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Polynomial values free of large prime factors

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Dedicated to Professor András Sárközy
on the occasion of his 60th Birthday

Abstract. For $F \in \mathbb{Z}[X]$, let $\Psi_F(x,y)$ denote the number of positive integers n not exceeding x such that $F(n)$ is free of prime factors $> y$. Our main purpose is to obtain lower bounds of the form $\Psi_F(x,y) \gg x$ for arbitrary F and for y equal to a suitable power of x. Our proofs rest on some results and methods of two articles by the third author concerning localization of divisors of polynomial values. Analogous results for the polynomial values at prime arguments are also obtained.

1. Introduction and statements of results

In this paper we investigate multiplicative properties of sequences

$$\{F(n)\}_{n=1}^{\infty}$$

for polynomials $F(X)$ with integer coefficients; in particular, we are concerned with the distribution of $P^+(F(n))$, where $P^+(m)$ denotes the largest prime factor of m, with the convention that $P^+(1) = 1$. If we let $\Psi(x,y)$ denote the number of integers $n \leqslant x$ such that $P^+(n) \leqslant y$, then it is well-known that, in a large range for (x,y),

$$\Psi(x,y) = \{1 + o(1)\}\varrho(u)x \tag{1·1}$$

where u is defined by $x = y^u$ and ϱ is the Dickman function. In particular, for a fixed real number $0 < \alpha < 1$, the integers free of prime factors exceeding x^α comprise a positive proportion of the integers up to x. When $F(X)$ is a polynomial of degree 1, an asymptotic formula for the number of integers $n \leqslant x$ with $P^+(F(n)) \leqslant x^{1/u}$ was first established, for fixed F and u, in the work of Buchstab [3] on numbers without large prime factors in arithmetic progressions. Later work has provided

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results having some uniformity in the coefficients of the linear polynomial—see the work of Hildebrand and the third author [10] (Section 6) for a discussion of such results.

Concerning the values of a polynomial $F(X)$ of degree $g \geqslant 2$ that are free of large prime factors, Schinzel [14] (Theorem 15) showed that

$$P^+(F(n)) \leqslant n^{g-1-\delta(g)} \tag{1·2}$$

holds for infinitely many integers n, with some $\delta(g) \in]0,1[$ such that $\delta(g) \sim 2/g$ for large g. This result is non-trivial because $F(n)$ has order of magnitude n^g. He also showed [14] (Theorem 13) that if F has the special form $F(X) = aX^g + b$ for some non-zero integers a, b, and $g \geqslant 2$, then for any positive real number α there are infinitely many integers n for which $P^+(F(n)) \leqslant n^\alpha$. The same conclusion also holds, by work of Balog and Wooley [2] (extending a result of Eggleton and Selfridge [5]), when $F(X) = \prod_{1\leqslant i\leqslant t}(a_iX^{g_i} + b_i)$ with arbitrary parameter t and integer coefficients a_i, b_i.(1)

The proofs of these results do not give very strong estimates for how many such values are taken by F. If we define the counting function

$$\Psi_F(x,y) = \#\{1 \leqslant n \leqslant x \colon P^+(F(n)) \leqslant y\},$$

then presumably, for any fixed polynomial $F(X)$ and positive real number α, we should have $\Psi_F(x, x^\alpha) \sim c(F,\alpha)x$ for some positive constant $c(F,\alpha)$. However, the argument of Schinzel in [14] only yields, when $\delta < \delta(g)$,

$$\Psi_F\big(x, x^{g-1-\delta}\big) \gg x^{\beta(\delta)}$$

for rather small values of $\beta(\delta)$,(2) and the lower bounds provided by the Balog–Wooley approach are clearly of the order of an iterated logarithm.

When $F(X)$ is a linear polynomial, Buchstab's work cited above gives the asymptotic formula $\Psi_F(x, x^\alpha) \sim \varrho(1/\alpha)x$, directly extending (1·1). Some results of Hmyrova [11], [12] and Timofeev [17] give upper bounds for $\Psi_F(x, x^\alpha)$ for polynomials of arbitrary degree; however, little progress has been achieved in the direction of establishing a lower bound of the presumed order of magnitude, viz.

$$\Psi_F(x, x^\alpha) \asymp_{F,\alpha} x, \tag{1·3}$$

for polynomials of degree at least 2. Generalizing a result of Hildebrand [8], Balog and Ruzsa [1] proved that (1·3) holds for any $\alpha > 0$ when $F(X) = X(aX + b)$ for positive integers a and b. For $F(X) = \prod_{1\leqslant j\leqslant k}(X + j)$ with some $k \geqslant 2$, Hildebrand [9] established (1·3) provided $\alpha > e^{-1/(k-1)}$. The first author [4]

1. Theorem 2 of [2] is only stated for a product of linear polynomials, however Lemma 2.2 of the same paper provides the result mentioned here with arbitrary degrees g_i.
2. The first step of the iteration in [14] provides $\beta\,(1/(g-1)) = 1/(g-1)$ when $g \geqslant 4$.

considered the polynomial $F(X) = X^2 + 1$ and showed in this case that (1·3) holds for $\alpha > \frac{149}{179}$.

Our approach in this work is based on the fact that an upper bound for $P^+(F(n))$ may be derived from the information that $F(n)$ has localized divisors in suitable intervals. Such problems have been considered in [15] and [16], from which we borrow some techniques and results. The following theorem is proved along these lines and provides a lower bound of the form (1·3) for an arbitrary polynomial.

Theorem 1.1. *Let F be a polynomial with integer coefficients. Let g be the largest of the degrees of the irreducible factors of F and let k be the number of distinct irreducible factors of $F(X)$ of degree g. Given any positive real number ε, the estimate*

$$\Psi_F(x,y) \asymp x \tag{1·4}$$

holds for all large x provided $y \geqslant x^{g+\varepsilon-1/k}$.

We note that, by the definition of g, the bound $P^+(F(n)) \ll x^g$ holds for all $n \leqslant x$. When $g = 1$ or $g = 2$ and $k = 1$, our result is weaker than some of the theorems mentioned previously. Actually, our main purpose in this work is to provide a basic bench mark to stimulate further research. Theorem 1.1 may be regarded as a dual version of theorems 9.7 and 10.1 of Halberstam and Richert [6] showing, that, for any polynomial F without fixed prime divisor, there exists a constant $r = r(F)$ such that F represents infinitely many integers with at most r prime factors. In the case when F is an irreducible polynomial of degree g, these results state that $r = g + 1$ is admissible.

A natural variant of Theorem 1.1 concerns polynomial values at prime arguments. Defining

$$\Psi_F^*(x,y) = \#\big\{p \leqslant x \colon P^+(F(p)) \leqslant y\big\}$$

(here and subsequently the letter p denotes a prime number), then presumably

$$\Psi_F^*(x,x^\alpha) \asymp_{F,\alpha} x/\log x \tag{1·5}$$

for any positive α. We establish a lower bound of this form for $\alpha > g - 1/(2k)$.

Theorem 1.2. *Let $F(X)$, g, and k be as in Theorem 1.1, with $F(0) \neq 0$ if $g = k = 1$, and let ε be a positive real number. Then the estimate*

$$\Psi_F^*(x,y) \asymp \frac{x}{\log x} \tag{1·6}$$

holds for all large x provided $y \geqslant x^{g+\varepsilon-1/2k}$.

The hypothesis $F(0) \neq 0$ is necessary in the case $g = k = 1$: if $F(0) = 0$ then X divides $F(X)$, in which case $P^+(F(p)) \geqslant p$ always when p is prime and hence $\Psi_F^*(x,y)$ cannot exceed $\pi(y)$.

In the case $F(X) = X + a$ for some non-zero integer a, Friedlander [7] shows that the lower bound (1·5) holds for any α greater than $1/(2\sqrt{e}) \approx 0.30326$. This corresponds to a larger range for y than that provided by Theorem 1.2. The aforementioned work of Hmyrova contains upper bounds for $\Psi_F^*(x,y)$ as well as for $\Psi_F(x,y)$, but no non-trivial lower bound for $\Psi_F^*(x,x^{g-\delta})$ had been shown when $F(X)$ is non-linear and δ is any fixed positive number.

Finally, we establish a theorem that "interpolates" between Schinzel's result (1·2) and Theorem 1.1.

Theorem 1.3. *Let $F \in \mathbb{Z}[X]$ such that $gk > 1$ and let β be any constant exceeding $\log 4 - 1$. There is a constant $K = K_F$ such that*

$$\Psi_F(x,y) \gg x/(\log x)^{\beta k}$$

provided $y \geqslant Kx^{g-1/k}$.

In the case $k = 1$, this result is a comparatively simple consequence of a lower bound established by the third author in [16] for the number of integers n such that $F(n)$ has at least one divisor in a prescribed range.

2. Proofs

Let $F(X) \in \mathbb{Z}[X]$ with factorization of the form

$$F(X) = G(X) \prod_{1 \leqslant j \leqslant k} F_j(X)^{\alpha_j}, \tag{2·1}$$

where the F_j are distinct irreducible polynomials of degree $g \geqslant 1$, the α_j are positive integers and all irreducible factors of $G(X)$ have degree $< g$. We denote by $\Psi_F(x,y)$ the number of integers $n \leqslant x$ with $P^+(F(n)) \leqslant y$. For k-dimensional vectors $\boldsymbol{y} := (y_1, \ldots, y_k)$ and $\boldsymbol{z} := (z_1, \ldots, z_k)$ with positive coordinates, the quantity $H_F(x; \boldsymbol{y}, \boldsymbol{z})$ is defined as the number of $n \leqslant x$ such that $F_j(n)$ has at least one divisor d_j with $y_j < d_j \leqslant z_j$ for all j, $1 \leqslant j \leqslant k$.

We note at the outset that there exists a constant K, depending only on F, such that

$$\Psi_F(x,y) \geqslant H_F(x; \boldsymbol{w}, \boldsymbol{z}) \qquad \text{if} \quad Kx^g/y < w_j \leqslant z_j \leqslant y \quad (1 \leqslant j \leqslant k). \tag{2·2}$$

Indeed, if n is counted by $H_F(x; \boldsymbol{w}, \boldsymbol{z})$, then $F_j(n) = d_j d_j'$ with $Kx^g/y < d_j \leqslant y$ for each j, whence

$$|d_j'| \leqslant |F_j(n)| y/(Kx^g) \leqslant y,$$

so the largest prime factor of $F(n)$ does not exceed y.

2.1. Proof of Theorem 1.1

Let $\delta \in]0, \frac{1}{2}[$. We shall show that, when x is sufficiently large,

$$H_F(x; \boldsymbol{w}, \boldsymbol{z}) \gg x \tag{2·3}$$

for $z_j = x^{(1-\delta)/k}$ and $w_j := z_j^{1-\delta}$ $(1 \leqslant j \leqslant k)$, so that (2·2) will yield the required lower bound on taking $y = x^{g+\varepsilon-1/k}$ and δ sufficiently small in terms of ε and K.

There exists an integer M, depending only on F, such that the congruences

$$F_i(n) \equiv 0 \,(\mathrm{mod}\, p), \qquad F_j(n) \equiv 0 \,(\mathrm{mod}\, p)$$

are incompatible whenever $i \neq j$ and $p \nmid M$ — see, e.g., [15], p. 411. Put $D := D^* F(1) M$, where D^* is the discriminant of $F^*(X) := \prod_{1 \leqslant j \leqslant k} F_j(X)$. Let $\varrho^*(t)$ denote the number of roots of $F^*(X)$ modulo t, and, for $1 \leqslant j \leqslant k$, let $\varrho_j(t)$ denote the number of roots of $F_j(X)$ modulo t. We remark that

$$\varrho^*(p) \leqslant \min(kg, p-1)$$

for all j whenever $p \nmid D$. We make the convention that the letter t, with or without a subscript, denotes, generally, an integer coprime to D.

Let $\boldsymbol{z}$, $\boldsymbol{w}$ be as above, and put $v := x^\alpha$, where α is a small parameter at our disposal. Here and subsequently, x is assumed to be sufficiently large. Put

$$G_j := \gcd\big\{F_j(n) \,:\, n \geqslant 1\big\} = \mathrm{lcm}\,\big\{p^\nu \,:\, \varrho_j\big(p^\nu\big) = p^\nu\big\}.$$

Let $P^-(n)$ denote, generically, the smallest prime factor of an integer n, with the convention that $P^-(1) = 1$. Since the quantity

$$B_F(n) := \prod_{j=1}^{k} \sum_{\substack{t_j \mid F_j(n) \\ w_j < t_j \leqslant z_j \\ P^-(F_j(n)/G_j t_j) > v}} 1 \tag{2·4}$$

is $\ll 2^{kg/\alpha}$ for $n \leqslant x$, we have

$$\begin{aligned} H_F(x; \boldsymbol{w}, \boldsymbol{z}) &\gg \sum_{n \leqslant x} B_F(n) \\ &= \sum_{w_1 < t_1 \leqslant z_1} \cdots \sum_{w_k < t_k \leqslant z_k} \sum_{\substack{n \leqslant x \\ F_j(n) \equiv 0 \,(\mathrm{mod}\, t_j)\ (1 \leqslant j \leqslant k) \\ P^-(F_j(n)/G_j t_j) > v\ (1 \leqslant j \leqslant k)}} 1. \end{aligned} \tag{2·5}$$

The inner sum is empty if the t_j are not mutually coprime. Otherwise, a standard application of Selberg's sieve for the set

$$\Big\{F^*(n) / \textstyle\prod_{1 \leqslant j \leqslant k} (t_j G_j) \,:\, n \leqslant x,\, F_j(n) \equiv 0 \,(\mathrm{mod}\, t_j)\ (1 \leqslant j \leqslant k)\Big\}$$

(see e.g. [15], Lemma 3.4, for a similar bound with slightly different assumptions) gives that, for small enough $\alpha = \alpha(\varepsilon, \delta, F)$, it is

$$\asymp \prod_{j=1}^{k} \frac{\varrho_j(t_j)}{\varphi^*(t_j)} \frac{x}{(\log x)^k}$$

uniformly in $t_j \in]w_j, z_j]$ $(1 \leqslant j \leqslant k)$, with $\varphi^*(t) := t\prod_{p|t}\big\{1-(\varrho^*(p)-1)/(p-1)\big\}$. It follows from this and (2·5) that

$$(2\cdot6)\qquad H_F(x;\boldsymbol{w},\boldsymbol{z}) \gg \frac{x}{(\log x)^k}\sum_{w_1<t_1\leqslant z_1}\ \sum_{\substack{w_2<t_2\leqslant z_2\\(t_1,t_2)=1}}\cdots\sum_{\substack{w_k<t_k\leqslant z_k\\(t_k,t_1\cdots t_{k-1})=1}}\ \prod_{1\leqslant j\leqslant k}\frac{\varrho_j(t_j)}{\varphi^*(t_j)}.$$

From the prime ideal theorem, it can be shown (see, e.g., [16], Lemma 2.1 for details in a very similar case) that

$$\sum_{\substack{t\leqslant T\\(t,b)=1}}\varrho_j(t) \asymp \frac{\varphi_j(b)}{b}T,\qquad \sum_{\substack{t\leqslant T\\(t,b)=1}}\frac{\varrho_j(t)\varphi(t)}{t} \asymp \frac{\varphi_j(b)}{b}T$$

uniformly for $T \geqslant 1$ and $b \ll \mathrm{e}^T$, where the implied constants depending only on F, $\varphi(t)$ is the usual Euler function and $\varphi_j(t) := t\prod_{p|t}\big(1-\varrho_j(p)/p\big)$. These estimates enable us, via partial summation, to evaluate successively the t_j-sums appearing in (2·6). Indeed, let S be the k-fold sum appearing on the left of (2·6). By the above, and since $\varphi^*(t) \leqslant t$ for all t, we see that the inner t_k-sum is, ignoring the factor depending on the t_j with $j < k$,

$$\geqslant \sum_{\substack{w_k<t_k\leqslant z_k\\(t_k,t_1\cdots t_{k-1})=1}}\frac{\varrho_k(t_k)}{t_k} \gg \frac{\varphi_k(t_1\cdots t_{k-1})}{t_1\cdots t_{k-1}}\log x = \prod_{j=1}^{k-1}\frac{\varphi_k(t_j)}{t_j}\log x.$$

Using now the inequality $t\varphi_k(t) \geqslant \varphi(t)\varphi^*(t)$ with $t = t_{k-1}$[3], we get

$$\frac{S}{\log x} \gg \sum_{w_1<t_1\leqslant z_1}\ \sum_{\substack{w_2<t_2\leqslant z_2\\(t_1,t_2)=1}}\cdots\sum_{\substack{w_{k-1}<t_{k-1}\leqslant z_{k-1}\\(t_{k-1},t_1\cdots t_{k-2})=1}}\ \prod_{1\leqslant j\leqslant k-2}\frac{\varrho_j(t_j)\varphi_k(t_j)}{t_j\varphi^*(t_j)}L_{k-1}(t_{k-1}),$$

with

$$L_{k-1}(t) := \varrho_{k-1}(t)\varphi(t)/t^2$$

We now estimate the inner-sum as before and insert the inequality

$$\varphi_k(t)\varphi_{k-1}(t) \geqslant \varphi^*(t)\varphi(t)$$

with $t = t_{k-2}$. Iterating the procedure,[4] we obtain (2·3) as required. □

3. This follows from the fact that $(t_{k-1}, D) = 1$ and the definitions of the ϱ-functions.
4. At step ℓ, we need the inequality $\prod_{k-\ell<j\leqslant k}\varphi_j(t) \geqslant t^{\ell-2}\varphi^*(t)\varphi(t)$ with $t = t_{k-\ell}$.

2.2. Proof of Theorem 1.2

The method of the previous section provides an estimate of comparable strength in the case of prime arguments.

Let $\delta \in]0, \frac{1}{2}[$. Let $H_F^*(x; \boldsymbol{y}, \boldsymbol{z})$ be the number of primes p such that $F_j(p)$ has at least one divisor d_j with $y_j < d_j \leqslant z_j$ for all j, $1 \leqslant j \leqslant k$, so that, as before,

$$\Psi_F^*(x,y) \geqslant H_F^*(x; \boldsymbol{w}, \boldsymbol{z}) \tag{2·7}$$

if $\boldsymbol{w}$, $\boldsymbol{z}$ satisfy the condition in (2·2). We shall show that

$$H_F^*(x; \boldsymbol{w}, \boldsymbol{z}) \gg x/\log x \tag{2·8}$$

for $z_j = x^{(1-\delta)/2k}$ and $w_j := z_j^{1-\delta}$ $(1 \leqslant j \leqslant k)$, so that (2·7) will yield the required lower bound on taking $y = x^{g+\varepsilon-1/2k}$ and δ sufficiently small in terms of ε and K.

Let M and D be defined as in the proof of Theorem 1.1. We now adopt the convention that the letter t, with or without a subscript, denotes, generally, an integer coprime to $E := DF^*(0)\prod_{p \leqslant kg+2} p$, which is non-zero by hypothesis. We note that $\varrho^*(p) \leqslant \min(kg, p-2)$ whenever $p \nmid E$.

Let $\boldsymbol{z}$, $\boldsymbol{w}$ be as above, and put $v := x^\alpha$, where α is a small parameter at our disposal. Put $G_j^* := \gcd\{F_j(p) : p \text{ prime}\}$. We define a function $B_F^*(p)$ analogous to $B_F(n)$ in (2·4) by

$$B_F^*(p) := \prod_{j=1}^{k} \sum_{\substack{t_j \mid F_j(p) \\ w_j < t_j \leqslant z_j \\ P^-(F_j(p)/G_j^* t_j) > v}} 1. \tag{2·9}$$

Again, $B_F^*(p)$ is uniformly bounded for all $p \leqslant x$, and we have

$$H_F^*(x; \boldsymbol{w}, \boldsymbol{z}) \gg \sum_{p \leqslant x} B_F^*(p) = \sum_{w_1 < t_1 \leqslant z_1} \cdots \sum_{w_k < t_k \leqslant z_k} \sum_{\substack{p \leqslant x \\ F_j(p) \equiv 0 \,(\mathrm{mod}\, t_j)\ (1 \leqslant j \leqslant k) \\ P^-(F_j(p)/G_j^* t_j) > v\ (1 \leqslant j \leqslant k)}} 1. \tag{2·10}$$

The inner sum is empty if the t_j are not mutually coprime. As before, we may apply Selberg's sieve but now need the Bombieri–Vinogradov theorem to bound on average the remainder in the sieve assumptions. We obtain that, for small enough $\alpha = \alpha(\varepsilon, \delta, F)$,

$$\asymp \frac{x}{(\log x)^{k+1}} \prod_{j=1}^{k} \frac{\varrho_j(t_j)}{\varphi^{**}(t_j)}$$

uniformly in $t_j \in]w_j, z_j]$ $(1 \leqslant j \leqslant k)$, with $\varphi^{**}(t) := t\prod_{p \mid t}\{1 - \varrho^*(p)/(p-1)\}$. We deduce as before that (2·8) holds. □

2.3. Proof of Theorem 1.3

Assume first that $k=1$, i.e. F has only one irreducible factor of highest degree. The result stated then follows immediately from (2·2) and the estimate

$$H_F\big(x;\tfrac12 x,x\big)\geqslant x/(\log x)^\beta$$

proved in [16].(5) When $k>1$, the technique of [16] extends without difficulty to yield

$$H_F(x;\boldsymbol{w},\boldsymbol{z})\geqslant x/(\log x)^{\beta k} \tag{2·11}$$

for $w_j:=\frac12 x^{1/k}$, $z_j:=x^{1/k}$. Assuming this temporarily, the required result follows from (2·2) as previously.

We now briefly outline the proof of (2·11), which is very similar to that given in [16] for the case $k=1$. Let Δ denote the Erdős–Hooley function, defined as

$$\Delta(n):=\sup_{u\in\mathbb{R}}\#\big\{d|n:\mathrm{e}^u<d\leqslant \mathrm{e}^{u+1}\big\}\qquad(n\geqslant 1).$$

The first and crucial step is to show that the upper bound

$$\sum_{n\leqslant x}\prod_{1\leqslant j\leqslant k}\Delta\big(F_j(n)\big)^t\ll x(\log x)^{k(2^t-t-1)+o(1)} \tag{2·12}$$

holds for each fixed $t\geqslant 1$ as $x\to\infty$. This can be established by a straightforward reappraisal of the computations in [16], but it is even simpler to appeal to Corollary 3 of [13] which provides a ready-to-use estimate: indeed, it implies(6) that the left-hand side of (2·12) is

$$\ll x\prod_{1\leqslant j\leqslant k}\Big\{\prod_{p\leqslant x}\Big(1-\frac{\varrho_j(p)}{p}\Big)\sum_{n\leqslant x}\frac{\varrho_j(n)\Delta(n)^t}{n}\Big\},$$

from which the required result follows by the prime ideal theorem and Lemma 2.2 of [16] which states that each inner n-sum is $\ll(\log x)^{2^t-t+o(1)}$.

Next, we introduce, as in [16],

$$\Delta^*(n,y):=\#\{d|n\,:\,y<d\leqslant 2y\}$$

and put $T(n):=\prod_{1\leqslant j\leqslant k}\Delta^*\big(F_j(n);w_j\big)$. Following [16] (page 223), we infer from (2·12) that, for any $\beta>\log 4-1$

$$\sum_{\substack{n\leqslant x\\ T(n)>(\log x)^{k\beta}}}T(n)=o(x)$$

and hence that

$$H_F(x;\boldsymbol{w},\boldsymbol{z})\geqslant\sum_{\substack{n\leqslant x\\ T(n)\leqslant(\log x)^{k\beta}}}\frac{T(n)}{(\log x)^{k\beta}}\gg\frac{x}{(\log x)^{k\beta}}.$$

□

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5. Here $H_F(x;y,z)$ denotes the number of integers $n\leqslant x$ such that $F(n)$ has at least one divisor in $]y,z]$.
6. Taking $y=x$ and $F_j=\Delta^t$ for each j in Corollary 3 of [13]. The extra condition required in [13] that no F_j should have a fixed prime divisor can be ignored here since the implicit constant in (2·12) is allowed to depend on the F_j.

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