Advances in Differential Equations Volume xx, Number xxx, , Pages xx–xx

ANALYSIS OF STRONG SOLUTIONS FOR THE EQUATIONS MODELING THE MOTION OF A RIGID-FLUID SYSTEM IN A BOUNDED DOMAIN

Takéo Takahashi
Institut Elie Cartan, Faculté des Sciences
BP239 54506 Vandoeuvre-lès-Nancy, Cedex, France
and
INRIA Lorraine, Projet CORIDA

(Submitted by: Juan-Luis Vázquez)

Abstract. In this paper, we study a fluid–rigid-body interaction problem. The motion of the fluid is modeled by the Navier-Stokes equations, written in an unknown bounded domain depending on the displacement of the rigid body. Our main result yields existence and uniqueness of strong solutions. In the two-dimensional case, the solutions are global provided that the rigid body does not touch the boundary. In the three-dimensional case, we obtain local-in-time existence and global existence for small data. Moreover, we prove an asymptotic stability result.

1. Introduction

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with a regular boundary $\partial\mathcal{O}$ (of class C^2). We consider a solid occupying the domain $B(t) \subset \mathcal{O}$ and surrounded by a viscous homogeneous fluid within the domain $\Omega(t) = \mathcal{O}\backslash B(t)$.

We shall assume that the motion of the fluid is described by the classical Navier-Stokes equations, whereas the motion of the rigid body is governed by the balance equations for linear and angular momentum (Newton's laws). Hence, in the case of two space dimensions, we can write the full system of equations modeling the motion of the fluid and the rigid body as

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \qquad x \in \Omega(t), t \in [0, T], \tag{1.1}$$

$$\operatorname{div} u = 0, \qquad x \in \Omega(t), t \in [0, T], \tag{1.2}$$

$$u = 0, \qquad x \in \partial\mathcal{O},\ t \in [0, T], \tag{1.3}$$

$$u = h'(t) + \omega(t)(x - h(t))^{\perp}, \qquad x \in \partial B(t),\ t \in [0, T], \tag{1.4}$$

Accepted for publication: October 2003.
AMS Subject Classifications: 35Q30, 76D03, 76D05.

$$Mh''(t) = -\int_{\partial B(t)} \sigma n \,\mathrm{d}\Gamma + \rho \int_{B(t)} f(t)\,\mathrm{d}x, \quad t \in [0,T], \tag{1.5}$$

$$J\omega'(t) = -\int_{\partial B(t)} (x-h(t))^{\perp} \cdot \sigma n \,\mathrm{d}\Gamma + \rho \int_{B(t)} (x-h(t))^{\perp} \cdot f(x,t)\,\mathrm{d}x, \quad t \in [0,T], \tag{1.6}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega(0), \tag{1.7}$$

$$h(0) = h_0 \in \mathbb{R}^2,\ h'(0) = h_1 \in \mathbb{R}^2,\ \omega(0) = \omega_0 \in \mathbb{R}. \tag{1.8}$$

In the above system the unknows are $u(x,t)$ (the Eulerian velocity field of the fluid), $p(x,t)$ (the pressure of the fluid), $h(t)$ (the position of the mass center of the rigid body) and $\omega(t)$ (the angular velocity of the rigid body). The domain $B(t)$ is defined by $B(t) = \{R_{-\theta(t)}y + h(t),\ y \in B(0)\}$, where

$$\theta(t) = \int_0^t \omega(s)\,\mathrm{d}s \tag{1.9}$$

and R_θ is the rotation matrix corresponding to the angle θ.

The constants M and J are the mass and the moment of inertia of the rigid body. Moreover, $f(x,t)$ is the force acting on the fluid.

For all $x = \binom{x_1}{x_2}$, we denote by $x^\perp$ the vector $x^\perp = \binom{x_1}{-x_2}$. We have also denoted by w' and w'' the derivatives of a function w depending only on the time t. If $x, y \in \mathbb{R}^2$, then $x \cdot y$ defines the inner product of x and y and $|x|$ stands for the corresponding norm. Moreover, we have denoted by $\partial B(t)$ the boundary of the rigid body at instant t and by $n(x,t)$ the unit normal to $\partial B(t)$ at the point x directed to the interior of the rigid body.

The positive constant ν is the viscosity of the fluid.

The stress tensor (also called the Cauchy stress) is defined by

$$\sigma(x,t) = -p(x,t)\,\mathrm{Id} + 2\nu D(u), \tag{1.10}$$

where Id is the identity matrix and $D(u)$ is the tensor field defined by

$$D(u)_{k,l} = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big).$$

In the major part of the paper, we consider the two-dimensional problem given by the above system. Some of the result and methods will be extended to the three-dimensional case. In recent years, several papers concerning the existence of weak solutions (in two or three dimensions) have been published in the literature. We mention here Desjardins and Esteban [4] and [5]; Conca, San Martin, and Tucsnak [3]; Gunzburger, Lee, and Seregin [13]; Hoffmann and Starovoitov [16] and [17]; San Martin, Starovoitov, and Tucsnak [21] (with the domain of the fluid bounded); and Serre [22], Judakov [19], and Silvestre [23] (in the case where the fluid–rigid-body system fills the whole space). The stationary problem was studied in [22] and in Galdi [10]. More recently, the question of existence of global weak solutions in three space dimensions has been investigated in Feireisl [7], [8].

Most of the above references are based upon a weak formulation similar to the one introduced for nonhomogeneous fluids in [20]. We notice that uniqueness of weak solutions is an open question, even in the two-dimensional case.

On the other hand, as far as we know, only a few results on strong solutions are available in the literature. A local (in time) existence result of strong solutions was proved in Grandmont and Maday [12] provided that the fluid–rigid-body system occupies a bounded domain and the inertia of the rigid body is large enough with respect to the inertia of the fluid. In the case of the fluid–rigid-body system filling the whole space, existence and uniqueness of strong solutions have been proved in Takahashi and Tucsnak [25] for an infinite cylinder and in two space dimensions; a similar result has been proved in Silvestre and Galdi [11] for a rigid body having an arbitrary form.

A one-dimensional version of the problem tackled in this paper was studied in Vázquez and Zuazua [27] where the asymptotic behavior of solutions has also been investigated.

The aim of this paper is to prove existence and uniqueness of strong solutions in the case of a bounded domain without the hypothesis of [12] about the inertia of the rigid body.

We use the same method as in [25] but the change of variables is more complicated in our case and we need more estimates to prove local existence.

The plan of this paper is as follows: in Section 2 we introduce some notation and we state the main result. In Section 3, we give the main steps of the proof of the main result, whereas Section 4 is devoted to the change of variables we use to solve our problem. In Section 5, we study a linear problem in a fixed domain associated to our problem. The local-in-time result is proved in Section 6, and the main result is proved in Section 7. Section 8 is devoted to a result of asymptotic stability which gives, in particular, conditions that are sufficient to prevent collisions. We give some extensions in the case of three space dimensions in the last section.

2. Notation and main result

In the sequel, we denote $\Omega = \Omega(0)$ and $B = B(0)$. We first define the function spaces $L^2(0,T;H^2(\Omega(t)))$, $H^1(0,T;L^2(\Omega(t)))$, $C([0,T];H^1(\Omega(t)))$, and $L^2(0,T;H^1(\Omega(t)))$, which will be extensively used in the sequel. Suppose that there exists a C^∞ diffeomorphism ψ from Ω on $\Omega(t)$ such that the derivatives

$$\frac{\partial^{i+\alpha_1+\alpha_2}\psi}{\partial t^i \partial y_1^{\alpha_1} \partial y_2^{\alpha_2}}, \quad i \leqslant 1, \quad \alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 0$$

exist and are continuous. For all functions $u(\cdot,t) : \Omega(t) \to \mathbb{R}^2$, we denote $U(y,t) = u(\psi(y,t),t)$. Then the functions spaces introduced above are defined by

$$\begin{aligned} L^2(0,T;H^2(\Omega(t))) &= \{u : U \in L^2(0,T;H^2(\Omega))\}, \\ H^1(0,T;L^2(\Omega(t))) &= \{u : U \in H^1(0,T;L^2(\Omega))\}, \\ C([0,T];H^1(\Omega(t))) &= \{u : U \in C([0,T];H^1(\Omega))\}, \\ L^2(0,T;H^1(\Omega(t))) &= \{u : U \in L^2(0,T;H^1(\Omega))\}. \end{aligned}$$

As usual, we denote by $H^m(\Omega)$ the Sobolev space formed by the functions in $L^2(\Omega)$ which have distributional derivatives, up to the order m, in $L^2(\Omega)$. Denote

$$\mathcal{H}^i(t) = [H^i(\Omega(t))]^2, \quad \mathcal{H}^i = \mathcal{H}^i(0), \quad \mathcal{H}^i(\mathcal{O}) = [H^i(\mathcal{O})]^2,$$
$$\mathcal{L}^i(t) = [L^i(\Omega(t))]^2, \quad \mathcal{L}^i = \mathcal{L}^i(0), \quad \mathcal{L}^i(\mathcal{O}) = [L^i(\mathcal{O})]^2,$$

and

$$\mathcal{U}(0,T;\Omega(t)) = L^2(0,T;\mathcal{H}^2(t)) \cap H^1(0,T;\mathcal{L}^2(t)) \cap C([0,T];\mathcal{H}^1(t)). \quad (2.1)$$

We define the concept of strong solutions of (1.1)–(1.8) as follows:

Definition 2.1. Suppose that $T > 0$. A quadruplet (u,p,h,ω) is called a strong solution of (1.1)–(1.8) if

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T,\mathbb{R}^2) \times H^1(0,T,\mathbb{R}),$$

and if the distance from $B(t)$ to $\partial\mathcal{O}$ is positive and (1.1)–(1.8) are satisfied almost everywhere in $(0,T)$ and in $\Omega(t)$ or in the trace sense.

The main result of the paper is

Theorem 2.2. *Suppose that* $f \in L^2_{loc}(0,\infty;[W^{1,\infty}(\mathcal{O})]^2)$, $u_0 \in \mathcal{H}^1$ *and that*

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \Omega, \\ u_0(x) = 0 & \text{on } \partial\mathcal{O}, \\ u_0(x) = h_1 + \omega_0 (x-h_0)^\perp & \text{on } \partial B, \\ \operatorname{dist}(B,\partial\mathcal{O}) > 0. \end{cases}$$

Then there exists a maximal $T_0 > 0$ *such that equations* (1.1)–(1.8) *admit a unique strong solution*

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T,\mathbb{R}^2) \times H^1(0,T,\mathbb{R}),$$

for all $T \in (0,T_0)$. *Moreover, one of the following alternatives holds true:*

(1) $T_0 = \infty$; *i.e., the solution is global.*

(2) $\lim_{t\to T_0} \operatorname{dist}(B(t), \partial\mathcal{O}) = 0$.

Remark 2.3. The existence of solutions of (1.1)–(1.8) with initial data satisfying the assumptions in Theorem 2.2 has been investigated in [4] in two or three space dimensions. The novelty brought in by our result consists in H^2 regularity with respect to the space variable and in the uniqueness of the solution.

3. Main steps of the proof of Theorem 2.2

For the sake of simplicity, we prove Theorem 2.2 in the case of $f = 0$. We can obviously suppose that $h_0 = 0$.

The first step in the proof of Theorem 2.2 is to reduce the system (1.1)–(1.8) to a problem in the cylindrical domain $\Omega \times (0, T)$. For this, we use a change of variables, which depends on h and on ω and which is inspired by Inoue and Wakimoto [18] (this change of variables is described in the next section). We then get equations of the form

$$\begin{aligned}
\frac{\partial U}{\partial t} - \nu[\mathbf{L}U] + [\mathbf{M}U] + [\mathbf{N}U] + [\mathbf{G}P] &= 0, && \text{in } \Omega \times [0, T],\\
\operatorname{div} U &= 0, && \text{in } \Omega \times [0, T],\\
U(y, t) &= 0, && \text{on } \partial\mathcal{O} \times [0, T],\\
U(y, t) &= R_\theta h' + \omega y^\perp, && \text{on } \partial B \times [0, T],\\
M h''(t) &= -\int_{\partial B} \Sigma n \, \mathrm{d}\Gamma, && t \in [0, T],\\
J\omega'(t) &= -\int_{\partial B} y^\perp \cdot \Sigma n \, \mathrm{d}y && t \in [0, T],\\
U(y, 0) &= U_0, && \text{in } \Omega,\\
h(0) = 0,\ h'(0) &= h_1,\ \omega(0) = \omega_0,
\end{aligned}$$

where $\Sigma = -P\,\mathrm{Id} + 2\nu D(U)$. The unknows of this system are $U(y, t)$, $P(y, t)$, $h(t)$, and $\omega(t)$. $[\mathbf{L}U]$ is the transform of Δu and $[\mathbf{M}U]$ is a linear term containing U and ∇U, whereas $[\mathbf{N}U]$ is a nonlinear term corresponding to $(u \cdot \nabla)u$ in equation (1.1). All the coefficients are regular with respect to the time t so $[\mathbf{L}U]$ is close to ΔU and $[\mathbf{G}P]$ is close to ∇P for small t. Hence we can solve the previous system by searching for the solution as a fixed point of the mapping

$$\mathcal{N} : (W, Q, h, \omega) \mapsto (U, P, \tilde{h}, \tilde{\omega}),$$

where $(U, P, \tilde{h}, \tilde{\omega})$ satisfies

$$\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = F, \qquad \text{in } \Omega \times [0, T],$$

$$
\begin{aligned}
\operatorname{div} U = 0, &\qquad \text{in } \Omega \times [0,T],\\
U(y) = 0, &\qquad y \in \partial\mathcal{O},\ t \in [0,T],\\
U(y) = \widetilde{H}'(t) + \tilde{\omega}(t) y^{\perp}, &\qquad y \in \partial B,\ t \in [0,T],\\
M\widetilde{H}''(t) = -\int_{\partial B} \Sigma n \,\mathrm{d}\Gamma + F_M, &\qquad t \in [0,T],\\
J\tilde{\omega}'(t) = -\int_{\partial B} y^{\perp} \cdot \Sigma n \,\mathrm{d}y, &\qquad t \in [0,T],\\
U(x,0) = u_0(x), &\qquad x \in \Omega,\\
&\widetilde{H}(0) = 0,\ \widetilde{H}'(0) = h_1,\ \tilde{\omega}(0) = \omega_0,
\end{aligned}
$$

and where

$$
F = \nu[(\mathbf{L} - \Delta)W] - [\mathbf{M}W] + [(\nabla - \mathbf{G})Q] + [\mathbf{N}W],
$$

$$
F_M = M\omega(t) R_\theta h'(t), \qquad \tilde{h}(t) = \int_0^t R_{-\theta} \widetilde{H}'(s) \,\mathrm{d}s.
$$

For T_0 small enough, we show that there exists a closed ball $\mathcal{K}$ (in an appropriate Banach space) such that $\mathcal{N}$ maps $\mathcal{K}$ into $\mathcal{K}$ and such that the restriction of $\mathcal{N}$ to this ball is a contraction. This will prove the local (in time) existence and uniqueness of the strong solution. The last step is to extend our solution on $[0,T]$ for T such that the distance between $B(t)$ and $\partial\mathcal{O}$ is positive for all $t \in [0,T]$.

4. The transformed equations

In this section, we consider a fixed pair $(h,\omega) \in H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})$. Moreover, in the sequel of this work, we fix ε such that $0 < \varepsilon < \operatorname{dist}(B, \partial\mathcal{O})$.

4.1. **The change of variables.** We remark that, by a classical Sobolev embedding theorem, we have that $h \in C^1([0,T];\mathbb{R}^2)$ and $\omega \in C([0,T];\mathbb{R})$. Define $V_R : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$ by

$$
V_R(x,t) = h'(t) + \omega(t)(x - h(t))^{\perp}, \quad \forall t \in [0,T], \quad \forall x \in \mathbb{R}^2. \tag{4.1}
$$

We clearly have that for all $t \in [0,T]$, the function $V_R(\cdot,t)$ is of class $C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ and that for all $x \in \mathbb{R}^2$, the function $V_R(x,\cdot)$ is in $H^1(0,T;\mathbb{R}^2)$. We denote by

$$
\begin{cases}
V_{R_1}(x_2,t) = h_1'(t) + \omega(t)(x_2 - h_2(t))\\
V_{R_2}(x_1,t) = h_2'(t) - \omega(t)(x_1 - h_1(t))
\end{cases}
$$

the components of V_R, and for all $\mu > 0$, we denote

$$
\mathcal{O}_\mu = \{x \in \mathcal{O} : \operatorname{dist}(x, \partial\mathcal{O}) > \mu\}.
$$

If $\varepsilon > 0$, then it is well-known that there exists $\xi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ with compact support contained in $\mathcal{O}_{\varepsilon/2}$ and equal to 1 in $\overline{\mathcal{O}}_\varepsilon$. In the sequel, we use the functions $w : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$

$$w(x_1, x_2, t) = -\int_0^{x_1} V_{R2}(s,t)\, ds + \int_0^{x_2} V_{R1}(s,t)\, ds \tag{4.2}$$

and $\Lambda : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$

$$\Lambda(x_1, x_2, t) = \begin{pmatrix} \frac{\partial \xi}{\partial x_2} w + \xi V_{R1} \\ -\frac{\partial \xi}{\partial x_1} w + \xi V_{R2} \end{pmatrix}. \tag{4.3}$$

Notice that w is the stream function associated to the rigid velocity field V_R; i.e., $V_R = (\nabla w)^\perp$. One can easily check that Λ is continuous, and that for all $t \in [0,T]$, the function $\Lambda(\cdot, t)$ is of class $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and that for all $x \in \mathbb{R}^2$, $\Lambda(x, \cdot)$ is of class $H^1(0,T;\mathbb{R}^2)$. Moreover, we have

Lemma 4.1. *The mapping Λ defined by (4.3) has the following properties:*
- (i) $\Lambda = 0$ *outside* $\mathcal{O}$,
- (ii) $\operatorname{div} \Lambda = 0$ *in* $\mathbb{R}^2$,
- (iii) $\Lambda(x,t) = V_R(x,t)$ *if* $x \in B(t)$ *and if the distance between* $B(t)$ *and* $\partial\mathcal{O}$ *is larger than* ε.

Proof. Assertion (i) simply follows from the fact that the support of ξ is contained in $\mathcal{O}$. In order to show (ii), we notice that

$$\frac{\partial w}{\partial x_1} = -V_{R_2} \quad \frac{\partial w}{\partial x_2} = V_{R_1}.$$

Thus,

$$\begin{aligned} \operatorname{div} \Lambda &= \frac{\partial}{\partial x_1}\left(\frac{\partial \xi}{\partial x_2} w + \xi V_{R_1}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial \xi}{\partial x_1} w + \xi V_{R_2}\right) \\ &= \frac{\partial^2 \xi}{\partial x_1 \partial x_2} w + \frac{\partial w}{\partial x_1}\frac{\partial \xi}{\partial x_2} + \frac{\partial \xi}{\partial x_2} V_{R_1} + \xi \frac{\partial V_{R_1}}{\partial x_1} \\ &\quad - \frac{\partial^2 \xi}{\partial x_1 \partial x_2} w - \frac{\partial w}{\partial x_2}\frac{\partial \xi}{\partial x_1} + \frac{\partial \xi}{\partial x_2} V_{R_2} + \xi \frac{\partial V_{R_2}}{\partial x_2} = 0. \end{aligned}$$

Thus, we have showed that (ii) holds. Finally, we notice that if $B(t) \subset \mathcal{O}_\varepsilon$, then

$$\xi = 1, \quad \frac{\partial \xi}{\partial x_2} = \frac{\partial \xi}{\partial x_1} = 0 \quad \text{in } B(t).$$

Hence, by using relation (4.3), we obtain assertion (iii) in the lemma. □

Next, we consider the initial-value problem

$$X'(t) = \Lambda(X(t), t), \quad X(0) = y, \tag{4.4}$$

with Λ given in (4.3).

Lemma 4.2. *For all $T > 0$ and for all $y \in \mathbb{R}^2$, the initial-value problem* (4.4) *admits a unique solution $X(y,t)$ on $[0,T]$. Moreover, the derivatives*

$$\frac{\partial^{i+\alpha_1+\alpha_2} X}{\partial t^i \partial y_1^{\alpha_1} \partial y_2^{\alpha_2}}, \quad i \leqslant 1, \quad \alpha_1, \alpha_2 \in \mathbb{N}$$

exist and they are continuous. Moreover, the mapping $y \mapsto X(y,t)$ is a diffeomorphism from $\mathcal{O}$ onto itself and a diffeomorphism from Ω onto $\Omega(t)$. Its inverse $Y(x,t)$ is such that its derivatives

$$\frac{\partial^{i+\alpha_1+\alpha_2} Y}{\partial t^i \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \quad i \leqslant 1, \quad \alpha_1, \alpha_2 \in \mathbb{N}$$

exist and are continuous.

Proof. Since Λ is of class C^1, by the classical Cauchy-Lipschitz theorem, it follows that (4.4) admits a unique maximal solution $X(y;t)$ on a subinterval of $[0,T]$. Since $\Lambda = 0$ on the boundary of $\mathcal{O}$, it follows that $X(y,t) \in \overline{\mathcal{O}}$ for all t. Since $\overline{\mathcal{O}}$ is compact, the maximal solution is global (i.e., defined on $[0,T]$).

The global existence and uniqueness of the solution of (4.4) imply that $X(\cdot,t)$ is a bijection from $\mathbb{R}^2$ onto $\mathbb{R}^2$. Since $X(\cdot,t)_{|\mathcal{O}^c} = Id_{|\mathcal{O}^c}$, it follows that $X(\cdot,t)$ is a bijection from $\mathcal{O}$ onto $\mathcal{O}$.

By using a classical result (see, for instance, Hartman [15, p. 95]), we obtain that the derivatives

$$\frac{\partial^{i+\alpha_1+\alpha_2} X}{\partial t^i \partial y_1^{\alpha_1} \partial y_2^{\alpha_2}}, \quad i \leqslant 1, \quad \alpha_1, \alpha_2 \in \mathbb{N} \tag{4.5}$$

exist and they are continuous. Since the inverse $Y(\cdot,t)$ satisfies the similar initial-value problem,

$$Y'(s) = -\Lambda(Y(s), t-s), \quad Y(0) = x, \tag{4.6}$$

it follows that $X(\cdot,t)$ is a C^∞ diffeomorphism.

On the other hand, if $\operatorname{dist}(\partial B(t), \partial\mathcal{O}) > \varepsilon$, we easily check that, for all $y \in B$, the function

$$\tilde{X}(y,t) = h(t) + R_{-\theta(t)} y \tag{4.7}$$

is a solution of (4.4). Therefore, by using again the uniqueness of the solution of (4.4), we get that $X(\cdot,t)(B) \subset B(t)$ and similarly that $Y(\cdot,t)(B(t)) \subset B$. Hence $X(\cdot,t)(B) = B(t)$ and $X(\cdot,t) : \Omega \to \Omega(t)$ is a diffeomorphism. $\square$

Our change of variables satisfies the following useful property, which follows from the condition $\operatorname{div}\Lambda = 0$ via a classical result of Liouville (see, for instance, Arnold [2, p. 249]):

Lemma 4.3. *Let X be as in Lemma 4.2 and let*

$$J_X = \left(\frac{\partial X_i}{\partial y_j}\right)_{i,j}$$

be the Jacobian matrix of the transform $y \mapsto X(y)$. Then

$$\det J_X(y,t) = 1, \quad \forall t \in [0,T], \quad \forall y \in \mathbb{R}^2.$$

4.2. **The equations in a cylindrical domain.** By the previous transform, we reduce our problem to a problem in a cylindrical domain. In the sequel, we denote $\Omega_T = \Omega \times [0,T]$ and $\mathcal{O}_T = \mathcal{O} \times [0,T]$. Define

$$U(y,t) = J_Y(X(y,t),t)u(X(y,t),t);$$

i.e.,

$$U_i(y,t) = \sum_{j=1}^{2} \frac{\partial Y_i}{\partial x_j}(X(y,t),t)u_j(X(y,t),t)$$

and $P(y,t) = p(X(y,t),t)$, with X and Y as in Lemma 4.2.

By using Lemma 4.3, we obtain the following result (see, for instance, [18, Proposition 2.4]).

Lemma 4.4. *Suppose that X and U are as above. Then,*

$$\operatorname{div} U(y,t) = \operatorname{div} u(X(y,t),t) \quad \forall (y,t) \in \Omega_T.$$

In order to write down the equations satisfied by $U(y,t)$ and $P(y,t)$ we define

$$[\mathbf{L}U]_i = \sum_{j,k} \frac{\partial}{\partial y_j}(g^{jk}\frac{\partial U_i}{\partial y_k}) + 2\sum_{j,k,l} g^{kl}\Gamma^i_{jk}\frac{\partial U_j}{\partial y_l} + \sum_{j,k,l}\left\{\frac{\partial}{\partial y_k}(g^{kl}\Gamma^i_{jl}) + \sum_m g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right\}U_j, \quad (4.8)$$

$$[\mathbf{N}U]_i = \sum_j U_j\frac{\partial U_i}{\partial y_j} + \sum_{j,k}\Gamma^i_{jk}U_jU_k, \quad (4.9)$$

$$[\mathbf{M}U]_i = \sum_j \frac{\partial Y_j}{\partial t}\frac{\partial U_i}{\partial y_j} + \sum_{j,k}\left\{\Gamma^i_{jk}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_j}\right\}U_j, \quad (4.10)$$

$$[\mathbf{G}P]_i = \sum_{j=1}^{2} g^{ij} \frac{\partial P}{\partial y_j}, \tag{4.11}$$

where we have denoted (see for instance [6])

$$g^{ij} = \sum_k \frac{\partial Y_i}{\partial x_k} \frac{\partial Y_j}{\partial x_k} \quad \text{(metric contravariant tensor)}, \tag{4.12}$$

$$g_{ij} = \sum_k \frac{\partial X_i}{\partial y_k} \frac{\partial X_j}{\partial y_k} \quad \text{(metric covariant tensor)}, \tag{4.13}$$

and

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left\{ \frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} + \frac{\partial g_{ij}}{\partial y_l} \right\} \quad \text{(Christoffel symbol)}. \tag{4.14}$$

We also denote

$$\theta(t) = \int_0^t \omega(s)\,\mathrm{d}s \tag{4.15}$$

and let

$$R_{\theta(t)} = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix} \tag{4.16}$$

be the rotation matrix of angle $\theta(t)$. Then,

Proposition 4.5. *Suppose that* $h \in H^2(0,T;\mathbb{R}^2)$ *and* $\omega \in H^1(0,T;\mathbb{R})$ *is such that* $B(t) \subset \mathcal{O}_\varepsilon$ *for all* $t \in [0,T]$. *Then*

$$(u,p) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t)))$$

satisfies equations (1.1)–(1.4) *and* (1.7) *if and only if*

$$(U,P) \in \mathcal{U}(0,T;\Omega) \times L^2(0,T;H^1(\Omega))$$

satisfies the following equations:

$$\frac{\partial U}{\partial t} - \nu[\mathbf{L}U] + [\mathbf{M}U] + [\mathbf{N}U] + [\mathbf{G}P] = 0 \qquad \textit{in } \Omega_T, \tag{4.17}$$

$$\operatorname{div} U = 0, \qquad \textit{in } \Omega_T, \tag{4.18}$$

$$U = 0, \qquad \textit{in } \partial\mathcal{O} \times [0,T], \tag{4.19}$$

$$U(y,t) = R_{\theta(t)} h'(t) + \omega(t) y^\perp, \qquad \textit{on } \partial B \times [0,T], \tag{4.20}$$

$$U(y,0) = u_0(y), \qquad \textit{in } \Omega. \tag{4.21}$$

Proof. The equivalence of (1.1) and (4.17) has been established in Theorem 2.5 from Inoue–Wakimoto [18]. The equivalence of (1.2) and (4.18) follows from Lemma 4.3. The facts that (1.3) is equivalent to (4.19) and that (1.7) is equivalent to (4.21) follow directly from the change of variables. We still have to show that (1.4) is equivalent to (4.20).

Since $B(t) \subset \mathcal{O}_\varepsilon$, a simple calculation shows that

$$X(y,t) = h(t) + R_{-\theta(t)} y, \quad \forall y \in \partial B. \tag{4.22}$$

Therefore, for all $y \in \partial B$, we have that

$$J_X(y,t) = R_{-\theta(t)}, \quad J_Y(X(y,t),t) = R_{\theta(t)}.$$

In particular,

$$J_Y(X(y,t),t)\,(X(y,t) - h(t))^\perp = R_{\theta(t)} \left(R_{-\theta(t)} y\right)^\perp = y^\perp.$$

This concludes the proof of the proposition. □

In order to transform the equations satisfied by h and ω, we define

$$\Sigma = -P\mathrm{Id} + 2\nu D(U).$$

Then, the following result holds:

Proposition 4.6. *Suppose that $h \in H^2(0,T;\mathbb{R}^2)$ and $\omega \in H^1(0,T;\mathbb{R})$ is such that $B(t) \subset \mathcal{O}_\varepsilon$. Suppose also*

$$(u,p) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))).$$

Then we have that

$$\int_{\partial B(t)} \sigma n \,\mathrm{d}\Gamma = R_{-\theta(t)} \left(\int_{\partial B} \Sigma n \,\mathrm{d}\Gamma \right), \tag{4.23}$$

$$\int_{\partial B(t)} (x - h(t))^\perp \cdot \sigma n \,\mathrm{d}x = \int_{\partial B} y^\perp \cdot \Sigma n \,\mathrm{d}y. \tag{4.24}$$

Proof. By applying a classical change-of-variables formula for curvilinear integrals (see, for instance, Gurtin [14, p. 51]) and (4.22), it follows that for any continuous function φ, we have

$$\int_{\partial B(t)} \varphi(x) n(x) \,\mathrm{d}x = \int_{\partial B} \varphi(X(y,t)) R_{-\theta} n(y) \,\mathrm{d}y,$$

where $R_{\theta(t)}$ is defined by (4.15) and (4.16). On the other hand, since $B(t) \subset \mathcal{O}_\varepsilon$, we have that

$$\forall x \in B(t), \quad u(x,t) = R_{-\theta(t)} U(Y(x,t),t),$$

which implies that

$$\forall x \in B(t), \quad \nabla u(x,t) = R_{-\theta(t)} \nabla U(Y(x,t),t) J_Y(x,t),$$

and hence

$$\forall x \in B(t), \quad \nabla u(X(y,t),t) = R_{-\theta(t)} \nabla U(y,t) R_{\theta(t)}.$$

Therefore, we obtain that

$$\forall x \in B(t), \quad D(u)(X(y,t),t) = R_{-\theta(t)} D(U)(y,t) R_{\theta(t)},$$

which proves (4.23). In order to prove (4.24), we notice that

$$-\int_{\partial B(t)} (x-h(t))^{\perp} \cdot \sigma n \, \mathrm{d}x = \int_{\partial B(t)} \sigma (x-h(t))^{\perp} \cdot n \, \mathrm{d}x$$
$$= \int_{\partial B} \left(R_{-\theta(t)} \Sigma y^{\perp} \right) \cdot \left(R_{-\theta(t)} n \right) \mathrm{d}y = \int_{\partial B} y^{\perp} \cdot \Sigma n \, \mathrm{d}y. \qquad \square$$

5. Some background on linear problems.

We first consider a linear system coupling Stokes-type equations in a fixed domain to a system of O.D.E.

$$\frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = F, \qquad \text{in } \Omega \times [0,T], \tag{5.1}$$
$$\operatorname{div} U = 0, \qquad \text{in } \Omega \times [0,T], \tag{5.2}$$
$$U(y,t) = 0, \qquad y \in \partial\mathcal{O},\ t \in [0,T], \tag{5.3}$$
$$U(y,t) = h'(t) + \omega(t) y^{\perp}, \qquad y \in \partial B,\ t \in [0,T], \tag{5.4}$$
$$M h''(t) = -\int_{\partial B} \Sigma n \, \mathrm{d}\Gamma + F_M, \qquad t \in [0,T], \tag{5.5}$$
$$J \omega'(t) = -\int_{\partial B} y^{\perp} \cdot \Sigma n \, \mathrm{d}y + F_J, \qquad t \in [0,T], \tag{5.6}$$
$$U(x,0) = u_0(x), \qquad x \in \Omega, \tag{5.7}$$
$$h(0) = 0,\ h'(0) = h_1,\ \omega(0) = \omega_0, \tag{5.8}$$

where $\Sigma(y,t) = -P(y,t)\mathrm{Id} + 2\nu D(U)(y,t)$. We suppose $B \subset \mathcal{O}_\varepsilon$. U, P, h, and ω are the unknows of this system, whereas u_0, h_1, ω_0, F, F_M, and F_J are given. The gravity center of B is in $y = 0$.

The study of a similar system has already been made in [25]. We will follow the same approach based on the theory of semigroups (see also [27] for a similar approach).

We also define

$$\mathbb{H} = \{\phi \in \mathcal{L}^2(\mathcal{O}) : \operatorname{div}(\phi) = 0 \quad \text{in } \mathcal{O}, \quad D(\phi) = 0 \quad \text{in } B, \quad \phi \cdot n = 0 \quad \text{on } \partial\mathcal{O}\}, \tag{5.9}$$

$$\mathbb{V} = \{\phi \in \mathcal{H}^1_0(\mathcal{O}) : \operatorname{div}(\phi) = 0 \quad \text{in } \mathcal{O}, \quad D(\phi) = 0 \quad \text{in } B\}. \tag{5.10}$$

We know (see, for instance, Lemma 1.1 of [26, p. 18]) that for any $\phi \in \mathbb{H}$, there exist $V_\phi \in \mathbb{R}^2$ and $\omega_\phi \in \mathbb{R}$ such that

$$\phi(y) = V_\phi + \omega_\phi y^\perp, \quad \forall y \in B.$$

We define an inner product in $\mathcal{L}^2(\mathcal{O})$ by

$$(\psi, \phi)_{\mathcal{L}^2(\mathcal{O})} = \int_\Omega (\psi \cdot \phi)\,\mathrm{d}y + \int_B (\rho\psi \cdot \phi)\,\mathrm{d}y$$

where $\rho > 0$ is the density of the rigid body. The associated norm is equivalent to the usual norm of $\mathcal{L}^2(\mathcal{O})$. If ψ and $\phi \in \mathbb{H}$, then

$$(\psi, \phi)_{\mathcal{L}^2(\mathcal{O})} = \int_\Omega (\psi \cdot \phi)\,\mathrm{d}y + MV_\phi \cdot V_\psi + J\omega_\phi\omega_\psi. \tag{5.11}$$

Since Ω is of class C^2, we can use the following result on the Stokes equations (see, for instance, Galdi [9, Theorem 5.1, p. 232]):

Theorem 5.1. *For any $f \in \mathcal{L}^2$ and $v_* \in \mathcal{H}^{3/2}(\partial\Omega)$, with*

$$\int_{\partial\Omega} v_* \cdot n\,\mathrm{d}\Gamma = 0,$$

there exists one and only one pair $(v, p) \in \mathcal{H}^2 \times H^1(\Omega)$ such that

$$\begin{aligned} -\nu\Delta v + \nabla p = f &\quad \textit{in} \quad \Omega,\\ \operatorname{div} v = 0 &\quad \textit{in} \quad \Omega,\\ v = v_* &\quad \textit{on} \quad \partial\Omega,\\ \int_\Omega p\,\mathrm{d}y = 0. & \end{aligned}$$

In addition, there exists a constant $c > 0$ such that

$$\|v\|_{\mathcal{H}^2} + \|p\|_{H^1(\Omega)} \leqslant c(\|f\|_{\mathcal{L}^2} + \|v_*\|_{\mathcal{H}^{3/2}(\partial\Omega)}).$$

Above we denoted $\mathcal{H}^{3/2}(\partial\Omega) = \left[H^{3/2}(\partial\Omega)\right]^2$. In order to solve (5.1)–(5.8) we use a semigroup approach. Define

$$D(A) = \{\phi \in \mathcal{H}^1_0(\mathcal{O}) : \phi_{|\Omega} \in \mathcal{H}^2, \quad \operatorname{div}(\phi) = 0 \quad \text{in } \mathcal{O}, \quad D(\phi) = 0 \quad \text{in } B\} \tag{5.12}$$

and the operators

$$\mathcal{A}u = \begin{cases} -\nu\Delta u \quad \text{in } \Omega,\\ \dfrac{2\nu}{M}\displaystyle\int_{\partial B} D(u)n\,\mathrm{d}\Gamma + \left[\dfrac{2\nu}{J}\displaystyle\int_{\partial B} y^\perp \cdot D(u)n\,\mathrm{d}y\right] y^\perp \quad \text{in } B, \end{cases} \tag{5.13}$$

and

$$Au = \mathbb{P}\mathcal{A}u, \tag{5.14}$$

defined for all $u \in D(A)$, where $\mathbb{P}$ is the orthogonal projector from $\mathcal{L}^2(\mathcal{O})$ on $\mathbb{H}$ ($\mathbb{H}$ is clearly a closed subspace of $\mathcal{L}^2(\mathcal{O})$) and where, in the expression of $\mathcal{A}u$, $D(u)$ represents the trace of the restriction of $D(u)$ to $\Omega = \mathcal{O} \setminus B$. The simple result below will be used several times in the sequel.

Lemma 5.2. *Suppose that $u \in \mathbb{V}$, where $\mathbb{V}$ is defined by* (5.10)*. Then*

$$\forall u \in D(A), \quad \|\nabla u\|^2_{[L^2(\mathcal{O})]^4} = 2\|D(u)\|^2_{[L^2(\Omega)]^4}.$$

The following proposition can be proven in the same way as Proposition 4.2 in [25], except we use Theorem 5.1 instead of Theorem 3.2 in [25].

Proposition 5.3. *The operator A defined by* (5.12)*,* (5.13)*, and* (5.14) *is self-adjoint and positive. Consequently $-A$ is the generator of a contraction semigroup in $\mathbb{H}$. Moreover, there exists a constant $C > 0$ such that for any $u \in D(A)$, we have*

$$\|u\|_{[H^2(\Omega)]^2} \leqslant C\|Au\|_{[L^2(\Omega)]^2}. \tag{5.15}$$

In the following, we denote by $\mathcal{U}$ the Banach space

$$\mathcal{U} = L^2(0,T;\mathcal{H}^2) \cap C([0,T];\mathcal{H}^1) \cap H^1(0,T;\mathcal{L}^2), \tag{5.16}$$

endowed with the norm

$$\|U\|_{\mathcal{U}} = \|U\|_{L^2(0,T;\mathcal{H}^2)} + \|U\|_{L^\infty(0,T;\mathcal{H}^1)} + \|U\|_{H^1(0,T;\mathcal{L}^2)}.$$

As in [25], we can write the system (5.1)–(5.8) in the form

$$U' + AU = \mathbb{P}F, \quad U(0) = u_0,$$

and by using the previous proposition, we can prove the following result:

Corollary 5.4. *Suppose that $F \in L^2(0,T;\mathcal{L}^2)$, that $F_M \in L^2(0,T;\mathbb{R}^2)$, that $F_J \in L^2(0,T;\mathbb{R})$, that $u_0 \in \mathcal{H}^1(\mathcal{O})$, and that*

$$\begin{aligned} \operatorname{div} u_0 &= 0, && \text{in } \Omega, \\ u_0(y) &= 0, && y \in \partial\mathcal{O}, \\ u_0(y) &= h_1 + \omega_0 y^\perp, && y \in \partial B. \end{aligned}$$

Then the system (5.1)–(5.8) *admits a unique solution (U, P, h, ω) with*

$$U \in L^2(0,T;\mathcal{H}^2) \cap C([0,T];\mathcal{H}^1) \cap H^1(0,T;\mathcal{L}^2),$$

$$P \in L^2(0,T;H^1(\Omega)), \quad h \in H^2(0,T;\mathbb{R}^2), \quad \omega \in H^1(0,T;\mathbb{R}).$$

Moreover, there exists a positive constant K such that

$$\begin{aligned} &\|U\|_{\mathcal{U}} + \|\nabla P\|_{L^2(0,T;\mathcal{L}^2)} + \|h'\|_{H^1(0,T;\mathbb{R}^2)} + \|\omega\|_{H^1(0,T;\mathbb{R})} \\ &\leqslant K\left(\|u_0\|_{\mathcal{H}^1(\mathcal{O})} + \|F\|_{L^2(0,T;\mathcal{L}^2)} + \|F_M\|_{L^2(0,T;\mathbb{R}^2)} + \|F_J\|_{L^2(0,T;\mathbb{R})}\right). \end{aligned} \tag{5.17}$$

The constant K depends only on Ω and T and is nondecreasing with respect to T.

6. Local existence.

6.1. **Introduction.** The aim of this section is to prove a local-in-time existence result for the system (1.1)–(1.8). Denote

$$H(t)=\int_0^t R_{\theta(s)}h'(s)\,\mathrm{d}s.$$

Then $H'(t)=R_{\theta(t)}h'(t)$ and

$$H''(t)=R_{\theta(t)}h''(t)-\omega(t)R_{\theta(t)}h'(t)^{\perp}=R_{\theta(t)}h''(t)-\omega(t)H'(t)^{\perp}. \tag{6.1}$$

Then, by Propositions 4.5 and 4.6 the local-in-time existence for the system (1.1)–(1.8) is equivalent to the local existence of a solution for

$$\frac{\partial U}{\partial t}-\nu[\mathbf{L}U]+[\mathbf{M}U]+[\mathbf{N}U]+[\mathbf{G}P]=0, \qquad \text{in } \Omega_T, \tag{6.2}$$

$$\operatorname{div} U=0, \qquad \text{in } \Omega_T, \tag{6.3}$$

$$U=0, \qquad \text{in } \partial\mathcal{O}\times[0,T], \tag{6.4}$$

$$U(y,t)=H'(t)+\omega'(t)y^{\perp}, \qquad \text{in } \partial B\times[0,T], \tag{6.5}$$

$$MH''(t)=-\int_{\partial B}\Sigma n\,\mathrm{d}\Gamma+M\omega(t)H'(t)^{\perp}, \qquad t\in[0,T], \tag{6.6}$$

$$J\omega'(t)=-\int_{\partial B}y^{\perp}\cdot\Sigma n\,\mathrm{d}y, \qquad t\in[0,T], \tag{6.7}$$

$$U(y,0)=u^0(y), \qquad \text{in } \Omega, \tag{6.8}$$

$$H(0)=0,\ H'(0)=h_1,\ \omega(0)=\omega_0. \tag{6.9}$$

The main result of this section is

Proposition 6.1. *Suppose that $C_1>0$, that $u_0\in\mathcal{H}^1$, that $0<\varepsilon<\operatorname{dist}(B,\partial\mathcal{O})$, and that*

$$\begin{cases} \operatorname{div} u_0=0 & \textit{in } \Omega,\\ u_0(x)=0 & \textit{on } \partial\mathcal{O},\\ u_0(x)=h_1+\omega_0x^{\perp} & \textit{on } \partial B,\\ \|u_0\|_{\mathcal{H}^1}\leqslant C_1. \end{cases}$$

Then there exists $T_0>0$, depending only on C_1 and ε, such that equations (1.1)–(1.8) admit a unique strong solution (u,p,h,ω) such that $u\in\mathcal{U}(0,T_0;\Omega(t))$, $p\in L^2(0,T_0;H^1(\Omega(t)))$, $h\in H^2(0,T_0,\mathbb{R}^2)$, and $\omega\in H^1(0,T_0,\mathbb{R})$.

The solution of the above problem can be seen as a fixed point of the mapping $\mathcal{N} : (W, Q, h, \omega) \mapsto (U, P, \tilde{h}, \tilde{\omega})$ from

$$\mathcal{U} \times L^2(0,T; H^1(\Omega)) \times H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})$$

into itself, where $(U, P, \tilde{h}, \tilde{\omega})$ satisfies

$$\begin{aligned}
\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = F, &\qquad \text{in } \Omega \times [0,T],\\
\operatorname{div} U = 0, &\qquad \text{in } \Omega \times [0,T],\\
U(y,t) = 0, &\qquad y \in \partial\mathcal{O},\ t \in [0,T],\\
U(y,t) = \widetilde{H}'(t) + \tilde{\omega}(t) y^{\perp}, &\qquad y \in \partial B,\ t \in [0,T],\\
M\widetilde{H}''(t) = -\int_{\partial B} \Sigma n \,\mathrm{d}\Gamma + F_M, &\qquad t \in [0,T],\\
J\tilde{\omega}'(t) = -\int_{\partial B} y^{\perp} \cdot \Sigma n \,\mathrm{d}y, &\qquad t \in [0,T],\\
U(x,0) = u_0(x), &\qquad x \in \Omega,\\
\widetilde{H}(0) = 0,\ \widetilde{H}'(0) = h_1,\ \tilde{\omega}(0) = \omega_0,&
\end{aligned}$$

where

$$F = \nu[(\mathbf{L} - \Delta)W] - [\mathbf{M}W] + [(\nabla - \mathbf{G})Q] - [\mathbf{N}W], \tag{6.10}$$

$$F_M = M\omega(t) R_\theta h'(t)^{\perp}, \tag{6.11}$$

and

$$\tilde{h} = \int_0^t R_{-\theta} \widetilde{H}'(s) \,\mathrm{d}s,$$

with the operators $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{G}$ and the function θ defined from (h, ω) by (4.8)–(4.15).

Let $T > 0$. Define

$$\begin{aligned}
\mathcal{K} = \{&(W, Q, h, \omega) \in \mathcal{U} \times L^2(0,T; H^1(\Omega)) \times H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})\\
&\|W\|_{\mathcal{U}} + \|Q\|_{L^2(0,T;H^1(\Omega))} + \|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})} \leqslant R\}.
\end{aligned} \tag{6.12}$$

We clearly have that $\mathcal{K}$ is a closed subset of

$$\mathcal{U} \times L^2(0,T; H^1(\Omega)) \times H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R}).$$

By Corollary 5.4, $\mathcal{N}$ maps from

$$\mathcal{U} \times L^2(0,T; H^1(\Omega)) \times H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})$$

into itself. We will prove that for T small enough and R large enough, we have that $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$ and that $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction.

6.2. **Estimates on the coefficients.** Consider $(W, Q, h, \omega) \in \mathcal{K}$ where $\mathcal{K}$ is defined by (6.12). In this subsection, we prove some estimates in order to prove that for T small enough and R large enough, we have that $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$. The estimates are not always sharp but sufficient to conclude.

In the sequel, we denote by K_0 a quantity satisfying the following condition:

(C1) K_0 is a positive function of ω_0, h_1, $\|u_0\|_{\mathcal{H}^1}$, T, and R which is nondecreasing with respect to T, R, $\|u_0\|_{\mathcal{H}^1}$, $|h_0|$, $|h_1|$, and $|\omega_0|$.

Similarly we denote by C_0 a quantity satisfying the following condition:

(C2) C_0 is a positive function of ω_0, h_1, $\|u_0\|_{\mathcal{H}^1}$, and T which is nondecreasing with respect to T, $\|u_0\|_{\mathcal{H}^1}$, $|h_0|$, $|h_1|$, and $|\omega_0|$. We also denote by C an independent constant of ω_0, h_1, $\|u_0\|_{H^1(\Omega)}$, and R.

Then, by using (6.12) and the Cauchy-Schwarz inequality, we easily check the following:

Lemma 6.2. *Suppose that* $(W, Q, h, \omega) \in \mathcal{K}$. *Then there exists* C_0 *satisfying* (C2) *such that*

$$\|\omega\|_{L^\infty(0,T;\mathbb{R})} \leqslant C_0 + T^{1/2}R, \quad \|h'\|_{L^\infty(0,T;\mathbb{R}^2)} \leqslant C_0 + T^{1/2}R,$$
$$\|h\|_{L^\infty(0,T;\mathbb{R}^2)} \leqslant C_0 + T^{1/2}R.$$

There also exist C_0 *satisfying* (C2) *and a constant* K_0 *satisfying* (C1) *such that the function* V_R, *defined by* (4.1), *satisfies*

$$\|V_R\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0, \quad \left\|\frac{\partial V_R}{\partial x}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0.$$

We next estimate the solution of the Cauchy problem (4.4).

Lemma 6.3. *Suppose that* $(W, Q, h, \omega) \in \mathcal{K}$. *Then there exist* C_0 *satisfying* (C2) *and a constant* K_0 *satisfying* (C1) *such that the function* Λ *defined by* (4.3) *satisfies the following relations:*

$$\|\Lambda\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0, \quad \left\|\frac{\partial \Lambda}{\partial x}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0,$$

$$\left\|\frac{\partial^2 \Lambda}{\partial x_i \partial x_j}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0, \quad \left\|\frac{\partial^3 \Lambda}{\partial x_i \partial x_j \partial x_k}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0.$$

Proof. From (4.2), it follows that

$$w(x,t) = -h_2'(t)x_1 + \omega(t)\frac{x_1^2}{2} - \omega(t)h_1(t)x_1 + h_1'(t)x_2 + \omega(t)\frac{x_2^2}{2} - \omega(t)h_2(t)x_2.$$

The above relation combined with the estimates on h', h, ω, and V_R in Lemma 6.2 imply that

$$\|w\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0, \quad \left\|\frac{\partial w}{\partial x}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0 + T^{1/2}K_0,$$

$$\left\|\frac{\partial^2 w}{\partial x_i \partial x_j}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0+T^{1/2}K_0, \quad \left\|\frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k}\right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant C_0+T^{1/2}K_0,$$

for some constant K_0 satisfying (C1) and C_0 satisfying (C2). So by using (4.3), the previous lemma, and the fact that

$$\frac{\partial^2 V_R}{\partial x_i \partial x_j} = 0,$$

we obtain the conclusion. □

The result below yields estimates of the change-of-variable mappings X and Y.

Lemma 6.4. *Suppose that* $(W, Q, h, \omega) \in \mathcal{K}$. *Then the functions* X *and* Y *defined by* (4.4) *and by Lemma* 4.2 *satisfy the following inequalities:*

$$\left\|\frac{\partial Y_i}{\partial x_j}\right\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0, \quad \left\|\frac{\partial X_i}{\partial y_j}\right\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0,$$

$$\left\|\frac{\partial^2 Y_i}{\partial x_j \partial x_k}\right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0, \quad \left\|\frac{\partial^2 X_i}{\partial y_j \partial y_k}\right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0,$$

$$\left\|\frac{\partial^3 Y_i}{\partial x_j \partial x_l \partial x_k}\right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0, \quad \left\|\frac{\partial^3 X_i}{\partial y_j \partial y_l \partial y_k}\right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0.$$

Proof. By using a classical result (see, for instance, [15, p. 95]), we have that $z(y,t) = \frac{\partial X}{\partial y_j}(y,t)$ satisfies

$$\frac{\partial z}{\partial t}(y,t) = J(y,t)z(y,t), \quad z(0) = e_j,$$

where $J(y,t) = (\frac{\partial \Lambda}{\partial x}(X(y,t),t))$ and where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Therefore, we have that

$$z(y,t) = e_j + \int_0^t J(y,s)z(y,s)ds, \tag{6.13}$$

which implies, by using Gronwall's lemma and Lemma 6.3, that

$$\|z\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant K_0.$$

By using Lemma 6.3, we obtain the other estimates on X. The proof of the estimates on Y and its derivatives is similar, so it is left to the reader. □

The above lemma allows us to get several new estimates:

Corollary 6.5. *Suppose that* $(W, Q, h, \omega) \in \mathcal{K}$. *Then there exists a constant* K_0 *satisfying* (C1) *such that*

$$\left\| \frac{\partial Y_m}{\partial x_l} - \delta_{m,l} \right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0, \quad \left\| \frac{\partial X_m}{\partial y_l} - \delta_{m,l} \right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0, \tag{6.14}$$

$$\left\| g^{ml} - \delta_{m,l} \right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0, \quad \left\| g_{ml} - \delta_{m,l} \right\|_{L^\infty(\mathcal{O}_T)} \leqslant TK_0. \tag{6.15}$$

Proof. Since

$$\frac{\partial X_m}{\partial y_l}(0, y) = \delta_{m,l}, \quad \frac{\partial Y_m}{\partial x_l}(0, x) = \delta_{m,l},$$

by using the mean-value theorem and the previous lemma, we get (6.14). The estimates on g^{ml} and on g_{ml} follow then by (4.12) and (4.13). □

Corollary 6.6. *Suppose that* $(W, Q, h, \omega) \in \mathcal{K}$, *and that* $\mathbf{L}$, $\mathbf{M}$, *and* $\mathbf{G}$ *are given by* (4.8), (4.9), *and* (4.11). *Then there exists a constant* K_0 *satisfying* (C1) *such that*

(i) $\|\nu[(\mathbf{L} - \Delta)W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant TK_0$

(ii) $\|[\mathbf{M}W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant T^{1/2}K_0$

(iii) $\|[(\nabla - \mathbf{G})Q]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant TK_0$.

Proof. From (4.8), it follows that

$$[\mathbf{L}W]_i = \sum_{k,j} g^{jk} \frac{\partial^2 W_i}{\partial y_j \partial y_k} + \sum_{k,j} \frac{\partial (g^{jk})}{\partial y_j} \frac{\partial W_i}{\partial y_k} + 2 \sum_l g^{kl} \Gamma^i_{jk} \frac{\partial W_j}{\partial y_l} + \sum_{k,l} \left\{ \frac{\partial}{\partial y_l} (g^{kl} \Gamma^i_{jl}) + \sum_m g^{kl} \Gamma^m_{jl} \Gamma^i_{km} \right\} W_j. \tag{6.16}$$

By using Lemma 6.4, we deduce that the coefficients of the second and first derivatives of W are bounded in $L^\infty(0, T; L^\infty(\mathcal{O}))$ by K_0. On the other hand,

$$\|W\|_{L^2(0,T;\mathcal{H}^1)} \leqslant T^{1/2} \|W\|_{L^\infty(0,T;\mathcal{H}^1)} \leqslant T^{1/2} R. \tag{6.17}$$

Thus,

$$\|[(\mathbf{L} - \Delta)W]_i\|_{L^2(0,T;\mathcal{L}^2)} \leqslant \left\| \sum_{k,j} (g^{jk} - \delta_{j,k}) \frac{\partial^2 W_i}{\partial y_j \partial y_k} \right\|_{L^2(0,T;\mathcal{L}^2)} + K_0 T^{1/2} R,$$

and assertion (i) can be deduced from Corollary 6.5. The proof of assertion (ii) is similar: we use Lemma 6.4 and relation (6.17). Since

$$[(\nabla - \mathbf{G})Q]_i = \sum_j \left(\delta_{i,j} - g^{ij} \right) \frac{\partial Q}{\partial x_j},$$

the relation (iii) is a consequence of (6.15). □

The following useful lemma has been proven in [25].

Lemma 6.7. *Let $\mathcal{U}$ be the space defined by* (5.16)*. Then, for any $V, W \in \mathcal{U}$, we have $(W \cdot \nabla)V \in L^{5/2}(0,T;\mathcal{L}^2)$, and for all $i, j \in \{1,2\}$, we have that $W_i V_j \in L^\infty(0,T;\mathcal{L}^2)$. Finally, the following estimates hold:*

$$\|(W \cdot \nabla)V\|_{L^{5/2}(0,T;\mathcal{L}^2)} \leqslant C\|W\|_{L^\infty(0,T;\mathcal{H}^1)}\|V\|^{1/5}_{L^\infty(0,T;\mathcal{H}^1)}\|V\|^{4/5}_{L^2(0,T;\mathcal{H}^2)},$$

$$\|W_i V_j\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\|W\|_{L^\infty(0,T;\mathcal{H}^1)}\|V\|_{L^\infty(0,T;\mathcal{H}^1)}.$$

The previous lemma allows us to get estimates on the nonlinear term $[\mathbf{N}W]$.

Corollary 6.8. *There exists a constant K_0 satisfying* (C1) *such that for any $(W, Q, h, \omega) \in \mathcal{K}$, we have that*

$$\|[\mathbf{N}W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant T^{1/10}K_0.$$

Proof. By using the Lemma 6.7, we have that

$$\|(W \cdot \nabla)W\|_{L^{5/2}(0,T;\mathcal{L}^2)} \leqslant C\|W\|^{6/5}_{L^\infty(0,T;\mathcal{H}^1)}\|W\|^{4/5}_{L^2(0,T;\mathcal{H}^2)} \leqslant K_0.$$

Thus, by Hölder's inequality, we deduce that

$$\|(W \cdot \nabla)W\|_{L^2(0,T;\mathcal{L}^2)} \leqslant T^{1/10}\|(W \cdot \nabla)W\|_{L^{5/2}(0,T;\mathcal{L}^2)} \leqslant T^{1/10}K_0.$$

To conclude, we use Lemma 6.4 and Lemma 6.7. □

From the previous estimates (Corollary 6.8 and Corollary 6.6) and Corollary 5.4, we obtain the following result.

Corollary 6.9. *There exists a constant K_0 satisfying* (C1) *such that for any $(W, Q, h, \omega) \in \mathcal{K}$, the functions F and F_M, defined by* (6.10) *and* (6.11)*, satisfy*

$$\|F\|_{L^2(0,T;\mathcal{L}^2)} \leqslant K_0 T^{1/10}, \quad \|F_M\|_{L^2(0,T;\mathbb{R}^2)} \leqslant C_0 + K_0 T^{1/2}.$$

Consequently, for R large enough ($R > C_0$) and T small enough, $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$.

6.3. **Proof of the local existence result.** In order to apply the fixed-mapping theorem, we show that for R as in the previous section, and for T small enough, the mapping $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction. Let us consider $(W^1, Q^1, h^1, \omega^1)$, $(W^2, Q^2, h^2, \omega^2) \in \mathcal{K}$. In the sequel, we denote by Y^1, X^1, Γ^{1i}_{jk}, g^{1ij}, U^1, P^1, V^1_R, etc. the terms corresponding to $(W^1, Q^1, h^1, \omega^1)$ defined by (4.8)–(4.14) and $\mathcal{N}(W^1, Q^1, h^1, \omega^1) = (U^1, P^1, \tilde{h^1}, \tilde{\omega^1})$. We use a similar notation for $(W^2, Q^2, h^2, \omega^2) \in \mathcal{K}$. We denote $Y = Y^1 - Y^2$, $h = h^1 - h^2$, etc. The difference $U = U^1 - U^2$ satisfies

$$\begin{aligned}
\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = F, &\qquad \text{in } \Omega \times [0,T],\\
\operatorname{div} U = 0, &\qquad \text{in } \Omega \times [0,T],\\
U(y,t) = 0, &\qquad y \in \partial\mathcal{O},\ t \in [0,T],\\
U(y,t) = \widetilde{H}'(t) + \tilde{\omega}(t)y^\perp, &\qquad y \in \partial B,\ t \in [0,T],\\
M\widetilde{H}''(t) = -\int_{\partial B} \Sigma n \,\mathrm{d}\Gamma + F_M, &\qquad t \in [0,T],\\
J\tilde{\omega}'(t) = -\int_{\partial B} y^\perp \cdot \Sigma n \, dy, &\qquad t \in [0,T],\\
U(x,0) = 0(x), &\qquad x \in \Omega,\\
\widetilde{H}(0) = 0,\ \widetilde{H}'(0) &= 0,\ \tilde{\omega}(0) = 0,
\end{aligned}$$

where

$$\begin{aligned}
F = \nu[(\mathbf{L}^1 - \Delta)W] + \nu[\mathbf{L}W^2] - [\mathbf{M}^1 W] - [\mathbf{M}W^2] + [(\nabla - \mathbf{G}^1)Q] + [\mathbf{G}Q^2]\\
+ [\mathbf{N}^1 W^1] - [\mathbf{N}^2 W^2]. \quad (6.18)
\end{aligned}$$

$$F_M = M\omega^1 R^1_\theta[(h^1)']^\perp - M\omega^2 R^2_\theta[(h^2)']^\perp. \qquad (6.19)$$

As in the previous section, we first estimate the coefficients:

Lemma 6.10. *There exists a constant K_0 satisfying* (C1) *such that the functions $\omega = \omega^1 - \omega^2$, $h = h^1 - h^2$, and $V_R = V_R^1 - V_R^2$ satisfy*

$$\|\omega\|_{L^\infty(0,T;\mathbb{R})} \leqslant T^{1/2}\|\omega'\|_{L^2(0,T;\mathbb{R})},$$

$$\|h'\|_{L^\infty(0,T;\mathbb{R}^2)} \leqslant T^{1/2}\|h''\|_{L^2(0,T;\mathbb{R}^2)},$$

$$\|h\|_{L^\infty(0,T;\mathbb{R}^2)} \leqslant K_0 T^{1/2}\|h''\|_{L^2(0,T;\mathbb{R}^2)}$$

and

$$\|V_R\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant T^{1/2}K_0\{\|\omega'\|_{L^2(0,T;\mathbb{R})} + \|h''\|_{L^2(0,T;\mathbb{R}^2)}\}.$$

Proof. In fact,

$$\omega(t) = \omega^1(t) - \omega^2(t) = \Big(\omega_0 + \int_0^t (\omega^1)'(s)\,ds\Big) - \Big(\omega_0 + \int_0^t (\omega^2)'(s)\,ds\Big).$$

Thus,

$$\omega(t) = \int_0^t \left[(\omega^1)' - (\omega^2)'\right](s)\,ds = \int_0^t \omega'(s)\,ds.$$

The above relation and the Cauchy-Schwarz inequality yield the first estimate. The proof of the next two estimates is similar. For the last assertion of the lemma, we notice that

$$\begin{aligned} V_R(x,t) &= V_R^1(x,t) - V_R^2(x,t) \\ &= (h^1)'(t) + \omega^1(t)(x-h^1(t))^\perp - (h^2)'(t) - \omega^2(t)(x-h^2(t))^\perp \\ &= h'(t) + \omega(t)(x-h^2(t))^\perp - \omega^1(t)h(t), \end{aligned} \tag{6.20}$$

which implies the conclusion of the lemma. □

Lemma 6.11. *There exists a constant K_0 such that the function $\Lambda = \Lambda^1 - \Lambda^2$ satisfies*

$$\left\| \frac{\partial^{\alpha_1+\alpha_2}\Lambda}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} \right\|_{[L^\infty(O_T)]^2} \leqslant T^{1/2}K_0\{\|\omega'\|_{L^2(0,T;\mathbb{R})} + \|h''\|_{L^2(0,T;\mathbb{R}^2)}\}. \tag{6.21}$$

Proof. By using (4.3), we obtain that

$$\Lambda(x_1,x_2,t) = \begin{pmatrix} \frac{\partial\xi}{\partial x_2}w + \xi V_R^1 \\ -\frac{\partial\xi}{\partial x_1}w + \xi V_R^2 \end{pmatrix}, \tag{6.22}$$

where

$$w = w^1 - w^2 = -\int_0^{x_1} V_R^2(s,t)\,ds + \int_0^{x_2} V_R^1(s,t)\,ds.$$

The function w can also be written

$$\begin{aligned} w(x,t) = -h_2'(t)x_1 + \omega(t)\frac{x_1^2}{2} - \omega^1(t)h_1(t)x_1 - \omega(t)h_1^2(t)x_1 \\ + h_1'(t)x_2 + \omega(t)\frac{x_2^2}{2} - \omega^1(t)h_2(t)x_2 - \omega(t)h_2^2(t)x_2. \end{aligned}$$

By differentiating the above formula with respect to the space variables and by using (6.20), we obtain (6.21). □

Lemma 6.12. *There exists a constant K_0 such that the functions $X = X^1 - X^2$ and $Y = Y^1 - Y^2$ satisfy*

$$\left\| \frac{\partial^{\alpha_1+\alpha_2}X}{\partial y_1^{\alpha_1}\partial y_2^{\alpha_2}} \right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant K_0T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}), \tag{6.23}$$

$$\left\| \frac{\partial^{\alpha_1+\alpha_2}Y}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} \right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant K_0T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}). \tag{6.24}$$

Proof. For the sake of simplicity, we give only the proof of (6.23) in the particular case $\alpha_1 = 1, \alpha_2 = 0$ or $\alpha_1 = 0, \alpha_2 = 1$. The proof of (6.23) and

(6.24) in the general case can be obtained by iterating the same procedure. $X = X^1 - X^2$ satisfies the equations

$$X'(t) = \Lambda^1(X^1(t), t) - \Lambda^1(X^2(t), t) + \Lambda(X^2(t), t), \quad X(0) = 0.$$

If we integrate the above relation with respect to time, we obtain

$$X(t) = \int_0^t \Lambda^1(X^1(s), s) - \Lambda^1(X^2(s), s) + \Lambda(X^2(s), s)\, ds,$$

which implies that

$$|X(t)| \leqslant \int_0^t |\Lambda^1(X^1(s), s) - \Lambda^1(X^2(s), s)| + |\Lambda(X^2(s), s)|\, ds.$$

By using the mean-value theorem, and Lemma 6.3, we obtain that

$$|\Lambda^1(X^1(t), t) - \Lambda^1(X^2(t), t)| \leqslant K_0 |X(t)|,$$

for a constant K_0 satisfying the condition (C1).

On the other hand, by Lemma 6.11 and the previous relation, we have that

$$\int_0^t |\Lambda(X^2(s), s)| ds \leqslant T^{3/2} K_0 \{\|\omega'\|_{L^2(0,T;\mathbb{R})} + \|h''\|_{L^2(0,T;\mathbb{R}^2)}\}.$$

By using Gronwall's lemma, we deduce that

$$|X(t)| \leqslant T^{3/2} K_0 \{\|\omega'\|_{L^2(0,T;\mathbb{R})} + \|h''\|_{L^2(0,T;\mathbb{R}^2)}\} \exp(K_0 T). \tag{6.25}$$

By using (6.13), we have that

$$\frac{\partial X}{\partial y_i}(y, t) = \int_0^t J^1(y, s) \frac{\partial X^1}{\partial y_i}(y, s) - J^2(y, s) \frac{\partial X^2}{\partial y_i}(y, s) ds.$$

The above relation, Lemma 6.3, and Lemma 6.4 yield

$$\left\| \frac{\partial X}{\partial y_i}(t) \right\|_{[L^\infty(\mathcal{O})]^2} \leqslant K_0 \left\| J^1(s) - J^2(s) \right\|_{[L^\infty(\mathcal{O})]^2} + K_0 \int_0^t \left\| \frac{\partial X}{\partial y_i}(s) \right\|_{[L^\infty(\mathcal{O})]^2} ds.$$

The above relation combined with (6.21) and with (6.25) yields

$$\left\| \frac{\partial X}{\partial y_i} \right\|_{[L^\infty(\mathcal{O}_T)]^2} \leqslant K_0 T^{1/2} (\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}). \qquad \square$$

The following lemma is a simple consequence of Lemma 6.12, of Lemma 6.3, and of the relations (4.12), (4.13), and (4.14).

Lemma 6.13. *There exists a constant K_0 such that the functions $g^{m,k} = g^{m,k,1} - g^{m,k,2}$ and $\Gamma^j_{mk} = \Gamma^{j1}_{mk} - \Gamma^{j2}_{mk}$ satisfy the following inequalities:*

$$\|g^{m,k}\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0 T^{1/2} (\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}),$$

$$\|g_{m,k}\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0 T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}),$$

$$\Big\|\frac{\partial g^{mk}}{\partial y_l}\Big\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0 T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}),$$

$$\Big\|\frac{\partial g_{mk}}{\partial x_l}\Big\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0 T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}),$$

$$\|\Gamma^j_{mk}\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0 T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}),$$

$$\Big\|\frac{\partial \Gamma^j_{mk}}{\partial y_i}\Big\|_{L^\infty(\mathcal{O}_T)} \leqslant K_0 T^{1/2}(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}).$$

The following result also holds.

Proposition 6.14. *There exists a constant K_0 satisfying* (C1) *such that for any $(W,Q)\in\mathcal{U}\times L^2(0,T;H^1(\Omega))$, we have*

$$\|[\mathbf{L}W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant K_0 T^{1/2}\|W\|_{\mathcal{U}}\left(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}\right),$$

$$\|[\mathbf{M}W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant K_0 T^{1/2}\|W\|_{\mathcal{U}}\left(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}\right),$$

$$\|[\mathbf{G}Q]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant K_0 T\|\nabla Q\|_{L^2(0,T;\mathcal{L}^2)}\times\left(\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})}\right).$$

Proof. From (6.16), it follows that

$$\begin{aligned}
[\mathbf{L}W]_i = \sum_{j,k} g^{jk}\frac{\partial^2 W_i}{\partial y_j\partial y_k} &+ \sum_{j,k}\frac{\partial(g^{jk})}{\partial y_j}\frac{\partial W_i}{\partial y_k}\\
&+ 2\sum_{j,k,l}\Big[(g^{kl})^1\Gamma^i_{jk} + g^{kl}(\Gamma^i_{jk})^2\Big]\frac{\partial W_j}{\partial y_l}\\
+\sum_{j,k,l}\Big\{\frac{\partial}{\partial y_k}\Big((g^{kl})^1\Gamma^i_{jl} &+ g^{kl}(\Gamma^i_{jl})^2\Big) + \sum_m\Big[g^{kl}(\Gamma^m_{jl})^1(\Gamma^i_{km})^1\\
&+ (g^{kl})^2\Gamma^m_{jl}(\Gamma^i_{km})^1 + (g^{kl})^2(\Gamma^m_{jl})^2\Gamma^i_{km}\Big]\Big\}W_j.
\end{aligned}$$

Since

$$\|W\|_{L^2(0,T;\mathcal{H}^1)} \leqslant T^{1/2}\|W\|_{L^\infty(0,T;\mathcal{H}^1)} \leqslant T^{1/2}\|W\|_{\mathcal{U}},$$

by using the Lemma 6.13 and Lemma 6.4, we deduce the first assertion of the proposition. The two other assertions are proved similarly. □

We are now in a position to estimate the difference $[\mathbf{N}^1W^1]-[\mathbf{N}^2W^2]$.

Proposition 6.15. *There exists a constant K_0 satisfying* (C1) *such that for any* $(W^1, Q^1, h^1, \omega^1), (W^2, Q^2, h^2, \omega^2) \in \mathcal{K}$, *we have that*

$$\|[\mathbf{N}^1 W^1] - [\mathbf{N}^2 W^2]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant T^{1/10} K_0 \left\{\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})} + \|W\|_{\mathcal{U}}\right\}.$$

Proof. We clearly have that

$$(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2 = (W^1 \cdot \nabla)W + (W \cdot \nabla)W^2.$$

By using Lemma 6.7, we obtain that

$$\|(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2\|_{[L^{5/2}(\Omega_T)]^2} \leqslant C\left(\|W^1\|_{\mathcal{U}}\|W\|_{\mathcal{U}} + \|W\|_{\mathcal{U}}\|W^2\|_{\mathcal{U}}\right),$$

which implies that

$$\|(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2\|_{L^{5/2}(0,T;\mathcal{L}^2)} \leqslant CR\|W\|_{\mathcal{U}}.$$

By using Hölder's inequality, the above relation implies

$$\|(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2\|_{L^2(0,T;\mathcal{L}^2)} \leqslant T^{1/10} CR\|W\|_{\mathcal{U}}.$$

On the other hand,

$$\sum_{jk} \Gamma^{1i}_{jk} W^1_j W^1_k - \sum_{jk} \Gamma^{2i}_{jk} W^2_j W^2_k = \sum_{jk} \Gamma^{i}_{jk} W^1_j W^1_k + \Gamma^{2i}_{jk} W_j W^1_k + \Gamma^{2i}_{jk} W^2_j W_k,$$

so, by using Lemma 6.4, Lemma 6.13, and Lemma 6.7, we obtain that

$$\Big\| \sum_{jk} \Gamma^{1i}_{jk} W^1_j W^1_k - \sum_{jk} \Gamma^{2i}_{jk} W^2_j W^2_k \Big\|_{L^\infty(0,T;L^2(\Omega))} \leqslant K_0\{\|h''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega'\|_{L^2(0,T;\mathbb{R})} + \|W\|_{\mathcal{U}}\}. \qquad \square$$

As a consequence, we obtain the following result.

Corollary 6.16. *There exists a constant K_0 satisfying* (C1) *such that for any* $(W^1, Q^1, h^1, \omega^1), (W^2, Q^2, h^2, \omega^2) \in \mathcal{K}$, *the functions F and F_M defined by* (6.18) *and* (6.19) *satisfy*

$$\|F\|_{L^2(0,T;\mathcal{L}^2)} \leqslant K_0 T^{1/10}\left(\|W\|_{\mathcal{U}} + \|Q\|_{L^2(0,T;H^1(\Omega))} + \|h\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega\|_{H^1(0,T;\mathbb{R})}\right),$$

$$\|F_M\|_{L^2(0,T;\mathbb{R}^2)} \leqslant K_0 T^{1/2}\left(\|h\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega\|_{H^1(0,T;\mathbb{R})}\right).$$

In particular,

$$\|U\|_{\mathcal{U}} + \|P\|_{L^2(0,T;H^1(\Omega))} + \|\tilde{h}\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}\|_{H^1(0,T;\mathbb{R})} \leqslant K_0 T^{1/10}\left(\|W\|_{\mathcal{U}} + \|Q\|_{L^2(0,T;H^1(\Omega))} + \|h\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega\|_{H^1(0,T;\mathbb{R})}\right).$$

Proof. By using Lemma 6.4 and Corollary 6.5,

$$\|\nu[(\mathbf{L}-\Delta)W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant TK_0\|W\|_{\mathcal{U}},$$
$$\|[\mathbf{M}W]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant T^{1/2}K_0\|W\|_{\mathcal{U}},$$
$$\|[(\nabla-\mathbf{G})Q]\|_{L^2(0,T;\mathcal{L}^2)} \leqslant TK_0\|Q\|_{L^2(0,T;H^1(\Omega))}$$

(same proof as for Corollary 6.6).

Therefore by using Propositions 6.14 and 6.15, we obtain the estimates on F. For F_M we write

$$F_M = M\omega R_\theta^1[(h^1)']^\perp + M\omega^2 R_\theta[(h^2)']^\perp + M\omega^2 R_\theta^2(h')^\perp$$

and

$$\|R_\theta\|_{L^2(0,T;\mathbb{R}^4)} \leqslant C\|\theta\|_{L^\infty(0,T;\mathbb{R})} \leqslant CT^{1/2}\|\omega\|_{L^\infty(0,T;\mathbb{R})}. \qquad \square$$

We can now prove the main result of this section.

Proof of Proposition 6.1**.** By using Corollary 6.9, we have that for R large enough ($R > C_0$) and T small enough, $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$. By using Corollary 6.16, we have that $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction provided that T is small enough. Thus, we have proved that $\mathcal{N}$ has a fixed point, and consequently that (6.2)–(6.9) has a solution for T small enough as we have seen in Subsection 6.1. $\square$

7. Proof of the main result.

According to Proposition 6.1, in order to get global existence, it suffices to show that the H^1 norm of u does not blow up in finite time. In order to obtain this result, we need several estimates.

Lemma 7.1. *Suppose that* $T > 0$ *and that*

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T,\mathbb{R}^2) \times H^1(0,T,\mathbb{R})$$

is a strong solution of equations (1.1)–(1.8)*. Then, the following assertions hold.*

(1) *There exists a constant* $C_0 > 0$ *that depends only on* $\|u_0\|_{\mathcal{H}^1}$ *such that*
$$\|u\|^2_{L^\infty(0,T;\mathcal{L}^2(\mathcal{O}))} + \|u\|^2_{L^2(0,T;\mathcal{H}^1(\mathcal{O}))} \leqslant C_0, \tag{7.1}$$
where we have extended u *to a function of* $\mathcal{H}^1(\mathcal{O})$ *by*
$$u(x,t) = h'(t) + (x-h(t))^\perp\,\omega(t) \quad \textit{in } B(t).$$

(2) *There exists* $d \in \mathbb{R}^+$ *such that*
$$\lim_{t\to T} \operatorname{dist}(B(t),\partial\mathcal{O}) \to d.$$

Proof. (1) It can be easily checked that the solution u satisfies the weak formulation in [4]. This fact, combined with the estimates in Section 4.2 in [4], yields (7.1).

(2) By the first part of the lemma, there exists a constant C_0 such that for all $t \in [0,T]$,

$$|h'(t)| \leqslant C_0.$$

Therefore, by using a classical result (see, for instance, Arnaudiès-Fraysse [1, Theorem V.1.8]), we have that $\lim_{t\to T} h(t)$ exists. Similarly, the function θ defined by (1.9) has a limit in T. This yields the second assertion of the lemma. □

By using again the fact that our concept of solution satisfies the weak formulation in [4], we obtain the following consequence of the estimates from Section 4.2 in [4].

Lemma 7.2. *Suppose that* $T > 0$ *and that*

$$(u, p, h, \omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T,\mathbb{R}^2) \times H^1(0,T,\mathbb{R})$$

is a strong solution of equations (1.1)–(1.8)*, and suppose moreover that for any* $t \in [0,T]$*,* $B(t) \subset \mathcal{O}_\varepsilon$*. Then, there exists a constant* C_0 *that depends only on* $\|u_0\|_{\mathcal{H}^1}$ *such that*

$$\|u(t)\|_{\mathcal{H}^1(t)} \leqslant C_0 \quad \forall t \in [0,T].$$

We can now prove the main result of this paper.

Proof of Theorem 2.2**.** According to Proposition 6.1, problem (1.1)–(1.8) admits a unique maximal solution. By using Proposition 6.1 and Lemma 7.1, in order to get the conclusion, it suffices to show that if $T < \infty$, any strong solution on $[0,T)$ such that

$$\lim_{t\to T} \operatorname{dist}(B(t), \partial\mathcal{O}) > 0 \tag{7.2}$$

is not maximal. Since the function $t \mapsto \operatorname{dist}(B(t), \partial\mathcal{O})$ is continuous, (7.2) implies the existence of $\varepsilon > 0$ such that for all $t \in [0,T]$, we have that

$$\operatorname{dist}(B(t), \partial\mathcal{O}) > \varepsilon.$$

Therefore, Lemma 7.2 yields the existence of a constant C_0 such that

$$\|u(t)\|_{\mathcal{H}^1(t)} \leqslant C_0 \quad \forall t \in [0,T].$$

The above relation and Proposition 6.1 imply that we can extend the solution to $[0,T']$ with $T' > T$.

The uniqueness follows directly from the local uniqueness result in Proposition 6.1 □

Remark 7.3. *The above proof can be adapted for the case of several rigid bodies. Suppose that we have* n *rigid bodies* $B^1, \dots, B^n$. *Then, condition* (2) *in Theorem* 2.2 *should be replaced by*

$$\lim_{t\to T_0} \min\left\{ \min_{i=1,\dots,n} \operatorname{dist}(B^i(t), \partial\mathcal{O}), \min_{i\neq j} \operatorname{dist}(B^i(t), B^j(t))\right\} = 0.$$

8. Lack of collisions and asymptotic stability

The aim of this section is the study of the initial and boundary problem (1.1)–(1.8) in the case when the forces f are vanishing for $t \to \infty$ and with small initial data.

Proposition 8.1. *Suppose that* $u_0 \in \mathcal{H}^1$ *and that*

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \Omega, \\ u_0(x) = 0 & \text{on } \partial\mathcal{O}, \\ u_0(x) = h_1 + \omega_0 (x - h_0)^{\perp} & \text{on } \partial B, \\ \operatorname{dist}(B, \partial\mathcal{O}) > 0. \end{cases}$$

Suppose moreover that there exists $T > 0$ *such that* $f(t) = 0$, $\forall t \geqslant T$. *If* $\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \int_0^T \|f\|_{\mathcal{L}^2(\mathcal{O})} ds$ *is small enough, then the solution of* (1.1)–(1.8) *is global* (*in particular,*

$$\operatorname{dist}(B(t), \partial\mathcal{O}) > 0, \quad \forall t > 0).$$

Moreover, there exist two constants $c_a, c_b > 0$ *depending only on the geometry, on the viscosity of the fluid, and on the density of the rigid body such that*

$$\int_{\Omega(t)} |u(t)|^2 \,\mathrm{d}x + M|h'(t)|^2 + J|\omega(t)|^2 \leqslant c_a \left(\|u_0\|^2_{\mathcal{L}^2(\mathcal{O})} + \int_0^T \|f\|^2_{\mathcal{L}^2(\mathcal{O})} ds \right) e^{-c_b t}. \quad (8.1)$$

Proof. By Theorem 2.2, there exists a unique maximal solution (u, p, h, ω) of equations (1.1)–(1.8) on $(0, T_0)$.

Moreover, one of the following alternatives holds true:

(1) $T_0 = \infty$; i.e., the solution is global.
(2) $\lim_{t\to T_0} \operatorname{dist}(B(t), \partial\mathcal{O}) = 0$.

Set

$$\widetilde{\rho}(x,t) = \begin{cases} 1 & \text{if } x \in \Omega(t), \\ \rho & \text{if } x \in B(t). \end{cases}$$

By taking the inner product of (1.1) with $u(t)$ and by integrating on $\Omega(t)$, we get that

$$\int_{\Omega(t)} \frac{\partial u}{\partial t}(t)\cdot u(t)dx - \nu\int_{\Omega(t)} \Delta u(t)\cdot u(t)dx + \int_{\Omega(t)} [(u(t)\cdot\nabla)u(t)]\cdot u(t)dx$$
$$+\int_{\Omega(t)} \nabla p(t)\cdot u(t)dx = \int_{\Omega(t)} f(t)\cdot u(t)dx, \quad \text{a.e. in}\,(0,T).$$

By Reynolds' transport theorem ([14, p. 78]), we have

$$\int_{\Omega(t)} \frac{\partial u}{\partial t}(t)\cdot u(t)dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} |u(t)|^2dx - \frac{1}{2}\int_{\partial B(t)} |u(t)|^2(u(t)\cdot n)dx. \quad (8.2)$$

On the other hand, standard calculations show that

$$-\nu\int_{\Omega(t)} \Delta u(t)\cdot u(t)dx = 2\nu\int_{\Omega(t)}|D(u(t))|^2dx - 2\nu\int_{\partial B(t)} u(t)\cdot[D(u(t))n]\,\mathrm{d}\Gamma, \quad (8.3)$$

$$\int_{\Omega(t)} [(u(t)\cdot\nabla)u(t)]\cdot u(t)dx = \int_{\partial B(t)} \frac{1}{2}|u(t)|^2(u\cdot n)\,\mathrm{d}\Gamma, \quad (8.4)$$

and that

$$\int_{\Omega(t)} \nabla p(t)\cdot u(t)dx = \int_{\partial B(t)} p(t)(u(t)\cdot n)\,\mathrm{d}\Gamma. \quad (8.5)$$

By using (1.5), (1.6), and (8.2)–(8.5), we obtain that for almost every $t \in (0,T_0)$ we have

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\Omega(t)} |u(t)|^2dx + M|h'(t)|^2 + J|\omega(t)|^2\Big) + 2\nu\int_{\Omega(t)} |D(u(t))|^2dx$$
$$= \int_{\Omega(t)} f(t)\cdot u(t)dx + \rho\int_{B(t)} f(t)dx\cdot h'(t) + \rho\int_{B(t)} (x-h(t))^{\perp} f(x,t)dx\,\omega(t). \quad (8.6)$$

Therefore, by using the definition of $\widetilde{\rho}$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathcal{O}} \widetilde{\rho}(t)|u(t)|^2dx\Big) + 2\nu\int_{\mathcal{O}} |D(u(t))|^2dx = \int_{\mathcal{O}} \widetilde{\rho}(t)f(t)\cdot u(t)dx, \quad (8.7)$$

almost everywhere in $(0,T)$. The above relation, Lemma 5.2, and Poincaré's inequality imply the existence of constants $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{d}{dt}\Big(\int_{\mathcal{O}} \widetilde{\rho}(t)|u(t)|^2dx\Big) \leqslant -c_1\Big(\int_{\mathcal{O}} \widetilde{\rho}(t)|u(t)|^2dx\Big) + c_2\int_{\mathcal{O}} |f(t)|^2dx,$$

almost everywhere in $(0,T)$. We deduce that

$$\frac{d}{dt}\Big(e^{c_1t}\int_{\mathcal{O}} \widetilde{\rho}(t)|u(t)|^2dx\Big) \leqslant c_2e^{c_1t}\|f(t)\|^2_{\mathcal{L}^2(\mathcal{O})}, \quad \text{a.e. in}\,(0,T). \quad (8.8)$$

If we integrate the above relation with respect to time on $[0,t]$, we obtain the existence of a constant $c_3 > 0$ such that

$$\int_{\mathcal{O}} \widetilde{\rho}(t)|u(t)|^2 dx \leqslant c_3 e^{-c_1 t}\|u_0\|^2_{\mathcal{L}^2(\mathcal{O})} + c_2 \int_0^t e^{-c_1(t-s)}\|f(s)\|^2_{\mathcal{L}^2(\mathcal{O})} ds,$$

$\forall t \in (0,T_0)$. We deduce from the above relation the existence of a positive constant c_4 depending only on the geometry, on the viscosity of the fluid and on the density of the rigid body such that

$$\|u(t)\|_{\mathcal{L}^2(\mathcal{O})} \leqslant c_4 \left(\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \|f\|_{\mathcal{L}^2((0,T)\times\mathcal{O})}\right), \quad \forall t \in (0,T_0). \tag{8.9}$$

Hence, we obtain that

$$|h(t)| \leqslant c_5 T \left(\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \|f\|_{\mathcal{L}^2((0,T)\times\mathcal{O})}\right), \quad \forall t \in (0, \min(T_0,T)),$$

and

$$|\theta(t)| \leqslant c_5 T \left(\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \|f\|_{\mathcal{L}^2((0,T)\times\mathcal{O})}\right), \quad \forall t \in (0, \min(T_0,T)),$$

where $\theta(t) = \int_0^t \omega(s) ds$. The above relations imply that for

$$\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \|f\|_{\mathcal{L}^2((0,T)\times\mathcal{O})}$$

small enough, we have that $T_0 > T$, by using again Theorem 2.2. By integrating (8.8) on $[T,t]$ and the hypothesis on f, we obtain that

$$\int_{\mathcal{O}} \widetilde{\rho}(t)|u(t)|^2 dx \leqslant c_6 e^{-c_1 t}\|u(T)\|^2_{\mathcal{L}^2(\mathcal{O})}, \quad \forall t \in [T,T_0), \tag{8.10}$$

which implies that

$$|h'(t)| \leqslant c_7 e^{-c_8 t}\|u(T)\|_{\mathcal{L}^2(\mathcal{O})}, \quad \forall t \in [T,T_0).$$

By integrating the above relation, we deduce that

$$|h(t)| \leqslant c_7 \frac{1}{c_8} c_4 \left(\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \|f\|_{\mathcal{L}^2((0,T)\times\mathcal{O})}\right), \quad \forall t \in [T,T_0).$$

We can estimate in the same way the function $\theta(t)$. Hence, for $\|u_0\|_{\mathcal{L}^2(\mathcal{O})} + \|f\|_{\mathcal{L}^2((0,T)\times\mathcal{O})}$ small enough, there exists $\varepsilon > 0$ such that for all $t \in [0,T_0)$, we have $\operatorname{dist}(B(t), \partial\mathcal{O}) > \varepsilon$. Therefore, by using Theorem 2.2, we obtain that $T_0 = \infty$; i.e., the solution is global. Then, by using (8.10) and (8.9), we deduce (8.1). □

9. Some extensions to the three-dimensional case.

In this section, we give some extensions of the results in the previous sections to the three-dimensional case.

In the case of a fluid–rigid-body system in a bounded domain of $\mathbb{R}^3$, the equations modeling the motion can be written as follows.

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \qquad x \in \Omega(t), t \in [0,T], \tag{9.1}$$

$$\operatorname{div} u = 0, \qquad x \in \Omega(t), t \in [0,T], \tag{9.2}$$

$$u = 0, \qquad x \in \partial\mathcal{O},\ t \in [0,T], \tag{9.3}$$

$$u = h'(t) + \omega(t) \wedge (x - h(t)), \qquad x \in \partial B(t),\ t \in [0,T], \tag{9.4}$$

$$Mh''(t) = -\int_{\partial B(t)} \sigma n \,\mathrm{d}\Gamma + \rho\int_{B(t)} f(t)dx, \quad t \in [0,T], \tag{9.5}$$

$$J\omega'(t) = J\omega\wedge\omega - \int_{\partial B(t)}(x-h(t))\wedge \sigma n\,\mathrm{d}\Gamma$$
$$+ \rho\int_{B(t)}(x-h(t))\wedge f(x,t)dx, \quad t\in[0,T], \tag{9.6}$$

$$u(x,0) = u_0(x), \quad x \in \Omega(0), \tag{9.7}$$

$$h(0) = 0,\ h'(0) = h_1 \in \mathbb{R}^3,\ \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{9.8}$$

We use the same notation as in the previous sections, except for the dimension: for example, here ω is a vector of $\mathbb{R}^3$ and $\mathcal{H}^i(t) = [H^i(\Omega(t))]^3$, and so on.

The local existence result in Theorem 2.2 can be generalized as follows:

Theorem 9.1. *Suppose that $f \in L^2_{loc}(0,\infty;\left[W^{1,\infty}(\mathcal{O})\right]^3)$, $u_0 \in \mathcal{H}^1$, and that*

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \Omega,\\ u_0(x) = 0 & \text{on } \partial\mathcal{O},\\ u_0(x) = h_1 + \omega_0\wedge x & \text{on } \partial B,\\ \operatorname{dist}(B,\partial\mathcal{O}) > 0. \end{cases}$$

Consider C_1 and ε, two positive constants such that $\|u_0\|_{\mathcal{H}^1} \leqslant C_1$ and such that $0 < \varepsilon < \operatorname{dist}(B,\partial\mathcal{O})$. Then there exists $T_0 > 0$, such that equations (9.1)–(9.8) *admit a unique strong solution*

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;H^1(\Omega(t))) \times H^2(0,T,\mathbb{R}^3) \times H^1(0,T,\mathbb{R}^3),$$

for all $T \in (0,T_0)$.

Proof. The proof of Theorem 9.1 is very similar to the proof of Theorem 2.2, so we only sketch here the proof.

We transform our equations with almost the same change of variables: First we consider $\xi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ with compact support contained in $\mathcal{O}_{\varepsilon/2}$ and equal to 1 in $\overline{\mathcal{O}}_\varepsilon$. Then we use the functions $w : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$, given by

$$\begin{aligned} w_1(x_2,x_3,t) &= \omega_1(\frac{x_2^2}{2} - h_2x_2 - \frac{x_3^2}{2} + h_3x_3) \\ w_2(x_1,x_3,t) &= \omega_2(\frac{x_3^2}{2} - h_3x_3 - \frac{x_1^2}{2} + h_1x_1) \\ w_3(x_1,x_2,t) &= \omega_3(\frac{x_1^2}{2} - h_1x_1 - \frac{x_2^2}{2} + h_2x_2), \end{aligned} \tag{9.9}$$

and $\Lambda : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$,

$$\Lambda(x_1,x_2,,x_3,t) = \begin{pmatrix} \xi V_{R1} + \frac{\partial \xi}{\partial x_2} w_3 - \frac{\partial \xi}{\partial x_3} w_2 \\ \xi V_{R2} + \frac{\partial \xi}{\partial x_3} w_1 - \frac{\partial \xi}{\partial x_1} w_3 \\ \xi V_{R3} + \frac{\partial \xi}{\partial x_1} w_2 - \frac{\partial \xi}{\partial x_2} w_1 \end{pmatrix}. \tag{9.10}$$

We have the same linear system as in Section 5, the same estimates as in Section 6 (notice that Lemma 6.7 holds since the Sobolev embedding result that we used is valid in the three-dimensional case). This proves the local existence of solutions of the system (9.1)–(9.8). $\square$

As a consequence of Theorem 9.1 we obtain

Corollary 9.2. *Suppose that the hypothesis of Theorem* 9.1 *holds and that there exists $T > 0$ such that $f(t) = 0$, $\forall t \geqslant T$. If $\|f\|_{\mathcal{L}^2((0,\infty)\times\mathcal{O})} + \|u_0\|_{\mathcal{H}^1(\mathcal{O})}$ is small enough, then the solution of* (9.1)–(9.8) *is global* (*in particular,* $\operatorname{dist}(B(t), \partial\mathcal{O}) > 0$ *for all* $t > 0$).

Moreover, there exist two constants $c_a, c_b > 0$ depending only on the geometry, on the viscosity of the fluid, and on the density of the rigid body such that

$$\int_{\Omega(t)} |u(t)|^2 dx + M|h'(t)|^2 + J|\omega(t)|^2 \leqslant c_a \left(\|u_0\|^2_{\mathcal{L}^2(\mathcal{O})} + \int_0^T \|f\|^2_{\mathcal{L}^2(\mathcal{O})} ds \right) e^{-c_b t}. \tag{9.11}$$

Proof. The method used in the proof of Proposition 8.1 can not be directly applied in the three-dimensional case since it is based on Theorem 2.2, which is valid only in the two-dimensional case. However, a proof can be obtained by using the following ideas. First, by using Theorem 9.1, we consider the maximal solution of (9.1)–(9.8) on $[0, T_{max})$. Using calculations identical to those in Proposition 8.1, we get that for

$$\|f\|_{\mathcal{L}^2((0,\infty)\times\mathcal{O})} + \|u_0\|_{\mathcal{L}^2(\mathcal{O})}$$

small enough, there exists $\varepsilon > 0$ such that for all $t \in [0, T_{max})$, we have that $\operatorname{dist}(B(t), \partial\mathcal{O}) > \varepsilon$. The main difference with respect to the two-dimensional case is that in the three-dimensional case we have to show that the H^1 norm of u does not blow up in finite time. This fact is proved in [4] provided

$$\|f\|_{\mathcal{L}^2((0,\infty)\times\mathcal{O})} + \|u_0\|_{\mathcal{H}^1(\mathcal{O})}$$

is small enough. We can now conclude that for small data the solution is global. From now on, the proof of Proposition 8.1 can be adapted to prove the asymptotic stability. □

References

[1] J.-M. Arnaudiès and H. Fraysse, "Cours de Mathématiques," Vol. 2, Dunod, Paris, 1988.

[2] V.I. Arnold, "Ordinary Differential Equations," Springer-Verlag, Berlin, 1992.

[3] C. Conca, J. San Martín, and M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid,* Comm. Partial Differential Equations, 25 (2000), 1019–1042.

[4] B. Desjardins and M.J. Esteban, *Existence of weak solutions for the motion of rigid bodies in a viscous fluid,* Arch. Ration. Mech. Anal., 146 (1999), 59–71.

[5] B. Desjardins and M.J. Esteban, *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models,* Comm. Partial Differential Equations, 25 (2000), 1399–1413.

[6] L.P. Eisenhart, "Riemannian Geometry," Princeton University Press, Princeton, N.J., 1949.

[7] E. Feireisl, *On the motion of rigid bodies in a viscous compressible fluid,* Arch. Ration. Mech. Anal., 167 (2003), 281–308.

[8] E. Feireisl, *On the motion of rigid bodies in a viscous fluid,* Appl. Math., 47 (2002), 463–484.

[9] G.P. Galdi, "An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. I," Springer-Verlag, New York, 1994.

[10] G.P. Galdi, *On the steady self-propelled motion of a body in a viscous incompressible fluid,* Arch. Ration. Mech. Anal., 148 (1999), 53–88.

[11] G.P. Galdi and A.L. Silvestre, *Strong solutions to the problem of motion of a rigid body in a Navier-Stokes liquid under the action of prescribed forces and torques,* Nonlinear Problems in Mathematical Physics and Related Topics I, Kluwer/Plenum, New York, 1 (2002), 121–144.

[12] C. Grandmont and Y. Maday, *Existence for an unsteady fluid-structure interaction problem,* M2AN Math. Model. Numer. Anal., 34 (2000), 609–636.

[13] M.D. Gunzburger, H.-C. Lee, and G.A. Seregin, *Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions,* J. Math. Fluid Mech., 2 (2000), 219–266.

[14] M.E. Gurtin, "An Introduction to Continuum Mechanics," Academic Press Inc., Harcourt Brace Jovanovich Publishers, New York, 1981.

[15] P. Hartman, "Ordinary Differential Equations," Birkhäuser, Boston, Mass., second edition, 1982.

[16] K.-H. Hoffmann and V.N. Starovoitov, *On a motion of a solid body in a viscous fluid. Two-dimensional case,* Adv. Math. Sci. Appl., 9 (1999), 633–648.

[17] K.-H. Hoffmann and V.N. Starovoitov, *Zur Bewegung einer Kugel in einer zähen Flüssigkeit,* Doc. Math., 5 (2000), 15–21.

[18] A. Inoue and M. Wakimoto, *On existence of solutions of the Navier-Stokes equation in a time dependent domain,* J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), 303–319.

[19] N.V. Judakov, *The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid,* Dinamika Splošn. Sredy, 255 (1974), 249–253.

[20] P.-L. Lions, "Mathematical Topics in Fluid Mechanics," Vol. 1, Oxford Lecture Series in Mathematics and its Applications, New York, 1996.

[21] J. San Martín, V. Starovoitov, and M. Tucsnak, *Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid,* Arch. Rational Mech. Anal., 161 (2002), 113–147.

[22] D. Serre, *Chute libre d'un solide dans un fluide visqueux incompressible. Existence,* Japan J. Appl. Math., 4 (1987), 99–110.

[23] A.L. Silvestre, *On the self-propelled motion of a rigid body in a viscous liquid and on the attainability of steady symmetric self-propelled motions,* J. Math. Fluid Mech., 4 (2002), 285–326.

[24] A.L. Silvestre, *On the slow motion of a self-propelled rigid body in a viscous incompressible fluid*, J. Math. Anal. Appl., 274 (2002), 203–227.

[25] T. Takahashi and M. Tucsnak, *Global strong solutions for the two dimensional motion of an infinite cylinder in a viscous fluid,* J. Math. Fluid Mech., to appear.

[26] R. Temam, "Problèmes Mathématiques en Plasticité," Gauthier-Villars, Montrouge, 1983.

[27] J.L. Vázquez and E. Zuazua, *Large time behavior for a simplified 1d model of fluid-solid interaction,* preprint.