D.V. Chudnovsky et al. (eds),
Number Theory (New York Seminar 1989–90),
(Springer Verlag, 1991), pp. 119–128.

The set of multiples of a short interval

R.R. Hall and G. Tenenbaum

1. Introduction

Following recent work [11, 14, 15], we denote by $H(x,y,z)$ the number of integers n not exceeding x and having at least one divisor in the interval $(y,z]$. Thus if $\mathcal{A} := (y,z]\cap\mathbb{Z}^+$ and $\mathcal{B}(\mathcal{A})$ is the set of multiples of $\mathcal{A}$, then $H(x,y,z)$ is the counting function of $\mathcal{B}(\mathcal{A})$. To determine the asymptotic behaviour of this quantity with good precision is a difficult and interesting sieve problem with many applications in number theory — see in particular chap. 2 of [11].

We are concerned here with the case $z \leqslant 2y$, when we call $\mathcal{A}$ a short interval. The other case will not be mentioned further and we refer the reader to [11, 14] for corresponding results.

Let us assume throughout that $3 \leqslant y \leqslant z \leqslant \sqrt{x}$, and define $\beta = \beta(y,z)$ implicitly from the relation

$$z = y + y(\log y)^{-\beta}. \tag{1}$$

The second author discovered in [14] a *threshold* in the behaviour of $H(x,y,z)$ at the point $\beta = \log 4 - 1$. It may be as well to explain the term threshold here. It is the analogue, for an arithmetical quantity, of a critical value which represents a phase transition in Physics ! More precisely, we say that β_0 is a threshold if the function H under study behaves, as a function of β, in a given way when β lies on one side of β_0, but requires, to be described on the other side, a formula of a different shape. As it often occurs in Physics, there is usually an additional feature : from the two typical formulae, one is fairly simple and natural, whereas the other turns out to be more complicated and does not lend itself to obvious heuristic explanation.

Examples of thresholds may be found in Erdős [2, 3], Erdős–Rényi [7], Erdős–Hall [4], Hall [9]. In some of these cases [2, 4, 7, 9] it has been found that if β is allowed to converge towards the threshold in a suitable manner, involving the introduction of a further parameter ξ, then the ratio between the function studied and its simple approximation may be described by a probability distribution function. In the four cases listed above, see Erdős–Hall [5, 4, 6] and Hall–Tenenbaum [10], respectively. In the Erdős–Rényi example, one encounters a Poisson distribution ; in the other instances, it is Gaussian.

In accord with these ideas, we introduced in [11] the parameter ξ such that

$$\beta = \log 4 - 1 + \frac{\xi}{\sqrt{\log_2 y}}. \tag{2}$$

When $\xi \to +\infty$, $H(x,y,z)$ has a simple behaviour, namely

$$H(x,y,z) \sim x \sum_{y<d\leqslant z} \frac{1}{d} \sim x(\log y)^{-\beta} \quad (y \to \infty). \tag{3}$$

This was proved by Tenenbaum in [14] and may be heuristically explained by the fact that, when the interval is short enough, the various congruence conditions involved in the definition of $H(x,y,z)$ are essentially independent from a probabilistic viewpoint.
Now set

$$Q(t) := t\log t - t + 1 \qquad (t > 0), \tag{4}$$

$$G(\beta) := \begin{cases} \beta & (\beta \geqslant \log 4 - 1) \\ \beta + 2Q\Big(\dfrac{1+\beta}{\log 4}\Big) = Q\Big(\dfrac{1+\beta}{\log 2}\Big) & (\beta < \log 4 - 1). \end{cases} \tag{5}$$

Then Tenenbaum showed that for $\beta \geqslant 0$ we have

$$H(x,y,z) = x(\log x)^{-G(\beta)+o(1)} \tag{6}$$

and more precise bounds are given in [11], viz.

$$x(\log y)^{-G(\beta)}L^{-1} \ll H(x,y,z) \ll x(\log y)^{-G(\beta)}\big(1 + (-\xi)^+\big)^{-1} \tag{7}$$

where

$$L := \exp\big\{b\sqrt{\log_2 y \log_3 y}\big\}$$

and b is large enough.

In [11] (p. 31), we formulated the conjecture that if $x, y, z \to +\infty$ in such a way that ξ is fixed and $z \leqslant \sqrt{x}$, the limit

$$\Xi(\xi) := \lim x^{-1}H(x,y,z)(\log y)^{\beta} \tag{8}$$

exists and is a distribution function on the real line. It seems likely that Ξ would be continuous and < 1 for all ξ, so (3) would be false — this already follows from (7) for large negative ξ.

Our first result below is a step towards this conjecture. Although we still cannot prove the existence of Ξ, we can evaluate to within a constant factor the expression under the limit in (8).

Theorem 1. *Let $3 \leqslant y+1 \leqslant z \leqslant \sqrt{x}$ and β, ξ be defined by (1), (2). Then we have*

$$H(x,y,z) \asymp x(\log y)^{-G(\beta)}\big(1 + (-\xi)^+\big)^{-1} \tag{9}$$

uniformly under the condition

$$\xi \geqslant -C_0(\log_2 y)^{\frac{1}{6}} \tag{10}$$

where C_0 is an arbitrary positive constant.

Define the arithmetic functions

$$\tau(n;y,z) := \sum_{\substack{d\mid n\\ y<d\leqslant z}} 1, \qquad \chi(n;y,z) := \min\big(\tau(n;y,z),1\big).$$

Then $H(x,y,z) = \sum_{n\leqslant x} \chi(n;y,z)$, and (3) may be rewritten as

$$H(x,y,z) \sim \sum_{n\leqslant x} \tau(n;y,z). \tag{11}$$

In certain applications, it is of interest to have at hand weighted analogues of (11) — and to know the value of the corresponding threshold. The following result describes such a situation on the "smooth" side of the threshold. We set

$$\Omega(n,z) := \sum_{\substack{p^\nu\|n\\ p\leqslant z}} \nu, \quad H(x,y,z;t) := \sum_{n\leqslant x} \chi(n;y,z)t^{\Omega(n,z)}.$$

Theorem 2. *Let $0<\varepsilon\leqslant \frac12$, $\frac12\leqslant t\leqslant 1$, $3\leqslant y<z\leqslant\sqrt{x}$ and β be as in (1). Let ζ be such that*

$$\beta = t\log 4 - 1 + \frac{\zeta}{\sqrt{\log_2 y}} \tag{12}$$

and suppose in addition that

$$2Q(t) \leqslant (1-\varepsilon)\zeta\big/\sqrt{\log_2 y} \leqslant C_1. \tag{13}$$

Then we have

$$H(x,y,z;t) = (1+\eta)\sum_{n\leqslant x} \tau(n;y,z)t^{\Omega(n,z)} \tag{14}$$

where

$$\eta \ll \exp\big\{-\varepsilon^2\zeta^2/2t\big\} + (\log y)^{-\log 2}.$$

This includes (3), and is more precise : when $t=1$, $Q(t)=0$ and the extra condition (13) is nugatory. From (12) and (13) we must have

$$\beta > t\log 4 - 1 + 2Q(t). \tag{15}$$

The function on the right increases from 0 to $\log 4 - 1$ as t increases from $\frac12$ to 1. Therefore, we have a variable threshold, and an asymptotic formula for the weighted sum $H(x,y,z;t)$ is available for every $\beta>0$, with a suitable t. The heuristic argument exposed in [11] (pp. 30–31) suggests that η will tend to 0 whenever $\beta > t\log 4 - 1$, but this is not clear, and we leave the determination of this threshold for $t<1$ as an open problem which may well require an interesting new idea.

2. Proof of Theorem 1

By (7), we see that only the lower bound for $H(x, y, z)$ has to be established. Moreover, (3) implies the required estimate when $\xi \to +\infty$, and the proof given in [14] or [11] gives $H(x, y, z) \gg x(\log y)^{-\beta}$ for $\xi > \xi_0$. This is sufficient since $G(\beta) = \beta$ for $\xi \geqslant 0$.

Hence, we suppose throughout that $\xi \leqslant \xi_0$, where ξ_0 is a (possibly large) absolute constant, and set out to prove the lower bound contained in (9). Our argument is similar to that of [11] (§ 2.5), but involves a treatment with a specific technique.

We first introduce the function

$$\overline{\Omega}(n, z) := \sum_{\substack{p^\nu \| n \\ 2<p\leqslant z}} \nu.$$

It will be seen that it is important here to exclude the prime 2 in the prime factors counted by $\overline{\Omega}(n, z)$. The following result is a very special case of a theorem of Balazard [1]. In this simple context, it can also bee seen as a straightforward generalization of an estimate of Halász [8] and Sárközy [13].

Lemma 1. *Let E_k denote the set of integers $n \leqslant x$ such that $\overline{\Omega}(n, z) = k$. For fixed $\delta > 0$, and uniformly for large x, $3 \leqslant z \leqslant x$, $1 \leqslant k \leqslant (3 - \delta)\log_2 z$, we have*

$$\frac{x}{\log z}\,\frac{(\log_2 z)^{k-1}}{(k-1)!} \ll |E_k| \ll \frac{x}{\log z}\,\frac{(\log_2 z)^k}{k!}. \tag{16}$$

The idea of our lower bound method for $H(x, y, z)$ is easily explained. Put

$$R_k := \sum_{n\in E_k} \tau(n; y, z), \qquad S_k := \sum_{n\in E_k} \binom{\tau(n; y, z)}{2}.$$

Then we have for every $K \geqslant 1$

$$H(x, y, z) \geqslant \sum_{\substack{n\leqslant x \\ \overline{\Omega}(n,z)\leqslant K}} \tau(n; y, z)\big(2 - \tau(n; y, z)\big) = \sum_{k\leqslant K} (R_k - 2S_k).$$

We shall select K as large as possible under the constraint that $S_k = o(R_k)$ for $k \leqslant K$. Thus, we need a lower bound for R_k and an upper bound for S_k.

For $2 \leqslant k \leqslant (3 - \delta)\log_2 y$, we have, by Lemma 1,

$$\begin{aligned} R_k = \sum_{\substack{md\leqslant x \\ y<d\leqslant z \\ \overline{\Omega}(md,z)=k}} 1 &= \sum_{i+j=k} \sum_{\substack{y<d\leqslant z \\ \overline{\Omega}(d,z)=i}} \sum_{\substack{m\leqslant x/d \\ \overline{\Omega}(m,z)=j}} 1 \\ &\gg \frac{x}{\log y} \sum_{i+j=k} \frac{1}{(j-1)!}(\log_2 y)^{j-1} \sum_{\substack{y<d\leqslant z \\ \overline{\Omega}(d,z)=i}} \frac{1}{d}. \end{aligned}$$

The inner d-sum is certainly

$$\geqslant \sum_{\substack{t\leqslant\sqrt{y}\\ \Omega(t)=i-1}} \frac{1}{t} \sum_{y/t<p\leqslant z/t} \frac{1}{p} \gg \frac{1}{(\log y)^{1+\beta}}\,\frac{(\log_2 y)^{i-1}}{(i-1)!}$$

by partial summation, using the first estimate in (16). After change of variables $i\mapsto i+1$ and $j\mapsto j+1$, and summation over i,j, we obtain

$$\begin{aligned} R_k &\gg \frac{x}{(\log y)^{2+\beta}} \sum_{i+j=k-2} \frac{(\log_2 y)^{k-2}}{i!j!} = \frac{x}{(\log y)^{2+\beta}}\,\frac{(2\log_2 y)^{k-2}}{(k-2)!} \\ &\gg \frac{x}{(\log y)^{2+\beta}}\,\frac{(2\log_2 y)^{k}}{k!} \gg \frac{x}{(\log y)^{1+\beta+Q(\kappa)}}\,(\log_2 y)^{-5/2} \end{aligned} \tag{17}$$

where $\kappa := k/\log_2 y$ and we assume $\kappa \gg 1$. We have used Stirling's formula to obtain the last inequality.

The desired upper bound for S_k will be derived in a manner very similar to [11], § 2.5. We introduce a free parameter $v\in[0,1]$ and estimate

$$T(v) := \sum_{n\leqslant x} \binom{\tau(n;y,z)}{2} v^{\overline{\Omega}(n,z)}.$$

Arguing precisely as in [11], p. 39, we arrive at

$$T(v) \ll x(\log y)^A(\log_2 y)^B \tag{18}$$

where $A := 2v-2-2\beta+\max(2v-1,0)$, $B := 2-2v+\delta(2v)$ with $\delta(u) := 1$ or 0 according to whether $u=1$ or not. For any v, we have

$$S_k \leqslant v^{-k}T(v) \tag{19}$$

and we choose $v=1/2$ and combine the resulting inequality with (17) and (18) to deduce

$$R_kS_k^{-1} \gg (\log y)^{\beta-Q(\kappa)}(\log_2 y)^{-5/2}. \tag{20}$$

Put $\lambda := (1+\beta)/\log 2$, so that $\lambda\leqslant 5/2$ for large y. Given any X, $0\leqslant X\leqslant\sqrt{\log_2 y}$, we have by Taylor's formula

$$\begin{aligned} Q\Big(\lambda-\frac{X}{\log_2 y}\Big) &= Q(2)+Q'(2)\Big\{\frac{\xi}{\log 2\cdot\sqrt{\log_2 y}}-\frac{X}{\log_2 y}\Big\}+O\Big(\frac{\xi^2+1}{\log_2 y}\Big) \\ &= \beta-\frac{X\log 2}{\log_2 y}+O\Big(\frac{\xi^2+1}{\log_2 y}\Big) \end{aligned}$$

since $Q(2) = \log 4 - 1$, $Q'(2) = \log 2$. In particular, we have

$$\Big\{\beta - Q\Big(\lambda - \frac{X}{\log_2 y}\Big)\Big\} \log_2 y - \tfrac{5}{2} \log_3 y \to +\infty$$

for $X := c_1\xi^2 + c_2 \log_3 y$ with suitable absolute constants c_1, c_2. Since $Q(t)$ is increasing in the range $1 \leqslant t \leqslant 5/2$, we deduce from this and (20) that $S_k = o(R_k)$ for

$$\log_2 y \leqslant k \leqslant \lambda \log_2 y - X =: K.$$

We write $k =: 2\mu \log_2 y$ and note that $\mu \leqslant 1$ if y is large enough and $\xi \leqslant -1$. Now by (17) we have

$$\begin{aligned}\sum_{2+\log_2 y \leqslant k \leqslant K} R_k &\geqslant \frac{x}{(\log y)^{2+\beta}} \Big\{ \sum_{k \leqslant K} - \sum_{k \leqslant \log_2 y} \Big\} \frac{(2\log_2 y)^k}{k!} \\ &\gg \frac{x}{(\log y)^{\beta + 2Q(\mu)}} \ \big(1 + (-\xi)^+\big)^{-1}\end{aligned}$$

where the partial sums of the exponential series have been estimated according to Theorem (1.11) of Norton [12]. Indeed, when $-1 \leqslant \xi \leqslant \xi_0$, we have $\mu = 1 + O\big(1/\sqrt{\log_2 y}\big)$ and $Q(\mu) \ll 1/\log_2 y$. This shows in particular that the required estimate holds in the range $0 \leqslant \xi \leqslant \xi_0$, since then $G(\beta) = \beta$. When $\xi \leqslant 0$, the hypothesis $\xi \ll (\log_2 y)^{1/6}$ and the fact that $Q'(1 \pm \varepsilon) \ll \varepsilon$ yield

$$Q(\mu) = Q\big(\tfrac{1}{2}\lambda\big) + O\Big(\frac{1}{\log_2 y}\Big)$$

and, since $G(\beta) = \beta + 2Q\big(\tfrac{1}{2}\lambda\big)$, the desired bound follows again.

This completes the proof of Theorem 1.

3. Proof of Theorem 2

There are two cases, according as $\zeta > 2\sqrt{\log_2 y}$ or not. The former is easier and we begin with this. Clearly

$$H(x, y, z; t) \leqslant \sum_{n \leqslant x} \tau(n; y, z) t^{\Omega(n,z)} \tag{21}$$

Moreover

$$H(x, y, z; t) \geqslant \sum_{n \leqslant x} \tau(n; y, z) t^{\Omega(n,z)} - 2S$$

where

$$S := 2 \sum_{n \leqslant x} \binom{\tau(n; y, z)}{2} t^{\Omega(n,z)} = \sum_{y < d < d' \leqslant z} t^{\Omega([d,d'])} \sum_{r \leqslant x/[d,d']} t^{\Omega(r,z)}.$$

We put $m = (d, d')$, $d = mu$, $d' = mu'$, and notice that $m|(d'-d)$ whence $m < y - z$. The inner r-sum above is

$$\ll \frac{x}{[d,d']}(\log Z)^{t-1} \qquad \Big(Z := \min\big(z, \frac{x}{[d,d']}\big)\Big)$$

and we observe that since $dd' < z^2 \leqslant x$, we have $Z > m$. So

$$S \ll x \sum_{m<z-y} \frac{t^{\Omega(m)}}{m}(\log 2m)^{t-1}\Big\{\sum_{y/m<u<z/m} \frac{1}{u}\Big\}^2$$

where we have dropped the factor $t^{\Omega(u,m)}$ multiplying u^{-1} in the inner sum. Since

$$\sum_{y/m<u\leqslant z/m} \frac{1}{u} = \log\big(\frac{z}{y}\big) + O\big(\frac{m}{y}\big) \ll (\log y)^{-\beta},$$

we have

$$S \ll \frac{x}{(\log y)^{2\beta}} \sum_{m\leqslant z} \frac{t^{\Omega(m)}}{m}(\log 2m)^{t-1} \ll x(\log y)^{2t-2\beta}\log_2 y,$$

by partial summation — the factor $\log_2 y$ occuring only when $t = \frac{1}{2}$. Now

$$\sum_{n\leqslant x} \tau(n;y,z,t)t^{\Omega(n,z)} \gg x(\log y)^{2t-2-\beta} \tag{22}$$

uniformly in the given ranges for the variables (this is in fact the only place where we need the upper bound of (13)), and so

$$H(x,y,z;t) \geqslant \Big(1 + O\big((\log y)^{1-\beta}\log_2 y\big)\Big)\sum_{n\leqslant x}\tau(n;y,z)t^{\Omega(n,z)}.$$

When $\zeta > 2\sqrt{\log_2 y}$, we have $\beta > 1 + \log 2$. Together with the upper bound (21), this gives the result stated.

Next, we consider the case $\zeta \leqslant 2\sqrt{\log_2 y}$. We follow [11], § 2.5, quite closely. However the reader will need to see where the new condition (13) arises.

Let χ denote the characteristic function of the set of integers satisfying

$$\Omega(n,z) \leqslant \mathcal{Z} := 2t\log_2 y + \varepsilon\zeta\sqrt{\log_2 y},$$

and put

$$M_1 := \sum_{n\leqslant x}\big(1-\chi(n)\big)\tau(n;y,z)t^{\Omega(n,z)}.$$

We show that M_1 is relatively small for large ζ. For all $v \geqslant 1$, we have

$$M_1 \leqslant \sum_{n\leqslant x} v^{\Omega(n,z)-\mathcal{Z}}\tau(n;y,z)t^{\Omega(n,z)} \ll_{v_0} v^{-\mathcal{Z}}x(\log y)^{2vt-2-\beta}$$

provided $vt \leqslant v_0 < 2$. We put $v := \mathcal{Z}/(2t\log_2 y) =: 1+\delta$ (say) and we obtain

$$M_1 \ll x(\log y)^{2t-2-\beta-2tQ(1+\delta)}.$$

Now we have $Q(1+\delta) \leqslant -\frac{1}{3}\,\delta^2$ $(0 \leqslant \delta \leqslant 1)$ and

$$\delta = \frac{\varepsilon\zeta}{2t\sqrt{\log_2 y}} \leqslant 1, \quad vt = t + \tfrac{1}{2}\varepsilon\zeta \leqslant \tfrac{3}{2}.$$

Hence, we may fix $v_0 = \frac{3}{2}$ above and deduce that

$$M_1 \ll x(\log y)^{2t-2-\beta}\exp\Big\{-\tfrac{1}{6}\frac{\varepsilon^2\zeta^2}{t}\Big\} \qquad (\zeta > 0). \tag{23}$$

We set

$$M_2 := \sum_{n\leqslant x}\chi(n)\tau(n;y,z)t^{\Omega(n,z)}$$

and in view of (22) and (23), we find that

$$M_2 = \Big\{1 + O\big(e^{-\varepsilon^2\zeta^2/6t}\big)\Big\}\sum_{n\leqslant x}\tau(n;y,z)t^{\Omega(n,z)}.$$

We split M_2 into two parts, according as $\tau(n;y,z) \geqslant 1$ or not. Plainly, the first part does not exceed $H(x,y,z;t)$, that is

$$M_2 \leqslant H(x,y,z;t) + M_3 \tag{24}$$

where

$$\begin{aligned} M_3 := \sum_{\substack{n\leqslant x\\ \tau(n;y,z)>1}} \chi(n)\tau(n;y,z)t^{\Omega(n,z)} &\leqslant 2\sum_{n\leqslant x}\chi(n)\binom{\tau(n;y,z)}{2}t^{\Omega(n,z)}\\ &\leqslant 2v^{-\mathcal{Z}}\sum_{n\leqslant x}\chi(n)\binom{\tau(n;y,z)}{2}(vt)^{\Omega(n,z)}, \end{aligned}$$

the last inequality being valid provided $v \leqslant 1$. Arguing exactly as in [11], p. 39, and choosing $v = 1/(2t) \leqslant 1$, we arrive at

$$\begin{aligned} M_3 &\ll x(2t)^{\mathcal{Z}}(\log y)^{-2\beta-1}(\log_2 y)^2\\ &\ll x(\log y)^D e^{\varepsilon\zeta(\log 2t)\sqrt{\log_2 y}}(\log_2 y)^2, \end{aligned}$$

with

$$\begin{aligned} D &:= 2t\log(2t) - 2\beta - 1 \\ &= (2t - 2 - \beta) + (2t\log t - 2t + 2) + t\log 4 - \beta - 1 \\ &= 2t - 2 - \beta + 2Q(t) - \zeta/\sqrt{\log_2 y} \\ &\leqslant 2t - 2 - \beta - \varepsilon\zeta/\sqrt{\log_2 y} \end{aligned}$$

by the left hand side of (13). We obtain hence

$$S_3 \ll x(\log y)^{2t-2-\beta} e^{-\frac{1}{2}\varepsilon\zeta(1-\log 2t)\sqrt{\log_2 y}}$$

for $z \geqslant z_0(\varepsilon, \zeta)$. Since $\varepsilon < 1$, and $\zeta \leqslant 2\sqrt{\log_2 y}$, we have

$$\tfrac{1}{2}\varepsilon\zeta(1 - \log 2t) > \frac{\varepsilon^2\zeta^2}{6t}$$

and so

$$M_2 - M_3 = \Big\{1 + O\big(e^{-\varepsilon^2\zeta^2}\big)\Big\} \sum_{n \leqslant x} \tau(n; y, z) t^{\Omega(n,z)}.$$

By (24), the left hand side is a lower bound for $H(x, y, z; t)$ and this completes the proof.

References

[1] M. Balazard, "Remarques sur un théorème de Hálasz et Sárkőzy", *Bull. Soc. Math. de France*, **117** (1989), 387–413.

[2] P. Erdős, "On the distribution of the divisors of integers in the residue classes (mod d)", *Bull. Soc. Math. Grèce* **6** (1965), 27–36.

[3] P. Erdős, "Asymptotische Untersuchen über die Anzahl der Teiler von n", *Math. Annalen* **169** (1967), 230–238.

[4] P. Erdős and R.R. Hall, "Some distribution problems concerning the divisors of integers", *Acta Arithmetica* **26** (1974), 175–188.

[5] P. Erdős and R.R. Hall, "Proof of a conjecture about the distribution of divisors of integers in residue classes", *Math. Proc. Camb. Phil. Soc.* **79** (1976), 281–287.

[6] P. Erdős and R.R. Hall, "Some new results in probabilistic group theory", *Comment. Math. Helvetici* **53** (1987), 448–457.

[7] P. Erdős and A. Rényi, "Probabilistic methods in group theory", *J. d'Analyse Math.* **14** (1965), 127–138.

[8] G. Hálasz, "Remarks to my paper 'On the distribution of additive and the mean value of multiplicative functions' ", *Acta. Math. Acad. Sci. Hungar.* **23** (1972), 425–432.

[9] R.R. Hall, "The maximum value of the divisor function on short intervals", *Bull. London Math. Soc.* **13** (1981), 121–124.

[10] R.R. Hall and G. Tenenbaum, "On the local behaviour of some arithmetical functions", *Acta Arithmetica* **43** (1984), 375–390.

[11] R.R. Hall and G. Tenenbaum, *Divisors.* Cambridge Tracts in Mathematics No. 90 (Cambridge) 1988.

[12] K.K. Norton, "Estimates for partial sums of the exponential series", *J. Math. Analysis and Applications* **63** (1976), 265–296.

[13] A. Sárközy, Remarks to a paper of G. Halász, *Period. Math. Hungar.* **8** (1977), 135–150.

[14] G. Tenenbaum, "Sur la probabilité d'un entier possède un diviseur dans un intervalle donné", *Compositio Mathematica* **51** (1984), 243–263.

[15] G. Tenenbaum, "Un problème de probabilité conditionnelle en Arithmétique", *Acta Arithmetica* **49** (1987), 165–187.

Department of Mathematics
York University
Heslington, York YO15DD
England

Département de Mathématiques
Université de Nancy I
BP 239
54506 Vandœuvre Cedex
France