslant 1$, $x^\varepsilon \leqslant y \leqslant x$,*

$$\sum_{D<d\leqslant 2D} \Big(\Big[\frac{x+y}{d}\Big] - \Big[\frac{x}{d}\Big]\Big) \ll y\,\mathcal{L}(\log x)^{\sqrt{2}+o(1)} \qquad (x\to\infty).$$

This is an immediate consequence of Corollary 4 since the left-hand sum above equals

$$\sum_{D<d\leqslant 2D}\ \sum_{x/d<m\leqslant (x+y)/d} 1 = \sum_{x<n\leqslant x+y}\ \sum_{\substack{d\mid n\\ D<d\leqslant 2D}} 1 \leqslant \sum_{x<n\leqslant x+y} \Delta(n).$$

For our next corollary, we introduce, in the summation conditions of Theorem 1, a supplementary constraint on the largest prime factor of the polynomial values involved. This provides a gain corresponding roughly to the probabilistic expectation.

Corollary 6. *Let the hypotheses of Theorem 1 hold, but with $F \in \mathcal{M}_k(A,B,\varepsilon\delta/4)$ instead of $F \in \mathcal{M}_k(A,B,\varepsilon\delta/3)$. Furthermore, set $\gamma := \sum_{h=1}^r \gamma_h$ and let $\kappa > 0$ be given. Then we have uniformly for $x \geqslant c_0\|Q\|^\delta$, $x^{4g^2\varepsilon} \leqslant y \leqslant x$, and $2^{\kappa/2g^2\varepsilon} \leqslant z \leqslant x$*

$$\sum_{\substack{x<n\leqslant x+y\\ P^+(|Q_k(n)|)\leqslant z}} F\big(|Q_1(n)|,\dots,|Q_k(n)|\big) \ll y\mathrm{e}^{-\kappa u}\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{n\leqslant x} v(n;F,\boldsymbol{\varrho}) \tag{21}$$

where $u := (\log x)/\log z$.

We now insert Corollary 4 into the technique of [9] to obtain a lower bound for the greatest prime factor of polynomial values in certain short intervals.

Theorem 2. *Let $Q \in \mathbb{Z}[X]$ be irreducible and put $P_{x,y} := P^+(\prod_{x<n\leqslant x+y} Q(n))$ for $x \geqslant 1$, $y \geqslant 1$. Let $k \geqslant g$ be an arbitrary positive integer. Then, for any $\alpha < 2 - \log 4$ and $y = x^{g/k}$ we have*

$$P_{x,y} > y\exp\{(\log x)^\alpha\} \qquad (x > x_0(\alpha,Q)).$$

It would of course be desirable to relax the shape condition on y in this result. This would apparently require a completely different approach.

As observed earlier, we can also use the flexibility of the hypotheses in Theorem 1 to restrict the summation variable n to prime values with the expected saving.

Theorem 3. *Let the hypotheses of Theorem 1 hold and assume furthermore that $Q(0) \neq 0$. Then we have*

$$\sum_{x<p\leqslant x+y} F\big(|Q_1(p)|,\dots,|Q_k(p)|\big) \ll \frac{|Q(0)|}{\varphi(|Q(0)|)}\frac{y}{\log x}\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{n\leqslant x} v(n;F,\boldsymbol{\varrho}) \tag{22}$$

uniformly for $x \geqslant c_0\|Q\|^\delta$ and $x^{4g^2\varepsilon} \leqslant y \leqslant x$. The dependencies of the implicit or explicit constants are the same as in Theorem 1.

In the special case $k = 1$, $Q(X) = Q_1(X) = X + a$, $a \neq 0$, we get the following result.

Corollary 7. *Let $A \geqslant 1$, $B \geqslant 1$, $0 < \varepsilon < \frac{1}{8}$, $0 < \delta \leqslant 1$, and $F \in \mathcal{M}(A,B,\varepsilon\delta/3)$. Then we have*

$$\sum_{x<p\leqslant x+y} F(|p+a|) \ll \frac{|a|}{\varphi(|a|)}\frac{y}{(\log x)^2}\sum_{n\leqslant x}\frac{F(n)}{n}$$

uniformly for $x \geqslant c_0|a|^\delta$, $x^{4\varepsilon} \leqslant x \leqslant y$, where the implicit constant in the $\ll$ sign and the constant c_0 depend at most on A, B, ε, δ.

Specialising in the above $F = \Delta$, Hooley's function, and appealing to the necessary weighted average estimate for $\Delta(n)$ ([9], Lemma 2.2), we obtain the following.

Corollary 8. *We have*

$$\sum_{x<p\leqslant x+y}\Delta(|p+a|)\ll\frac{|a|}{\varphi(|a|)}\frac{y}{(\log x)^2}\sum_{n\leqslant x}\frac{\Delta(n)}{n}\ll\frac{y}{\log x}\mathcal{L}(\log x)^{\sqrt{2}+o(1)}$$

uniformly for $1\leqslant|a|\leqslant x$, $x^\varepsilon\leqslant y\leqslant x$.

This estimate averages well over a. We can also use the same procedure as in Corollary 5 to derive an average version of the Brun–Titchmarsh theorem which is, to our knowledge, far beyond the scope of other available techniques. We recall the classical notation

$$\pi(x;q,a)=\sum_{\substack{p\leqslant x\\ p\equiv a\,(\mathrm{mod}\,q)}}1.$$

Corollary 9. *Let* $\varepsilon\in]0,1[$. *Then we have, uniformly for* $K\geqslant 1$, $x^\varepsilon\leqslant y\leqslant x$

$$\max_{K\in\mathbb{R}^+}\sum_{K<q\leqslant 2K}\{\pi(x+y;q,a)-\pi(x;q,a)\}\ll\frac{y}{\log x}\mathcal{L}(\log x)^{\sqrt{2}+o(1)}.$$

We can also take, e.g., $a=-N$ and F equal to the characteristic function of those n with $P^-(n)>x$ in Corollary 7. This provides a Goldbach-type upper bound with one of the primes in a very short interval.

Corollary 10. *Let* $\varepsilon\in]0,1[$, $\delta\in]0,1]$. *Then we have, uniformly for* $N\geqslant 1$, $x\geqslant N^\delta$, $x^\varepsilon\leqslant y\leqslant x$,

$$\sum_{\substack{p+q=N\\ x<p\leqslant x+y}}1\ll\frac{N}{\varphi(N)}\frac{y}{(\log N)^2},$$

where p *and* q *denote prime numbers.*

4. Proof of Theorem 1

In this section we assume throughout that the hypotheses of Theorem 1 are fulfilled. The proof will require two preliminary estimates.

Lemma 1. *Let* ϑ_h $(1\leqslant h\leqslant r)$ *denote multiplicative arithmetic functions such that*

$$\sum_{p\leqslant x}\sum_{\nu\geqslant 1}\frac{|\vartheta_h(p^{\nu-1})-\vartheta_h(p^\nu)|}{p^\nu}\ll 1\qquad(1\leqslant h\leqslant r).\tag{23}$$

Define $\sigma_h(n):=\varrho_h(n)\vartheta_h(n)$ $(1\leqslant h\leqslant r)$. *Then we have uniformly for* $x>0$

$$\sum_{n\leqslant x}v(n;F,\boldsymbol{\sigma})\ll\sum_{n\leqslant x}v(n;F,\boldsymbol{\varrho}).\tag{24}$$

Proof. Write $\vartheta_h(n)=\sum_{d|n}\lambda_h(d)$, so that λ_h is multiplicative and, by (23), satisfies

$$\sum_{p\leqslant x}\sum_{\nu\geqslant 1}\frac{|\lambda_h(p^\nu)|}{p^\nu}\ll 1\qquad(1\leqslant h\leqslant r).\tag{25}$$

Writing in (16) each n_h as $n_h = m_h d_h$ and interchanging summations, we obtain that the left-hand side of (24) equals

$$\sideset{}{^\dagger}\sum_{d_1^{\gamma_1}\cdots d_r^{\gamma_r}\leqslant x}\prod_{h=1}^r\frac{\lambda_h(d_h)}{d_h}\sideset{}{^\dagger}\sum_{\substack{m_1^{\gamma_1}\cdots m_r^{\gamma_r}\leqslant x/d_1^{\gamma_1}\cdots d_r^{\gamma_r}\\(m_h,d_j)=1\ (1\leqslant h<j\leqslant r)}}\widetilde{F}(d_1m_1,\dots,d_rm_r)\prod_{h=1}^r\frac{\varrho_h(d_hm_h)}{m_h}.$$

We further decompose each m_h as $m_h = t_h\ell_h$ where $t_h|d_h^\infty$ and $(\ell_h, d_h) = 1$. Then $d_hm_h = d_ht_h\ell_h$, with $(\ell_h, d_ht_h) = 1$ and correspondingly $d'_hm'_h = d'_ht'_h\ell'_h$ with $(\ell'_h, d'_ht'_h) = 1$. Using (12) we thus obtain the upper bound

$$\sideset{}{^\dagger}\sum_{\ell_1^{\gamma_1}\cdots\ell_r^{\gamma_r}\leqslant x}\widetilde{F}(\ell_1,\dots,\ell_r)\prod_{h=1}^r\frac{\varrho_h(\ell_h)}{\ell_h}\sideset{}{^\dagger}\sum_{d_1^{\gamma_1}\cdots d_r^{\gamma_r}\leqslant x/\ell_1^{\gamma_1}\cdots\ell_r^{\gamma_r}}\prod_{h=1}^r\frac{|\lambda_h(d_h)|}{d_h}H(d_1,\dots,d_r)$$

with

$$H(d_1,\dots,d_r) := \sideset{}{^\dagger}\sum_{\substack{t_1,\dots,t_r\\t_h|d_h^\infty}}\widetilde{G}(d_1t_1,\dots,d_rt_r)\prod_{h=1}^r\frac{\varrho_h(d_ht_h)}{t_h}.$$

Using the observation (17) and extending the inner d-sum to infinity, we simplify this to

$$\leqslant\sideset{}{^\dagger}\sum_{\ell_1^{\gamma_1}\cdots\ell_r^{\gamma_r}\leqslant x}\widetilde{F}(\ell_1,\dots,\ell_r)\prod_{h=1}^r\frac{\varrho_h(\ell_h)}{\ell_h}\prod_{h=1}^r\sum_{d\geqslant1,\,t|d^\infty}\frac{|\lambda_h(d)|\varrho_h(dt)f_h(dt)}{dt}.$$

Each inner sum over d and t is

$$\leqslant\prod_p\Big\{\sum_{\nu=1}^\infty|\lambda(p^\nu)|\sum_{j\geqslant\nu}\frac{\varrho_h(p^j)f_h(p^j)}{p^j}\Big\}\ll1$$

if e.g. $\varepsilon < 1/g$. This easily follows from the bounds

$$\sup_{1\leqslant h\leqslant R,\,j\geqslant1}\varrho_h(p^j)\ll1,\qquad f_h(p^j)\leqslant(B+1)\min(A,p^\varepsilon)^{j\gamma_h}$$

and (25). The proof of Lemma 1 is therefore complete.

Lemma 2. *Set* $\gamma := \sum_{h=1}^r\gamma_h$. *Let* $\vartheta_h\in\mathcal{M}$ $(1\leqslant h\leqslant r)$, $\kappa>0$. *Then we have uniformly for* $x\geqslant z\geqslant 4^{\gamma\kappa}$

$$\sum_{n>x,\,P^+(n)\leqslant z}v(n;F,\boldsymbol{\vartheta})\ll\mathrm{e}^{-\kappa u}\sum_{n\leqslant z}v(n;F,\boldsymbol{\vartheta}),\tag{26}$$

where $u := (\log x)/\log z$.

Proof. The result is trivial if $F(1,\dots,1) = 0$ for (12) then implies that F vanishes identically. We may hence assume without loss of generality that $F(1,\dots,1) > 0$, and indeed that $F(1,\dots,1) = 1$ since, otherwise, we may consider instead the function $F^*(n_1,\dots,n_k) := F(n_1,\dots,n_k)/F(1,\dots,1)$ which also belongs to $\mathcal{M}_k(A,B,\varepsilon)$.

Let β satisfy $0<\beta<(1/\gamma)-2\varepsilon$ and put $\beta_h := \beta\gamma_h$. The quantity on the left-hand side of (26) does not exceed

$$\sideset{}{^\dagger}\sum_{\substack{n_1^{\gamma_1}\cdots n_r^{\gamma_r}>x\\P^+(n_1\cdots n_r)\leqslant z}}\widetilde{F}(n_1,\dots,n_r)\prod_{h=1}^r\frac{\vartheta_h(n_h)}{n_h}\Big(\frac{n_1^{\gamma_1}\cdots n_r^{\gamma_r}}{x}\Big)^\beta$$

$$\leqslant x^{-\beta}\sideset{}{^\dagger}\sum_{P^+(n_1\cdots n_r)\leqslant z}\widetilde{F}(n_1,\dots,n_r)\prod_{h=1}^r\frac{\vartheta_h(n_h)}{n_h}n_h^{\beta_h}.$$

Let ψ_h be the multiplicative function defined by $\psi_h(p^\nu) = p^{\beta_h\nu}(1-p^{-\beta_h})$, so that $n^{\beta_h} = \sum_{d|n}\psi_h(d)$. Writing $n_h = m_h d_h$ $(1 \leqslant h \leqslant r)$, substituting $n_h^{\beta_h}$ by $\sum_{d_h|n_h}\psi_h(d_h)$ and inverting the order of the summations, we get the bound

$$x^{-\beta} \sideset{}{^\dagger}\sum_{P^+(d_1\cdots d_r)\leqslant z} \prod_{h=1}^r \frac{\psi_h(d_h)}{d_h} \sideset{}{^\dagger}\sum_{P^+(m_1\cdots m_r)\leqslant z} \widetilde{F}(d_1m_1,\ldots,d_rm_r)\prod_{h=1}^r \frac{\vartheta_h(d_hm_h)}{m_h}.$$

Now write $m_h = \ell_h t_h$ with $(\ell_h, d_h) = 1$ and $t_h|d_h^\infty$ so that $d_hm_h = \ell_h t_h d_h$ with $(\ell_h, t_hd_h) = 1$. The bound is therefore

$$\leqslant x^{-\beta} S \sideset{}{^\dagger}\sum_{P^+(\ell_1\cdots\ell_r)\leqslant z} \widetilde{F}(\ell_1,\ldots,\ell_r)\prod_{h=1}^r \frac{\vartheta_h(\ell_h)}{\ell_h} = x^{-\beta} S \sum_{P^+(\ell)\leqslant z} v(\ell; F, \boldsymbol{\vartheta})$$

with

$$S := \sideset{}{^\dagger}\sum_{P^+(d_1\cdots d_r)\leqslant z} \prod_{h=1}^r \frac{\psi_h(d_h)}{d_h} \sum_{t_1|d_1^\infty,\ldots,t_r|d_r^\infty} \widetilde{G}(t_1d_1,\ldots,t_rd_r)\prod_{h=1}^r \frac{g_h(t_hd_h)}{t_h},$$

where, for each h, $g_h \in \mathcal{M}$ is the sub-multiplicative function defined by (11) for $k = 1$ and $F = \vartheta_h \in \mathcal{M}$. Using the bound (17), we obtain that

$$\begin{aligned} S &\leqslant \prod_{h=1}^r \sum_{P^+(d)\leqslant z}\sum_{t|d^\infty} \frac{\psi_h(d)f_h(td)g_h(td)}{td} \\ &\leqslant \prod_{h=1}^r \prod_{p\leqslant z}\Big\{1 + \sum_{\nu=1}^\infty\sum_{j=0}^\infty \frac{\psi_h(p^\nu)f_h(p^{\nu+j})g_h(p^{\nu+j})}{p^{\nu+j}}\Big\} \\ &= \prod_{h=1}^r \prod_{p\leqslant z}\Big\{1 + \sum_{\nu=1}^\infty \frac{f_h(p^\nu)g_h(p^\nu)(p^{\nu\beta_h}-1)}{p^\nu}\Big\} \ll 1, \end{aligned}$$

provided we choose $\beta_h < 1 - 2\varepsilon\gamma_h$. This inequality holds if we take $\beta := \kappa/\log z$ and $z \geqslant 4^{\kappa\gamma}$ since $\varepsilon < 1/8g^2 < 1/8\gamma$ and $\log 4 > \frac{4}{3}$.

We have thus shown that

$$\sum_{n>x,\, P^+(n)\leqslant z} v(n; F, \boldsymbol{\vartheta}) \ll \mathrm{e}^{-\kappa u} \sum_{P^+(n)\leqslant z} v(n; F, \boldsymbol{\vartheta}). \tag{27}$$

Let K denote a large constant and put $w = z^{1/K}$ so that $1 \leqslant w \leqslant z$. In the sum on the right, we may decompose uniquely $n = ab$ with $P^+(a) \leqslant w$, $w < P^-(b)$, $P^+(b) \leqslant z$. By (12), (16) and (17), we readily obtain that

$$v(ab; F, \boldsymbol{\vartheta}) \leqslant v(a; F, \boldsymbol{\vartheta}) \sideset{}{^\dagger}\sum_{b_1^{\gamma_1}\cdots b_r^{\gamma_r}=b} \frac{f_1(b_1)\cdots f_r(b_r)}{b_1\cdots b_r},$$

from which we infer in turn by a standard computation[2] that

$$\sum_{n>x,\, P^+(n)\leqslant z} v(n; F, \boldsymbol{\vartheta}) \ll_K \mathrm{e}^{-\kappa u} \sum_{P^+(n)\leqslant w} v(n; F, \boldsymbol{\vartheta}).$$

2. Involving, in particular, the fact that $\sum_b f_h(b)/b \ll_K 1$ for all h, $1 \leqslant h \leqslant r$.

If $w<2$, this simplifies directly to the required bound. Otherwise, we use (27) with $\kappa=\frac12$ and $u=K$ to obtain that

$$\begin{aligned}\sum_{P^+(n)\leqslant w} v(n;F,\boldsymbol{\vartheta}) &\leqslant \sum_{n\leqslant z} v(n;F,\boldsymbol{\vartheta}) + \sum_{\substack{n>z\\ P^+(n)\leqslant w}} v(n;F,\boldsymbol{\vartheta})\\ &\leqslant 2\sum_{n\leqslant z} v(n;F,\boldsymbol{\vartheta})\end{aligned}$$

for sufficiently large K. This implies the required bound and finishes the proof of Lemma 2.

Completion of the proof of Theorem 1.

As in the proof of Lemma 2, we may assume that $F(1,\dots,1)=1$. We also suppose that x is sufficiently large in terms of all the parameters on which the implicit constant of (18) is allowed to depend.

For each $n\in(x,x+y]$, let ξ_n denote the largest of the integers ξ such that

$$a_n(\xi) := \prod_{p^\nu\,\|\,Q(n),\,p\leqslant\xi} p^\nu \leqslant x^{3g^2\varepsilon}.$$

We put $a_n:=a_n(\xi_n)$, $b_n:=|Q(n)|/a_n$, $q_n:=P^-(b_n)$, $q_n^{e(n)}\|n$. Of course, we have

$$a_n q_n^{e(n)} > x^{3g^2\varepsilon}, \tag{28}$$

and $P^+(a_n)<q_n$. Define

$$a_{hn} := \prod_{p^\nu\,\|\,R_h(n),\,p\leqslant\xi_n} p^\nu, \qquad b_{hn} := \prod_{p^\nu\,\|\,R_h(n),\,p>\xi_n} p^\nu \qquad (1\leqslant h\leqslant r),$$

so that $a_n=\prod_{h=1}^r a_{hn}^{\gamma_h}$ and $b_n=\prod_{h=1}^r b_{hn}^{\gamma_h}$. From (9), we deduce that

$$a'_{jn} := \prod_{p^\nu\,\|\,Q_j(n),\,p\leqslant\xi_n} p^\nu = \prod_{h=1}^r a_{hn}^{\gamma_{jh}}, \quad b'_{jn} := \prod_{p^\nu\,\|\,Q_j(n),\,p\leqslant\xi_n} p^\nu = \prod_{h=1}^r a_{hn}^{\gamma_{jh}} \quad (1\leqslant j\leqslant k).$$

We furthermore observe that

$$\begin{aligned}F(Q_1(n),\dots,Q_k(n)) &= F(a'_{1n}b'_{1n},\dots,a'_{kn}b'_{kn})\\ &\leqslant F(a'_{1n},\dots,a'_{kn})G(b'_{1n},\dots,b'_{kn})\\ &= \widetilde F(a_{1n},\dots,a_{rn})\widetilde G(b_{1n},\dots,b_{rn})\end{aligned}\tag{29}$$

and that

$$\widetilde G(b_{1n},\dots,b_{rn}) \leqslant \min\{A^{\Omega(b_n)}, Bb_n^{\varepsilon\delta/3}\}. \tag{30}$$

Put

$$\varepsilon_1 := 3g^2\varepsilon, \qquad \varepsilon_2 := \tfrac12\varepsilon_1, \qquad \varepsilon_3 := \tfrac12 g\varepsilon.$$

We split the integers of $(x,x+y]$ into four disjoint classes, according to the conditions

(C_1) $\quad a_n\leqslant x^{\varepsilon_2},\ q_n>x^{\varepsilon_3},$

(C_2) $\quad a_n\leqslant x^{\varepsilon_2},\ q_n\leqslant x^{\varepsilon_3},$

(C_3) $\quad x^{\varepsilon_2}<a_n\leqslant x^{\varepsilon_1},\ P^+(a_n)\leqslant w,$

(C_4) $\quad x^{\varepsilon_2}<a_n\leqslant x^{\varepsilon_1},\ P^+(a_n)>w,$

where w is a parameter to be chosen later. Let S_i $(1 \leqslant i \leqslant 4)$ denote the contribution from the integers of C_i to the left-hand side of (18).

Estimation of S_1. We have $P^-(b_n) = q_n > x^{\varepsilon_3}$ for $n \in C_1$, and so

$$x^{\varepsilon_3 \Omega(b_n)} \leqslant b_n \leqslant |Q(n)| \ll \|Q\| x^g \ll x^{g+1/\delta},$$

hence $\Omega(b_n) \ll 1$. From (29) and (30) we therefore obtain that

$$S_1 \ll \sum_{m_1^{\gamma_1} \cdots m_r^{\gamma_r} \leqslant x^{\varepsilon_2}} \widetilde{F}(m_1, \ldots, m_r) \sum_{\substack{x<n\leqslant x+y \\ m_h | R_h(n)\ (1\leqslant h\leqslant r) \\ P^-(Q^*(n)/m_1\cdots m_r) > x^{\varepsilon_3}}} 1. \tag{31}$$

Consider the inner sum. When $r = 1$, i.e. $Q^* = R_1$, Brun's sieve easily yields the estimate

$$\ll \frac{y}{m_1} \varrho_1(m_1) \prod_{p \leqslant x^{\varepsilon_3},\, p \nmid m_1} \Big(1 - \frac{\varrho(p)}{p}\Big),$$

since $y/m_1 > x^{\varepsilon_2}$. For further details of this argument see e.g. [5], p. 264.

If $r \geqslant 2$, we must proceed more carefully, employing an argument analogous to that of [8], Lemma 3.4. Since R_h and R_i have no common zero for $h \neq i$, there exist $U, V \in \mathbb{Z}[X]$ such that

$$R_h(X)U(X) + R_i(X)V(X) = \mathcal{R}_{hi},$$

where $\mathcal{R}_{hi}$ is the resultant of R_h and R_i. Hence $(m_h, m_i) | \mathcal{R}_{hi}$ whenever $m_h | R_h(n)$, $m_i | R_i(n)$ and $h \neq i$. Further, if $\mathfrak{D}(T)$ denotes the discriminant of a polynomial T, we have that $\mathfrak{D}(R_h R_i) = \mathfrak{D}(R_h)\mathfrak{D}(R_i)\mathcal{R}_{hi}^2$ and hence $(m_h, m_i) | \mathfrak{D}(R_h R_i)$. Since $R_h R_i | Q^*$, we have that $\mathfrak{D}(R_h R_i) | \mathfrak{D}(Q^*) = D^*$ and so $(m_h, m_i) | D^*$ for any $h \neq i$. Proofs of these algebraic facts may be found e.g. in [6], pp. 443–453.

Writing $m_h = n_h d_h$ $(1 \leqslant h \leqslant r)$ where $d_h := (m_h, D^*)$, we have $(m_h, m_i) | d_h$ and so $(n_h, n_i) = 1$ for $h \neq i$. We estimate the inner sum of (31) by replacing the conditions $m_h | R_h(n)$ by $n_h | R_h(n)$ and observe that the sum is empty unless all d_h divide $\overline{D}$. Since the n_h are pairwise coprime and $y/(n_1 \cdots n_r) \geqslant y/a_n \geqslant x^{\varepsilon_2/2}$, Brun's (or Selberg's) sieve yields the estimate

$$S_1 \ll y \sum_{d_1 | \overline{D}, \ldots, d_r | \overline{D}} \prod_{h=1}^r f_h(d_h) \sum_{n_1, \ldots, n_r}^{\dagger} \widetilde{F}(n_1, \ldots, n_r) \prod_{h=1}^r \frac{\varrho_h(n_h)}{n_h} \prod_{\substack{p \leqslant x^{\varepsilon_3} \\ p \nmid n_1 \cdots n_r}} \Big(1 - \frac{\varrho(p)}{p}\Big),$$

where we have used (12) and (17). Since $\varrho(p) \leqslant \min(g, p-1)$ by our assumption that Q has no fixed prime factor, the last product is

$$\ll \prod_{h=1}^r \Big(\frac{\varphi(n_h)}{n_h}\Big)^g \prod_{p \leqslant x^{\varepsilon_3}} \Big(1 - \frac{\varrho(p)}{p}\Big).$$

Hence, writing

$$\vartheta_h(n) := \big(\varphi(n)/n\big)^g, \qquad \sigma_h(n) := \varrho_h(n)\vartheta_h(n) \qquad (1 \leqslant h \leqslant r) \tag{32}$$

we obtain

$$S_1 \ll y \prod_{p \leqslant x^{\varepsilon_3}} \Big(1 - \frac{\varrho(p)}{p}\Big) \sum_{n \leqslant x} v(n; F, \boldsymbol{\sigma}).$$

Since $\vartheta_h(n)$ trivially satisfies the requirements of Lemma 1, we obtain that this bound is compatible with (18).

Estimation of S_2. Let $n \in C_2$. Then we see from (28) that $Q(n)$ must be divisible by a prime power $q^e > x^{\varepsilon_1-\varepsilon_2} = x^{\varepsilon_2}$ with $q \leqslant x^{\varepsilon_3}$. Hence there is an h, $1 \leqslant h \leqslant r$, such that $q^\nu | R_h(n)$ with $q^\nu \geqslant q^{e/r} \geqslant x^{3\varepsilon_3}$. Let $\nu(q)$ denote the least integer such that $q^{\nu(q)} > x^{3\varepsilon_3}$. Then $\nu(q) \geqslant 3$ and $q^{\nu(q)-1} \leqslant x^{3\varepsilon_3}$ so $q^{\nu(q)} \leqslant q^{4\varepsilon_3} \leqslant y$. Since $q^{\nu(q)} | Q^*(n)$ we may hence write

$$S_2 \leqslant 2 \max_{n\leqslant 2x} F(|Q_1(n)|,\dots,|Q_k(n)|) \sum_{q\leqslant x^{\varepsilon_3}} \frac{y\varrho^*(q^{\nu(q)})}{q^{\nu(q)}}.$$

With the bound $F(|Q_1(n)|,\dots,|Q_k(n)|) \leqslant G(|Q_1(n)|,\dots,|Q_k(n)|) \leqslant B|Q(n)|^{\varepsilon\delta/3}$, we deduce that

$$S_2 \ll \|Q\|^{\varepsilon\delta/3} x^{\varepsilon\delta/3} \frac{y}{x^{3\varepsilon_3}} x^{\varepsilon_3} \ll yx^{-\varepsilon/3}, \tag{33}$$

taking account of the assumption that $\|Q\| \leqslant x^{1/\delta}$. Now we note that the right-hand side of (18) is $\gg y/(\log x)^g$ since $v(1;F;\boldsymbol{\varrho}) = F(1,\dots,1) = 1$ and $\varrho(p) \leqslant g$ for all p. Hence the estimate (33) for S_2 is also of the required order of magnitude.

Estimation of S_3 *and* S_4. We begin by an estimate which is common to S_3 and S_4. For all n in $C_3 \cup C_4$, we have

$$b_n = |Q(n)|/a_n \leqslant (g+1)\|Q\|(2x)^g/x^{\varepsilon_2} \leqslant x^{g+1/\delta}.$$

Since

$$P^+(a_n)^{\Omega(b_n)} < P^-(b_n)^{\Omega(b_n)} \leqslant b_n,$$

we deduce from (30) that

$$\widetilde{G}(b_{1n},\dots,b_{rn}) \ll x^{E(a_n)},$$

with

$$E(a) := \min\{g\varepsilon,\, s/\log P^+(a)\}, \qquad s := (g+1/\delta)\log A.$$

Therefore, using (12), we may write

$$S_3 + S_4 \ll \sum_{m_1^{\gamma_1}\cdots m_r^{\gamma_r} \leqslant x^{\varepsilon_1}} \widetilde{F}(m_1,\dots,m_r) x^{E(m'')} \sum_{\substack{x<n\leqslant x+y\\ m_h|R_h(n)\ (1\leqslant h\leqslant r)\\ P^-(Q^*(n)/m_1\cdots m_r)>P^+(m_1\cdots m_r)}} 1. \tag{34}$$

Employing the sieve as in S_1 to bound the inner sum, we arrive at

$$S_3 + S_4 \ll y \sum_{\substack{x^{\varepsilon_2}/\overline{D}^r<n\leqslant x^{\varepsilon_1}\\ (n,\overline{D})=1}} v(n;F,\boldsymbol{\sigma}) x^{E(n)} \prod_{p\leqslant P^+(n)} \Big(1-\frac{\varrho(p)}{p}\Big), \tag{35}$$

where $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_r)$ is defined by (32). Moreover, S_3 and S_4 correspond respectively to the extra conditions $P^+(n) \leqslant w$, $P^+(n) > w$.

Since $\widetilde{F}(n_1,\dots,n_r) \leqslant \widetilde{G}(n_1,\dots,n_r) \leqslant Bn^{\varepsilon\delta/3} \ll x^{\varepsilon/3}$, and $\varrho_h(n_h) \leqslant g^{\omega(n_h)}$ when $(n_h,\overline{D}) = 1$ we may write, using the trivial bound of 1 for the product over p in (35),

$$\begin{aligned} S_3 &\ll yx^{g\varepsilon} \sum_{\substack{n_1,\dots,n_r\\ x^{\varepsilon_2}/\overline{D}^r<n''\leqslant x^{\varepsilon_1}}} \widetilde{F}(n_1,\dots,n_r) \prod_{h=1}^r \frac{\varrho_h(n_h)}{n_h} \\ &\ll yx^{g\varepsilon+\varepsilon/3-\varepsilon_2/g} \Big(\sum_{n\leqslant x,\, P^+(n)\leqslant w} 1\Big)^r, \end{aligned}$$

where we have used the fact that $\prod_{h=1}^r n_h \geqslant (\prod_{h=1}^r n_h^{\gamma_h})^{1/g} \gg x^{\varepsilon_3/g}$. We now choose

$$w := 2Ag\overline{D}^r 4^{(s+1)/\varepsilon_2}. \tag{36}$$

This implies that the last sum over n is $\ll (\log x)^w$, and in turn shows that S_3 is of the required order of magnitude.

It remains to estimate S_4. We consider the sub-sum of (35) corresponding to $P^+(n) > w$, which we expand by writing $n = q^\nu m$ with $P^+(m) < q$. We observe that $v(n; F, \boldsymbol{\sigma}) \leqslant (Ag/q)^\nu v(m; F, \boldsymbol{\sigma})$ by (12) and (13), and hence we get

$$S_3 \ll y \prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)T_3$$

with

$$T_3 := \sum_{w<q\leqslant x^{\varepsilon_1},\, \nu\geqslant 1} x^{s/\log q}\Big(\frac{Ag}{q}\Big)^\nu \prod_{q<p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)^{-1} \sum_{\substack{x^{\varepsilon_2}/q^\nu \overline{D}^r < m\leqslant x^{\varepsilon_1}/q^\nu \\ P^+(m)\leqslant q}} v(m; F, \boldsymbol{\sigma}).$$

The last product over p is clearly $\ll (\log x/\log q)^g$. For each given q and ν, the inner m-sum may be bounded by applying Lemma 2 with $\kappa = (s+1)/\varepsilon_2$ and then Lemma 1, having checked that our choice for w guarantees that $w > 4^{\kappa g}$. We obtain

$$\begin{aligned} T_3 &\ll \sum_{w<q\leqslant x^{\varepsilon_1},\, \nu\geqslant 1} x^{-1/\log q}\Big(\frac{\log x}{\log q}\Big)^g\Big(\frac{Age^\kappa \overline{D}^r}{q}\Big)^\nu \sum_{m\leqslant x} v(m; F, \boldsymbol{\varrho}) \\ &\ll \sum_{q\leqslant x}\frac{x^{-1/2\log q}}{q}\sum_{m\leqslant x} v(m; F, \boldsymbol{\varrho}) \ll \sum_{m\leqslant x} v(m; F, \boldsymbol{\varrho}). \end{aligned}$$

This is also compatible with (18) and therefore completes the proof of Theorem 1.

5. Proofs of Corollaries 1 & 2

Put $n = mq + a$ and define $P_j \in \mathbb{Z}[X]$ by

$$P_j(X) = Q_j(qX+a) \qquad (1\leqslant j\leqslant k).$$

Then

$$\begin{aligned} &\sum_{\substack{x<n\leqslant x+y \\ n\equiv a \,(\mathrm{mod}\, q)}} F(|Q_1(n)|, \dots, |Q_k(n)|) \\ &\qquad = \sum_{(x-a)/q<m\leqslant (x-a)/q+y/q} F(|P_1(m)|, \dots, |P_k(m)|). \end{aligned} \tag{37}$$

Put $P := P_1 \cdots P_k$ and denote by P^* the squarefree kernel of P. Since each $R_h(qX+a)$ is also irreducible, we have that

$$P^*(X) = \prod_{h=1}^r R_h(qX+a) = Q^*(qX+a).$$

Let η_j $(1 \leqslant j \leqslant g^*)$ be the zeros of Q^*. Then the zeros of P^* are $(\eta_j - a)/q$ $(1 \leqslant j \leqslant g^*)$ and thus, denoting by D_1^* the discriminant of P^*, we have

$$D_1^* = \big(a_{g^*} q^{g^*}\big)^{2(g^*-1)} \prod_{1 \leqslant i < j \leqslant g^*} \Big(\frac{\eta_j - a}{q} - \frac{\eta_i - a}{q}\Big)^2 = q^{g^*(g^*-1)} D^*. \tag{38}$$

A simple computation using $(q, Q(a)) = 1$ shows that if ϱ'_h is the rho-function for $R_h(qX + a)$ then for p prime and $1 \leqslant h \leqslant r$ we have

$$\begin{aligned} &\varrho'_h(p) = 0 && \text{if} \quad p \mid q, \\ &\varrho'_h(p^\nu) = \varrho_h(p^\nu) && \text{if} \quad p \nmid q, \ \nu \geqslant 1, \end{aligned}$$

so that, writing $\varrho' = \varrho_P$,

$$\varrho'(p) = 0 \quad (\text{if } p \mid q), \qquad \varrho'(p^\nu) = \varrho(p^\nu) \quad (\text{if } p \nmid q, \ \nu \geqslant 1).$$

These facts imply that P has no fixed prime divisor. Moreover, setting $\boldsymbol{\varrho'} = (\varrho'_1, \ldots, \varrho'_r)$, we may write

$$\prod_{p \leqslant (x-a)/q} \Big(1 - \frac{\varrho'(p)}{p}\Big) = \prod_{\substack{p \leqslant (x-a)/q \\ p \nmid q}} \Big(1 - \frac{\varrho(p)}{p}\Big) \ll \prod_{p \leqslant x, \, p \nmid q} \Big(1 - \frac{\varrho(p)}{p}\Big),$$

since $(x - a)/q \gg x^\beta$ —see [5], Lemma 2(i) for more details of this argument. By (38), we also deduce that

$$\overline{D_1} := \prod_{\substack{p^\nu \| D_1^* \\ \varrho'(p) \neq 0}} p^\nu = \prod_{\substack{p^\nu \| D_1^*, \, p \nmid q \\ \varrho'(p) \neq 0}} p^\nu = \prod_{\substack{p^\nu \| D^*, \, p \nmid q \\ \varrho'(p) \neq 0}} p^\nu = \prod_{\substack{p^\nu \| D^*, \, p \nmid q \\ \varrho(p) \neq 0}} p^\nu$$

and hence $\overline{D_1} \mid \overline{D}$. A final observation is that

$$\sum_{n \leqslant (x-a)/q} v(n; F, \boldsymbol{\varrho'}) = \sum_{\substack{n \leqslant (x-a)/q \\ (n,q)=1}} v(n; F, \boldsymbol{\varrho}) \leqslant \sum_{\substack{n \leqslant x \\ (n,q)=1}} v(n; F, \boldsymbol{\varrho}).$$

We can now apply Theorem 1 to the right-hand side of (37) (with ε in Theorem 1 replaced by $\frac{1}{2}\varepsilon\beta$) to deduce that

$$\sum_{\substack{x < n \leqslant x+y \\ n \equiv a \,(\mathrm{mod}\, q)}} F(|Q_1(n)|, \ldots, |Q_k(n)|) \ll \frac{y}{q} \prod_{\substack{p \leqslant x \\ p \nmid q}} \Big(1 - \frac{\varrho(p)}{p}\Big) \sum_{\substack{n \leqslant x \\ (n,q)=1}} v(n; F, \boldsymbol{\varrho})$$

provided that (i) $y/q \geqslant \{(x - a)/q\}^{4g^2\varepsilon\beta/2}$, (ii) $y \leqslant x - a$ and (iii) $x \geqslant c_0 \|P\|^\delta$. It remains to confirm that these conditions hold. Now

$$(x - a)^{2g^2\varepsilon\beta} q^{1-2g^2\varepsilon\beta} \leqslant y^{\beta/2} y^{(1-\beta)(1-2g^2\varepsilon\beta)} \leqslant y^{1-\beta/2} \leqslant y,$$

so (i) holds. Next, using $\|P\| \leqslant q^g \|Q\|$ and $x \geqslant c_1 \|Q\|^{2\delta}$, we have that $x \geqslant c_1 \|P\|^{2\delta} q^{-2g\delta}$, hence $xq \geqslant c_1 \|P\|^{2\delta}$ and thus $x^2 \geqslant c_1 \|P\|^{2\delta}$. So the choice $c_1 = c_0^2$ suffices to confirm (iii). Finally, for (ii), it is easily checked that if $x - a < y \leqslant x$ then we have the trivial estimate

$$\begin{aligned} \sum_{2(x-a)/q < m \leqslant (x-a)/q + y/q} F(|P_1(m)|, \ldots, |P_k(m)|) &\ll \Big(\frac{y}{q} - \frac{x-a}{q} + 1\Big) \|P\|^{\varepsilon\beta\delta/6} \\ &\ll \frac{y^{1-\beta}}{q} \|P\|^{\varepsilon\beta\delta/6} \ll \frac{y}{q} \Big(\frac{\|P\|^{\varepsilon\delta/6}}{x}\Big)^\beta \ll \frac{y}{q} x^{-\beta/2} \end{aligned}$$

which is smaller than the required bound. The proof of Corollary 1 is thus complete.

Corollary 2 follows immediately from Corollary 1 on noticing that, when the Q_j are irreducible and mutually coprime we have $k = r$, $\gamma_j = 1$ and $n'_j = n_j$ for all j.

6. Proofs of Corollaries 3 & 4

Proof of Corollary 3. A quick computation confirms that the function F of k variables defined by

$$F(n_1,\dots,n_k)=\prod_{j=1}^k F_j(n_j)$$

belongs to $\mathcal{M}_k(A,B^k,\varepsilon\delta/3)$. Further the coprimality condition enables us to write each Q_j as $Q_j=\prod_{r_{j-1}<h\leqslant r_j}R_h^{\gamma_h}$ where $r_k=r$ is the number of irreducible factors of $Q=\prod_{h=1}^r R_h^{\gamma_h}$. Hence, in the notation of section 2,

$$\gamma_{jh}=\begin{cases}\gamma_h & \text{if } r_{j-1}<h\leqslant r_j\\ 0 & \text{otherwise}\end{cases}$$

for each $j\in[1,k]$, and with $r_0:=0$. Thus with $\varrho_h:=\varrho_{R_h}$, we obtain that

$$\begin{aligned} v(n;F,\boldsymbol{\varrho}) &= \sideset{}{^\dagger}\sum_{n_1^{\gamma_1}\cdots n_k^{\gamma_k}=n}\Big(\prod_{j=1}^k F_j\big(\textstyle\prod_{h=1}^r n_h^{\gamma_{jh}}\big)\Big)\displaystyle\prod_{h=1}^r\frac{\varrho_h(n_h)}{n_h}\\ &= \sideset{}{^\dagger}\sum_{n_1^{\gamma_1}\cdots n_k^{\gamma_k}=n}\Big(\prod_{j=1}^k F_j\big(\textstyle\prod_{r_{j-1}<h\leqslant r_j} n_h^{\gamma_h}\big)\Big)\displaystyle\prod_{h=1}^r\frac{\varrho_h(n_h)}{n_h}\\ &\leqslant \prod_{j=1}^k\ \sideset{}{^\dagger}\sum_{n_{r_{j-1}+1}^{\gamma_{j-1}+1}\cdots n_{r_j}^{\gamma_{r_j}}\leqslant x} F_j\big(\textstyle\prod_{r_{j-1}<h\leqslant r_j}n_h^{\gamma_h}\big)\displaystyle\prod_{r_{j-1}<h\leqslant r_j}\frac{\varrho_h(n_h)}{n_h}. \end{aligned}\tag{39}$$

At this stage, we observe that for all $j\in[1,k]$ and any $n_h\geqslant 1$ $(r_{j-1}<h\leqslant r_j)$ we have

$$\prod_{h=1}^r\frac{\varrho_h(n_h)}{n_h}\leqslant\frac{\varrho_{Q_j}\big(\prod_{r_{j-1}<h\leqslant r_j}n_h^{\gamma_h}\big)}{\prod_{r_{j-1}<h\leqslant r_j}n_h^{\gamma_h}}\tag{40}$$

To see this, consider a typical prime p dividing $\prod_{r_{j-1}<h\leqslant r_j}n_h$ then, since the n_h are mutually coprime, there is a unique index, say ℓ, such that $p|n_\ell$, and $p\nmid n_h$ for $h\neq\ell$. Let $p^\alpha\|n_\ell$. The contribution of p^α to the left-hand side of (40) is

$$\frac{\varrho_\ell(p^\alpha)}{p^\alpha}=\frac{\varrho_\ell(p)}{p^\alpha}$$

since $p\nmid D^*$ and $\mathfrak{D}(R_\ell)\mid D^*=\mathfrak{D}(Q^*)$. The contribution of p^α to the right-hand side of (40) is

$$\frac{\varrho_{Q_\ell}(p^{\alpha\gamma_\ell})}{p^{\alpha\gamma_\ell}}=\sum_{r_{j-1}<h\leqslant r_j}\frac{\varrho_h(p)}{p^{\lceil\alpha\gamma_\ell/\gamma_h\rceil}}\geqslant\frac{\varrho_\ell(p)}{p^\alpha},$$

where the equality is a classical fact about polynomial congruences — see e.g. [8], equation (2.11). Here and in the sequel we let $\lceil u\rceil$ denote the smallest integer larger than or equal to u. This implies (40).

Inserting (40) into (39), we obtain

$$v(n;F,\boldsymbol{\varrho})\leqslant\prod_{j=1}^k\sum_{n\leqslant x}\frac{F_j(n)\varrho_{Q_j}(n)}{n}$$

and the result now follows from Theorem 1.

Proof of Corollary 4. We have (see e.g. [2], Lemma 61.1)

$$\Delta(mn) \leqslant \tau(m)\Delta(n) \qquad ((m,n)=1).$$

Hence, given arbitrary $\varepsilon_0 \in]0,1[$, we have $\Delta^t \in \mathcal{M}(2^t, B, \varepsilon_0)$ for some $B = B(t,\varepsilon_0) \geqslant 1$ and we are in a position to apply Corollary 2 which implies that

$$\sum_{x<n\leqslant x+y} \Delta(|Q(n)|)^t \ll y \prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{n\leqslant x}\frac{\Delta(n)^t\varrho(n)}{n}.$$

Now Lemma 2.2 of Tenenbaum [9] states that

$$\sum_{n\leqslant x}\frac{\Delta(n)^t\varrho(n)}{n} \ll (\log x)^{\beta(t)}\mathcal{L}(\log x)^{\sqrt{2t}+o(1)}$$

and combining this with the classical estimate

$$\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big) \ll \frac{1}{\log x}$$

completes the proof of Corollary 4.

7. Proof of Corollary 6

As in the proof of Lemma 2, we assume $F(1,\dots,1)=1$. We next note that we may then restrict to the case when $z \geqslant z_0$ with arbitrary large $z_0 = z_0(\kappa,\varepsilon,\delta,g)$. Indeed, when $2^{\kappa/2g^2\varepsilon} \leqslant z < z_0$, the left-hand side of (21) is trivially $\ll (\log x)^{O(1)}$, which is of smaller order of magnitude than the right-hand side.

Let $\beta := \kappa/\log z$ and let $\chi(n)$ denote the completely multiplicative function defined by $\chi(p) = p^\beta$ if $p \leqslant z$, $\chi(p)=1$ otherwise, and put

$$F_1(n_1,\dots,n_k) := F(n_1,\dots,n_k)\chi(n_k).$$

We have $\chi(n) = n^\beta$ when $P^+(n) \leqslant z$. Since $|Q_k(n)| \gg x$ for $x < n \leqslant x+y$, the sum on the right-hand side of (21) is

$$\ll x^{-\beta}\sum_{x<n\leqslant x+y} F_1(|Q_1(n)|,\dots,|Q_k(n)|) = \mathrm{e}^{-\kappa u}\sum_{x<n\leqslant x+y} F_1(|Q_1(n)|,\dots,|Q_k(n)|).$$

With a suitable choice of z_0, we have $\beta < \frac{1}{12}\varepsilon\delta$, and it may be readily checked that $F_1 \in \mathcal{M}_k(Ae^\kappa, B, \varepsilon\delta/3)$. We may hence apply Theorem 1 to F_1, with the result that

$$\sum_{\substack{x<n\leqslant x+y\\ P^+(|Q_k(n)|)\leqslant z}} F\big(|Q_1(n)|,\dots,|Q_k(n)|\big) \ll y\mathrm{e}^{-\kappa u}\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{n\leqslant x} v(n;F_1,\boldsymbol{\varrho}).$$

Hence it only remains to prove that

$$\sum_{n\leqslant x} v(n;F_1,\boldsymbol{\varrho}) \ll \sum_{n\leqslant x} v(n;F,\boldsymbol{\varrho}). \tag{41}$$

To this end, we apply Lemma 1 with $\vartheta_h(n) = \chi(n)^{\gamma_{kh}}$ $(1 \leqslant h \leqslant r)$, observing that, with the notation of lemma 1, $v(n; F_1, \boldsymbol{\varrho}) = v(n; F, \boldsymbol{\sigma})$ for all n. We have for each h

$$\sum_{p\leqslant x}\sum_{\nu\geqslant 1}\frac{|\vartheta_h(p^{\nu-1}) - \vartheta_h(p^\nu)|}{p^\nu} = \sum_{p\leqslant z}\frac{1-p^{-\beta}}{p^{1-\beta}-1} \ll \beta\sum_{p\leqslant z}\frac{\log p}{p} \ll 1.$$

By Lemma 1, this implies (41) and thus completes the proof of Corollary 6.

Remark. In the case $k = 1$, $Q_1(n) = n$, $F = 1$, Corollary 6 yields

$$\Psi(x+y,z) - \Psi(x,z) \ll_\kappa y\mathrm{e}^{-\kappa u}$$

for all fixed $\kappa > 0$, $x^\varepsilon \leqslant y \leqslant x$ and $z \geqslant 4^{\kappa/\varepsilon}$. An inspection of the proof shows that, if we make explicit the dependence upon κ, the same technique will furnish a bound of the type $\ll yu^{-cu}$, where $c = c(\varepsilon)$ provided, say, $z \geqslant (\log x)^2$. Such result, which could similarly be derived in the general situation considered in this paper, is comparable with the best known upper bounds for the number of integers free of small prime factors in short intervals, due to Hildebrand [4]. However, the results in [4] are valid without any lower bound restriction on the variable y.

8. Proof of Theorem 2

We write $Q(n) = a_n b_n$ with $P^+(a_n) \leqslant Cy < P^-(b_n)$ for some large C. We have

$$\sum_{x<n\leqslant x+y}\log|Q(n)| \sim yg\log x = ky\log y$$

and

$$\begin{aligned}\sum_{x<n\leqslant x+y}\log a_n &= \sum_{p\leqslant Cy,\, p^\nu\leqslant x}\log p^\nu\{y\varrho_Q(p^\nu)/p^\nu + O(1)\}\\ &= (1+o(1))y\log y\end{aligned}$$

by the prime ideal theorem. Hence, since $y = x^{g/k}$,

$$\sum_{x<n\leqslant x+y}\log b_n \geqslant \{k-1+o(1)\}y\log y. \tag{42}$$

There is a constant A_0 such that $b_n \leqslant A_0x^g = A_0y^k$. Hence, if C is large enough, the condition $P^-(b_n) > Cy$ implies $\Omega(b_n) \leqslant k-1$. Moreover, if $Q(n)$ has a divisor in $(\frac{1}{2}y, y]$, then $b_n \leqslant 2A_0x^g/y = 2A_0y^{k-1} < (Cy)^{k-1}$, so $\Omega(b_n) \leqslant k-2$. Thus, letting $H_Q(x,y)$ denote the number of $n \in (x, x+y]$ such that $Q(n) \equiv 0 \,(\mathrm{mod}\, d)$ for some $d \in (\frac{1}{2}y, y]$, we obtain

$$\sum_{x<n\leqslant x+y}\log b_n \leqslant (k-1)(\log P_{x,y})\{y - H_Q(x,y)\} + (k-2)(\log P_{x,y})H_Q(x,y).$$

Taking (42) into account, we get for large x

$$P_{x,y} \geqslant y\exp\Big\{\frac{H_Q(x,y)\log y}{ky}\Big\}. \tag{43}$$

This is a short interval analogue of an inequality of Erdős and Schinzel [1] and constitutes the basis of our method. It seems that a new idea would be required to relax the shape condition imposed on y in the theorem.

We need a lower bound for $H_Q(x,y)$ and proceed as in [9], using Corollary 4. Writing $\Delta(m,y)=\sum_{d|m,\ \frac12 y<d\leqslant y}1$, we have for any positive η

$$H_Q(x,y)\geqslant(\log x)^{-\eta}\sum_{\substack{x<n\leqslant x+y\\ \Delta(|Q(n)|,y)\leqslant(\log x)^{\eta}}}\Delta(|Q(n)|,y). \tag{44}$$

Let us write the last n-sum as S_1-S_2 with

$$\begin{aligned}S_1=\sum_{x<n\leqslant x+y}\Delta(|Q(n)|,y)&\geqslant\sum_{\frac12 y<d\leqslant y}\varrho_Q(d)\Big[\frac{y}{d}\Big]\\ &\geqslant\tfrac12 y\sum_{\frac12 y<d\leqslant y}\frac{\varrho_Q(d)}{d}\gg y,\end{aligned}$$

using again the prime ideal theorem as in [9], Lemma 2.1. Then, for any $\eta>0$, we have

$$S_2=\sum_{\substack{x<n\leqslant x+y\\ \Delta(|Q(n)|,y)>(\log x)^{\eta}}}\Delta(|Q(n)|,y)\leqslant(\log x)^{-\varepsilon\eta}\sum_{x<n\leqslant x+y}\Delta(|Q(n)|)^{1+\varepsilon}$$

We choose $\eta>\log 4-1$ and define $\varepsilon>0$ by the relation $2^{\varepsilon}=(1+\eta)/\log 4$. Applying Corollary 4 with $t=1+\varepsilon$, we obtain $S_2\ll y(\log x)^{-\sigma+o(1)}$ with

$$\sigma=1+\eta\varepsilon-\beta(1+\varepsilon)=2\{2^{\varepsilon}\log(2^{\varepsilon})-2^{\varepsilon}+1\}>0.$$

Thus $S_2=o(y)$ and, by (44),

$$H_Q(x,y)\gg y/(\log x)^{\eta}.$$

By (43), this implies $P_{x,y}>y\exp\big\{(\log x)^{1-\eta_1}\big\}$ for any $\eta_1>\eta$ and large x. The required result follows since η, and hence η_1, may be taken arbitrarily close from $\log 4-1$.

9. Proof of Theorem 3

We may plainly assume that $c_0\geqslant 1$ and $F(1,\dots,1)=1$. We first observe that it is sufficient to bound the sub-sum of (22) where the variable p is further restricted by the condition $p\nmid Q(0)$. Indeed, the complementary contribution may be trivially bounded above by

$$\omega(|Q(0)|)B\sup_{x<p\leqslant 2x}|Q(p)|^{\varepsilon\delta/3}.$$

Since $|Q(0)|\leqslant\|Q\|\leqslant x^{1/\delta}$ and $|Q(p)|\leqslant\|Q\|(2x)^{g}$, the above bound is, for $x\geqslant x_0(\delta)$,

$$\leqslant B(\log x)(2^{g}x^{g+1/\delta})^{\varepsilon\delta/3}\ll_B(\log x)x^{\varepsilon(g+1)/3}\ll\sqrt{y}.$$

This is (with a lot to spare) of smaller order of magnitude than the right-hand side of (22).

Let $\chi(n)=1$ if $P^-(n)>x$, and $\chi(n)=0$ otherwise. We set

$$F_0(n_1,\dots,n_{k+1})=F(n_1,\dots,n_k)\chi(n_{k+1}).$$

It is readily checked that our hypothesis on F implies that $F_0\in\mathcal{M}_{k+1}(A,B,\varepsilon\delta/3)$.

Since $Q(0)\neq 0$, none of the $Q_j(X)$ is equal to X, so

$$\widehat{Q}(X):=X\prod_{j=1}^{k}Q_j(X)$$

has degree $g+1$ and has $r+1$ irreducible factors $R_j(X)$ $(1 \leqslant j \leqslant r+1)$, with $R_{r+1}(X) = X$. Of course $\|\widehat{Q}\| = \|Q\|$. We set $\widehat{\varrho} := \varrho_{\widehat{Q}}$, $\widehat{\boldsymbol{\varrho}} := (\varrho_1, \ldots, \varrho_{r+1})$ and note that $\varrho_{r+1} = \mathbf{1}$.

Now we make the observation that if the variable of summation n in Theorem 1 is restricted to values coprime to a fixed integer, say q, then the implicit constant in the $\ll$ sign only depends on $\overline{D}_q := \overline{D}/(\overline{D}, q)$. This may be easily checked by taking into account the extra condition $(n, q) = 1$ in the sieve arguments employed for the upper bounds of S_1, S_3, S_4 in section 4, so we omit the details.

Let $\widehat{D}^*$ denote the discriminant of $(\widehat{Q})^*$. Then $\widehat{D}^* = a_{g^*}^2 Q^*(0)^2 D^*$, hence $\overline{\widehat{D}}_q | \overline{D}$ when $q = Q(0)$. It follows that Theorem 1 (suitably modified as indicated above) yields the bound

$$\sum_{\substack{x<p\leqslant x+y \\ p\nmid Q(0)}} F\big(|Q_1(p)|, \ldots, |Q_k(p)|\big) \leqslant \sum_{\substack{x<n\leqslant x+y \\ n\nmid Q(0)}} F_0\big(|Q_1(n)|, \ldots, |Q_k(n)|\big)$$

$$\ll y \prod_{p\leqslant x} \Big(1 - \frac{\widehat{\varrho}(p)}{p}\Big) \sum_{n\leqslant x} v(n; F_0, \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{\varrho}}).$$

It remains to evaluate the right-hand side in terms of ϱ and $v(n; F, \boldsymbol{\varrho}) = v(n; F, \boldsymbol{Q}, \boldsymbol{\varrho})$.

We first note that, plainly, $\widehat{\varrho}(p) = 1 + \varrho(p)$ if $p \nmid Q(0)$ and $\widehat{\varrho}(p) = \varrho(p)$ if $p \mid Q(0)$. Therefore

$$\prod_{p\leqslant x} \Big(1 - \frac{\widehat{\varrho}(p)}{p}\Big) \ll \prod_{p\leqslant x,\, p\nmid Q(0)} \Big(1 - \frac{1}{p}\Big) \prod_{p\leqslant x} \Big(1 - \frac{\varrho(p)}{p}\Big)$$

$$\ll \frac{|Q(0)|}{\varphi(|Q(0)|)} \frac{1}{\log x} \prod_{p\leqslant x} \Big(1 - \frac{\varrho(p)}{p}\Big).$$

Next, we observe that $v(n; F_0, \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{\varrho}}) \leqslant \sum_{d|n} v(n/d; F, \boldsymbol{Q}, \boldsymbol{\varrho})\chi(d)/d$, hence

$$\sum_{n\leqslant x} v(n; F_0, \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{\varrho}}) \leqslant \sum_{d\leqslant x} \frac{\chi(d)}{d} \sum_{n\leqslant x} v(n; F, \boldsymbol{Q}, \boldsymbol{\varrho}) = \sum_{n\leqslant x} v(n; F, \boldsymbol{Q}, \boldsymbol{\varrho}),$$

since $\chi(1) = 1$ and $\chi(d) = 0$ whenever $2 \leqslant d \leqslant x$.

This completes the proof of Theorem 3.

References

[1] P. Erdős & A. Schinzel, On the greatest prime factor of $\prod_{k=1}^{x} f(k)$, *Acta Arith.* **55** (1990), 191–200.

[2] R.R. Hall & G. Tenenbaum, *Divisors*, Cambridge tracts in mathematics no. 90, Cambridge University Press (1988).

[3] C. Hooley, On a new technique and its applications to the theory of numbers, *Proc. London Math. Soc.* (3) **38** (1979), 115–151.

[4] A. Hildebrand, Integers free of large prime factors in short intervals, *Quart. J. Math. Oxford* (2), **36** (1985), 57–69.

[5] M. Nair, Multiplicative functions of polynomial values in short intervals, *Acta Arith.* **62** (1992), 257–269.

[6] L. Rédei, *Algebra* Vol 1, Pergamon Press (1967).

[7] P. Shiu, A Brun–Titchmarsh theorem for multiplicative functions, *J. reine angew. Math.* **313** (1980), 161–170.

[8] G. Tenenbaum, Sur une question d'Erdős et Schinzel, in : A. Baker, B. Bollobás, A. Hajnal (eds.) *A Tribute to Paul Erdős,* Cambridge University Press (1990), 405–443.

[9] G. Tenenbaum, Sur une question d'Erdős et Schinzel, II, *Inventiones Math.* **99** (1990), 215–224.

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