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Global Strong Solutions for the Two-Dimensional Motion of an Infinite Cylinder in a Viscous Fluid

Takéo Takahashi and Marius Tucsnak

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Abstract. In this paper, we consider a two-dimensional fluid-rigid body problem. The motion of the fluid is modelled by the Navier–Stokes equations, whereas the dynamics of the rigid body is governed by the conservation laws of linear and angular momentum. The rigid body is supposed to be an infinite cylinder of circular cross-section. Our main result is the existence and uniqueness of global strong solutions.

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1. Introduction

We consider the system composed by a homogeneous rigid body, represented by a closed disk $B(t) \subset \mathbb{R}^2$, moving in a viscous incompressible fluid which occupies the domain $\Omega(t)$. We suppose that the system fluid-rigid body fills the whole space, i.e. that we have $\Omega(t) = \mathbb{R}^2 \backslash B(t)$. Without loss, we assume that the disk is of radius 1 and the fluid is homogeneous and of density one.

We shall also assume that the motion of the fluid is described by the classical Navier–Stokes equations, whereas the motion of the rigid body will be governed by the balance equations for linear and angular momentum (Newton's laws).

Hence, we can write the full system of equations modelling the motion of the fluid and the rigid body as

$$\frac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla) U + \nabla P = F, \quad x \in \Omega(t), \ t \in [0,T], \tag{1.1}$$

$$\operatorname{div} U = 0, \quad x \in \Omega(t), \ t \in [0,T], \tag{1.2}$$

$$U = h'(t) + \omega(t)(x - h(t))^{\perp}, \quad x \in \partial B(t), t \in [0,T], \tag{1.3}$$

$$Mh''(t) = -\int_{\partial B(t)} \sigma n \, \mathrm{d}\Gamma + M\zeta, \qquad t \in [0,T], \tag{1.4}$$

$$J\omega'(t) = -\int_{\partial B(t)} (x - h(t))^{\perp} \cdot \sigma n \, \mathrm{d}\Gamma + J\eta, \qquad t \in [0,T], \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega(0), \tag{1.6}$$

$$h(0) = h_0 \in \mathbb{R}^2, \ h'(0) = h_1 \in \mathbb{R}^2, \ \omega(0) = \omega_0 \in \mathbb{R}. \tag{1.7}$$

In the above system the unknowns are $U(x,t)$ (the Eulerian velocity field of the fluid), $P(x,t)$ (the pressure of the fluid), $h(t)$ (the position of the mass center of the rigid body) and $\omega(t)$ (the angular velocity of the rigid body).

The constants M and J are mass and moment of inertia of the rigid body, respectively. Moreover, $F(x,t)$ is the force acting on the fluid, while $M\zeta$ and $J\eta$ are the force and the torque acting on the rigid body.

For all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we denote by $x^{\perp}$ the vector $x^{\perp} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$. We have also denoted by w' and w'' the derivatives of a function w depending only on the time t. If x, $y \in \mathbb{R}^2$, then $x \cdot y$ stands for the inner product of x and y and $|x|$ stands for the corresponding norm. Moreover we have denoted by $\partial B(t)$ the boundary of the rigid body at instant t and by $n(x,t)$ the unit normal to $\partial B(t)$ at the point x directed toward the interior of the rigid body.

The positive constant ν is the viscosity of the fluid.

The Cauchy stress tensor is defined by

$$\sigma(x,t) = -P(x,t)\,\mathrm{Id} + 2\nu D(U),$$

where Id is the identity matrix and $D(U)$ is the tensor field defined by

$$D(U)_{k,l} = \frac{1}{2}\left(\frac{\partial U_k}{\partial x_l} + \frac{\partial U_l}{\partial x_k}\right).$$

Our main result asserts the existence and uniqueness of strong solutions of (1.1)–(1.7). Our method, based on a suitable change of variables, works only in the case of a moving rigid disk. The existence of strong solutions in small time for a rigid body having an arbitrary form has been proved in [7]. The existence of weak solutions (in two or three dimensions) has been treated in Conca, San Martin and Tucsnak [2], Desjardins and Esteban [3] and [4], Gunzburger, Lee and Seregin [9], Hoffmann and Starovoitov [10] and [11], San Martin, Starovoitov and Tucsnak [15] (with the domain of the fluid bounded) and in Serre [16], Judakov [12] (in the case where the fluid-rigid body system fills the whole space). A local (in time) existence result of strong solutions was proved in Grandmont and Maday [8] provided that the fluid-rigid body system occupies a bounded domain and the inertia of the rigid body is large enough with respect to the inertia of the fluid. The stationary problem was studied in [16] and in Galdi [6]. A one-dimensional version of the problem takled in this paper was studied in Vázquez and Zuazua [20] where the asymptotic behavior of solutions has also been investigated.

The plan of the paper is as follows. In Section 2 we introduce some notation and state the main result. In Section 3 we furnish preliminary results, whereas in Section 4 we investigate the linearized problem. The main result is proved in Section 5. In Section 6, existence and uniqueness of weak solutions are proved, by passing to the limit in a sequence of strong solutions.

2. Statement of the main results

In order to transform equations (1.1)–(1.7) in a system written in a fixed domain, we perform the change of variables

$$y = x - h(t), \quad u(y,t) = U(y+h(t),t),$$

$$p(y,t) = P(y+h(t),t), \quad \sigma(y,t) = -p(y,t)\mathrm{Id} + 2\nu D(u)(y,t),$$

$$f(y,t) = F(y+h(t),t).$$

Set $B = B(0)$ and $\Omega = \mathbb{R}^2 \setminus B$. After some calculations (see for instance [16]), equations (1.1)–(1.7) yield the following system in the unknowns $u(y,t)$, $p(y,t)$, $h(t)$ and $\omega(t)$:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u - (h'(t)\cdot\nabla)u + \nabla p = f, \quad \text{in } \Omega\times[0,T], \tag{2.1}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega\times[0,T], \tag{2.2}$$

$$u(y) = h'(t) + \omega(t)y^{\perp}, \qquad y\in\partial B,\ t\in[0,T], \tag{2.3}$$

$$Mh''(t) = -\int_{\partial B}\sigma n\,\mathrm{d}\Gamma + M\zeta, \qquad t\in[0,T], \tag{2.4}$$

$$J\omega'(t) = -\int_{\partial B} y^{\perp}\cdot\sigma n dy + J\eta,\ t\in[0,T], \tag{2.5}$$

$$u(y,0) = u_0(y), \quad y\in\Omega, \tag{2.6}$$

$$h(0) = h_0\in\mathbb{R}^2,\ h'(0) = h_1\in\mathbb{R}^2,\ \omega(0) = \omega_0\in\mathbb{R}. \tag{2.7}$$

Denote by $\widehat{H}^1(\Omega)$ the homogeneous Sobolev space

$$\widehat{H}^1(\Omega) = \{u\in L^2_{loc}(\overline{\Omega}) \mid \nabla u\in[L^2(\Omega)]^2\},$$

where $u\in L^2_{loc}(\overline{\Omega})$ means that $u\in L^2(\Omega\cap B_0)$ for all open balls $B_0\subset\mathbb{R}^2$ with $B_0\cap\Omega\neq\emptyset$. We identify two functions of $\widehat{H}^1(\Omega)$ if they differ by a constant. The main result of this paper is the following.

Theorem 2.1. *Let* $T > 0$. *Suppose* $f \in L^2(0,T;[L^2(\Omega)]^2)$, $\zeta \in L^2(0,T,\mathbb{R}^2)$, $\eta \in L^2(0,T;\mathbb{R})$, $u_0 \in [H^1(\Omega)]^2$ *and that,*

$$\begin{aligned} \operatorname{div} u_0 &= 0, && \text{in } \ \Omega, \\ u_0(y) &= h_1 + \omega_0 y^{\perp}, && y \in \partial B. \end{aligned}$$

Then, there exists a unique strong solution (u, p, h, ω) *of* (2.1)–(2.7) *satisfying*

$$u \in L^2(0,T;[H^2(\Omega)]^2) \cap C([0,T];[H^1(\Omega)]^2) \cap H^1(0,T;[L^2(\Omega)]^2),$$
$$p \in L^2(0,T;\widehat{H}^1(\Omega)), \quad h \in H^2(0,T;\mathbb{R}^2), \quad \omega \in H^1(0,T;\mathbb{R}).$$

Remark 2.2. The function spaces in Theorem 2.1 are similar to those encountered in the global existence result given by Farwig and Sohr in [5] for the Navier–Stokes equations in a exterior domain.

Remark 2.3. In the more difficult case of several rigid bodies and a fluid filling a bounded domain it was shown in [3] that, for initial data having the same regularity as in the above theorem, there exists at least one solution having the regularity $u \in L^2\left(0,T;W_0^{1,p}\right)$, for all $p \geqslant 1$.

Following [2], we define a weak solutions of (2.1)–(2.7) as follows:

Definition 2.4. A triplet (u, h, ω) is called a weak solution of (2.1)–(2.7) if

$$u \in L^2(0,T;[H^1(\Omega)]^2) \cap L^\infty(0,T;[L^2(\Omega)]^2),$$
$$h \in W^{1,\infty}(0,T;\mathbb{R}^2), \quad \omega \in L^\infty(0,T;\mathbb{R}),$$

if

$$\begin{cases} \operatorname{div}(u) = 0 \quad \text{in} \quad \Omega, \\ u(y,t) = h'(t) + \omega(t) y^{\perp}, \quad \forall y \in \partial B, \end{cases}$$

and if for all triplets $(v, l, k) \in [H^1(\Omega)]^2 \times \mathbb{R}^2 \times \mathbb{R}$, such that

$$\begin{cases} \operatorname{div}(v) = 0 \quad \text{in} \quad \Omega, \\ v(y) = l + k y^{\perp}, \quad \forall y \in \partial B, \end{cases}$$

we have

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega u(t)\cdot v \, \mathrm{d}y + M h'(t)\cdot l + J\omega(t)k\right) &+ 2\nu \int_\Omega D(u(t)) : D(v) \, \mathrm{d}y \\ + \int_\Omega \left[(u(t)\cdot\nabla)\, u(t)\right]\cdot v \, \mathrm{d}y &- \int_\Omega \left[(h'(t)\cdot\nabla)\, u(t)\right]\cdot v \, \mathrm{d}y \\ &= \int_\Omega f(t)\cdot v \, \mathrm{d}y + M\zeta(t)\cdot l + J\eta(t)k, \quad \text{a.e. in } (0,T), \end{aligned} \tag{2.8}$$

and (2.6)–(2.7).

Above, we used the standard notation $M : N \equiv \mathrm{Trace}(M^T N)$, for second order tensors M, N, with M^T the transpose of M. Using Theorem 2.1, we can deduce the existence and the uniqueness of weak solutions. More precisely we have

Proposition 2.5. *Let* $T > 0$. *Assume* $f \in L^2(0,T;[L^2(\Omega)]^2)$, $\zeta \in L^2(0,T;\mathbb{R}^2)$, $\eta \in L^2(0,T;\mathbb{R})$ *and* $u_0 \in [L^2(\Omega)]^2$ *with* $\operatorname{div} u_0 = 0$ *in* Ω. *Then, the problem* (2.1)–(2.7) *has a unique weak solution on* $[0,T]$.

Remark 2.6. The existence of weak solutions for this problem was given, in slightly different spaces, in [16].

For the sake of simplicity, we prove Theorem 2.1 and Proposition 2.5 in the case when $f = 0$, $\zeta = 0$ and $\eta = 0$.

3. Preliminaries

We first recall several inequalities which are close to classical results (see, for instance, [18, pp. 11]). However, for the sake of completeness, we sketch here the proof.

Lemma 3.1. 1. *There exists a positive constant* C *such that*

$$\left|\int_\Omega [(u\cdot\nabla)\, v]\cdot w \,\mathrm{d}y\right| \leqslant C\|u\|^{1/2}_{[L^2(\Omega)]^2}\|u\|^{1/2}_{[H^1(\Omega)]^2}\|v\|_{[H^1(\Omega)]^2}\|w\|^{1/2}_{[L^2(\Omega)]^2}\|w\|^{1/2}_{[H^1(\Omega)]^2},$$

for all u, v, $w \in [H^1(\Omega)]^2$.

2. *There exists a positive constant* C *such that*

$$\left|\int_\Omega [(u\cdot\nabla)\, u]\cdot v \,\mathrm{d}y\right| \leqslant C\|u\|^{1/2}_{[L^2(\Omega)]^2}\|u\|_{[H^1(\Omega)]^2}\|u\|^{1/2}_{[H^2(\Omega)]^2}\|v\|_{[L^2(\Omega)]^2},$$

for all $u \in [H^2(\Omega)]^2$ *and for all* $v \in [L^2(\Omega)]^2$.

Proof. We use the Hölder's inequalities under the following form: for all functions $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $h \in L^r(\Omega)$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, we have that

$$\left|\int_\Omega fgh\,\mathrm{d}y\right| \leqslant \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}. \tag{3.1}$$

Thus, for all u, v, $w \in [H^1(\Omega)]^2$,

$$\left|\int_\Omega u_i\frac{\partial v_j}{\partial y_i}w_j\,\mathrm{d}y\right| \leqslant \|u_i\|_{L^4(\Omega)}\left\|\frac{\partial v_j}{\partial y_i}\right\|_{L^2(\Omega)}\|w_j\|_{L^4(\Omega)}, \tag{3.2}$$

and for all $u \in [H^2(\Omega)]^2$, $v \in [L^2(\Omega)]^2$,

$$\left|\int_\Omega u_i\frac{\partial u_j}{\partial y_i}v_j\,\mathrm{d}y\right| \leqslant \|u_i\|_{L^4(\Omega)}\left\|\frac{\partial u_j}{\partial y_i}\right\|_{L^4(\Omega)}\|v_j\|_{L^2(\Omega)}. \tag{3.3}$$

Moreover, using the continuous embedding of $H^{1/2}(\Omega)$ in $L^4(\Omega)$ and the interpolation inequality in Lions–Magenes [13], we have that

$$\forall z \in H^1(\Omega), \quad \|z\|_{L^4(\Omega)} \leqslant C\|z\|_{H^{1/2}(\Omega)} \leqslant C\|z\|^{1/2}_{L^2(\Omega)}\|z\|^{1/2}_{H^1(\Omega)}, \tag{3.4}$$

Relations (3.2) and (3.4) yield the first inequality stated in the lemma, whereas relations (3.3) and (3.4) imply the second inequality in the statement of the lemma. □

We next state the following direct consequence of Theorem 2.1 in [5]:

Theorem 3.2. *For every* $f \in [L^2(\Omega)]^2$, $\psi \in [H^2(\Omega)]^2$ *with* $\int_{\partial B} \psi \cdot n \, \mathrm{d}\Gamma = 0$, *there exists a unique pair* $(u, p) \in [H^2(\Omega)]^2 \times \widehat{H}^1(\Omega)$ *solving*

$$u - \nu\Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial B} = \psi|_{\partial B}. \tag{3.5}$$

Morover there exists a constant C *depending only on* Ω *such that*

$$\|u\|_{[H^2(\Omega)]^2} + \|\nabla p\|_{L^2(0,T;[L^2(\Omega)]^2)} \leqslant C\left(\|f\|_{[L^2(\Omega)]^2} + \|\psi\|_{[H^2(\Omega)]^2}\right).$$

We also recall the following well-known result:

Proposition 3.3. *Let* H *be a Hilbert space with inner product* $(\cdot,\cdot)$ *and norm* $\|\cdot\|$. *Let* $A : D(A) \to H$ *be a self-adjoint and accretive operator. Moreover, suppose that* $f \in L^2(0,T;H)$, $u_0 \in D(A^{1/2})$. *Then, the problem*

$$u' + Au = f, \quad u(0) = u_0 \tag{3.6}$$

has a unique solution

$$u \in L^2(0,T;D(A)) \cap C([0,T];D(A^{1/2})) \cap H^1(0,T;H). \tag{3.7}$$

Moreover, there exists a positive constant K *such that*

$$\begin{aligned} &\|u\|_{L^2(0,T;D(A))} + \|u\|_{L^\infty(0,T;D(A^{1/2}))} + \|u\|_{H^1(0,T;H)} \\ &\leqslant K\left(\|u\|_{D(A^{1/2})} + \|f\|_{L^2(0,T;H)}\right). \end{aligned} \tag{3.8}$$

The constant K *depends only on the operator* A *and on* T. *Moreover* K *is nondecreasing with respect to* T.

The proof of this proposition can be found, for instance, in Bensoussan et al. [1, Lemma 3.3 and Theorem 3.1].

4. Analysis of the linearized problem

In this section we consider the following initial and boundary value problem, obtained by neglecting the nonlinear terms in (2.1)–(2.7).

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = f, \quad \text{in } \Omega \times [0,T], \tag{4.1}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega \times [0,T], \tag{4.2}$$

$$u(y) = h'(t) + \omega(t)y^{\perp}, \qquad y \in \partial B,\ t \in [0,T], \tag{4.3}$$

$$Mh''(t) = -\int_{\partial B} \sigma n \,\mathrm{d}\Gamma, \qquad t \in [0,T], \tag{4.4}$$

$$J\omega'(t) = -\int_{\partial B} y^{\perp} \cdot \sigma y \,\mathrm{d}y, \qquad t \in [0,T], \tag{4.5}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{4.6}$$

$$h(0) = h_0 \in \mathbb{R}^2,\ h'(0) = h_1 \in \mathbb{R}^2,\ \omega(0) = \omega_0 \in \mathbb{R}, \tag{4.7}$$

where

$$\sigma(y,t) = -p(y,t)\mathrm{Id} + 2\nu D(u)(y,t).$$

We extend u to a function defined on the whole space by putting $u = h'(t)+\omega(t)y^{\perp}$ in B. Consider the following function spaces:

$$\mathcal{H} = \{\phi \in [L^2(\mathbb{R}^2)]^2 \mid \operatorname{div}(\phi) = 0 \quad \text{in } \mathbb{R}^2, \quad D(\phi) = 0 \quad \text{in } B\}, \tag{4.8}$$

$$\mathcal{V} = \{\phi \in [H^1(\mathbb{R}^2)]^2 \mid \operatorname{div}(\phi) = 0 \quad \text{in } \mathbb{R}^2, \quad D(\phi) = 0 \quad \text{in } B\}. \tag{4.9}$$

According to Lemma 1.1 in [19, pp. 18], for all $\phi \in \mathcal{H}$, there exists $V_\phi \in \mathbb{R}^2$ and $\omega_\phi \in \mathbb{R}$ such that

$$\phi(y) = V_\phi + \omega_\phi y^{\perp}, \quad \text{in } B.$$

Define an inner product on $[L^2(\mathbb{R}^2)]^2$ by

$$(\psi,\phi)_{[L^2(\mathbb{R}^2)]^2} = \int_\Omega (\psi\cdot\phi)\,\mathrm{d}y + \int_B (\rho\psi\cdot\phi)\,\mathrm{d}y$$

where $\rho > 0$ is the density of the rigid body. The corresponding norm is equivalent to the usual norm in $[L^2(\mathbb{R}^2)]^2$. If ψ and ϕ belong to $\mathcal{H}$, then a simple calculation shows that

$$(\psi,\phi)_{[L^2(\mathbb{R}^2)]^2} = \int_\Omega (\psi\cdot\phi)\,\mathrm{d}y + MV_\phi\cdot V_\psi + J\omega_\phi\omega_\psi \tag{4.10}$$

In order to solve (4.1)–(4.7) we use a semi-group approach. Define

$$D(A) = \{\phi \in [H^1(\mathbb{R}^2)]^2 \mid \phi_{|\Omega} \in [H^2(\Omega)]^2,\ \operatorname{div}(\phi) = 0 \text{ in } \mathbb{R}^2, \\ D(\phi) = 0 \text{ in } B\} \tag{4.11}$$

$$\mathcal{A}u = \begin{cases} -\nu\Delta u \text{ in } \Omega, \\ \dfrac{2\nu}{M}\displaystyle\int_{\partial B} D(u)n \,\mathrm{d}\Gamma + \left[\dfrac{2\nu}{J}\displaystyle\int_{\partial B} y^{\perp}\cdot D(u)n \,\mathrm{d}\Gamma\right] y^{\perp} \text{ in } B, \end{cases} \quad \forall u \in D(A), \tag{4.12}$$

and

$$\forall u \in D(A), \quad Au = \mathbb{P}\mathcal{A}u, \tag{4.13}$$

where $\mathbb{P}$ is the orthogonal projector from $[L^2(\mathbb{R}^2)]^2$ onto $\mathcal{H}$ ($\mathcal{H}$ is clearly a closed subspace of $[L^2(\mathbb{R}^2)]^2$) and where, in the expression of $\mathcal{A}u$, $D(u)$ represents the trace of the restriction of $D(u)$ to $\Omega = \mathbb{R}^2 \setminus B$. The simple result below will be used several times in the sequel.

Lemma 4.1. *Suppose that $u \in \mathcal{V}$, where $\mathcal{V}$ is defined by* (4.9)*. Then*

$$\forall u \in D(A), \quad \|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} = 2\|D(u)\|^2_{[L^2(\mathbb{R}^2)]^4}.$$

Proof. Let $u \in \mathcal{D}(\mathbb{R}^2)^2$ be a C^∞ function with compact support. Then, after some simple calculations, we get

$$\nabla u : \nabla u - 2D(u) : D(u) = \operatorname{div}\left((\nabla u)^T u - \operatorname{div}(u) u\right) + (\operatorname{div}(u))^2.$$

It follows that

$$\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} = 2\|D(u)\|^2_{[L^2(\mathbb{R}^2)]^4} + \|\operatorname{div}(u)\|^2_{L^2(\mathbb{R}^2)}$$

and thus, by density, the above inequality holds for all $u \in [H^1(\mathbb{R}^2)]^2$. Hence for all $u \in D(A)$, we have that

$$\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} = 2\|D(u)\|^2_{[L^2(\mathbb{R}^2)]^4}. \qquad \square$$

Proposition 4.2. *The operator A defined by* (4.11), (4.12) *and* (4.13) *is self-adjoint and positive. Moreover, for any $u \in D(A)$, we have that*

$$\|u\|_{[H^2(\Omega)]^2} \leqslant C\|(I + A)u\|_{[L^2(\Omega)]^2}. \tag{4.14}$$

Proof. Let u, $v \in D(A)$. According to (4.10) and (4.12), we have

$$\begin{aligned}(Au, v)_{[L^2(\mathbb{R}^2)]^2} &= (\mathcal{A}u, v)_{[L^2(\mathbb{R}^2)]^2} \\ &= -\nu\int_{\Omega} \Delta u \cdot v \,\mathrm{d}y + 2\nu V_v \cdot \int_{\partial B} D(u)n \mathrm{d}\Gamma + 2\nu\left[\int_{\partial B} n^{\perp}\cdot D(u)n \,\mathrm{d}\Gamma\right]\omega_v.\end{aligned}$$

On the other hand,

$$\Delta u \cdot v = 2\operatorname{div}(D(u)) \cdot v = 2\operatorname{div}(D(u)v) - 2D(u) : D(v)$$

thus, by (4.11), for all u, $v \in D(A)$, we have

$$(Au,v)_{[L^2(\mathbb{R}^2)]^2} = \int_\Omega 2\nu D(u) : D(v)\,\mathrm{d}y - \nu \int_{\partial B} 2D(u)v\cdot n\,\mathrm{d}\Gamma + 2\nu V_v \cdot \int_{\partial B} D(u)n\,\mathrm{d}\Gamma$$
$$+ 2\nu \left[\int_{\partial B} n^\perp \cdot D(u)n\,\mathrm{d}\Gamma\right]\omega_v = \int_\Omega 2\nu D(u) : D(v)\,\mathrm{d}y = (u, Av)_{[L^2(\mathbb{R}^2)]^2}. \quad (4.15)$$

It follows that A is symmetric. Moreover, for all $u \in D(A)$, we have

$$(Au,u)_{[L^2(\mathbb{R}^2)]^2} = \int_\Omega 2\nu |D(u)|^2\,\mathrm{d}y \geqslant 0.$$

Thus $-A$ dissipative. In order to prove A is self-adjoint, it suffices to check that $I + A : D(A) \to \mathcal{H}$ is onto (where I is the identity mapping). Let $f \in \mathcal{H}$. We have to show that there exists $u \in D(A)$ such that

$$(I + A)u = f, \quad (4.16)$$

which is equivalent to

$$(u,\phi)_{[L^2(\mathbb{R}^2)]^2} + (Au,\phi)_{[L^2(\mathbb{R}^2)]^2} = (f,\phi)_{[L^2(\mathbb{R}^2)]^2}, \quad \forall \phi \in \mathcal{H}.$$

By using (4.12) and (4.15), the above equality is equivalent to

$$(u,\phi)_{[L^2(\mathbb{R}^2)]^2} + \int_\Omega 2\nu D(u) : D(\phi)\,\mathrm{d}y = (f,\phi)_{[L^2(\mathbb{R}^2)]^2}, \quad (4.17)$$

for any $\phi \in \mathcal{V}$. The bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by

$$a(u,\phi) = (u,\phi)_{[L^2(\mathbb{R}^2)]^2} + \int_\Omega 2\nu D(u) : D(\phi)\,\mathrm{d}y, \quad \forall u, \phi \in \mathcal{V}$$

is continuous on the Hilbert space $\mathcal{V}$. Moreover, by using Lemma 4.1, a is elliptic in the sense that there exists $C > 0$ such that

$$a(u,u) \geqslant C\|u\|^2_{[H^1(\mathbb{R}^2)]^2}, \quad \forall u \in \mathcal{V}.$$

On the other hand, the mapping

$$\phi \mapsto (f,\phi)_{[L^2(\mathbb{R}^2)]^2}$$

is a linear and continuous form on $\mathcal{V}$, thus by the Lax–Milgram theorem, there exists a unique $u \in \mathcal{V}$ which satisfies (4.17). In particular, (4.17) holds if

$$\phi = \begin{cases} \widetilde{\phi} & \text{in } \Omega, \\ 0 & \text{in } B, \end{cases}$$

where $\widetilde{\phi}$ is a C^∞ function with compact support such that $\operatorname{div}(\widetilde{\phi}) = 0$. By using such ϕ, and according to Proposition 1.1 and Proposition 1.2 from Temam [17], there exists $p \in \mathcal{D}'(\Omega)$ such that

$$u(t) - \nu\Delta u(t) + \nabla p(t) = f(t), \quad \text{in } \mathcal{D}'(\Omega).$$

Since $u \in \mathcal{V}$, we have that $\operatorname{div}(u) = 0$ in Ω and there exists $l \in \mathbb{R}^2$ and $k \in \mathbb{R}$ with

$$u(y) = l + ky^{\perp}, \quad \forall y \in B.$$

The above relations imply that u satisfies the Stokes type system:

$$\begin{aligned} u - \nu \Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u(y) &= l + ky^{\perp} && \text{on } \partial B. \end{aligned}$$

Consider $\theta \in C^{\infty}(\mathbb{R}^2, \mathbb{R})$ with support contained in

$$B(0,2) = \{x \in \mathbb{R}^2 \mid |x| \leqslant 2\}$$

and equal to 1 in B. Then

$$\psi(y) = \theta(y)\left(l + ky^{\perp}\right) \in [H^2(\Omega)]^2, \tag{4.18}$$

and

$$\int_{\partial B} \psi \cdot n \, \mathrm{d}\Gamma = 0.$$

Thus, by Theorem 3.2, $u \in [H^2(\Omega)]^2$ and

$$\|u\|_{[H^2(\Omega)]^2} \leqslant C\left(\|f\|_{[L^2(\Omega)]^2} + \|\psi\|_{[H^2(\Omega)]^2}\right). \tag{4.19}$$

Thus by (4.11) and by (4.17), $u \in D(A)$ and $(I + A)u = f$.
If we take $\phi = u$ in (4.17), then we get that

$$\|u\|^2_{[L^2(\Omega)]^2} \leqslant \|f\|^2_{[L^2(\Omega)]^2},$$

which, in view of (4.10), implies that

$$M|l|^2 + Jk^2 \leqslant \|f\|^2_{[L^2(\Omega)]^2}.$$

The above equality and relation (4.18) yield

$$\|\psi\|_{[H^2(\Omega)]^2} \leqslant C\|f\|_{[L^2(\Omega)]^2}.$$

By using relation (4.19), we obtain (4.14). □

A consequence of Proposition 4.2 is:

Corollary 4.3. *Let $f \in L^2(0,T;[L^2(\Omega)]^2)$ and $u_0 \in [H^1(\Omega)]^2$ such that*

$$\begin{aligned} &\operatorname{div} u_0 = 0, && \text{in } \Omega, \\ &u_0(y) = h_1 + \omega_0 y^{\perp}, && y \in \partial B. \end{aligned}$$

Then the system (4.1)–(4.7) admits a unique solution (u, p, h, ω) with

$$u \in L^2(0,T;[H^2(\Omega)]^2) \cap C([0,T];[H^1(\Omega)]^2) \cap H^1(0,T;[L^2(\Omega)]^2),$$

$$p \in L^2(0,T;\widehat{H}^1(\Omega)), \quad h \in H^2(0,T;\mathbb{R}^2), \quad \omega \in H^1(0,T;\mathbb{R}).$$

Moreover, there exists a positive constant K such that

$$\|u\|_{L^2(0,T;[H^2(\Omega)]^2)}+\|u\|_{L^\infty(0,T;[H^1(\Omega)]^2)}+\|u\|_{H^1(0,T;[L^2(\Omega)]^2)}+\|\nabla p\|_{L^2(0,T;[L^2(\Omega)]^2)}$$
$$+\|h'\|_{H^1(0,T;\mathbb{R}^2)}+\|\omega\|_{H^1(0,T;\mathbb{R})}\leqslant K\left(\|u_0\|_{[H^1(\mathbb{R}^2)]^2}+\|f\|_{L^2(0,T;[L^2(\Omega)]^2)}\right). \quad (4.20)$$

The constant K depends only on Ω and on T and it is non-decreasing with respect to T.

Proof. We apply Proposition 3.3. We begin to observe that

$$D(A^{1/2})=\{\phi\in[H^1(\mathbb{R}^2)]^2 \mid \operatorname{div}(\phi)=0 \quad \text{in } \mathbb{R}^2, \quad D(\phi)=0 \quad \text{in } B\}=\mathcal{V}.$$

Indeed, according to (4.15), the graph's norm of $D(A^{1/2})$ is

$$\begin{aligned}\|u\|^2_{D(A^{1/2})} &= (u,u)_{[L^2(\mathbb{R}^2)]^2}+(Au,u)_{[L^2(\mathbb{R}^2)]^2}\\ &= (u,u)_{[L^2(\mathbb{R}^2)]^2}+2\nu(D(u),D(u))_{[L^2(\mathbb{R}^2)]^4}.\end{aligned}$$

Thus by Lemma 4.1, the norm of $D(A^{1/2})$ is equivalent to the norm of $[H^1(\mathbb{R}^2)]^2$. Moreover, if we extend u_0, f to $\mathbb{R}^2$ by

$$\begin{aligned}u_0(y)&=h_1+\omega_0 y^\perp, && \forall y\in B,\\ f(y)&=0, && \forall y\in B,\end{aligned}$$

then $u_0\in D(A^{1/2})$ and $f\in L^2\left(0,T;\left[L^2(\mathbb{R}^2)\right]^2\right)$. According to Proposition 3.3, the Cauchy problem

$$u'+Au=\mathbb{P}f, \quad (4.21)$$
$$u(0)=u_0 \quad (4.22)$$

admits a unique solution

$$u\in L^2(0,T;D(A))\cap C([0,T];D(A^{1/2}))\cap H^1(0,T;\mathcal{H}).$$

Since $u\in H^1(0,T;\mathcal{H})$, there exist $V\in H^1(0,T;\mathbb{R}^2)$ and $\omega\in H^1(0,T;\mathbb{R})$ such that $u(y,t)=V(t)+\omega(t)y^\perp$ for all $y\in B$ and for all $t\in[0,T]$. If we denote

$$h(t)=h_0+\int_0^t V(s)\,ds,$$

then $h\in H^2(0,T;\mathbb{R}^2)$ and $u(y,t)=h'(t)+\omega(t)y^\perp$ for all $y\in B$ and for all $t\in[0,T]$.

Let $\phi\in\mathcal{H}$. By taking the inner product of (4.21) with ϕ, we obtain

$$(u'(t),\phi)_{[L^2(\mathbb{R}^2)]^2}+(\mathcal{A}u(t),\phi)_{[L^2(\mathbb{R}^2)]^2}=(f(t),\phi)_{[L^2(\mathbb{R}^2)]^2}$$

for almost all $t\in[0,T]$. Thus

$$\int_\Omega u'(t)\cdot\phi\,\mathrm{d}y+Mh''(t)\cdot V_\phi+J\omega'(t)\omega_\phi+\int_\Omega(-\nu\Delta u)\cdot\phi\,\mathrm{d}y$$
$$+2\nu\left(\int_{\partial B}D(u)n\,\mathrm{d}\Gamma\right)\cdot V_\phi+2\nu\left(\int_{\partial B}n^\perp\cdot D(u)n\,\mathrm{d}\Gamma\right)\omega_\phi=\int_\Omega f(t)\cdot\phi\,\mathrm{d}y, \quad (4.23)$$

for all $\phi \in \mathcal{H}$ and for almost all $t \in [0, T]$. In particular, (4.23) holds if

$$\phi = \begin{cases} \psi & \text{in } \Omega, \\ 0 & \text{in } B, \end{cases}$$

where ψ is a C^∞ function with compact support such that $\operatorname{div} \psi = 0$. For such ϕ, (4.23) gives

$$\int_\Omega (u'(t) - \nu \Delta u(t) - f(t)) \cdot \psi \, \mathrm{d}y = 0,$$

for all C^∞ function ψ with compact support such that $\operatorname{div} \psi = 0$. By using Proposition 1.1 and Proposition 1.2 from Temam [17], there exists $p \in L^2(0, T; \widehat{H}^1(\Omega))$ such that

$$u'(t) - \nu \Delta u(t) + \nabla p(t) = f(t), \quad \text{in } \Omega.$$

Hence we have showed that u satisfies (4.1). Moreover, the above relation implies that

$$\int_\Omega (u'(t) - \nu \Delta u(t) + \nabla p(t) - f(t)) \cdot \phi = 0, \quad \forall \phi \in \mathcal{H},$$

which yields

$$\int_\Omega (u'(t) - \nu \Delta u(t) - f(t)) \cdot \phi \mathrm{d}y = - \int_{\partial B} p(t) n \cdot \phi \mathrm{d}\Gamma, \quad \forall \phi \in \mathcal{H}.$$

The above equality and (4.23) imply that

$$Mh''(t) \cdot V_\phi + J\omega'(t)\omega_\phi + 2\nu \left(\int_{\partial B} D(u) n \, \mathrm{d}\Gamma \right) \cdot V_\phi$$
$$+ 2\nu \left(\int_{\partial B} n^\perp \cdot D(u) n \, \mathrm{d}\Gamma \right) \omega_\phi = - \int_{\partial B} p(t) n \cdot \phi \mathrm{d}\Gamma, \quad \forall \phi \in \mathcal{H},$$

from which we deduce that

$$Mh''(t) = - \int_{\partial B} \sigma n \, \mathrm{d}\Gamma,$$
$$J\omega'(t) = - \int_{\partial B} y^\perp \cdot \sigma y dy.$$

Hence (u, p, h, ω) is a solution of the system (4.1)–(4.7).

In order to prove uniqueness, it suffices to remark that solutions of the system (4.1)–(4.7) satisfy the system (4.21)–(4.22) whose solutions are unique by Proposition 3.3.

Relation (4.20) is a direct consequence of (3.8). □

5. Proof of the main result

The aim of this section is to prove Theorem 2.1. We first give the following local in time existence result.

Proposition 5.1. *For every* $u_0 \in [H^1(\Omega)]^2$ *such that*

$$\begin{aligned} \operatorname{div} u_0 &= 0, & &\text{in } \ \Omega, \\ u_0(y) &= h_1 + \omega_0 y^{\perp}, & &y \in \partial B, \end{aligned}$$

there exists $T_0 > 0$ *such that equations* (2.1)–(2.7) *admit a strong solution* (u, p, h, ω) *for any* $T < T_0$*. Moreover, we can choose* T_0 *such that one of the following alternatives holds true:*

(i) $T_0 = +\infty$

(ii) *the function* $t \to \|u(t)\|_{[H^1(\mathbb{R}^2)]^2}$ *is not bounded on* $[0, T_0)$.

In order to prove the above proposition, we need the following result.

Lemma 5.2. *For all*

$$v, w \in L^2(0,T;[H^2(\Omega)]^2) \cap L^\infty(0,T;[H^1(\Omega)]^2) \cap H^1(0,T;[L^2(\Omega)]^2),$$

we have $(w \cdot \nabla)v \in L^{5/2}(0,T;[L^2(\Omega)]^2)$ *and*

$$\begin{aligned} &\|(w \cdot \nabla)v\|_{L^{5/2}(0,T;[L^2(\Omega)]^2)} \\ &\leqslant C\|w\|_{L^\infty(0,T;[H^1(\Omega)]^2)} \|v\|^{1/5}_{L^\infty(0,T;[H^1(\Omega)]^2)} \|v\|^{4/5}_{L^2(0,T;[H^2(\Omega)]^2)}, \end{aligned} \tag{5.1}$$

where C *is a positive constant depending only on* Ω.

Proof. By a classical Sobolev embedding theorem, we have that

$$H^1(\Omega) \subset L^q(\Omega), \quad \forall q \geqslant 2,$$

thus, for almost every t,

$$w_i(t) \in L^5(\Omega), \quad \frac{\partial v_j}{\partial y_i}(t) \in L^2(\Omega), \quad \frac{\partial v_j}{\partial y_i}(t) \in L^4(\Omega),$$

and there exists a constant $C > 0$ such that

$$\|w_i(t)\|_{L^5(\Omega)} \leqslant C\|w_i(t)\|_{H^1(\Omega)}, \quad \left\|\frac{\partial v_j}{\partial y_i}(t)\right\|_{L^2(\Omega)} \leqslant C\|v_i(t)\|_{H^1(\Omega)},$$

$$\left\|\frac{\partial v_j}{\partial y_i}(t)\right\|_{L^4(\Omega)} \leqslant C\|v_i(t)\|_{H^2(\Omega)}.$$

Thus, by the Hölder's inequality,

$$\begin{aligned} \int_\Omega [w_i(y,t)]^2 \left[\frac{\partial v_j}{\partial y_i}(y,t)\right]^2 dy &= \int_\Omega [w_i(y,t)]^2 \left[\frac{\partial v_j}{\partial y_i}(y,t)\right]^{2/5} \left[\frac{\partial v_j}{\partial y_i}(y,t)\right]^{8/5} dy \\ &\leqslant \|\, [w_i(t)]^2 \,\|_{L^{5/2}(\Omega)} \left\| \left[\frac{\partial v_j}{\partial y_i}(t)\right]^{2/5} \right\|_{L^5(\Omega)} \left\| \left[\frac{\partial v_j}{\partial y_i}(t)\right]^{8/5} \right\|_{L^{5/2}(\Omega)} \end{aligned}$$

$$\begin{aligned}
&= \|w_i(t)\|^2_{L^5(\Omega)} \left\|\frac{\partial v_j}{\partial y_i}(t)\right\|^{2/5}_{L^2(\Omega)} \left\|\frac{\partial v_j}{\partial y_i}(t)\right\|^{8/5}_{L^4(\Omega)} \\
&\leqslant C\left(\|w_i(t)\|^2_{H^1(\Omega)}\|v_i(t)\|^{2/5}_{H^1(\Omega)}\|v_i(t)\|^{8/5}_{H^2(\Omega)}\right),
\end{aligned}$$

We have thus proved that there exists a constant $C > 0$ such that

$$\|(w(t)\cdot\nabla)v(t)\|_{[L^2(\Omega)]^2} \leqslant C\left(\|w(t)\|_{[H^1(\Omega)]^2}\|v(t)\|^{1/5}_{[H^1(\Omega)]^2}\|v(t)\|^{4/5}_{[H^2(\Omega)]^2}\right), \quad \text{a.e. in } (0,T).$$

We deduce that

$$\begin{aligned}
&\|(w\cdot\nabla)v\|^{5/2}_{L^{5/2}(0,T;[L^2(\Omega)]^2)} \\
&\quad = \int_0^T \|(w\cdot\nabla)v\|^{5/2}_{[L^2(\Omega)]^2} \\
&\quad \leqslant C\int_0^T \|w(t)\|^{5/2}_{[H^1(\Omega)]^2}\|v(t)\|^{1/2}_{[H^1(\Omega)]^2}\|v(t)\|^{2}_{[H^2(\Omega)]^2}\,dt \\
&\quad \leqslant C\|w\|^{5/2}_{L^\infty(0,T;[H^1(\Omega)]^2)}\|v\|^{1/2}_{L^\infty(0,T;[H^1(\Omega)]^2)}\|v\|^{2}_{L^2(0,T;[H^2(\Omega)]^2)},
\end{aligned}$$

and the proof of the lemma is achieved. □

Proof of Proposition 5.1. Denote by $\mathcal{Z}$ the mapping

$$\begin{aligned}
\mathcal{Z} : L^2(0,T;[L^2(\Omega)]^2) &\to L^2(0,T;[L^2(\Omega)]^2) \\
f &\mapsto -(u\cdot\nabla)u + (h'(t)\cdot\nabla)u
\end{aligned}$$

where (u, p, h, ω) is the solution of (4.1)–(4.7). Denote

$$B(0,R) = \{u \in L^2(0,T;[L^2(\Omega)]^2) \mid \|u\|_{L^2(0,T;[L^2(\Omega)]^2)} \leqslant R\}.$$

The proof of the local in time existence will be divided into 3 steps.

The first step is to show that for T small enough and for $R > 0$ large enough, $\mathcal{Z}$ maps $B(0,R)$ into itself. In the sequel, we denote by C_0 a constant depending only on $\Omega, |h_1|, |h_0|, |\omega_0|, \|u_0\|_{[H^1(\Omega)]^2}$ which is non-decreasing with respect to $\|u_0\|_{[H^1(\mathbb{R}^2)]^2}$ and to $|h_0|$. Moreover, in the rest of this paper, we denote by C a constant depending only on Ω and on T_a. Let $f \in B(0,R)$, $T_a > 0$ and $T \leqslant T_a$. By using Corollary 4.3, we have that

$$\begin{aligned}
&\|u\|_{L^2(0,T;[H^2(\Omega)]^2)} + \|u\|_{L^\infty(0,T;[H^1(\Omega)]^2)} + \|u\|_{H^1(0,T;[L^2(\Omega)]^2)} + \|h\|_{H^2(0,T;\mathbb{R}^2)} \\
&\qquad \leqslant C(R + \|u_0\|_{[H^1(\Omega)]^2} + |h_1| + |h_0| + |\omega_0|) \leqslant C(C_0 + R). \qquad (5.2)
\end{aligned}$$

By using Lemma 5.2, we have that $(u\cdot\nabla)u \in L^{5/2}(0,T;[L^2(\Omega)]^2)$ and that

$$\|(u\cdot\nabla)u\|_{L^{5/2}(0,T;[L^2(\Omega)]^2)} \leqslant C\|u\|^{6/5}_{L^\infty(0,T;[H^1(\Omega)]^2)}\|u\|^{4/5}_{L^2(0,T;[H^2(\Omega)]^2)} \leqslant C(C_0+R)^2. \qquad (5.3)$$

On the other hand,

$$\|(h'(t)\cdot\nabla)\,u(t)\|_{[L^2(\Omega)]^2} \leqslant C|h'(t)|\,\|\nabla u(t)\|_{[L^2(\Omega)]^4}, \quad \forall t\in(0,T)$$

thus $(h'(t)\cdot\nabla)\,u \in L^\infty(0,T;[L^2(\Omega)]^2)$. By using the continuous imbedding of $H^1(0,T;\mathbb{R}^2)$ in $L^\infty(0,T;\mathbb{R}^2)$, we get that

$$\begin{aligned}\|(h'(t)\cdot\nabla)\,u\|_{L^\infty(0,T;[L^2(\Omega)]^2)} &\leqslant C\|h'\|_{L^\infty(0,T;\mathbb{R}^2)}\|u\|_{L^\infty(0,T;[H^1(\Omega)]^2)}\\ &\leqslant C\|h\|_{H^2(0,T;\mathbb{R}^2)}\|u\|_{L^\infty(0,T;[H^1(\Omega)]^2)}.\end{aligned}$$

By (5.2) and the definition of C_0, it follows that

$$\|(h'(t)\cdot\nabla)\,u\|_{L^{5/2}(0,T;[L^2(\Omega)]^2)} \leqslant C\,\|(h'(t)\cdot\nabla)\,u\|_{L^\infty(0,T;[L^2(\Omega)]^2)} \leqslant C(C_0+R)^2.$$

Thus, we have shown that

$$\|\mathcal{Z}(f)\|_{L^{5/2}(0,T;[L^2(\Omega)]^2)} \leqslant C(C_0+R)^2.$$

Hence, by Hölder's inequality, $\mathcal{Z}$ maps $L^2(0,T;[L^2(\Omega)]^2)$ into itself and

$$\|\mathcal{Z}(f)\|_{L^2(0,T;[L^2(\Omega)]^2)} \leqslant T^{1/10}C(C_0+R)^2.$$

Therefore for $R > C_0$ and $T \leqslant \left(\frac{1}{4CR}\right)^{10}$, $\mathcal{Z}$ maps $B(0,R)$ into $B(0,R)$.

The second step is to prove that $\mathcal{Z}|_{B(0,R)}$ is a contraction for T small enough. Let f^1, $f^2 \in B(0,R)$. Denote by (u^1,p^1,h^1,ω^1) and (u^2,p^2,h^2,ω^2) the corresponding solutions. We put $f=f^1-f^2$, $u=u^1-u^2$ and $h=h^1-h^2$. Then

$$\mathcal{Z}(f^1)-\mathcal{Z}(f^2) = -(u\cdot\nabla)u^1-(u^2\cdot\nabla)u+(h'\cdot\nabla)u^1+((h^2)'\cdot\nabla)u \tag{5.4}$$

and the difference $u=u_1-u_2$ satisfies

$$\begin{aligned}u'+Au &= \mathbb{P}f,\\ u(0) &= 0.\end{aligned}$$

According to Corollary 4.3, we have that

$$\begin{aligned}\|u\|_{L^2(0,T;[H^2(\Omega)]^2)} + \|u\|_{L^\infty(0,T;[H^1(\Omega)]^2)} + \|u\|_{H^1(0,T;[L^2(\Omega)]^2)}&\\ + \|h\|_{H^2(0,T;\mathbb{R}^2)} \leqslant C\|f\|_{L^2(0,T;[L^2(\Omega)]^2)}&.\end{aligned}$$

The above relation combined to Lemma 5.2, and to relations (5.4), (5.2) implies that

$$\begin{aligned}\|\mathcal{Z}(f^1)-\mathcal{Z}(f^2)\|_{L^{5/2}(0,T;[L^2(\Omega)]^2)} &\leqslant C(C_0+R)\|f\|_{L^2(0,T;[L^2(\Omega)]^2)}\\ &\leqslant CR\|f\|_{L^2(0,T;[L^2(\Omega)]^2)}.\end{aligned}$$

By Hölder's inequality, we obtain that

$$\|\mathcal{Z}(f^1)-\mathcal{Z}(f^2)\|_{L^2(0,T;[L^2(\Omega)]^2)} \leqslant T^{1/10}CR\|f\|_{L^2(0,T;[L^2(\Omega)]^2)}.$$

Thus for $R > C_0$ and $T \leqslant \left(\frac{1}{4CR}\right)^{10}$, the function $\mathcal{Z}$ is a contraction from $B(0,R)$ into $B(0,R)$. Hence, by using the Banach fixed point theorem we obtain the local existence of a solution of (2.1)–(2.7)on $[0,T]$ for $T \leqslant \left(\frac{1}{CR}\right)^{10}$.

The third step consists in showing that one of the alternatives (i) or (ii) holds. The result follows in a standard manner from the fact that the local existence time T_1 obtained at the previous steps is uniform with respect to u_0 provided that $\|u_0\|_{[H^1(\mathbb{R}^2)]^2} \leqslant M$. □

Proof of Theorem 2.1. With the notations of Section 4, (2.1)–(2.7) can be written as

$$u' + Au = \mathbb{P}f \quad \text{on } [0,T], \tag{5.5}$$

$$u(0) = u_0, \tag{5.6}$$

with

$$f = \begin{cases} -(u\cdot\nabla)u + (h'(t)\cdot\nabla)u & \text{in } \Omega, \\ 0 \quad \text{in } B. \end{cases}$$

We first show that the solutions of (2.1)–(2.7) are global in time. By Proposition 5.1, it suffices to show that there exists $M > 0$ such that for almost every $t \in [0, T_0)$, $\|u(t)\|_{[H^1(\mathbb{R}^2)]^2} \leqslant M$. Let u be a solution of (5.5)–(5.6) on $[0,T]$. By taking the inner product of (5.5) by $u(t) \in [L^2(\mathbb{R}^2)]^2$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2} + (u(t), Au(t)) = (f(t), u(t)), \quad \text{a.e. in } (0,T). \tag{5.7}$$

Since

$$\forall w \in [H^1(\Omega)]^2, \quad \int_\Omega [(w\cdot\nabla)\,u]\cdot u\,\mathrm{d}y = \frac{1}{2}\int_{\partial B} |u|^2 (w\cdot n)\,\mathrm{d}\Gamma$$

and

$$\forall y \in \partial B, \quad u(t)\cdot y = h'(t)\cdot y,$$

we get that

$$(f(t), u(t)) = \int_\Omega [-(u(t)\cdot\nabla)u(t) + (h'(t)\cdot\nabla)u(t)]\cdot u(t)\,\mathrm{d}y = 0, \quad \text{a.e. in } (0,T).$$

Therefore by integrating (5.7) with respect to t, we obtain that

$$\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2} + 2\int_0^t (u(s), Au(s))\,ds \leqslant \|u_0\|^2_{[L^2(\mathbb{R}^2)]^2}, \quad \text{a.e. in } (0,T), \tag{5.8}$$

which can also be written as

$$\begin{aligned} \|u(t)\|^2_{[L^2(\Omega)]^2} + M|h'(t)|^2 + J|\omega(t)|^2 + 2\int_0^t (u(s), Au(s))\,ds \\ \leqslant \|u_0\|^2_{[L^2(\Omega)]^2} + M|h_1|^2 + J|\omega_0| = C_0, \quad \text{a.e. in } (0,T). \end{aligned} \tag{5.9}$$

(Here, as in the proof of Proposition 5.1, we denote by C_0 a positive quantity depending only on $\Omega, |h_1|, |h_0|, |\omega_0|$ and $\|u_0\|_{[H^1(\Omega)]^2}$, which is non-decreasing with respect to $\|u_0\|_{[H^1(\mathbb{R}^2)]^2}$ and to $|h_0|$.)

The above relation and Corollary 4.1 imply that

$$\begin{aligned}\int_0^t \|u(s)\|^2_{[H^1(\Omega)]^2}\,ds &\leqslant \int_0^t \left(\|u(s)\|^2_{[L^2(\Omega)]^2} + \frac{1}{\nu}(u,Au)(s)\right)ds \\ &\leqslant CC_0(T+1) \leqslant C_0, \quad \text{a.e. in } (0,T).\end{aligned} \tag{5.10}$$

On the other hand, by taking the inner product of (5.5) by $Au(t) \in [L^2(\mathbb{R}^2)]^2$, we obtain that

$$(u'(t), Au(t)) + (Au(t), Au(t)) = (\mathbb{P}f(t), Au(t)), \quad \text{a.e. in } (0,T).$$

The above relation and Lemma 3.1 yield

$$\frac{1}{2}\frac{d}{dt}(u(t), Au(t)) + \|Au(t)\|^2_{[L^2(\mathbb{R}^2)]^2} = \int_\Omega f(t)\cdot Au(t)\,\mathrm{d}y$$

$$= \int_\Omega [(u(t)\cdot\nabla)\,u(t)]\cdot(Au(t))\,\mathrm{d}y - \int_\Omega [(h'(t)\cdot\nabla)\,u(t)]\cdot(Au(t))\,\mathrm{d}y$$

$$\begin{aligned}&\leqslant C\|u(t)\|^{1/2}_{[L^2(\Omega)]^2}\|u(t)\|_{[H^1(\Omega)]^2}\|u(t)\|^{3/2}_{[H^2(\Omega)]^2} \\ &\qquad + \|\,(h'(t)\cdot\nabla)\,u(t)\|_{[L^2(\Omega)]^2}\|u(t)\|_{[H^2(\Omega)]^2}, \quad \text{a.e. in } (0,T).\end{aligned} \tag{5.11}$$

By Proposition 4.2,

$$c\|u\|_{[H^2(\Omega)]^2} \leqslant \|(I+A)u\|_{[L^2(\mathbb{R}^2)]^2},$$

which clearly gives

$$c\|u\|^2_{[H^2(\Omega)]^2} \leqslant \|Au\|^2_{[L^2(\mathbb{R}^2)]^2} + \|u\|^2_{[L^2(\mathbb{R}^2)]^2} + 2(u, Au). \tag{5.12}$$

Relations (5.11), (5.12) and the inequality

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a,b \geqslant 0, \quad \text{and} \quad \forall p,q > 1 \quad \text{such that} \quad \frac{1}{p}+\frac{1}{q} = 1 \tag{5.13}$$

imply that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(u(t), Au(t)) + c\|u(t)\|^2_{[H^2(\Omega)]^2} \\ &\quad\leqslant C\|u(t)\|^2_{[L^2(\Omega)]^2}\|u(t)\|^4_{[H^1(\Omega)]^2} + \frac{c}{4}\|u(t)\|^2_{[H^2(\Omega)]^2} + \|\,(h'(t)\cdot\nabla)\,u(t)\|^2_{[L^2(\Omega)]^2} \\ &\qquad + \frac{c}{4}\|u(t)\|^2_{[H^2(\Omega)]^2} + \|u(t)\|^2_{[L^2(\Omega)]^2} + 2(u(t), Au(t)), \quad \text{a.e. in } (0,T).\end{aligned} \tag{5.14}$$

By (5.9), we have that

$$\|\,(h'(t)\cdot\nabla)\,u(t)\|^2_{[L^2(\Omega)]^2} \leqslant CC_0\|u(t)\|^2_{[H^1(\Omega)]^2}, \qquad \|u(t)\|^2_{[L^2(\Omega)]^2} \leqslant C_0$$

thus, (5.14) implies that

$$\begin{aligned}\frac{d}{dt}(u(t), Au(t)) + c\|u(t)\|^2_{[H^2(\Omega)]^2} &\leqslant CC_0\|u(t)\|^4_{[H^1(\Omega)]^2} \\ &\quad + CC_0\|u(t)\|^2_{[H^1(\Omega)]^2} + C_0 + C\|u(t)\|^2_{[H^1(\Omega)]^2}, \quad \text{a.e. in } (0,T).\end{aligned}$$

By Corollary 4.1, we have that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{[H^1(\Omega)]^2} + c\|u(t)\|^2_{[H^2(\Omega)]^2} \leqslant CC_0\|u(t)\|^4_{[H^1(\Omega)]^2}\\ + CC_0\|u(t)\|^2_{[H^1(\Omega)]^2} + C_0 + \|u(t)\|^2_{[H^1(\Omega)]^2}, \quad \text{a.e. in } (0,T).\end{aligned}$$

If we integrate the above relation with respect to t, we get that

$$\begin{aligned}\|u(t)\|^2_{[H^1(\Omega)]^2} + c\int_0^t \|u(s)\|^2_{[H^2(\Omega)]^2}\, ds \leqslant C_0T\\ + \int_0^t \left[CC_0\|u(s)\|^2_{[H^1(\Omega)]^2} + CC_0 + 1\right] \|u(s)\|^2_{[H^1(\Omega)]^2}\, ds, \quad \text{a.e. in } (0,T)\end{aligned}$$

which, combined to the Gronwall's lemma and to (5.10), gives that

$$\|u(t)\|^2_{[H^1(\Omega)]^2} \leqslant C_0T\exp\left(CC_0^2(T+1) + (CC_0+1)T\right), \quad \text{a.e. in } (0,T). \tag{5.15}$$

Hence we proved the global in time existence of strong solutions of (2.1)–(2.7).

In order to prove uniqueness, we consider u^1 and u^2 two strong solutions of (2.1)–(2.7). Then, the difference $u = u^1 - u^2$ satisfies

$$u' + Au = \mathbb{P}g \quad \text{on } [0,T_0], \tag{5.16}$$

$$u(0) = 0, \tag{5.17}$$

with

$$g = \begin{cases} -(u\cdot\nabla)u^1 - (u^2\cdot\nabla)u + (h'\cdot\nabla)u^1 + ((h^2)'\cdot\nabla)u & \text{in } \quad \Omega,\\ 0 \quad \text{in} \quad B \end{cases}$$

(we have denoted $h = h^1 - h^2$). By taking the inner product of (5.16) by $u(t)$, we get

$$\frac{d}{dt}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2} + (u(t), Au(t)) = (g(t), u(t)). \tag{5.18}$$

with

$$(g(t), u(t)) = \int_\Omega \left[(u(t)\cdot\nabla)u^1(t)\right]\cdot u(t)\,\mathrm{d}y + \int_\Omega \left[(h'(t)\cdot\nabla)u^1(t)\right]\cdot u(t)\,\mathrm{d}y.$$

By using Lemma 3.1, we obtain that

$$\begin{aligned}(g(t), u(t)) \leqslant \|u(t)\|_{[L^2(\Omega)]^2}\|u(t)\|_{[H^1(\Omega)]^2}\|u^1(t)\|_{[H^1(\Omega)]^2}\\ + \|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}\|u^2(t)\|_{[H^1(\Omega)]^2}.\end{aligned}$$

The above relation combined to Corollary 4.1 and to (5.18) yields

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2} + \nu\|u(t)\|^2_{[H^1(\Omega)]^2}\\ \leqslant \nu\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2} + c\|u^1\|^2_{L^\infty(0,T;[H^1(\Omega)]^2)}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}\\ + \frac{\nu}{2}\|u(t)\|^2_{[H^1(\Omega)]^2} + \|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}\|u^2\|_{L^\infty(0,T;[H^1(\Omega)]^2)}.\end{aligned}$$

By integrating with respect to time and by using Gronwall's Lemma, we get that $u^1 = u^2$. This ends up the proof of Theorem 2.1. □

6. Existence and uniqueness of weak solutions

In the present section, we give a proof of Proposition 2.5. After the appropriate choice of the functions spaces, this proof is similar to the proof of the similar result for the Navier–Stokes system (see Prodi–Lions [14]).

Proof of Proposition 2.5. The existence part. Consider $T > 0$ and $u_0 \in [L^2(\Omega)]^2$ such that $\operatorname{div} u_0 = 0$ in Ω. Consider also h_0, $h_1 \in \mathbb{R}^2$ and $\omega_0 \in \mathbb{R}$. We extend u_0 to a function defined on $\mathbb{R}^2$ by

$$u_0(y) = h_1 + \omega_0 y^\perp, \quad \forall y \in B.$$

Then $u_0 \in \mathcal{H}$ and there exists a sequence $(u_0^n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $u_0^n \in D(A^{1/2})$ and such that

$$u_0^n \to u_0 \quad \text{in} \quad L^2(\mathbb{R}^2)$$

(with A defined by (4.11)–(4.13) and $\mathcal{H}$ by (4.8)).

By Theorem 2.1, for all $n \in \mathbb{N}$ there exists a unique function

$$u^n \in L^2(0,T;D(A)) \cap C([0,T];D(A^{1/2})) \cap H^1(0,T;\mathcal{H})$$

such that

$$(u^n)' + Au^n = \mathbb{P}f^n, \quad u^n(0) = u_0^n, \tag{6.1}$$

with

$$f^n = \begin{cases} (u^n \cdot \nabla)\, u^n - ((h^n)' \cdot \nabla)\, u^n & \text{in} \quad \Omega, \\ 0 & \text{on} \quad B, \end{cases} \tag{6.2}$$

and where $\mathbb{P}$ is the orthogonal projector from $[L^2(\mathbb{R}^2)]^2$ onto $\mathcal{H}$.

The rest of the proof will be given in several steps.

The first step is to show that there exists $u \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;D(A^{1/2}))$ such that

$$u^n \to u \quad \text{in} \quad L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;D(A^{1/2})) \quad \text{strongly.}$$

By taking the inner product of (6.1) with $\phi \in \mathcal{H}$, we obtain

$$((u^n)', \phi) + (Au^n, \phi) = (f^n, \phi). \tag{6.3}$$

If we replace ϕ by $u^n(t)$, we get as in the previous section

$$\|u^n\|^2_{L^\infty(0,T;[L^2(\mathbb{R}^2)]^2)} + \|u^n\|^2_{L^2(0,T;[H^1(\Omega)]^2)} \leqslant M,$$

with $M > 0$ a constant that does not depend on n.

By (6.3), the difference $u^n - u^m$ satisfies

$$((u^n - u^m)', \phi) + (A(u^n - u^m), \phi) = (f^n - f^m, \phi),$$

and by replacing ϕ by $(u^n - u^m)(t) \in \mathcal{H}$ we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u^n - u^m\|^2_{[L^2(\mathbb{R}^2)]^2} + (A(u^n - u^m), u^n - u^m) = (f^n - f^m, u^n - u^m). \tag{6.4}$$

By (6.2), we have that

$$\begin{aligned} f^n - f^m &= (u^n \cdot \nabla)\, u^n - (u^m \cdot \nabla)\, u^m - ((h^n)' \cdot \nabla)\, u^n + ((h^m)' \cdot \nabla)\, u^m \\ &= ((u^n - u^m) \cdot \nabla)\, u^n + (u^m \cdot \nabla)\,(u^n - u^m) \\ &\qquad - ((h^n - h^m)' \cdot \nabla)\, u^n - ((h^m)' \cdot \nabla)\,(u^n - u^m). \end{aligned}$$

The above relation implies that

$$\begin{aligned} (f^n - f^m, u^n - u^m) = &\int_\Omega [((u^n - u^m) \cdot \nabla)\, u^n] \cdot (u^n - u^m)\,\mathrm{d}y \\ &+ \int_\Omega [(u^m \cdot \nabla)\,(u^n - u^m)] \cdot (u^n - u^m)\,\mathrm{d}y \\ &- \int_\Omega \left[\left((h^n - h^m)' \cdot \nabla\right) u^n\right] \cdot (u^n - u^m)\,\mathrm{d}y \\ &+ \int_\Omega [((h^m)' \cdot \nabla)\,(u^n - u^m)] \cdot (u^n - u^m)\,\mathrm{d}y. \end{aligned}$$

On the other hand, for all $w \in [H^1(\Omega)]^2$, we have that

$$\int_\Omega [(w \cdot \nabla)\,(u^n - u^m)] \cdot (u^n - u^m)\,\mathrm{d}y = \frac{1}{2}\int_{\partial B} |u^n - u^m|^2 (w \cdot n)\,\mathrm{d}\Gamma$$

and

$$(h^m)'(y) \cdot n = u^m(y) \cdot n, \quad \forall y \in \partial B,$$

which imply that

$$\begin{aligned} (f^n - f^m, u_n - u_m) = &\int_\Omega [((u^n - u^m) \cdot \nabla)\, u^n] \cdot (u^n - u^m)\,\mathrm{d}y \\ &- \int_\Omega \left[\left((h^n - h^m)' \cdot \nabla\right) u^n\right] \cdot (u^n - u^m)\,\mathrm{d}y. \end{aligned}$$

By using Lemma 3.1, we deduce that

$$\begin{aligned} &|(f^n - f^m, u_n - u_m)| \\ &\leqslant C\|u^n - u^m\|_{[L^2(\Omega)]^2} \|u^n\|_{[H^1(\Omega)]^2} \|u^n - u^m\|_{[H^1(\Omega)]^2} \\ &\qquad + |(h^n - h^m)'|\, \|u^n\|_{[H^1(\Omega)]^2} \|u^n - u^m\|_{[H^1(\Omega)]^2} \\ &\leqslant C\|u^n - u^m\|^2_{[L^2(\Omega)]^2} \|u^n\|^2_{[H^1(\Omega)]^2} + \frac{\nu}{2}\|u^n - u^m\|^2_{[H^1(\Omega)]^2} \\ &\qquad + C\|u^n\|_{[H^1(\Omega)]^2} \|u^n - u^m\|^2_{[H^1(\mathbb{R}^2)]^2}. \end{aligned}$$

By using (6.4),

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^n - u^m\|^2_{[L^2(\mathbb{R}^2)]^2} + \frac{\nu}{2}\|u^n - u^m\|^2_{[H^1(\mathbb{R}^2)]^2} \\ &\qquad \leqslant \nu\|u^n - u^m\|^2_{[L^2(\mathbb{R}^2)]^2} + C\|u^n\|^2_{[H^1(\mathbb{R}^2)]^2} \|u^n - u^m\|^2_{[L^2(\mathbb{R}^2)]^2} \\ &\qquad\qquad + C\|u^n\|_{[H^1(\mathbb{R}^2)]^2} \|u^n - u^m\|^2_{[L^2(\mathbb{R}^2)]^2}. \end{aligned}$$

If we integrate the above relation with respect to t, we get that

$$\|(u^n - u^m)(t)\|^2_{[L^2(\mathbb{R}^2)]^2} + \nu \int_0^t \|(u^n - u^m)(s)\|^2_{[H^1(\mathbb{R}^2)]^2}\, ds \leqslant \|u_0^n - u_0^m\|^2_{[L^2(\mathbb{R}^2)]^2} + \int_0^t \theta(s) \|(u^n - u^m)(s)\|^2_{[L^2(\mathbb{R}^2)]^2}\, ds \quad (6.5)$$

with

$$\theta(s) = 2\nu + C\|u^n(s)\|^2_{[H^1(\mathbb{R}^2)]^2} + C\|u^n(s)\|_{[H^1(\mathbb{R}^2)]^2}.$$

By using the Gronwall's Lemma, we obtain

$$\|(u^n - u^m)(t)\|^2_{[L^2(\mathbb{R}^2)]^2} \leqslant \|u_0^n - u_0^m\|^2_{[L^2(\mathbb{R}^2)]^2} \exp\left(\int_0^t \theta(s)\, ds \right). \quad (6.6)$$

By using the fact that $(\|u_0^n\|_{[H^1(\mathbb{R}^2)]^2})_{n\in\mathbb{N}}$ is a bounded sequence, and relation (5.9), it follows that there exists a constant $M_1 > 0$ such that

$$\int_0^t \theta(s)\, ds \leqslant 2\nu T + CM + CMT^{1/2} = M_1. \quad (6.7)$$

The above relation and (6.6) imply that

$$\|(u^n - u^m)(t)\|^2_{[L^2(\mathbb{R}^2)]^2} \leqslant \|u_0^n - u_0^m\|^2_{[L^2(\mathbb{R}^2)]^2} e^{M_1}, \quad (6.8)$$

whereas relations (6.5), (6.7) and (6.8) yield

$$\|u^n - u^m\|^2_{L^2(0,T;[H^1(\mathbb{R}^2)]^2)} \leqslant 2\|u_0^n - u_0^m\|^2_{[L^2(\mathbb{R}^2)]^2} \left(1 + M_1 e^{M_1}\right). \quad (6.9)$$

Gathering relations (6.8) and (6.9) we obtain that (u_n) is a Cauchy sequence in

$$L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;D(A^{1/2}))$$

which is a Banach space. Thus there exists a function

$$u \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;D(A^{1/2}))$$

such that

$$u_n \to u \quad \text{in} \quad L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;D(A^{1/2})) \quad \text{strongly}. \quad (6.10)$$

The second step is to show that the function u defined in the first step is a weak solution of (2.1)–(2.7). If u_n is the function defined by (6.1) then for all $\phi \in \mathcal{V}$, we have

$$\frac{d}{dt}(u^n(t), \phi) + 2\nu(D(u^n(t)), D(\phi)) + \int_\Omega [(u^n(t) \cdot \nabla)\, u^n(t)] \cdot \phi \, \mathrm{d}y - \int_\Omega [((h^n)'(t) \cdot \nabla)\, u^n(t)] \cdot \phi \, \mathrm{d}y = 0, \quad \text{a.e. in } (0,T). \quad (6.11)$$

By (6.10), it follows that

$$\frac{d}{dt}(u^n(t), \phi) \to \frac{d}{dt}(u(t), \phi) \quad \text{in} \quad \mathcal{D}'(0,T), \quad (6.12)$$

$$2\nu(D(u^n(t)), D(\phi)) \to 2\nu(D(u(t)), D(\phi)) \quad \text{in} \quad L^2(0,T). \quad (6.13)$$

On the other hand,

$$\begin{aligned}\int_0^T &\left|\int_\Omega [(u^n\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y - \int_\Omega [(u\cdot\nabla)\,u]\cdot\phi\,\mathrm{d}y\right|\,\mathrm{d}t\\ &= \int_0^T \left|\int_\Omega [((u^n-u)\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y - \int_\Omega [(u\cdot\nabla)\,(u^n-u)]\cdot\phi\,\mathrm{d}y\right|\,\mathrm{d}t\\ &\leqslant \int_0^T \left|\int_\Omega [((u^n-u)\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y\right|\,\mathrm{d}t + \int_0^T \left|\int_\Omega [(u\cdot\nabla)\,(u^n-u)]\cdot\phi\,\mathrm{d}y\right|\,\mathrm{d}t.\end{aligned}$$

Therefore, by using Lemma 3.1, we deduce that

$$\begin{aligned}\int_0^T &\left|\int_\Omega [(u^n\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y - \int_\Omega [(u\cdot\nabla)\,u]\cdot\phi\,\mathrm{d}y\right|\,\mathrm{d}t\\ &\leqslant C\int_0^T \|u^n-u\|_{[H^1(\Omega)]^2}\|u^n\|_{[H^1(\Omega)]^2}\|\phi\|_{[H^1(\Omega)]^2}\\ &\qquad + \|u^n-u\|_{[H^1(\Omega)]^2}\|u\|_{[H^1(\Omega)]^2}\|\phi\|_{[H^1(\Omega)]^2}\,\mathrm{d}t.\end{aligned}$$

The above relation and the Cauchy–Schwarz inequality yield

$$\begin{aligned}\int_0^T &\left|\int_\Omega [(u^n\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y - \int_\Omega [(u\cdot\nabla)\,u]\cdot\phi\,\mathrm{d}y\right|\,\mathrm{d}t\\ &\leqslant C\left(\|\phi\|_{[H^1(\Omega)]^2}\|u^n-u\|_{L^2(0,T;[H^1(\Omega)]^2)}\|u^n\|_{L^2(0,T;[H^1(\Omega)]^2)}\right.\\ &\qquad \left.+\|\phi\|_{[H^1(\Omega)]^2}\|u^n-u\|_{L^2(0,T;[H^1(\Omega)]^2)}\|u\|_{L^2(0,T;[H^1(\Omega)]^2)}\right).\end{aligned}$$

In turn, this relation and (6.10) imply that

$$\int_\Omega [(u^n\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y \to \int_\Omega [(u\cdot\nabla)\,u]\cdot\phi\,\mathrm{d}y \quad \text{in} \quad L^1(0,T;\mathbb{R}). \tag{6.14}$$

Similarly, we obtain that

$$\int_\Omega [((h^n)'\cdot\nabla)\,u^n]\cdot\phi\,\mathrm{d}y \to \int_\Omega [(h'(s)\cdot\nabla)\,u]\cdot\phi\,\mathrm{d}y \quad \text{in} \quad L^1(0,T;\mathbb{R}). \tag{6.15}$$

Relations (6.12), (6.13), (6.14) and (6.15) imply that u satisfies

$$\begin{aligned}\frac{d}{dt}\,(u(t),\phi) + 2\nu\int_\Omega D(u(t)) : D(\phi)\,\mathrm{d}y + \int_\Omega [(u(t)\cdot\nabla)\,u(t)]\cdot\phi\,\mathrm{d}y&\\ - \int_\Omega [(h'(t)\cdot\nabla)\,u(t)]\cdot\phi\,\mathrm{d}y = 0, \quad \forall\phi\in\mathcal{V}.&\end{aligned} \tag{6.16}$$

This ends up the proof of the existence of a weak solution.

The uniqueness part. Let u^1 and u^2 be two weak solutions of the system

(2.1)–(2.7). Then u^1 and u^2 satisfy

$$\frac{d}{dt}(u^i(t),\phi)+2\nu\int_\Omega D(u^i(t)):D(\phi)\,\mathrm{d}y \\ +\int_\Omega \big[(u^i(t)\cdot\nabla)u^i\big]\cdot\phi+\big[((h^i)'(t)\cdot\nabla)u^i(t)\big]\cdot\phi\,\mathrm{d}y=0$$

and the difference $u=u^1-u^2$ satisfies

$$\frac{d}{dt}(u(t),\phi)+2\nu\int_\Omega D(u(t)):D(\phi)\,\mathrm{d}y+\int_\Omega \big[(u(t)\cdot\nabla)u^1(t)\big]\cdot\phi+\big[(u^2(t)\cdot\nabla)u(t)\big]\cdot\phi \\ -\big[(h'(t)\cdot\nabla)u^1(t)\big]\cdot\phi-\big[((h^2)'(t)\cdot\nabla)u(t)\big]\cdot\phi\,\mathrm{d}y=0. \quad (6.17)$$

Since

$$\int_\Omega \big[(u^2(t)\cdot\nabla)u(t)\big]\cdot u(t)-\big[((h^2)'(t)\cdot\nabla)u(t)\big]\cdot u(t)\,\mathrm{d}y=0, \quad \text{a.e. in } (0,T),$$

if we take $\phi=u(t)$ in relation (6.17), we get that

$$\frac{1}{2}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}+2\nu\int_0^t \|D(u(s))\|^2_{L^2(\mathbb{R}^2)^4}\,\mathrm{d}s \\ +\int_\Omega \big[(u(t)\cdot\nabla)u^1(t)\big]\cdot u(t)-\big[(h'(t)\cdot\nabla)u^1(t)\big]\cdot u(t)\,\mathrm{d}y=0.$$

By integrating with respect to time the above inequality and by using Lemma 3.1 we obtain that

$$\frac{1}{2}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}+2\nu\int_0^t \|D(u(s))\|^2_{L^2(\mathbb{R}^2)^4}\,\mathrm{d}s \\ \leqslant \int_0^t \|u(s)\|_{[L^2(\Omega)]^2}\|u(s)\|_{[H^1(\Omega)]^2}\|u^1(s)\|_{[H^1(\Omega)]^2}\,\mathrm{d}s \\ +\int_0^t |h'(s)|\times\|u(s)\|_{[L^2(\Omega)]^2}\|u^1(s)\|_{[H^1(\Omega)]^2}\,\mathrm{d}s.$$

Hence, by using Lemma 4.1, we get that

$$\frac{1}{2}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}+\nu\int_0^t \|\nabla u(s)\|^2_{[L^2(\mathbb{R}^2)]^4}\,\mathrm{d}s+\nu\int_0^t \|u(s)\|^2_{[L^2(\mathbb{R}^2)]^2}\,\mathrm{d}s \\ \leqslant \nu\int_0^t \|u(s)\|^2_{[L^2(\mathbb{R}^2)]^2}\,\mathrm{d}s+\nu\int_0^t \|u(s)\|^2_{[H^1(\mathbb{R}^2)]^2}\,\mathrm{d}s \\ +C\int_0^t \|u(s)\|^2_{[L^2(\mathbb{R}^2)]^2}\|u^1(s)\|^2_{[H^1(\Omega)]^2}\,\mathrm{d}s+C\int_0^t \|u(s)\|^2_{[L^2(\mathbb{R}^2)]^2}\|u^1(s)\|_{[H^1(\Omega)]^2}\,\mathrm{d}s,$$

The relation (5.9) and the above inequality imply that

$$\frac{1}{2}\|u(t)\|^2_{[L^2(\mathbb{R}^2)]^2}\leqslant(\nu+C)\int_0^t \|u(s)\|^2_{[L^2(\mathbb{R}^2)]^2}\,\mathrm{d}s.$$

By using Gronwall's Lemma, we conclude that $u = 0$. This completes the proof of Theorem 2.5.

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T. Takahashi and M. Tucsnak
Institut Elie Cartan
Faculté des Sciences, BP239
54506 Vandoeuvre-lès-Nancy, Cedex
France
e-mail: Takeo.Takahashi@iecn.u-nancy.fr
Marius.Tucsnak@iecn.u-nancy.fr
and
INRIA Lorraine, Projet CORIDA

(accepted: September 19, 2002)