Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Functional Analysis 226 (2005) 193–229
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator

K. Ramdani, T. Takahashi, G. Tenenbaum, M. Tucsnak*

Institut Elie Cartan, Université Henri Poincaré Nancy 1, BP 239, Vandœuvre lès Nancy 54506, France

Received 10 August 2004; accepted 16 February 2005
Communicated by P. Malliavin
Available online 27 April 2005

Abstract

Let A be a possibly unbounded skew-adjoint operator on the Hilbert space X with compact resolvent. Let C be a bounded operator from $\mathscr{D}(A)$ to another Hilbert space Y. We consider the system governed by the state equation $\dot{z}(t)=Az(t)$ with the output $y(t)=Cz(t)$. We characterize the exact observability of this system only in terms of C and of the spectral elements of the operator A. The starting point in the proof of this result is a Hautus-type test, recently obtained in Burq and Zworski (J. Amer. Soc. 17 (2004) 443–471) and Miller (J. Funct. Anal. 218 (2) (2005) 425–444). We then apply this result to various systems governed by partial differential equations with observation on the boundary of the domain. The Schrödinger equation, the Bernoulli–Euler plate equation and the wave equation in a square are considered. For the plate and Schrödinger equations, the main novelty brought in by our results is that we prove the exact boundary observability for an arbitrarily small observed part of the boundary. This is done by combining our spectral observability test to a theorem of Beurling on nonharmonic Fourier series and to a new number theoretic result on shifted squares.

MSC: 93C25; 93B07; 93C20; 11N36

Keywords: Boundary exact observability; Boundary exact controllability; Hautus test; Schrödinger equation; Plate equation; Wave equation

* Corresponding author. Fax: +33 3 83 68 45 34.
E-mail address: marius.tucsnak@iecn.u-nancy.fr (M. Tucsnak).

0022-1236/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2005.02.009

1. Introduction and statement of the main results

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$, and let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator. Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$ and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an observation operator. According to Stone's theorem, A generates a strongly continuous group of isometries in X denoted $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$. This paper is concerned with infinite-dimensional observation systems described by the equations

$$\dot{z}(t) = Az(t), \quad z(0) = z_0, \tag{1.1}$$

$$y(t) = Cz(t). \tag{1.2}$$

Here, a dot denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, $z(t)$ is called the state at time t and y is the output function. Such systems are often used as models of vibrating systems (e.g., the wave equation), electromagnetic phenomena (Maxwell's equations) or in quantum mechanics (Schrödinger's equation). In several particular cases we will also consider the control system which is the dual of (1.1), (1.2). However, in order to avoid technicalities, we do not use the general form of the dual control system and we do not detail the duality arguments (we refer, for instance, to [25] for a brief discussion of these issues). By a solution of (1.1) we mean that $z(t) = \mathbb{T}_t z_0$ (this is a mild solution). In order to give a sense to (1.2), we make the assumption that C is an admissible observation operator in the following sense (see [26]):

Definition 1.1. The operator C in system (1.1)–(1.2) is an admissible observation operator if for every $T > 0$ there exists a constant $K_T \geqslant 0$ such that

$$\int_0^T \|y(t)\|_Y^2 \,\mathrm{d}t \;\leqslant\; K_T^2 \|z_0\|_X^2 \quad \forall\, z_0 \in \mathcal{D}(A). \tag{1.3}$$

If C is *bounded*, i.e. if it can be extended such that $C \in \mathcal{L}(X, Y)$, then C is clearly an *admissible observation operator*.

Definition 1.2. System (1.1)–(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$\int_0^T \|y(t)\|_Y^2 \,\mathrm{d}t \;\geqslant\; k_T^2 \|z_0\|_X^2 \quad \forall\, z_0 \in \mathcal{D}(A). \tag{1.4}$$

System (1.1)–(1.2) is *exactly observable* if it is exactly observable in some time $T > 0$.

The exact observability property is dual to the exact controllability property, as it has been shown in [9]. By using the above duality, the exact controllability of a system governed by partial differential equations reduces to the observability estimate (1.4) (called “inverse inequality” in [18]). Most of the literature tackling exact observability and exact controllability for systems governed by partial differential equations is based on a *time domain approach*. This means that one considers directly solutions of (1.1) (or of a dual equation) which are manipulated in various ways: nonharmonic Fourier series ([2] and references therein), multipliers method [16,18] or microlocal analysis techniques [6].

Only few papers in the area of controllability and observability of systems governed by partial differential equations have considered a *frequency domain approach*, related to the classical Hautus test in the theory of finite dimensional systems (see [10]). Roughly speaking, a frequency domain test for the observability of (1.1)–(1.2) is formulated only in terms of the operators A, C and of a parameter (the frequency). This means that the time t does not appear in such a test and that we do not have to solve an evolution equation. In the case of a bounded observation operator C, such frequency domain methods have been proposed in [19,20]. In the case of an unbounded observation operator C a Hautus-type test has been recently obtained in [8,21].

The aim of this paper is to use Hautus-type tests in order to characterize the exact observability property only in terms of C and of the spectral elements of the operator A. This will be done provided that the operator A has a compact resolvent and therefore, that the spectrum of A is formed only by eigenvalues. More precisely, since A is skew-adjoint, it follows that the spectrum of A is given by $\sigma(A) = \left\{ i\mu_n \mid n \in \Lambda \right\}$ with $\Lambda = \mathbb{Z}^*$ or $\mathbb{N}^*$ and where $(\mu_n)_{n\in\Lambda}$ is a sequence of real numbers.

The main result of this paper reads as follows:

Theorem 1.3. *Assume that A is skew-adjoint with compact resolvent and that the operator C is admissible for system* (1.1)–(1.2). *Moreover, assume that $(\Phi_n)_{n\in\Lambda}$ is an orthonormal sequence of eigenvectors of A associated to the eigenvalues $(i\mu_n)_{n\in\Lambda}$.*

For $\omega \in \mathbb{R}$ and $\varepsilon > 0$, set

$$J_\varepsilon(\omega) = \{m \in \Lambda \text{ such that } |\mu_m - \omega| < \varepsilon\}. \tag{1.5}$$

Then system (1.1), (1.2) *is exactly observable if and only if one of the following equivalent assertions holds*:

(1) *There exists $\varepsilon > 0$ and $\delta > 0$ such that for all $\omega \in \mathbb{R}$ and for all $z = \sum_{m\in J_\varepsilon(\omega)} c_m \Phi_m$*:

$$\|Cz\|_Y \geqslant \delta \|z\|_X. \tag{1.6}$$

(2) *There exists $\varepsilon > 0$ and $\delta > 0$ such that for all $n \in \mathbb{Z}^*$ and for all $z = \sum_{m\in J_\varepsilon(\mu_n)} c_m \Phi_m$*:

$$\|Cz\|_Y \geqslant \delta \|z\|_X. \tag{1.7}$$

Remark 1.4. The above theorem can be seen as a generalization of several results in the literature. More precisely, in the particular case of a bounded observation operator C, the result in Theorem 1.3 follows, via a standard argument, from Theorem 3.2 in [20]. For unbounded C, but with the additional assumption that the sequence (μ_n) satisfies the gap condition (i.e., there exists $\gamma > 0$ such that $|\mu_n - \mu_m| > \gamma$ for all $m, n \in \Lambda$, $m \neq n$), the necessity of condition (1.6) in Theorem 1.3 is a consequence of Theorem 4.4 from Russell and Weiss [23].

An important part of this paper is devoted to the application of the spectral criteria in Theorem 1.3 to systems governed by partial differential equations. The Schrödinger equation, the Bernoulli–Euler plate equation and the wave equation in a square are considered. For the plate and Schrödinger equations, the main novelty brought in by our results is that we show that the exact observability property can hold for an arbitrarily small observed part of the boundary. More precisely, in the case of the plate equation, our observability result implies the following exact controllability result.

Theorem 1.5. *Consider the square* $\Omega = (0,\pi) \times (0,\pi)$ *and let* Γ *be an open subset of* $\partial\Omega$. *Consider the following control problem*:

$$\ddot{w} + \Delta^2 w = 0, \quad x \in \Omega, \ t > 0, \tag{1.8}$$

$$w(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \tag{1.9}$$

$$\Delta w(x,t) = 0, \quad x \in \partial\Omega \setminus \Gamma, \ t > 0, \tag{1.10}$$

$$\Delta w(x,t) = u, \quad x \in \Gamma, \ t > 0, \tag{1.11}$$

$$w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), \quad x \in \Omega, \tag{1.12}$$

where the input is the function $u \in L^2(0,T; L^2(\Gamma))$. *Then the following assertions are equivalent*:

(1) *For all* $T > 0$, *the above system is exactly controllable in* $H_0^1(\Omega) \times H^{-1}(\Omega)$ *in time T. This means that, for all* $(w_0, w_1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$, *we can find* $u \in L^2(0,T; L^2(\Gamma))$ *such that*

$$w(x,T) = 0, \quad \dot{w}(x,T) = 0 \quad \forall\, x \in \Omega.$$

(2) *The control region* Γ *contains both a horizontal and a vertical segment of nonzero length*.

The proof of the above result is based on a consequence of Theorem 1.3 combined to a theorem of Beurling on nonharmonic Fourier series and to a new number theoretic result (a theorem on shifted squares). Let us mention that in the case of a control acting in an arbitrary open subset of the square Ω, an exact controllability result for the plate equation has been given in [13].

Moreover, we consider a system governed by the wave equation in a square. We give a very simple proof (it uses only Parseval's theorem) of the boundary observability of this system.

The paper is organized as follows. Section 2 is devoted to the proof of our main result, namely Theorem 1.3. In Section 3, this result is applied to study the boundary observability of Shrödinger equation in a square. The case of a Dirichlet boundary observation and the Neumann one are successively considered. In Section 4, we derive the counterpart of Theorem 1.3 for second-order systems (see Proposition 4.5). Thanks to this result, we tackle in Section 5 the problem of the boundary observability for the Bernoulli–Euler plate equation in a square. A second application of the spectral criteria provided by Proposition 4.5 is detailed in Section 6. This application concerns the boundary observability of the wave equation in a square. Finally, Section 7 is devoted to the proof of Proposition 7.1, which is one of the main ingredients used to establish our observability results in Sections 3 and 5.

2. Proof of Theorem 1.3

The basic tool in the proof of Theorem 1.3 is a recent Hautus-type test. This result, given in [21], concerns the observability of systems with skew-adjoint generator and with unbounded observation operator. We first recall this result (see [8,21] for the proof).

Theorem 2.1. *Assume that A is a skew-adjoint operator in the Hilbert space X and that $C : \mathcal{D}(A) \to Y$ is an admissible observation operator. Then system* (1.1), (1.2) *is exactly observable if and only if there exists a constant $\delta > 0$ such that*

$$\|(A - i\omega I)z\|_X^2 + \|Cz\|_Y^2 \;\geqslant\; \delta\|z\|_X^2 \quad \forall\, \omega \in \mathbb{R} \quad \forall\, z \in \mathcal{D}(A). \tag{2.1}$$

In order to prove Theorem 1.3, we need the following consequence of the admissibility property.

Lemma 2.2. *Assume that the operators A and C are as in Theorem* 2.1. *Then there exists $M > 0$ such that*

$$\left\| C(A - I - i\omega I)^{-1} \right\|_{\mathcal{L}(X,Y)} \;\leqslant\; M \quad \forall\, \omega \in \mathbb{R}. \tag{2.2}$$

Proof. Our proof is a slight variation of the proof of Proposition 2.3 in [23].

Let us fix $T > 0$ and $z \in X$. Then, for any $n \in \mathbb{N}^*$ we have that

$$\int_0^{nT} \|e^{-t/2} C\mathbb{T}_t z\|_Y^2 \,\mathrm{d}t \;\leqslant\; \sum_{k=0}^{n-1} e^{-kT} \int_{kT}^{(k+1)T} \|C\mathbb{T}_t z\|_Y^2 \,\mathrm{d}t. \tag{2.3}$$

By using definition (1.3) of admissibility, combined to the fact that $\mathbb{T}$ is a group of isometries we have that

$$\int_{kT}^{(k+1)T} \|C\mathbb{T}_t z\|_Y^2 \,\mathrm{d}t \;\leqslant\; K_T^2 \|z\|_X^2 \quad \forall\, k \in \mathbb{N}.$$

The above inequality combined to (2.3) implies that

$$\int_0^\infty \|e^{-t/2} C\mathbb{T}_t z\|_Y^2 \,\mathrm{d}t \;\leqslant\; \frac{K_T^2}{1-e^{-T}} \|z\|_X^2 \quad \forall\, z \in X. \tag{2.4}$$

On the other hand,

$$\|C(A - I - i\omega I)^{-1} z\|_Y^2 = \left\| \int_0^\infty e^{-t} C\mathbb{T}_t z \,\mathrm{d}t \right\|_Y^2,$$

which, by applying the Cauchy–Schwartz inequality, yields:

$$\|C(A - I - i\omega I)^{-1} z\|_Y^2 \;\leqslant\; \left(\int_0^\infty e^{-t} \,\mathrm{d}t \right) \left(\int_0^\infty \|e^{-t/2} C\mathbb{T}_t z\|_Y^2 \,\mathrm{d}t \right).$$

Combined to (2.4), the above relation clearly implies the desired conclusion (2.2), with $M = \dfrac{K_T}{\sqrt{1-e^{-T}}}$. $\square$

We will also need the following result which can be seen as a generalization of Lemma 4.6 in [23]:

Lemma 2.3. *Assume that the operators A and C are as in Lemma* 2.2. *For each $\varepsilon > 0$ and $\omega \in \mathbb{R}$, we define the subspace $V(\omega) \subset X$ by*

$$V(\omega) = \{\Phi_m \mid m \in J_\varepsilon(\omega)\}^\perp, \tag{2.5}$$

where $J_\varepsilon(\omega)$ is defined in (1.5). *We denote by A_ω the part of A in $V(\omega)$, i.e.,*

$$A_\omega : \mathcal{D}(A) \cap V(\omega) \to V(\omega)$$

and

$$A_\omega z = Az \quad \forall\, z \in \mathcal{D}(A) \cap V(\omega).$$

Then, there exists $M > 0$ such that

$$\|C(A_\omega - i\omega I)^{-1}\|_{\mathcal{L}(V(\omega),Y)} \;\leqslant\; M \quad \forall\, \omega \in \mathbb{R}. \tag{2.6}$$

Proof. Given $\omega \in \mathbb{R}$, set $s = 1 + i\omega$. Then, thanks to the resolvent identity, we have

$$(A_\omega - i\omega I)^{-1} = (A_\omega - sI)^{-1}\left[I - (A_\omega - i\omega I)^{-1}\right]. \tag{2.7}$$

We first show that

$$\|(A_\omega - i\omega I)^{-1}\|_{\mathcal{L}(V(\omega))} \leqslant \frac{1}{\varepsilon}. \tag{2.8}$$

Indeed, let $f = \sum\limits_{m \notin J_\varepsilon(\omega)} f_m \Phi_m$ be an element of $V(\omega)$. Then

$$\|(A_\omega - i\omega I)^{-1} f\|^2 = \sum_{m \notin J_\varepsilon(\omega)} \frac{|f_m|^2}{|\mu_m - \omega|^2}.$$

The above relation and the fact that $|\mu_m - \omega| \geqslant \varepsilon$ for $m \notin J_\varepsilon(\omega)$ clearly imply (2.8).

On the other hand, we clearly have

$$\|C(A_\omega - sI)^{-1}\|_{\mathcal{L}(V(\omega),Y)} \leqslant \|C(A - sI)^{-1}\|_{\mathcal{L}(X,Y)}$$

and thus, by using Lemma 2.2, we obtain that there exists a constant $M > 0$ such that

$$\|C(A_\omega - sI)^{-1}\|_{\mathcal{L}(V(\omega),Y)} \leqslant M \quad \forall\, \omega \in \mathbb{R}.$$

The above relation, (2.7) and (2.8) yield then (2.6). $\square$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We first show that assertions (1) and (2) in Theorem 1.3 are equivalent. It is clear that assertion (1) implies assertion (2) (take $\omega = \mu_n$). Conversely, assume that assertion (2) holds true for some $\varepsilon > 0$, and let $\omega \in \mathbb{R}$. Then, either $J_{\varepsilon/2}(\omega)$ is empty, or there exists $n \in J_{\varepsilon/2}(\omega)$ and in this latter case, one can easily check that $J_{\varepsilon/2}(\omega) \subset J_\varepsilon(\mu_n)$. Consequently, in both cases, assertion (1) holds true.

It remains to show that the exact observability of system (1.1), (1.2) is equivalent to assertion (1). To achieve this, we use the characterization of exact observability provided by Theorem 2.1.

Assume that system (1.1), (1.2) is exactly observable. By Theorem 2.1, there exists a constant $\delta > 0$ such that

$$\|(A - i\omega I)z\|_X^2 + \|Cz\|_Y^2 \geqslant \delta\|z\|_X^2 \tag{2.9}$$

for all $\omega \in \mathbb{R}$, and for all $z \in \mathcal{D}(A)$.

On the other hand, for $z = \sum_{m \in J_\varepsilon(\omega)} c_m \Phi_m$ and for ε small enough, we have that

$$\|(A - i\omega I)z\|_X^2 = \sum_{m \in J_\varepsilon(\omega)} |i(\mu_m - \omega)c_m|^2 \leqslant \varepsilon^2 \|z\|_X^2 \leqslant \frac{\delta}{2} \|z\|_X^2. \tag{2.10}$$

By applying (2.9) to $z = \sum_{m \in J_\varepsilon(\omega)} c_m \Phi_m$ and by using (2.10), we obtain that assertion (1) holds.

Let us now assume that system (1.1), (1.2) is not exactly observable. Then, by Theorem 2.1, condition (2.1) is not satisfied, i.e., there exists sequences $(\omega_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ and $(z_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(A)$ such that

$$\|z_n\|_X = 1 \quad \forall\, n \in \mathbb{N} \tag{2.11}$$

and satisfying

$$\lim_{n \to \infty} \|(A - i\omega_n)z_n\|_X = 0, \qquad \lim_{n \to \infty} \|Cz_n\|_Y = 0. \tag{2.12}$$

We introduce then the following orthogonal decomposition of $z_n = \sum_{m \in \Lambda} c_m^n \Phi_m$:

$$z_n = z_n^0 + \widetilde{z}_n \tag{2.13}$$

with

$$z_n^0 = \sum_{m \in J_\varepsilon(\omega_n)} c_m^n \Phi_m, \quad \widetilde{z}_n = \sum_{m \notin J_\varepsilon(\omega_n)} c_m^n \Phi_m,$$

where $\varepsilon > 0$ and $J_\varepsilon(\omega)$ is defined for all $\omega \in \mathbb{R}$ by relation (1.5). Let us prove that the sequences $(\omega_n)_{n \in \mathbb{N}}$ and $(z_n^0)_{n \in \mathbb{N}}$ contradict assertion (1). First of all, we note that the orthogonality of $(A - i\omega_n)z_n^0$ and $(A - i\omega_n)\widetilde{z}_n$ implies that

$$\|(A - i\omega_n)z_n\|_X^2 \geqslant \|(A - i\omega_n)\widetilde{z}_n\|_X^2 = \sum_{m \notin J_\varepsilon(\omega_n)} \|(\mu_m - \omega_n)c_m^n\|^2 \geqslant \varepsilon^2 \|\widetilde{z}_n\|_X^2.$$

The above relation and (2.12) imply that

$$\lim_{n \to \infty} \|(A - i\omega_n)\widetilde{z}_n\|_X = 0 \tag{2.14}$$

and that

$$\lim_{n\to\infty} \|\widetilde{z}_n\|_X = 0.$$

Thanks to (2.11) and (2.13), the above relation yields

$$\lim_{n\to\infty} \|z_n^0\|_X = 1. \tag{2.15}$$

On the other hand, (2.13) implies that

$$\|Cz_n^0\|_Y \leqslant \|Cz_n\|_Y + \|C\widetilde{z}_n\|_Y. \tag{2.16}$$

Moreover, using the notation of Lemma 2.3, we have

$$C\widetilde{z}_n = C(A_{\omega_n} - i\omega_n)^{-1}(A_{\omega_n} - i\omega_n)\widetilde{z}_n.$$

Consequently, Lemma 2.3 implies that there exists $M > 0$ such that

$$\|C\widetilde{z}_n\|_Y \leqslant M\|(A_{\omega_n} - i\omega_n)\widetilde{z}_n\|_X \quad \forall\, n \in \mathbb{N}.$$

The above relation and (2.14) imply that

$$\lim_{n\to\infty} \|C\widetilde{z}_n\|_Y = 0.$$

This fact, together with (2.12) and (2.16) yield

$$\lim_{n\to\infty} \|Cz_n^0\|_Y = 0.$$

The above relation and (2.15) show that the sequences $(\omega_n)_{n\in\mathbb{N}}$ and $(z_n^0)_{n\in\mathbb{N}}$ contradict assertion (1) in Theorem 1.3. $\square$

3. Boundary observability of the Schrödinger equation in a square

3.1. Dirichlet boundary observation

Consider the square $\Omega = (0,\pi)\times(0,\pi)$ and let Γ be an open subset of $\partial\Omega$. We consider the following initial and boundary value problem:

$$\dot{z} + i\Delta z = 0, \quad x \in \Omega,\ t \geqslant 0, \tag{3.1}$$

$$\frac{\partial z}{\partial \nu} = 0, \quad x \in \partial\Omega, \ t \geqslant 0, \tag{3.2}$$

$$z(x, 0) = z_0(x), \quad x \in \Omega \tag{3.3}$$

with the output

$$y = z_{|\Gamma}. \tag{3.4}$$

This system can be described by equations of form (1.1), (1.2), if we introduce the appropriate spaces and operators. We first define the state space $X = L^2(\Omega)$ and the operator $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \left\{ \varphi \in H^2(\Omega) \ \middle| \ \frac{\partial \varphi}{\partial \nu} = 0 \right\}, \tag{3.5}$$

$$A\varphi = -i\Delta\varphi \quad \forall\, \varphi \in \mathcal{D}(A). \tag{3.6}$$

We next define the output space $Y = L^2(\Gamma)$ and the observation operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$

$$C\varphi = \varphi_{|\Gamma} \quad \forall\, \varphi \in \mathcal{D}(A). \tag{3.7}$$

Proposition 3.1. *With the above notation, C is an admissible observation operator. In other words, for all* $T \geqslant 0$ *there exists a constant* $K_T > 0$ *such that the if* $z, \ y$ *satisfy* (3.1)–(3.4) *then*

$$\int_0^T \int_\Gamma |y|^2 \, \mathrm{d}\Gamma \, \mathrm{d}t \ \leqslant \ K_T^2 \|z_0\|^2_{L^2(\Omega)} \quad \forall\, z_0 \in \mathcal{D}(A).$$

We skip the proof of the above result since it can be easily obtained from the Fourier series expansion of the solution of (3.1)–(3.3).

The main result in this subsection is:

Proposition 3.2. *For any nonempty open subset* Γ *of* $\partial\Omega$, *the system described by* (3.1)–(3.4) *is exactly observable. In other words there exists* $T > 0$ *and a constant* $k_T > 0$ *such that if* $z, \ y$ *satisfy* (3.1)–(3.4) *then*

$$\int_0^T \int_\Gamma |z|^2 \, \mathrm{d}\Gamma \, \mathrm{d}t \ \geqslant \ k_T^2 \|z_0\|^2_{L^2(\Omega)} \quad \forall\, z_0 \in \mathcal{D}(A).$$

Proof. We have seen in Proposition 3.1 that C is an admissible observation operator for (3.1)–(3.4) in the sense of Definition 1.1. On the other hand, A is clearly skew-adjoint. Moreover, since the imbedding $H^1(\Omega) \subset L^2(\Omega)$ is compact, A has a compact resolvent.

Consequently, according to Theorem 1.3, it suffices to check that the operators A and C defined by (3.5), (3.6) and (3.7) satisfy condition (2) in Theorem 1.3.

The eigenvalues of A are

$$\mu_{m,n} = i(m^2 + n^2) \quad \forall\, m, n \in \mathbb{N}^*.$$

A corresponding orthonormal basis of $X = L^2(\Omega)$ formed by eigenfunctions of A is

$$\Phi_{m,n}(x_1, x_2) = \frac{2}{\pi} \cos(mx_1) \cos(nx_2) \quad \forall\, m, n \in \mathbb{N}^*.$$

In order to check that condition (2) in Theorem 1.3 holds, we have to show that there exists $\varepsilon, \delta > 0$ such that for all $(q, r) \in \mathbb{N}^* \times \mathbb{N}^*$ and for all $z = \sum_{(m,n)\in J_\varepsilon(\mu_{q,r})} c_{m,n}\Phi_{m,n}$, we have

$$\|Cz\|_Y^2 = \int_\Gamma \left| \sum_{(m,n)\in J_\varepsilon(\mu_{q,r})} c_{m,n}\Phi_{m,n} \right|^2 \mathrm{d}\Gamma \geqslant \delta \sum_{(m,n)\in J_\varepsilon(\mu_{q,r})} |c_{m,n}|^2, \tag{3.8}$$

where

$$J_\varepsilon(\mu_{q,r}) = \{(m,n) \in \mathbb{N}^* \times \mathbb{N}^*;\ |(m^2 + n^2) - q^2 - r^2| < \varepsilon\}.$$

It is clear that if we choose $\varepsilon < 1$ then

$$J_\varepsilon(\mu_{q,r}) = \{(m,n) \in \mathbb{N}^* \times \mathbb{N}^*;\ m^2 + n^2 = q^2 + r^2\}.$$

Moreover, without loss of generality, we can assume that there exists $\alpha,\ \beta \in (0, \pi)$ with $\alpha < \beta$ and $(\alpha, \beta) \times \{0\} \subset \Gamma$. Let S denote the set of squares of positive integers. For $q,\ r \in \mathbb{N}^*$ we set

$$\Lambda_{qr} = \{m \in \mathbb{N}^* \mid q^2 + r^2 - m^2 \in S\} \tag{3.9}$$

and for $m \in \Lambda_{qr}$ we put

$$f(m) = \sqrt{q^2 + r^2 - m^2}. \tag{3.10}$$

We have

$$\|Cz\|_Y^2 \geqslant \frac{4}{\pi^2}\int_\alpha^\beta \left|\sum_{m\in\Lambda_{qr}} c_{m,f(m)}\cos(mx_1)\right|^2 \mathrm{d}x_1. \tag{3.11}$$

By using Proposition 7.1, the above relation implies that there exists a constant $\delta > 0$ such that

$$\frac{4}{\pi^2}\int_\alpha^\beta \left|\sum_{m\in\Lambda_{qr}} c_{m,f(m)}\cos(mx_1)\right|^2 \mathrm{d}x_1 \geqslant \delta \sum_{m^2+n^2=q^2+r^2} |c_{m,n}|^2. \tag{3.12}$$

From (3.11) and (3.12), we clearly get the desired estimate (3.8). □

By a standard duality argument, the above proposition implies that the following exact controllability holds.

Corollary 3.3. *For any nonempty open subset* Γ *of* $\partial\Omega$, *the system*

$$\begin{aligned} \dot z + i\Delta z &= 0, \quad x\in\Omega,\ t>0,\\ \frac{\partial z}{\partial\nu} &= 0, \quad x\in\partial\Omega\setminus\Gamma,\ t>0,\\ \frac{\partial z}{\partial\nu} &= u\in L^2(0,T;L^2(\Gamma)), \quad x\in\Gamma,\ t>0,\\ z(x,0) &= z_0(x), \quad x\in\Omega. \end{aligned}$$

is exactly controllable in some time T *in the state space* $L^2(\Omega)$.

3.2. Neumann boundary observation

The example studied in this subsection differs from the case considered in the previous one only by the boundary condition and the choice of the observation operator. We prove that, in order to get exact observability we need a supplementary assumption on the observed part of the boundary.

Consider the square $\Omega = (0,\pi)\times(0,\pi)$ and let Γ be an open nonempty subset of $\partial\Omega$. We consider the following initial and boundary value problem:

$$\dot z + i\Delta z = 0, \quad x\in\Omega,\ t\geqslant 0, \tag{3.13}$$

$$z = 0, \quad x\in\partial\Omega,\ t\geqslant 0, \tag{3.14}$$

$$z(x,0) = z_0(x), \quad x\in\Omega \tag{3.15}$$

with the output

$$y(t) = \left.\frac{\partial z}{\partial \nu}\right|_{\Gamma}. \tag{3.16}$$

The system can be described by equations of form (1.1), (1.2), if we introduce the appropriate spaces and operators. Indeed, let us first define the state space $X = H_0^1(\Omega)$ and the operator $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \{\varphi \in H^3(\Omega) \cap H_0^1(\Omega) \mid \Delta\varphi = 0 \quad \text{on} \quad \partial\Omega\}, \tag{3.17}$$

$$A\varphi = -i\Delta\varphi \quad \forall\, \varphi \in \mathcal{D}(A). \tag{3.18}$$

Next, we define the output space $Y = L^2(\Gamma)$ and the corresponding observation operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$ by

$$C\varphi = \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma} \quad \forall\, \varphi \in \mathcal{D}(A). \tag{3.19}$$

Proposition 3.4. *With the above notation, C is an admissible observation operator, i.e. for all $T \geqslant 0$ there exists a constant $K_T > 0$ such that if $z,\ y$ satisfy* (3.13)–(3.16) *then*

$$\int_0^T \int_\Gamma |y|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \;\leqslant\; K_T^2 \|z_0\|^2_{L^2(\Omega)} \quad \forall\, z_0 \in \mathcal{D}(A).$$

The above result is classical (see, for instance, [17]), so we skip its proof.

The observability properties of system (3.13)–(3.16) are different from those encountered in the study of system (3.1)–(3.4). More precisely, if we denote by $\Gamma_1 = ([0,\pi] \times \{0\}) \cup ([0,\pi] \times \{\pi\})$ the horizontal part of $\partial\Omega$ and by $\Gamma_2 = (\{0\} \times [0,\pi]) \cup (\{\pi\} \times [0,\pi])$ the vertical part of $\partial\Omega$, then the following result holds:

Proposition 3.5. *The system described by* (3.13)–(3.16) *is exactly observable if and only if $\Gamma \cap \Gamma_i \neq \emptyset$, for $i \in \{1,2\}$. In other words the following assertions are equivalent*:

(1) *There exists $T > 0$ and a constant $k_T > 0$ such that for all $z,\ y$ satisfying* (3.13)–(3.16) *we have*

$$\int_0^T \int_\Gamma |y|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \;\geqslant\; k_T^2 \|z_0\|^2_{H^1(\Omega)} \quad \forall\, z_0 \in \mathcal{D}(A).$$

(2) *The control region Γ contains both a horizontal and a vertical segment of nonzero length.*

Proof. We have seen in Proposition 3.1 that, for any open subset Γ of $\partial\Omega$, C is an admissible observation operator for (3.1)–(3.4) in the sense of Definition 1.1. Moreover, A is clearly skew-adjoint and it has compact resolvent. Therefore, we can apply condition (2) in Theorem 1.3.

The eigenvalues of A are

$$\mu_{m,n} = i(m^2 + n^2) \quad \forall\, m, n \in \mathbb{N}^*.$$

A corresponding family of normalized (in $X = H_0^1(\Omega)$) eigenfunctions are

$$\Phi_{m,n}(x_1, x_2) = \frac{2}{\pi\sqrt{m^2 + n^2}} \sin(mx_1)\sin(nx_2) \quad \forall\, m, n \in \mathbb{N}^*.$$

We first show the necessity of condition $\Gamma \cap \Gamma_i \neq \emptyset$ for $i = 1, 2$. Indeed, if this condition fails then we can assume, without loss of generality, that $\Gamma \subset \Gamma_1$. We notice that

$$\|C\Phi_{n,1}\|_Y^2 \leqslant \int_{\Gamma_1} \left|\frac{\partial \Phi_{n,1}}{\partial \nu}\right|^2 \mathrm{d}\Gamma = \frac{8}{\pi^2}\frac{1}{1+n^2}\int_0^\pi \sin^2(nx_1)\,\mathrm{d}x_1. \tag{3.20}$$

Consequently,

$$\lim_{n\to\infty} \|C\Phi_{n,1}\|_Y = 0,$$

which contradicts condition (2) in Theorem 1.3.

We next show that condition $\Gamma \cap \Gamma_i \neq \emptyset$ for $i = 1, 2$ implies that the operators A and C defined by (3.17), (3.18) and (3.19) satisfy condition (2) in Theorem 1.3. In this case, without loss of generality we can assume that

$$\Gamma \supset \big([\alpha_1, \beta_1] \times \{0\}\big) \cup \big(\{0\} \times [\alpha_2, \beta_2]\big)$$

with $0 < \alpha_i < \beta_i < \pi$, for $i \in \{1, 2\}$.

For $q,\ r \in \mathbb{N}^*$, we recall the notation

$$J_\varepsilon(\mu_{q,r}) = \{(m, n) \in \mathbb{N}^* \times \mathbb{N}^* \ ; \ |(m^2 + n^2) - q^2 - r^2| < \varepsilon\}.$$

It is clear that if we choose $\varepsilon < 1$, then

$$J_\varepsilon(\mu_{q,r}) = \{(m, n) \in \mathbb{N}^* \times \mathbb{N}^* \ ; \ m^2 + n^2 = q^2 + r^2\}.$$

If $z = \sum_{(m,n)\in J_\varepsilon(\mu_{q,r})} c_{m,n}\Phi_{m,n}$ then

$$\|Cz\|_Y^2 \geqslant \frac{4}{\pi^2}\left[\int_{\alpha_1}^{\beta_1}\left|\sum_{m\in\Lambda_{qr}} \frac{f(m)c_{m,f(m)}}{\sqrt{m^2+f(m)^2}}\sin(mx_1)\right|^2 \mathrm{d}x_1 \right.$$
$$\left. + \int_{\alpha_2}^{\beta_2}\left|\sum_{n\in\Lambda_{qr}} \frac{f(n)c_{f(n),n}}{\sqrt{f(n)^2+n^2}}\sin(nx_2)\right|^2 \mathrm{d}x_2\right], \tag{3.21}$$

where Λ_{qr} and f are defined in (3.9) and in (3.10). On the other hand, by using Proposition 7.1, we obtain that there exists a constant $\delta > 0$ such that

$$\begin{aligned}\int_{\alpha_1}^{\beta_1}\left|\sum_{n\in\Lambda_{qr}} \frac{f(m)c_{m,f(m)}}{\sqrt{m^2+f^2(m)}}\sin(mx_1)\right|^2 \mathrm{d}x_1 \;&\geqslant\; \delta\sum_{m\in\Lambda_{qr}}\frac{f^2(m)}{f^2(m)+m^2}|c_{m,f(m)}|^2 \\ &=\; \delta\sum_{m^2+n^2=q^2+r^2}\frac{n^2}{m^2+n^2}|c_{m,n}|^2\end{aligned}$$

and

$$\begin{aligned}\int_{\alpha_2}^{\beta_2}\left|\sum_{n\in\Lambda_{qr}} \frac{f(n)c_{f(n),n}}{\sqrt{f^2(n)+n^2}}\sin(nx_2)\right|^2 \mathrm{d}x_2 \;&\geqslant\; \delta\sum_{n\in\Lambda_{qr}}\frac{f^2(n)}{f^2(n)+n^2}|c_{f(n),n}|^2 \\ &=\; \delta\sum_{m^2+n^2=q^2+r^2}\frac{m^2}{m^2+n^2}|c_{m,n}|^2.\end{aligned}$$

By taking the sum of the two above inequalities we get

$$\int_{\alpha_1}^{\beta_1}\left|\sum_{m\in\Lambda_{qr}} \frac{f(m)c_{m,f(m)}}{\sqrt{m^2+f^2(m)}}\sin(mx_1)\right|^2 \mathrm{d}x_1 + \int_{\alpha_2}^{\beta_2}\left|\sum_{n\in\Lambda_{qr}} \frac{f(n)c_{f(n),n}}{\sqrt{f^2(n)+n^2}}\sin(nx_2)\right|^2 \mathrm{d}x_2$$
$$\geqslant \delta\sum_{m^2+n^2=q^2+r^2}|c_{m,n}|^2. \tag{3.22}$$

By using (3.21) and (3.22) we obtain that $\|Cz\|_Y \geqslant \delta\|z\|_X$ which concludes the proof. □

By a standard duality argument, the above proposition implies that the following exact controllability holds.

Corollary 3.6. *With the notation in Proposition* 3.5, *the system*

$$\begin{aligned} \dot{z} + i\Delta z &= 0, \quad x \in \Omega,\ t > 0, \\ z &= 0, \quad x \in \partial\Omega \setminus \Gamma,\ t > 0, \\ z &= u \in L^2(0, T; L^2(\Gamma)), \quad x \in \Gamma,\ t > 0, \\ z(x, 0) &= z_0(x), \quad x \in \Omega \end{aligned}$$

is exactly controllable in some time $T > 0$ (*in the state space* $X = H^{-1}(\Omega)$) *if and only if* $\Gamma \cap \Gamma_i \neq \emptyset$, *for* $i \in \{1, 2\}$.

Remark 3.7. It can be shown, by using techniques similar to those in [16,17], that the observability and the controllability results in this section hold for any $T > 0$.

4. Frequency domain tests for the exact observability of second-order systems

In this section we investigate an important particular case fitting in the framework of Theorem 1.3. This case is obtained by considering second-order evolution equations occurring in the study of vibrating systems. More precisely, let H be a Hilbert space equipped with the norm $\|\cdot\|$ and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint, positive and boundedly invertible operator, with compact resolvent. Consider the initial value problem:

$$\ddot{w}(t) + A_0 w(t) = 0, \tag{4.1}$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1, \tag{4.2}$$

which can be seen as a generic model for the free vibrations of elastic structures such as strings, beams, membranes, plates or three-dimensional elastic bodies. Moreover, let $C_0 \in \mathcal{L}\left(\mathcal{D}(A_0^{\frac{1}{2}}), Y\right)$ be an observation operator. We first show the equivalence of two conditions which will be used to define a concept of admissibility for observed systems described by second-order differential equations.

Proposition 4.1. *With the above notation, the following conditions are equivalent*:

(1) *For every* $T > 0$ *there exists a constant* $K_T \geqslant 0$ *such that the solutions* w *of* (4.1), (4.2) *satisfy*

$$\int_0^T \|C_0 \dot{w}(t)\|_Y^2 \, \mathrm{d}t \leqslant K_T^2 \left(\|w_0\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} + \|w_1\|^2 \right) \quad \forall (w_0, w_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}). \tag{4.3}$$

(2) *For every* $T > 0$ *there exists a constant* $K_T \geqslant 0$ *such that the solutions* w *of* (4.1), (4.2) *satisfy*

$$\int_0^T \|C_0 w(t)\|_Y^2 \, \mathrm{d}t \leqslant K_T^2 \left(\|w_0\|^2 + \|w_1\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})^*} \right) \quad \forall w_0 \in \mathcal{D}(A_0^{\frac{1}{2}}) \ \forall w_1 \in H, \tag{4.4}$$

where $\mathcal{D}(A_0^{\frac{1}{2}})^*$ *stands for the dual space of* $\mathcal{D}(A_0^{\frac{1}{2}})$ *with respect to the pivot space* H.

Proof. We first show that assertion (1) implies assertion (2). If $w_0 \in \mathcal{D}(A_0^{\frac{1}{2}})$, $w_1 \in H$ then the solution w of (4.1), (4.2) satisfies $w \in C([0,T]; \mathcal{D}(A_0^{\frac{1}{2}})) \cap C^1([0,T], H)$. Define

$$v(t) = \int_0^t w(s)\, \mathrm{d}s - A_0^{-1} w_1.$$

Clearly, we have

$$\ddot{v}(t) + A_0 v(t) = 0,$$

$$v(0) = -A_0^{-1} w_1 \in \mathcal{D}(A_0), \qquad \dot{v}(0) = w_0 \in \mathcal{D}(A_0^{\frac{1}{2}}).$$

Since we supposed that assertion (1) holds true it follows that

$$\begin{aligned} \int_0^T \|C_0 w(s)\|_Y^2 \, \mathrm{d}s &= \int_0^T \|C_0 \dot{v}(s)\|_Y^2 \, \mathrm{d}s \\ &\leqslant K_T \left(\|A_0^{-1} w_1\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} + \|w_0\|_H^2 \right) \end{aligned}$$

for all $(w_0, w_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}})$.

Since A_0 is an isometry from $\mathcal{D}(A_0^{\frac{1}{2}})$ onto $\mathcal{D}(A_0^{\frac{1}{2}})^*$, the above inequality implies that assertion (2) holds true.

We still have to show that assertion (2) implies assertion (1). First, assume that $w_0 \in \mathcal{D}(A_0^{\frac{3}{2}})$, $w_1 \in \mathcal{D}(A_0)$. Then, the solution w of system (4.1), (4.2) satisfies $w \in C^1([0,T]; \mathcal{D}(A_0)) \cap C^2([0,T]; \mathcal{D}(A_0^{\frac{1}{2}}))$. If we set $v(t) = \dot{w}(t)$ then

$v \in C([0,T], \mathcal{D}(A_0)) \cap C^1([0,T], \mathcal{D}(A_0^{\frac{1}{2}}))$ satisfies

$$\ddot{v}(t) + A_0 v(t) = 0,$$

$$v(0) = w_1 \in \mathcal{D}(A_0), \quad \dot{v}(0) = -A_0 w_0 \in \mathcal{D}(A_0^{\frac{1}{2}}).$$

Since we supposed that assertion (2) holds, we deduce that

$$\begin{aligned} \int_0^T \|C_0 \dot{w}(s)\|_Y^2 \, \mathrm{d}s &= \int_0^T \|C_0 v(s)\|_Y^2 \, \mathrm{d}s \\ &\leqslant K_T \left(\|w_1\|^2 + \|A_0 w_0\|^2_{[\mathcal{D}(A_0^{\frac{1}{2}})]^*} \right) \\ &= K_T \left(\|w_1\|^2 + \|w_0\|^2_{[\mathcal{D}(A_0^{\frac{1}{2}})]} \right) \end{aligned}$$

for all $w_0 \in \mathcal{D}(A_0^{\frac{3}{2}})$, $w_1 \in \mathcal{D}(A_0)$. A density argument shows that assertion (1) holds. $\square$

In the remaining part of this paper we consider systems of form (4.1), (4.2) with one of the two following outputs:

$$y = C_0 w, \tag{4.5}$$

or

$$y = C_0 \dot{w}, \tag{4.6}$$

We are now in a position to give a definition of the admissibility for second-order problems:

Definition 4.2. C_0 is an admissible observation operator for (4.1), (4.2) if it satisfies one of the equivalent conditions in Proposition 4.1.

We next state the equivalence of two conditions which will be used in order to define a concept of exact observability for observed systems described by second-order differential equations.

Proposition 4.3. *With the notation in Proposition* 4.1, *the following conditions are equivalent*:

(1) *For every* $T > 0$ *there exists a constant* $k_T > 0$ *such that the solutions* w *of* (4.1), (4.2) *satisfy*

$$\int_0^T \|C_0\dot{w}(t)\|_Y^2\,\mathrm{d}t \geqslant k_T^2\left(\|w_0\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} + \|w_1\|^2\right) \quad \forall (w_0, w_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}). \tag{4.7}$$

(2) *For every* $T > 0$ *there exists a constant* $k_T > 0$ *such that the solutions* w *of* (4.1), (4.2) *satisfy*

$$\int_0^T \|C_0 w(t)\|_Y^2\,\mathrm{d}t \;\geqslant\; k_T^2\left(\|w_0\|^2 + \|w_1\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})^*}\right) \quad \forall w_0 \in \mathcal{D}(A_0^{\frac{1}{2}}) \;\; \forall w_1 \in H. \tag{4.8}$$

We skip the proof of the above result since it is completely similar to the proof of Proposition 4.1.

Definition 4.4. The system described by (4.1), (4.2) and (4.5) is exactly observable in time T if it satisfies one of the equivalent conditions in Proposition 4.3.

We can now state the main result of this section.

Proposition 4.5. *Let* $A_0 : \mathcal{D}(A_0) \to H$ *be a self-adjoint, positive and boundedly invertible operator, with compact resolvent and let* $C_0 \in \mathcal{L}(\mathcal{D}(A_0^{\frac{1}{2}}), Y)$ *be an admissible observation operator for* (4.1)–(4.2). *Let us denote by* $(\lambda_n)_{n\in\mathbb{N}^*}$ *the increasing sequence formed by the eigenvalues of* $A_0^{\frac{1}{2}}$ *and by* $(\phi_n)_{n\in\mathbb{N}^*}$ *a corresponding sequence of eigenvectors, forming an orthonormal basis of H. For all* $\omega > 0$ *and all* $\varepsilon > 0$, *let us define the set*

$$I_\varepsilon(\omega) = \{m \in \mathbb{N}^* \text{ such that } |\lambda_m - \omega| < \varepsilon\}. \tag{4.9}$$

Then, the following propositions are equivalent:

(i) *System* (4.1)–(4.5) *is exactly observable*.
(ii) *There exists a constant* $\delta > 0$ *such that*

$$\forall\, \varphi \in \mathcal{D}(A_0),\;\; \forall \omega > 0 : \|(\omega^2 - A_0)\varphi\|^2 + \|\omega\, C_0\varphi\|_Y^2 \;\geqslant\; \delta\|\omega\varphi\|^2. \tag{4.10}$$

(iii) *There exists* $\varepsilon > 0$ *and* $\delta > 0$ *such that for all* $\omega > 0$ *and all* $\varphi = \sum_{m\in I_\varepsilon(\omega)} c_m\phi_m$:

$$\|C_0\varphi\|_Y \;\geqslant\; \delta\|\varphi\|. \tag{4.11}$$

(iv) *There exists* $\varepsilon > 0$ *and* $\delta > 0$ *such that for all* $n \in \mathbb{N}^*$ *and all* $\varphi = \sum_{m \in I_\varepsilon(\lambda_n)} c_m \phi_m$:

$$\|C_0\varphi\|_Y \geqslant \delta\|\varphi\|. \tag{4.12}$$

Remark 4.6. The fact that condition (ii) in the above proposition is equivalent to the exact observability can be seen as a generalization of Theorem 3.4 in [19], where a similar Hautus-type result has been proved in the case of a *bounded* observation operator C_0.

Proof of Proposition 4.5. It can be easily checked that system (4.1)–(4.5) can be written in form (1.1)–(1.2) provided that we define the state of the system by $z(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix}$ and that we make the following choice of spaces and operators:

$$X = \mathcal{D}\left(A_0^{\frac{1}{2}}\right) \times H, \quad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & C_0 \end{pmatrix}. \tag{4.13}$$

The Hilbert space X is endowed here with the norm $\|\cdot\|_X$ defined by

$$\|z\|_X^2 = \|A_0^{\frac{1}{2}}\varphi\|^2 + \|\psi\|^2 \quad \forall\, z = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in X.$$

- (i) $\Rightarrow$ (ii). By Theorem 2.1 there exists a constant $\delta > 0$ such that

$$\|(A - i\omega)z\|_X^2 + \|Cz\|_Y^2 \geqslant \delta\|z\|_X^2 \quad \forall\, \omega \in \mathbb{R} \quad \forall\, z \in \mathcal{D}(A). \tag{4.14}$$

Taking in the above relation $z = \begin{pmatrix} \varphi \\ i\omega\varphi \end{pmatrix}$, where $\varphi \in \mathcal{D}(A_0^{\frac{1}{2}})$, we obtain that

$$\|Cz\|_Y = \|\omega C_0\varphi\|_Y, \qquad \|z\|_X \geqslant \|\omega\varphi\|$$

while

$$\|(A - i\omega)z\|_X = \|(\omega^2 - A_0)\varphi\|.$$

Therefore, (4.14) implies (4.10), and (ii) holds true.

• (ii) $\Rightarrow$ (iii). Let $\varepsilon < \dfrac{\lambda_1}{2}$. Then it is easy to see that if $\omega < \varepsilon$, then $I_\varepsilon(\omega) = \emptyset$ and (iii) holds. If $\omega \geqslant \varepsilon$ then for all $\varphi = \sum_{m \in I_\varepsilon(\omega)} c_m \phi_m$, we have

$$\left\|(\omega^2 - A_0)\varphi\right\|^2 = \sum_{m \in I_\varepsilon(\omega)} \left|\omega^2 - \lambda_m^2\right|^2 |c_m|^2 \leqslant \varepsilon^2 \sum_{m \in I_\varepsilon(\omega)} (\omega + \lambda_m)^2 |c_m|^2 \leqslant 9\varepsilon^2 \|\omega\varphi\|^2.$$

Consequently, for ε small enough and for $\varphi = \sum_{m \in I_\varepsilon(\omega)} c_m \phi_m$, we have

$$\left\|(\omega^2 - A_0)\varphi\right\|^2 \leqslant \frac{\delta}{2} \|\omega\varphi\|^2.$$

By applying condition (ii) to $\varphi = \sum_{m \in I_\varepsilon(\omega)} c_m \phi_m$ and by using the above equation, we obtain (iii).

• (iii) $\Rightarrow$ (iv). This implication obviously holds (take $\omega = \lambda_n$).

• (iv) $\Rightarrow$ (i). In order to prove this assertion we use Theorem 1.3. Suppose that (iv) holds true. Without loss of generality, we can assume that the constant ε in (iv) satisfies $\varepsilon < \lambda_1$.

Let A and C be defined by (4.13), it can be easily checked that the eigenvalues of A are $(i\mu_n)_{n \in \mathbb{Z}^*}$ where

$$\mu_n = \begin{cases} \lambda_n & \text{if } n \in \mathbb{N}^*, \\ -\lambda_{-n} & \text{if } (-n) \in \mathbb{N}^*. \end{cases}$$

If we set $\phi_{-n} = \phi_n$, for all $n \in \mathbb{N}^*$, then an orthonormal family (in X) of eigenvectors $(\Phi_n)_{n \in \mathbb{Z}^*}$ of A is given by

$$\Phi_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{1}{i\mu_n}\phi_n \\ \phi_n \end{pmatrix} \quad \forall\, n \in \mathbb{Z}^*.$$

In order to prove (i) it suffices, by Theorem 1.3, to show that there exists $\varepsilon > 0$ and $\delta > 0$ such that for all $n \in \mathbb{Z}^*$ and for all

$$z = \sum_{m \in J_\varepsilon(\mu_n)} c_m \Phi_m = \frac{1}{\sqrt{2}} \begin{pmatrix} \displaystyle\sum_{m \in J_\varepsilon(\mu_n)} \frac{1}{i\mu_m} c_m \phi_m \\ \displaystyle\sum_{m \in J_\varepsilon(\mu_n)} c_m \phi_m \end{pmatrix} \tag{4.15}$$

we have

$$\|Cz\|_Y \geqslant \delta \|z\|_X.$$

Let us consider first the case where $\mu_n > 0$ in (4.15). Then, $\mu_n = \lambda_n$, and thus we have $J_\varepsilon(\mu_n) = I_\varepsilon(\lambda_n)$ (since $\varepsilon < \lambda_1$). Let us denote by φ the second component of z

$$\varphi = \frac{1}{\sqrt{2}} \sum_{m \in I_\varepsilon(\lambda_n)} c_m \phi_m.$$

Then, it can be easily checked that

$$Cz = C_0 \varphi$$

and that

$$\|z\|_X = \sqrt{2} \|\varphi\|.$$

Consequently, by applying then (iv) to φ we get that

$$\|Cz\|_Y \geqslant \frac{\delta}{\sqrt{2}} \|z\|_X.$$

The case $\mu_n < 0$ can be treated similarly, and the proof is thus complete. $\square$

5. Boundary observability for the Bernoulli–Euler plate equation in a square

Consider the square $\Omega = (0, \pi) \times (0, \pi)$ and let Γ be an open subset of $\partial\Omega$. We consider the following initial and boundary value problem:

$$\ddot{w} + \Delta^2 w = 0, \quad x \in \Omega, \ t \geqslant 0, \tag{5.1}$$

$$w(x,t) = \Delta w(x,t) = 0, \quad x \in \partial\Omega, \ t \geqslant 0, \tag{5.2}$$

$$w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), \quad x \in \Omega \tag{5.3}$$

with the output

$$y(t) = \left. \frac{\partial \dot{w}}{\partial \nu} \right|_\Gamma . \tag{5.4}$$

System (5.1)–(5.4) can be written in form (4.1), (4.2), (4.5). More precisely, we define

$$H = H_0^1(\Omega), \quad \mathcal{D}(A_0) = \{\varphi \in H^5(\Omega) \cap H_0^1(\Omega) \mid \Delta\varphi = \Delta^2\varphi = 0 \text{ on } \partial\Omega\},$$
$$Y = L^2(\Gamma), \quad A_0\varphi = \Delta^2\varphi \quad \forall\, \varphi \in \mathcal{D}(A_0).$$

With the above choice of spaces and operators, one can easily check that A_0 is self-adjoint, positive, boundedly invertible and that

$$\mathcal{D}(A_0^{\frac{1}{2}}) = \{\varphi \in H^3(\Omega) \cap H_0^1(\Omega) \mid \Delta\varphi = 0 \quad \text{on} \quad \partial\Omega\}.$$

Moreover, the dual space of $\mathcal{D}(A_0^{\frac{1}{2}})$ with respect to the pivot space H is

$$\mathcal{D}(A_0^{\frac{1}{2}})^* = H^{-1}(\Omega).$$

The output operator corresponding to (5.4) is

$$C_0\varphi = \left.\frac{\partial\varphi}{\partial\nu}\right|_\Gamma \quad \forall\, \varphi \in \mathcal{D}(A_0^{\frac{1}{2}}).$$

Proposition 5.1. *With the above notation, $C_0 \in \mathcal{L}(\mathcal{D}(A_0^{\frac{1}{2}}), Y)$ is an admissible observation operator, i.e. for all $T \geqslant 0$ there exists a constant $K_T > 0$ such that if $w,\ y$ satisfy* (5.1)–(5.4) *then*

$$\int_0^T \int_\Gamma |y|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \;\leqslant\; K_T^2 \left(\|w_0\|^2_{H^3(\Omega)} + \|w_1\|^2_{H^1(\Omega)}\right)$$

for all $(w_0, w_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}})$.

The above result is classical and for its proof we refer, for instance, to [18, p. 287].

In order to state the observability properties of system (5.1)–(5.4), let us denote by $\Gamma_1 = ([0,\pi] \times \{0\}) \cup ([0,\pi] \times \{\pi\})$ the horizontal part of $\partial\Omega$ and by $\Gamma_2 = (\{0\} \times [0,\pi]) \cup (\{\pi\} \times [0,\pi])$ its vertical part. Then, the following result holds:

Proposition 5.2. *The system described by* (5.1)–(5.4) *is exactly observable if and only if $\Gamma \cap \Gamma_i \neq \emptyset$, for $i \in \{1, 2\}$. In other words, the following statements are equivalent*

(1) *There exists $T > 0$ and a constant $k_T > 0$ such that if z, y satisfy* (5.1)–(5.4) *then*

$$\int_0^T \int_\Gamma |y|^2 \,\mathrm{d}\Gamma\,\mathrm{d}t \geqslant k_T^2 \left(\|w_0\|^2_{H^3(\Omega)} + \|w_1\|^2_{H^1(\Omega)}\right)$$
$$\forall\, (w_0, w_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}).$$

(2) *The control region* Γ *contains both a horizontal and a vertical segment of nonzero length*.

Proof. By Proposition 5.1, C_0 is an admissible observation operator for the system described by (5.1)–(5.4). Moreover, the imbedding $\mathcal{D}(A_0^{\frac{1}{2}}) \subset H$ is clearly compact. Consequently, we can apply Proposition 4.5.

The eigenvalues of $A_0^{\frac{1}{2}}$ are

$$\lambda_{m,n} = m^2 + n^2 \quad \forall\, m, n \in \mathbb{N}^*.$$

A corresponding family of normalized (in $H = H_0^1(\Omega)$) eigenfunctions are

$$\phi_{m,n}(x) = \frac{2}{\pi\sqrt{m^2+n^2}} \sin(mx_1) \sin(nx_2) \quad \forall\, m, n \in \mathbb{N}^* \quad \forall\, x = (x_1, x_2) \in \Omega.$$

We first show the necessity of condition $\Gamma \cap \Gamma_i = \emptyset$ for $i = 1, 2$. If this condition fails then we can assume, without loss of generality, that $\Gamma \subset \Gamma_1$. We notice that

$$\|C_0\phi_{n,1}\|_Y^2 \leqslant \int_{\Gamma_1} \left|\frac{\partial \phi_{n,1}}{\partial \nu}\right|^2 \mathrm{d}\Gamma = \frac{8}{\pi^2}\frac{1}{1+n^2}\int_0^\pi \sin^2(nx_1)\,\mathrm{d}x_1. \tag{5.5}$$

Consequently,

$$\lim_{n\to\infty} \|C_0\phi_{n,1}\|_Y = 0,$$

which contradicts condition (iii) in Proposition 4.5.

In the remaining part of the proof, we show that if $\Gamma \cap \Gamma_i \neq \emptyset$, for $i \in \{1, 2\}$, then the operators A_0 and C_0 satisfy condition (iv) in Proposition 4.5. More precisely, we prove that for all $\varepsilon \in (0, 1)$ there exists $\delta > 0$ such that, for all $q,\ r \in \mathbb{N}^*$ and for all $\varphi = \sum_{(m,n)\in I_\varepsilon(\lambda_{q,r})} a_{m,n}\phi_{m,n}$, we have

$$\|C_0\varphi\|_Y^2 = \int_\Gamma \left| \sum_{(m,n)\in I_\varepsilon(\lambda_{q,r})} a_{m,n} \frac{\partial \phi_{m,n}}{\partial \nu}\right|^2 \mathrm{d}\Gamma \geqslant \delta \left(\sum_{(m,n)\in I_\varepsilon(\lambda_{q,r})} |a_{m,n}|^2 \right), \tag{5.6}$$

where

$$I_\varepsilon(\lambda_{q,r}) = \{(m,n) \in \mathbb{N}^* \times \mathbb{N}^* \ \mid m^2 + n^2 = q^2 + r^2\}.$$

Without loss of generality, we can assume that

$$\Gamma \supset \big([\alpha_1, \beta_1] \times \{0\}\big) \cup \big(\{0\} \times [\alpha_2, \beta_2]\big)$$

with $0 < \alpha_i < \beta_i < \pi$, for $i \in \{1, 2\}$. Then, we have

$$\|C_0\varphi\|_Y^2 \geqslant \frac{4}{\pi^2}\left[\int_{\alpha_1}^{\beta_1}\left|\sum_{m\in\Lambda_{qr}} \frac{f(m)a_{m,f(m)}}{\sqrt{m^2+f^2(m)}}\sin(mx_1)\right|^2 \mathrm{d}x_1 + \int_{\alpha_2}^{\beta_2}\left|\sum_{n\in\Lambda_{qr}} \frac{f(n)a_{f(n),n}}{\sqrt{f^2(n)+n^2}}\sin(nx_2)\right|^2 \mathrm{d}x_2\right], \tag{5.7}$$

where the set Λ_{qr} and the function f are defined in (3.9) and (3.10). The desired inequality (5.6) follows now directly from (3.22) which was established in the proof of Proposition 3.5. □

We conclude this section by remarking that Theorem 1.5 stated in Section 1 follows directly from the results already proved in this section. More precisely, the fact that the system is exactly controllable is some time $T > 0$ follows from Proposition 5.2 by a standard duality argument. Showing that T can be chosen arbitrarily small can be achieved by slightly adapting a classical argument (see for instance [16, p. 81] or the appendix written by Zuazua in [18]).

6. Boundary observability of the wave equation in a square

In this section, we consider the problem of observability of the wave equation with Neumann boundary observation for the wave equation. This problem has been tackled by a large number of papers by using various methods (see for instance [18] and references therein). However, besides the one-dimensional case, no direct Fourier series-based proof seems to exist in the literature. We give such a proof in the case where the space domain is a square. If we except the use of Proposition 4.5, the basic ingredients of the proof are very simple (we only need Parseval's theorem).

Consider the square $\Omega = (0, \pi) \times (0, \pi)$ and let $\Gamma = ([0, \pi] \times \{0\}) \cup (\{0\} \times [0, \pi])$. We consider the following initial and boundary value problem:

$$\ddot{w} - \Delta w = 0, \quad x \in \Omega,\ t \geqslant 0, \tag{6.1}$$

$$w = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \tag{6.2}$$

$$w(x, 0) = w_0(x), \quad \dot{w}(x, 0) = w_1(x), \quad x \in \Omega \tag{6.3}$$

with the output

$$y = \left.\frac{\partial w}{\partial \nu}\right|_{\Gamma}. \tag{6.4}$$

System (6.1)–(6.4) can be written in form (4.1)–(4.5) if we introduce the following notation:

$$H = H_0^1(\Omega), \quad \mathcal{D}(A_0) = \{\varphi \in H^3(\Omega) \cap H_0^1(\Omega) \mid \Delta\varphi = 0 \quad \text{on} \quad \partial\Omega\}, \quad Y = L^2(\Gamma),$$

$$A_0\varphi = -\Delta\varphi \quad \forall\, \varphi \in \mathcal{D}(A),$$

$$C_0\varphi = \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma} \quad \forall\, \varphi \in \mathcal{D}(A).$$

One can easily check that, with the above choice of the spaces and operators, we have that A_0 is self-adjoint, positive and boundedly invertible and

$$\mathcal{D}(A_0^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega), \quad \mathcal{D}(A_0^{\frac{1}{2}})^* = L^2(\Omega).$$

Proposition 6.1. *With the above notation, $C_0 \in \mathcal{L}(\mathcal{D}(A_0^{\frac{1}{2}}), Y)$ is an admissible observation operator, i.e. for all $T \geqslant 0$ there exists a constant $K_T > 0$ such that if $w, \ y$ satisfy* (6.1)–(6.4) *then*

$$\int_0^T \int_\Gamma |y|^2 \,\mathrm{d}\Gamma\, \mathrm{d}t \leqslant K_T^2 \left(\|w_0\|^2_{H_0^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right)$$

for all $(w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

The above proposition is classical (see, for instance, [18, p. 44]), so we skip the proof.

The main result of this section is the following.

Theorem 6.2. *The system described by* (6.1)–(6.4) *is exactly observable. In other words, there exists $T > 0$ and $k_T > 0$ such that*

$$\int_0^T \int_\Gamma |y|^2 \,\mathrm{d}\Gamma\, \mathrm{d}t \geqslant k_T^2 \left(\|w_0\|^2_{H_0^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right)$$

for all $(w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

Proof. By Proposition 6.1, C_0 is an admissible observation operator for the system described by (6.1)–(6.4). Moreover, A_0 is clearly self-adjoint, positive, and boundedly invertible, whereas the resolvent of A_0 is clearly compact. Consequently, we can apply Proposition 4.5.

The eigenvalues of $A_0^{\frac{1}{2}}$ are

$$\lambda_{m,n} = \sqrt{m^2+n^2} \quad \forall\, m,n \in \mathbb{N}^*.$$

A corresponding family of normalized (in $H = H_0^1(\Omega)$) eigenfunctions are

$$\phi_{m,n}(x) = \frac{2}{\pi\sqrt{m^2+n^2}} \sin(mx_1)\sin(nx_2) \quad \forall m,n \in \mathbb{N}^* \ \forall\, x=(x_1,x_2)\in\Omega. \tag{6.5}$$

In the remaining part of the proof, we show that the operators A_0 and C_0 satisfy condition (iii) in Proposition 4.5. More precisely, we prove that there exists $\varepsilon, \delta > 0$ such that for all $\omega > 0$ and for all $\varphi = \sum_{(m,n)\in I_\varepsilon(\omega)} a_{m,n}\phi_{m,n}$, we have

$$\|C_0\varphi\|_Y^2 = \int_\Gamma \left| \sum_{(m,n)\in I_\varepsilon(\omega)} a_{m,n} \frac{\partial \phi_{m,n}}{\partial \nu} \right|^2 \mathrm{d}\Gamma \geqslant \delta \left(\sum_{(m,n)\in I_\varepsilon(\omega)} |a_{m,n}|^2 \right), \tag{6.6}$$

where

$$I_\varepsilon(\omega) = \{(m,n)\in\mathbb{N}^*\times\mathbb{N}^* \ ; \ |\lambda_{m,n}-\omega| < \varepsilon\}.$$

Let us introduce some notation. We first set

$$K_\varepsilon(\omega) = \left\{ m\in\mathbb{N}^* \mid \exists n\in\mathbb{N}^* \text{ with } (m,n)\in I_\varepsilon(\omega)\right\}.$$

It is clear that if $m\in K_\varepsilon(\omega)$ then $m < \omega+\varepsilon$. For $m\in K_\varepsilon(\omega)$ we introduce the set $L(m)$ defined by

$$L(m) = \left\{n\in\mathbb{N}^* \mid (m,n)\in I_\varepsilon(\omega)\right\} = \left\{ n\in\mathbb{N}^* \;\middle|\; |\sqrt{m^2+n^2}-\omega| \leqslant \varepsilon \right\}. \tag{6.7}$$

Then, we have

$$L(m) = \left\{ n\in\mathbb{N}^* \mid \sqrt{(\omega-\varepsilon)^2-m^2} \leqslant n \leqslant \sqrt{(\omega+\varepsilon)^2-m^2} \right\} \tag{6.8}$$

if $m \leqslant \omega - \varepsilon$ and

$$L(m) = \left\{ n \in \mathbb{N}^* \mid n \leqslant \sqrt{(\omega+\varepsilon)^2 - m^2} \right\}$$

if $\omega - \varepsilon < m < \omega + \varepsilon$. By using (6.5) and the above notation we get that

$$\begin{aligned}\|C_0\varphi\|_Y^2 = {} & \frac{4}{\pi^2}\int_0^\pi \left| \sum_{m\in K_\varepsilon(\omega)} \left[\sum_{n \in L(m)} \frac{n a_{mn}}{\sqrt{m^2+n^2}} \right] \sin(m x_1) \right|^2 \mathrm{d}x_1 \\ & + \frac{4}{\pi^2}\int_0^\pi \left| \sum_{n\in K_\varepsilon(\omega)} \left[\sum_{m \in L(n)} \frac{m a_{mn}}{\sqrt{m^2+n^2}} \right] \sin(n x_2) \right|^2 \mathrm{d}x_2. \end{aligned} \tag{6.9}$$

We are going to prove that if $\varepsilon \in (0, \frac{1}{10})$ and if $m \in K_\varepsilon(\omega)$ satisfies $m < (\omega+\varepsilon)/\sqrt{2}$, then the cardinal κ_m of the set $L(m)$ defined in (6.7) satisfies $\kappa_m = 1$.
Assume that $\varepsilon \in (0, \frac{1}{10})$ and $m \in K_\varepsilon(\omega)$ satisfies $m < (\omega+\varepsilon)/\sqrt{2}$. We first remark that

$$(\omega+\varepsilon)/\sqrt{2} \leqslant \omega - \varepsilon. \tag{6.10}$$

Indeed, if the above inequality is not satisfied, then we get that $\omega < 9\varepsilon$, and consequently, $\omega + \varepsilon < 10\varepsilon < 1$. On the other hand, the fact that $m \in K_\varepsilon(\omega)$ implies that $m < \omega + \varepsilon < 1$, which is a contradiction.

We have thus shown that (6.10) holds. Consequently, $L(m)$ satisfies (6.8), and its cardinal κ_m satisfies

$$\begin{aligned}\kappa_m & \leqslant \sqrt{(\omega+\varepsilon)^2 - m^2} - \sqrt{(\omega-\varepsilon)^2 - m^2} + 1 \\ & = \frac{4\omega\varepsilon}{\sqrt{(\omega+\varepsilon)^2 - m^2} + \sqrt{(\omega-\varepsilon)^2 - m^2}} + 1. \end{aligned} \tag{6.11}$$

On the other hand, since $m \leqslant (\varepsilon+\omega)/\sqrt{2}$, we have that

$$(\omega+\varepsilon)^2 - m^2 > \frac{\omega^2}{2}$$

and therefore, by (6.11), we obtain that

$$\kappa_m \leqslant 4\sqrt{2}\varepsilon + 1.$$

Since $\varepsilon \in (0, \frac{1}{10})$, the above relation implies that $\kappa_m = 1$ for all $m \in K_\varepsilon(\omega)$ such that $m \leqslant (\varepsilon+\omega)/\sqrt{2}$. The unique element of $L(m)$ is then denoted by ℓ_m. This fact, combined to (6.9) and to the orthogonality of the family $(\sin(mx))_{m\geqslant 1}$ in $L^2(0,\pi)$, yields the existence of a constant $\delta > 0$ such that

$$\|C_0\varphi\|_Y^2 \geqslant \frac{2}{\pi} \sum_{\substack{m\in K_\varepsilon(\omega)\\ m\leqslant(\omega+\varepsilon)/\sqrt{2}}} \left|\frac{\ell_m a_{m\ell_m}}{\sqrt{m^2+\ell_m^2}}\right|^2 + \frac{2}{\pi} \sum_{\substack{n\in K_\varepsilon(\omega)\\ n\leqslant(\omega+\varepsilon)/\sqrt{2}}} \left|\frac{\ell_n a_{\ell_n n}}{\sqrt{\ell_n^2+n^2}}\right|^2.$$

The above relation and the fact that there exists $C > 0$ such that for $m \leqslant (\varepsilon+\omega)/\sqrt{2}$, we have

$$\frac{\ell_m}{\sqrt{m^2+\ell_m^2}} \geqslant C,$$

implies the existence of $\delta > 0$ such that

$$\|C_0\varphi\|_Y^2 \geqslant \delta \left(\sum_{\substack{m\in K_\varepsilon(\omega)\\ m\leqslant(\omega+\varepsilon)/\sqrt{2}}} \left|a_{m\ell_m}\right|^2 + \sum_{\substack{n\in K_\varepsilon(\omega)\\ n\leqslant(\omega+\varepsilon)/\sqrt{2}}} \left|a_{\ell_n n}\right|^2 \right). \tag{6.12}$$

Using the fact that for all $(m,n) \in I_\varepsilon(\omega)$, we have either $m \leqslant \dfrac{\varepsilon+\omega}{\sqrt{2}}$ or $n \leqslant \dfrac{\varepsilon+\omega}{\sqrt{2}}$, we obtain that

$$\sum_{(m,n)\in I_\varepsilon(\omega)} |a_{mn}|^2 \leqslant \sum_{\substack{m\in K_\varepsilon(\omega)\\ m\leqslant(\omega+\varepsilon)/\sqrt{2}}} \left|a_{m\ell_m}\right|^2 + \sum_{\substack{n\in K_\varepsilon(\omega)\\ n\leqslant(\omega+\varepsilon)/\sqrt{2}}} \left|a_{\ell_n n}\right|^2.$$

The desired inequality (6.6) follows then from the above relation, together with relation (6.12). $\square$

7. An Ingham–Beurling-type result and a theorem on shifted squares

The following result plays a central rôle in the proof of the observability results in Sections 3 and 5.

Proposition 7.1. *For $q,\ r \in \mathbb{N}^*$, we set*

$$\Lambda_{qr} = \{m \in \mathbb{N}^* \mid q^2 + r^2 - m^2 \in S\},$$

where S denotes the set of squares of positive integers. Then, for any nonempty interval I, there exists a constant $\delta > 0$, depending only on I, such that the inequality

$$\int_I \left| \sum_{n \in \Lambda_{qr}} a_n \, e^{inx} \right|^2 \mathrm{d}x \;\geqslant\; \delta \sum_{n \in \Lambda_{qr}} |a_n|^2,$$

holds for all sequence $(a_n) \subset l^2(\mathbb{C})$.

The main ingredients of the proof of the above result are a version of a famous theorem of Beurling [7] on nonharmonic Fourier series, and a number theoretic theorem concerning shifted squares.

Let us first state the version of Beurling's result given in Theorem 1.5 in [5]. For the proof, we refer to [5].

Theorem 7.2. *Let $(\lambda_n)_{n \in \mathbb{Z}}$ be a strictly increasing sequence of real numbers such that*

$$\lambda_{n+1} - \lambda_n \;\geqslant\; \gamma' \quad \forall\, n \in \mathbb{Z}$$

for some $\gamma' > 0$. Moreover, assume that exists $\gamma \geqslant \gamma'$ and $M \in \mathbb{N}^*$ such that

$$\lambda_{n+M} - \lambda_n \;\geqslant\; \gamma M \quad \forall\, n \in \mathbb{N}.$$

Then, for any interval I of length $l(I) > \dfrac{2\pi}{\gamma}$, there exists $\delta > 0$ depending only on γ, γ', M and I, such that

$$\int_I \left| \sum_{n \in \mathbb{Z}} a_n \, e^{i\lambda_n x} \right|^2 \mathrm{d}x \;\geqslant\; \delta \sum_{n \in \mathbb{Z}} |a_n|^2$$

holds for all sequence $(a_n) \subset l^2(\mathbb{C})$.

Note that, due to a misprint, the condition $l(I) > \dfrac{2\pi}{\gamma}$ in the above theorem has been written $l(I) > 2\pi\gamma$ in [5].

Remark 7.3. The result in Theorem 7.2 can be seen as a generalization of a classical inequality proved by Ingham [12]. For other generalizations and related questions we refer to Avdonin and Moran [1,3], Baiocchi et al. [4], Jaffard et al. [14] and Kahane [15].

Next, we give the second main ingredient of the proof of Proposition 7.1.

Theorem 7.4. *For positive integers M, N, V, let $Z = Z(M, N, V)$ denote the set of those integers n such that $M < n \leqslant M + N$ and $V - n^2$ is a square. For a suitable positive absolute constant $C \geqslant 1$, we have*

$$|Z| \leqslant C\sqrt{N \log(2N)},$$

where $|Z|$ denotes the cardinality of the set Z.

For the sake of clarity, we postpone the proof of this theorem to the end of this section.

A useful consequence of Theorem 7.4 is the following.

Corollary 7.5. *Let $M \in \mathbb{N}^*$ and $\lambda_0 < \lambda_1 < \cdots < \lambda_M$ be $M + 1$ consecutive elements of Λ_{qr}. Then, we have*

$$\lambda_M - \lambda_0 \geqslant \frac{M^2}{2C^2 \log(2M)}, \tag{7.1}$$

where C is the constant appearing in Theorem 7.4.

Proof. For $N \in \mathbb{N}^*$, denote by $U(N)$ the cardinal number of the set

$$\{j \geqslant 1 \mid \lambda_j \leqslant \lambda_0 + N\}.$$

By Theorem 7.4 we clearly have

$$U(N) \leqslant C\sqrt{N \log(2N)} \quad \forall N \in \mathbb{N}^*.$$

Consequently,

$$M = U(\lambda_M - \lambda_0) \leqslant C\sqrt{(\lambda_M - \lambda_0)\log[2(\lambda_M - \lambda_0)]}.$$

Now observe that, since $C \geqslant 1$, (7.1) plainly holds if $\lambda_M - \lambda_0 > M^2$. Otherwise we have

$$M^2 \leqslant C^2(\lambda_M - \lambda_0)\log(2M^2),$$

so that (7.1) is still valid. $\square$

We are now in a position to prove Proposition 7.1.

Proof of Proposition 7.1. Take $\gamma > \dfrac{2\pi}{l(I)}$, where $l(I)$ denotes the length of the interval I. Since $2\log(2M) \leqslant 3\sqrt{M}$ for all $M \geqslant 1$, Corollary 7.5 implies that if $M > 9C^4\gamma^2$ and if $\lambda_0 < \lambda_1 < \cdots < \lambda_M$ are $M+1$ consecutive elements of Λ_{qr}, then $\lambda_M - \lambda_0 \geqslant M\gamma$. Moreover, the distance between any two distinct elements of Λ_{qr} is at least one. Therefore, we can apply Theorem 7.2 to get the desired inequality. $\square$

In order to prove Theorem 7.4, we first introduce some notation.

For any prime number p, let $(\mathbb{Z}/p\mathbb{Z})^*$ be the (cyclic) multiplicative group of invertible residues modulo p and let Q_p denote the subset of $(\mathbb{Z}/p\mathbb{Z})^*$ comprising all nonzero quadratic residues. Recall that the Legendre symbol is the mapping from $\mathbb{Z}$ onto $\{-1, 0, 1\}$ defined by the formula

$$\left(\frac{n}{p}\right) := \begin{cases} 1 & \text{if } n \in Q_p \ (\mathrm{mod}\ p) \\ 0 & \text{if } p|n \\ -1 & \text{if } n \in (\mathbb{Z}/p\mathbb{Z})^* \setminus Q_p \ (\mathrm{mod}\ p) \end{cases} \qquad \forall\, n \in \mathbb{Z}.$$

A classical result states that, for all odd primes p and all integers n such that $p \nmid n$, we have

$$\left(\frac{n}{p}\right) \equiv n^{(p-1)/2} \ (\mathrm{mod}\ p). \tag{7.2}$$

This will be used in the proof of the following lemma. The result is known—see, for instance, [22, Exercise 3.3.20] or [11, Theorem 7.8.2]—but, for convenience of the reader, we provide a short proof.

Lemma 7.6. *For any odd prime p and all $a \in (\mathbb{Z}/p\mathbb{Z})^*$, we have*

$$\sum_{0 \leqslant n < p} \left(\frac{n^2 + a}{p}\right) = -1.$$

Proof. Denote the sum on the left by $S_p(a)$. By (7.2), we have

$$\begin{aligned} S_p(a) &\equiv \sum_{0 \leqslant n < p} (n^2 + a)^{(p-1)/2} \\ &\equiv \sum_{0 \leqslant n < p} \sum_{0 \leqslant j \leqslant (p-1)/2} \binom{(p-1)/2}{j} n^{2j} a^{(p-1)/2 - j} \\ &\equiv \sum_{0 \leqslant j \leqslant (p-1)/2} \binom{(p-1)/2}{j} a^{(p-1)/2 - j} \sum_{0 \leqslant n < p} n^{2j} \ (\mathrm{mod}\ p). \end{aligned}$$

Now observe that the inner sum is zero modulo p unless when $j = (p-1)/2$, in which case it is -1. This is a well-known consequence of the fact that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic and we omit the details. We thus obtain

$$S_p(a) \equiv -1 \pmod{p}.$$

Since $|S_p(a)| \leqslant p$, this leaves the two possibilities $S_p(a) = p - 1$ and $S_p(a) = -1$. However the former case can only happen if exactly one of the Legendre symbols is 0 while all others have value 1. If this holds, then we have, for some integer $h \in [0, p[$,

$$\left(\frac{h^2 + a}{p}\right) = 0.$$

Since $p \nmid a$, we must have $h \neq 0$. Thus $h \not\equiv p - h \pmod{p}$ and obviously

$$\left(\frac{(p-h)^2 + a}{p}\right) = 0,$$

a contradiction. Hence $S_p(a) = -1$, as required. $\square$

We can now embark on the proof of Theorem 7.4.

Proof of Theorem 7.4. Our initial strategy consists in showing that, for all primes p such that $p \equiv 3 \pmod{4}$, the subset E_p of $\mathbb{Z}/p\mathbb{Z}$ comprising those residue classes which contain at least one element of Z is small in size. We consider two cases, according to whether $p \mid V$ or not. To deal with the first instance, we observe that

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = -1,$$

so $-n^2$ is not a quadratic residue modulo p if $p \nmid n$. Thus $V - n^2$ can only be a square if it is divisible by p—and in fact by p^2. Therefore, we have

$$|E_p| = 1 \quad \text{if } p \mid V.$$

In the second case, we have

$$n \in E_p \Rightarrow \left(\frac{V - n^2}{p}\right) = 1 \text{ or } 0.$$

Since there are at most two solutions of the equation $V - n^2 \equiv 0 \pmod p$, we plainly derive

$$\begin{aligned} |E_p| &\leqslant 1 + \frac{1}{2} \sum_{0 \leqslant n < p} \left\{ 1 + \left(\frac{V - n^2}{p} \right) \right\} \\ &= \frac{1}{2}(p+2) + \frac{1}{2} \left(\frac{-1}{p} \right) \sum_{0 \leqslant n < p} \left(\frac{n^2 - V}{p} \right) \\ &= \frac{1}{2}(p+3), \end{aligned}$$

where, in the last stage, we have appealed to Lemma 7.6.

We have therefore shown that, for all primes $p \equiv 3 \pmod 4$, the set Z is excluded from $\frac{1}{2}(p-3)$ residue classes modulo p. By the large sieve (see e.g. [24, Corollary I.4.6.1]) this yields, for all $Q \geqslant 1$,

$$|Z| \leqslant (N + Q^2)/L \tag{7.3}$$

with

$$L = L(Q) := \sum_{1 \leqslant q \leqslant Q} g(q),$$

where

$$g(q) := \begin{cases} \mu(q)^2 \displaystyle\prod_{p \mid q} \frac{p-3}{p+3} & \text{if } p \mid q \Rightarrow p \equiv 3 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \qquad (\forall q \geqslant 1).$$

Here, as usual in number theory, the letter p denotes a generic prime number and $q \mapsto \mu(q)$ denotes the Möbius function.

It remains to evaluate L as a function of Q. To this end, we introduce the Dirichlet series associated to g, viz

$$G(s) := \sum_{q \geqslant 1} \frac{g(q)}{q^s} = \prod_{p \equiv 3 \pmod 4} \left(1 + \frac{p-3}{(p+3)p^s} \right),$$

where s is a complex parameter, with initially $\Re e s > 1$. We need to express this quantity in terms of the Riemann zeta function $\zeta(s)$. This can be achieved by introducing the unique nonprincipal character modulo 4, defined by $\chi(p) = (-1)^{(p-1)/2}$, and the corresponding L-function

$$L(s,\chi) := \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)^s} = \prod_{p\geqslant 3} \big(1-\chi(p)/p^s\big)^{-1}.$$

We have, still for $\Re e s > 1$,

$$\begin{aligned} G(s) &= \prod_{p\geqslant 3}\left(1+\frac{\{1-\chi(p)\}(p-3)}{2(p+3)p^s}\right) \\ &= \prod_{p\geqslant 3}(1-p^{-s})^{-1/2}(1-\chi(p)p^{-s})^{1/2}H(s) \\ &= \zeta(s)^{1/2}L(s,\chi)^{-1/2}(1-2^{-s})^{1/2}H(s) \end{aligned}$$

with

$$\begin{aligned} H(s) &:= \prod_{p\geqslant 3}(1-p^{-s})^{1/2}(1-\chi(p)p^{-s})^{-1/2}\left(1+\frac{\{1-\chi(p)\}(p-3)}{2(p+3)p^s}\right) \\ &= \prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(\frac{1-p^{-s}}{1+p^{-s}}\right)^{1/2}\left(1+\frac{p-3}{(p+3)p^s}\right) \\ &= \prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(1-p^{-2s}\right)^{-1/2}\left(1-\frac{6p^s+p-3}{(p+3)p^{2s}}\right). \end{aligned}$$

Since $L(s,\chi)^{-1/2}$ has analytic continuation in the region

$$\sigma \;\geqslant\; 1-c/\log(3+|\tau|) \quad (s=\sigma+i\tau)$$

for a suitable positive absolute constant c (see e.g. [24, notes on Sections II.8.2 and II.8.3]) and since the product $H(s)$ converges for $\sigma > \frac{1}{2}$ and is bounded in any half-plane $\sigma \geqslant \frac{1}{2}+\delta$ with $\delta > 0$, we are in a position to apply Selberg–Delange type estimates, as given in [24, Theorem III.5.3]. This yields

$$L(Q) = \sum_{q\leqslant Q} g(q) = \frac{AQ}{\sqrt{\log Q}}\left\{1+O\left(\frac{1}{\log Q}\right)\right\} \quad \forall\ Q\geqslant 2 \tag{7.4}$$

with

$$A := \frac{H(1)}{\sqrt{2\pi L(1,\chi)}} = \frac{\sqrt{2}}{\pi}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(1-p^{-2}\right)^{-1/2}\left(1-\frac{7p-3}{p^2(p+3)}\right).$$

Inserting (7.4) into (7.3) and selecting $Q := \sqrt{N}$ furnishes the bound

$$|Z| \leqslant B\sqrt{N\log N}\left\{1 + O\left(\frac{1}{\log N}\right)\right\} \quad \forall\, N \geqslant 2$$

with

$$B := \frac{\sqrt{2}}{A} = \pi \prod_{p\equiv 3\,(\mathrm{mod}\,4)} \left(1 - p^{-2}\right)^{1/2}\left(1 + \frac{7p-3}{(p-1)(p^2+4p-3)}\right) \approx 5.31259.$$

This finishes the proof of Theorem 7.4. $\square$

References

[1] S. Avdonin, W. Moran, Ingham-type inequalities and Riesz bases of divided differences, Internat. J. Appl. Math. Comput. Sci. 11 (2001) 803–820 Mathematical methods of optimization and control of large-scale systems (Ekaterinburg, 2000).

[2] S.-A. Avdonin, S.-A. Ivanov, Families of Exponentials—The Method of Moments in Controllability Problems for Distributed Parameter Systems, Cambridge University Press, Cambridge, 1995.

[3] S.A. Avdonin, W. Moran, Simultaneous control problems for systems of elastic strings and beams, Systems Control Lett. 44 (2001) 147–155.

[4] C. Baiocchi, V. Komornik, P. Loreti, Ingham type theorem and applications to control theory, Boll. Un. Mat. Ital. B 8 (1999) 33–63.

[5] C. Baiocchi, V. Komornik, P. Loreti, Ingham-Beurling type theorems with weakened gap conditions, Acta Math. Hungar. 97 (2002) 55–95.

[6] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control. Optim. 30 (1992) 1024–1065.

[7] A. Beurling, The collected works of Arne Beurling, in: L. Carleson, P. Malliavin, J. Neuberger, J. Wermer (Eds.), Harmonic Analysis, Vol. 2, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 1989.

[8] N. Burq, M. Zworski, Geometric control in the presence of a black box, J. Amer. Math. Soc. 17 (2004) 443–471 (electronic).

[9] S. Dolecki, D.L. Russell, A general theory of observation and control, SIAM J. Control Optim. 15 (1977) 185–220.

[10] M. Hautus, Controllability and observability conditions of linear autonomous systems, Nederl. Akad. Wetensch. Proc. Ser. A 72 (1969) 443–448.

[11] L.K. Hua, Introduction to Number Theory, Springer, Berlin, Heidelberg, NY, 1982.

[12] A. Ingham, Some trigonometrical inequalities with applications to the theory of series, Math. Z. 41 (1936) 367–379.

[13] S. Jaffard, Contrôle interne exact des vibrations d'une plaque rectangulaire, Portugal Math. 47 (1990) 423–429.

[14] S. Jaffard, M. Tucsnak, E. Zuazua, On a theorem of Ingham, J. Fourier Anal. Appl. 3 (1997) 577–582.

[15] J.-P. Kahane, Pseudo-périodicité et séries de Fourier lacunaires, Ann. Sci. École Norm. Sup. 79 (3) (1962) 93–150.

[16] V. Komornik, Exact Controllability and Stabilization—The Multiplier Method, Wiley and Masson, Chichester and Paris, 1994.

[17] G. Lebeau, Contrôle de l'équation de Schrödinger, J. Math. Pures Appl. 71 (9) (1992) 267–291.

[18] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Recherches en Mathématiques Appliquées, Vol. 8, Masson, Paris, 1988.
[19] K. Liu, Locally distributed control and damping for the conservative systems, SIAM J. Control Optim. 35 (1997) 1574–1590.
[20] K. Liu, Z. Liu, B. Rao, Exponential stability of an abstract nondissipative linear system, SIAM J. Control Optim. 40 (2001) 149–165.
[21] L. Miller, Controllability cost of conservative systems: resolvent condition and transmutation, J. Funct. Anal., 218 (2) (2005) 425–444.
[22] I. Niven, H.S. Zuckerman, H.L. Montgomery, An Introduction to the Theory of Numbers, fifth ed., Wiley, New York, 1991.
[23] D.L. Russell, G. Weiss, A general necessary condition for exact observability, SIAM J. Control Optim. 32 (1994) 1–23.
[24] G. Tenenbaum, Introduction to analytic and probabilistic number theory, Cambridge Studies in Advanced Mathematics, Vol. 46, Cambridge University Press, Cambridge, 1995.
[25] M. Tucsnak, G. Weiss, Simultaneous exact controllability and some applications, SIAM J. Control Optim. 38 (2000) 1408–1427.
[26] G. Weiss, Admissible observation operators for linear semigroups, Israel J. Math. 65 (1989) 17–43.