

# Exact controllability of a fluid–rigid body system

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## Abstract

This paper is devoted to the controllability of a 2D fluid–structure system. The fluid is viscous and incompressible and its motion is modelled by the Navier–Stokes equations whereas the structure is a rigid ball which satisfies Newton’s laws. We prove the local null controllability for the velocities of the fluid and of the rigid body and the exact controllability for the position of the rigid body. An important part of the proof relies on a new Carleman inequality for an auxiliary linear system coupling the Stokes equations with some ordinary differential equations.

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## Résumé

Cet article est consacré à la contrôlabilité d’un système fluide–structure bidimensionnel. Le fluide est visqueux et incompressible et son mouvement est modélisé par les équations de Navier–Stokes tandis que la structure est une boule rigide satisfaisant les lois de Newton. Nous démontrons la contrôlabilité locale à zéro pour les vitesses du fluide et du solide rigide et la contrôlabilité exacte pour la position du solide rigide. Une partie importante de la démonstration utilise une nouvelle inégalité de Carleman pour un système linéaire auxiliaire couplant les équations de Stokes avec des équations différentielles ordinaires.

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## 1. Introduction

Let  $\mathcal{C}$  be an open bounded set of  $\mathbb{R}^2$ , containing the open ball  $\mathcal{S}(t)$  of radius 1 moving into a viscous fluid which is occupying the domain  $\Omega(t) = \mathcal{C} \setminus \mathcal{S}(t)$ . Let  $\mathcal{O}$  be an open subset with  $\overline{\mathcal{O}} \subset \mathcal{C}$ . The fluid–rigid body system is controlled by a force field supported in  $\mathcal{O}$  and we suppose that  $\mathcal{O} \subset \Omega(t)$ . Then, the equations of motion of the fluid–structure system are:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu \Delta v + \nabla p + 1_{\mathcal{O}}u = 0, \quad t \in (0, T), \quad x \in \Omega(t), \quad (1.1)$$

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$$\operatorname{div} v = 0, \quad t \in (0, T), \quad x \in \Omega(t), \tag{1.2}$$

$$v(x, t) = 0, \quad x \in \partial\mathcal{C}, \quad t \in [0, T], \tag{1.3}$$

$$v(x, t) = h'(t) + \theta'(t)(x - h(t))^\perp, \quad x \in \partial\mathcal{S}(t), \quad t \in [0, T], \tag{1.4}$$

$$Mh''(t) = - \int_{\partial\mathcal{S}(t)} \sigma(v, p)n \, d\Gamma, \quad t \in (0, T), \tag{1.5}$$

$$J\theta''(t) = - \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \sigma(v, p)n \, d\Gamma, \quad t \in (0, T), \tag{1.6}$$

$$v(x, 0) = v^0(x), \quad x \in \Omega(0), \tag{1.7}$$

$$h(0) = h^0, \quad h'(0) = h^1, \quad \theta(0) = \theta^0, \quad \theta'(0) = \theta^1, \tag{1.8}$$

where

$$\sigma(v, p) = -p \operatorname{Id} + 2\nu D(v) \quad \text{and} \quad D(v)_{i,j} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

In the above system the unknowns are  $v(x, t)$  (the Eulerian velocity field of the fluid),  $p(x, t)$  (the pressure of the fluid),  $h(t)$  (the position of the center of the rigid ball) and  $\theta(t)$  (the angular of the rigid body). The function  $u(x, t)$  is the control of the system. The domain  $\mathcal{S}(t)$  is defined by:

$$\mathcal{S}(t) = B(h(t)),$$

where  $B(c) = \{x \in \mathbb{R}^2; |x - c| < 1\}$  denotes the open ball of  $\mathbb{R}^2$  centered in  $c \in \mathbb{R}^2$ . The constants  $M$  and  $J$  are the mass and the moment of inertia of the rigid body. For sake of simplicity, we assume that the rigid body is homogeneous and thus we have that

$$M = 2\pi\delta, \quad J = \delta \int_S |y|^2 \, dy,$$

where  $\delta > 0$  is the density of the rigid body. The positive constant  $\nu$  is the viscosity of the fluid.

For all  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ , we denote by  $x^\perp$  the vector  $x^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ . Moreover we denote by  $\partial\mathcal{S}(t)$  the boundary of the rigid body and by  $n(x, t)$  the unit normal to  $\partial\mathcal{S}(t)$  at the point  $x$  directed to the interior of the rigid body.

Assume that

$$\overline{\mathcal{S}(0)} \subset \mathcal{C} \setminus \overline{\mathcal{O}}, \tag{1.9}$$

then, for  $|h_T - h^0|$  small enough, we have that  $B(h_T) \subset \mathcal{C} \setminus \overline{\mathcal{O}}$ . Therefore, it is natural to wonder if with some control  $u$  we can have  $\mathcal{S}(T) = B(h_T)$ . In fact, we are going to look for a control such that the velocities of the fluid and of the rigid body are equal to 0 at time  $T$ . Our main result is the following:

**Theorem 1.1.** *Let  $T > 0$  and assume that (1.9) holds true. Suppose also that  $v^0 \in H^1(\Omega(0))$  and that*

$$\begin{cases} \operatorname{div} v^0 = 0 & \text{in } \Omega(0), \\ v^0(x) = h^1 + \theta^1(x - h^0)^\perp & (x \in \partial\mathcal{S}(0)), \\ v^0(x) = 0 & (x \in \partial\mathcal{C}). \end{cases}$$

Then there exists  $\varepsilon > 0$  such that if

$$\|v^0\|_{H^1(\Omega(0))} + |h^0 - h_T| + |h^1| + |\theta^0 - \theta_T| + |\theta^1| < \varepsilon,$$

then the system (1.1)–(1.8) is null controllable at time  $T$  in velocity and exactly controllable at time  $T$  for the position of the rigid body. More precisely, there exists  $u \in L^2(0, T; L^2(\mathcal{O}))$  such that

$$v(T) = 0, \quad h'(T) = 0, \quad \theta'(T) = 0,$$

and

$$h(T) = h_T, \quad \theta(T) = \theta_T.$$

We notice that this result is local in velocity and in position. Since we use the same method as in [10], it is natural to obtain, like in [10], a local result for the velocity. However, for the position of the rigid ball, we can improve the above theorem:

**Corollary 1.2.** *Let  $\theta^0, \theta_T \in \mathbb{R}$ . Assume that  $h^0, h_T \in \mathcal{C}$  are such that there exists a continuous curve  $\kappa : [0, 1] \rightarrow \mathcal{C}$  with the following property:*

$$\overline{B(\kappa(\lambda))} \subset \mathcal{C} \setminus \overline{\mathcal{O}} \quad (\lambda \in [0, 1]) \quad \text{and} \quad \kappa(0) = h_0, \quad \kappa(1) = h_T.$$

Then there exist  $\varepsilon > 0$  and  $T > 0$  such that if  $v^0 \in H^1(\Omega(0))$  and that

$$\begin{cases} \operatorname{div} v^0 = 0 & \text{in } \Omega(0), \\ v^0(x) = h^1 + \theta^1(x - h^0)^\perp & (x \in \partial\mathcal{S}(0)), \\ v^0(x) = 0 & (x \in \partial\mathcal{C}), \end{cases}$$

and if

$$\|v^0\|_{H^1(\Omega(0))} + |h^1| + |\theta^1| < \varepsilon,$$

then there exists  $u \in L^2(0, T; L^2(\mathcal{O}))$  such that the solution of (1.1)–(1.8) satisfies:

$$v(T) = 0, \quad h'(T) = 0, \quad \theta'(T) = 0$$

and

$$h(T) = h_T, \quad \theta(T) = \theta_T.$$

Since the proof of this corollary relies on some simple compactness arguments we will skip it.

As for many problems of interaction between a fluid and a structure, the main difficulties are that the system (1.1)–(1.8) is nonlinear, strongly coupled and that the domain of the fluid is an unknown function of the time. In particular we want to emphasize that we are going to control here a free boundary problem.

Several papers concerning the study of this kind of systems have been published in the last decade. More precisely, when the fluid is modelled by the Navier–Stokes equations, the following papers study the existence of solutions in the case of a bounded domain: [3,4,2,14,16,15,13,24,8,9,27] whereas [25,20,26,28,12] consider the case where the viscous fluid–rigid body system fills the whole space. The stationary problem was studied in [25] and in [11]. The asymptotic behavior of solutions when  $t \rightarrow \infty$  has been treated in some simplified models in [29] and in [21]. When the fluid is inviscid and modelled by the Euler equations, the existence of solutions was studied in [22].

Concerning the controllability results on fluid–structure interaction problem, there are very few articles in the literature. We want to mention a paper of Raymond and Vanninathan [23] about a simplified model where the fluid equations are replaced by the Helmholtz equations. In that case, the domain is supposed to be fixed but one of the difficulties comes from the fact that there is no control in the solid part. In a paper of Doubova and Fernández-Cara [5], there are also some control results for a 1D model. In that case, the domain is not fixed any more and the proof of the result is based on Carleman estimates. Our method is also based upon a Carleman inequality. However we treat the control of the position of the rigid body in a different way than in [5]. Finally, let us mention the paper of Boulakia and Osses [1] where the authors deal with the same problem except that they consider a body of arbitrary shape. They prove the local controllability of the system by using different methods (the Carleman estimates are obtained directly for a system where the domain of the fluid is not fixed). It should be noticed that with their method, an assumption on the smallness of the  $H^3$ -norm of the initial velocity of the fluid is needed. In our result, we only need to impose an assumption on the  $H^1$ -norm of the initial velocity of the fluid.

By translation and rotation we always can reduce the controllability problem to the case  $h_T = 0$  and  $\theta_T = 0$ . Therefore in all the sequence, we assume that  $h_T = 0$  and  $\theta_T = 0$ . Moreover we will denote:

$$\Omega = \Omega(T) \quad \text{and} \quad \mathcal{S} = \mathcal{S}(T),$$

and

$$Q = \Omega \times (0, T), \quad \Sigma = (\partial\Omega) \times [0, T], \quad \mathcal{C}_T = \mathcal{C} \times (0, T).$$

The proof of Theorem 1.1 relies mainly on a Carleman estimate for the following linear system:

$$\frac{\partial \tilde{w}}{\partial t} - \nu \Delta \tilde{w} + \nabla \tilde{q} = \tilde{f}, \quad \text{in } \Omega \times (0, T), \tag{1.10}$$

$$\operatorname{div} \tilde{w} = 0, \quad \text{in } \Omega \times (0, T), \tag{1.11}$$

$$\tilde{w}(y, t) = 0, \quad y \in \partial \mathcal{C}, \quad t \in [0, T], \tag{1.12}$$

$$\tilde{w}(y, t) = \tilde{g}(t) + \tilde{\omega}(t)y^\perp, \quad y \in \partial \mathcal{S}, \quad t \in [0, T], \tag{1.13}$$

$$M\tilde{g}'(t) = - \int_{\partial \mathcal{S}} \sigma(\tilde{w}, \tilde{q})n \, d\Gamma + \tilde{l}, \quad t \in (0, T), \tag{1.14}$$

$$J\tilde{\omega}'(t) = - \int_{\partial \mathcal{S}} y^\perp \cdot \sigma(\tilde{w}, \tilde{q})n \, d\Gamma + \tilde{k}, \quad t \in (0, T), \tag{1.15}$$

$$\tilde{w}(y, 0) = 0, \quad y \in \Omega, \quad \tilde{g}(0) = 0, \quad \tilde{\omega}(0) = 0. \tag{1.16}$$

The Carleman estimate we prove can be written under the following form:

**Theorem 1.3.** *Let  $T > 0$  and let  $\mathcal{O}$  be an open subset such that  $\overline{\mathcal{O}} \subset \Omega$ . Then there exists a constant  $C > 0$  such that all the smooth solutions of (1.10)–(1.15) satisfy the inequality:*

$$\begin{aligned} & \int_{\mathcal{O}} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} \, dy \, dt + \int_0^T \lambda^3 s^3 \tilde{\varphi}^3 (|\tilde{g}|^2 + |\tilde{\omega}|^2) e^{2s\tilde{\alpha}} \, dt \\ & \leq C \left( \int_{\mathcal{O} \times (0, T)} \lambda^5 s^{15/2} \hat{\varphi}^{15/2} |\tilde{w}|^2 e^{4s\hat{\alpha} - 2s\tilde{\alpha}} \, dy \, dt \right. \\ & \quad \left. + \int_{\mathcal{O}} \lambda^5 s^{15/2} \hat{\varphi}^{15/2} |\tilde{f}|^2 e^{4s\hat{\alpha} - 2s\tilde{\alpha}} \, dy \, dt + \int_0^T \lambda^5 s^{15/2} \hat{\varphi}^{15/2} (|\tilde{l}|^2 + |\tilde{k}|^2) e^{4s\hat{\alpha} - 2s\tilde{\alpha}} \, dt \right), \end{aligned} \tag{1.17}$$

for all  $\lambda \geq \lambda^*$  and for all  $s \geq s^*(T^4 + T^8)$  and where the functions  $\alpha, \hat{\alpha}, \tilde{\alpha}, \varphi, \tilde{\varphi}$  and  $\hat{\varphi}$  are given by (3.1)–(3.8).

We first give, in Section 2 some preliminary results. In Section 3, we prove the Carleman inequality given in Theorem 1.3. Then, in Section 4, we first give a link between controllability properties and Carleman estimates and then prove the controllability of an auxiliary linear system associated to (1.1)–(1.8). Finally, Section 5 is devoted to the proof of Theorem 1.1.

## 2. Preliminaries

Let us denote

$$H^{1,2}(\mathcal{Q}) = \left\{ v : \mathcal{Q} \rightarrow \mathbb{R}^2; \frac{\partial v}{\partial t}, \frac{\partial^i v}{\partial x^i} \in L^2(\mathcal{Q}) \text{ if } i \in \mathbb{N}^2 \text{ and } |i| \leq 2 \right\}.$$

The linear system (1.10)–(1.16) is well posed in the following sense:

**Lemma 2.1.** *Suppose that  $\tilde{f} \in L^2(\mathcal{Q})$ ,  $\tilde{l}, \tilde{k} \in L^2(0, T)$  be given functions. Then, there exists a unique solution to problem (1.10)–(1.16),*

$$(\tilde{w}, \tilde{q}) \in H^{1,2}(\mathcal{Q}) \times L^2(0, T; H^1(\Omega)), \quad (\tilde{g}, \tilde{\omega}) \in H^1(0, T) \times H^1(0, T),$$

and it satisfies the estimate:

$$\begin{aligned} & \|(\tilde{w}, \tilde{q})\|_{H^{1,2}(\mathcal{Q}) \times L^2(0, T; H^1(\Omega))} + \|(\tilde{g}, \tilde{\omega})\|_{H^1(0, T) \times H^1(0, T)} \\ & \leq C (\|\tilde{f}\|_{L^2(\mathcal{Q})} + \|\tilde{l}\|_{L^2(0, T)} + \|\tilde{k}\|_{L^2(0, T)}). \end{aligned} \tag{2.1}$$

The proof of this lemma is proved in [28] and is based on the semi-group theory. More precisely, we can define the following Hilbert spaces:

$$\mathbb{H} = \{v \in L^2(\mathcal{C}); \operatorname{div} v = 0 \text{ in } \mathcal{C}, D(v) = 0 \text{ in } \mathcal{S}, v \cdot n = 0 \text{ on } \partial\mathcal{C}\}, \tag{2.2}$$

$$\mathbb{V} = H_0^1(\mathcal{C}) \cap \mathbb{H}. \tag{2.3}$$

We know that for any  $v \in \mathbb{H}$ , there exist  $l_v \in \mathbb{R}^2$  and  $k_v \in \mathbb{R}$  such that

$$v(y) = l_v + k_v y^\perp, \quad \forall y \in \mathcal{S}.$$

We also define an inner product in  $L^2(\mathcal{C})$  by:

$$(v, w) = \int_{\Omega} v \cdot w \, dy + \int_{\mathcal{S}} \delta v \cdot w \, dy$$

where  $\delta > 0$  is the density of the rigid body. The associated norm is equivalent to the usual norm of  $L^2(\mathcal{C})$  and if  $v$  and  $w \in \mathbb{H}$ , then we have that

$$(v, w) = \int_{\Omega} v \cdot w \, dy + M l_v \cdot l_w + J k_v k_w. \tag{2.4}$$

The above spaces are natural for our analysis. In fact, from (1.11)–(1.13), it can be easily checked that if we extend the solution  $\tilde{w}$  to  $\mathcal{S}$  by:

$$\tilde{w}(y, t) = \tilde{g}(t) + \tilde{\omega}(t) y^\perp \quad (y \in \mathcal{S}, t \in [0, T]), \tag{2.5}$$

then  $\tilde{w} \in \mathbb{H}$ .

*In all the sequel, the solution  $\tilde{w}$  will be extended as above.*

We will also extend  $\tilde{f}$  to  $\mathcal{S}$  by:

$$\tilde{f}(y, t) = \tilde{l}(t) + \tilde{k}(t) y^\perp \quad (y \in \mathcal{S}, t \in [0, T]). \tag{2.6}$$

In order to solve (1.10)–(1.16) we consider the following operators:

$$D(A) = \{v \in \mathbb{V}; v|_{\Omega} \in H^2(\Omega)\}, \tag{2.7}$$

$$Av = \begin{cases} v \Delta v & \text{in } \Omega, \\ -\frac{2v}{M} \int_{\partial\mathcal{S}} D(v)n \, d\Gamma - \left[ \frac{2v}{J} \int_{\partial\mathcal{S}} y^\perp \cdot D(v)n \, d\Gamma \right] y^\perp & \text{in } \mathcal{S}, \end{cases} \quad \forall v \in D(A), \tag{2.8}$$

and

$$\forall v \in D(A), \quad Av = \mathbb{P}Av, \tag{2.9}$$

where  $\mathbb{P}$  is the orthogonal projector from  $L^2(\mathcal{C})$  on  $\mathbb{H}$  and where, in the expression of  $\mathcal{A}u$ ,  $D(u)$  represents the trace of the restriction of  $D(u)$  to  $\Omega$ .

**Proposition 2.2.** *The operator  $A$  defined by (2.7)–(2.9) is self-adjoint and  $m$ -dissipative. Consequently  $A$  is the generator of a contraction semi-group in  $\mathbb{H}$ . Moreover, there exists a constant  $C > 0$  such that for any  $v \in D(A)$ , we have:*

$$\|v\|_{H^2(\Omega)} \leq C \|Av\|_{L^2(\mathcal{C})}. \tag{2.10}$$

The proof of Lemma 2.1 relies on the fact that if we extend  $\tilde{w}$  and  $\tilde{f}$  to  $\mathcal{C}$  by:

$$\tilde{w} = \tilde{g} + \tilde{\omega} \times y^\perp \quad \forall (y, t) \in \mathcal{S} \times (0, T)$$

and

$$\tilde{f} = \tilde{l} + \tilde{k} \times y^\perp \quad \forall (y, t) \in \mathcal{S} \times (0, T),$$

then the system (1.10)–(1.16) can be written under the form:

$$\begin{cases} (\tilde{w})' = A(\tilde{w}) + \mathbb{P}(\tilde{f}), \\ (\tilde{w})(0) = 0, \end{cases}$$

where  $\mathbb{P}(\tilde{f}) \in L^2(0, T; \mathbb{H})$ . Lemma 2.1 is thus a consequence of classical results on the semi-group theory.

### 3. The Carleman inequality

This section is devoted to the proof of Theorem 1.3. Let  $\mathcal{O}_0 \Subset \mathcal{O}$  be a nonempty open set and  $\{\partial\mathcal{S}\}_\varepsilon$  a neighborhood of  $\partial\mathcal{S}$  in  $\Omega$ . Then there exists a function  $\psi \in C^2(\overline{\Omega})$  such that

$$\psi(y) > 0 \quad \forall y \in \Omega, \quad \psi|_{\partial\Omega} = 0, \quad |\nabla\psi(y)| > 0 \quad \forall y \in \Omega \setminus \mathcal{O}_0, \tag{3.1}$$

$$\psi(y) = |y|^2 - 1 \quad \forall y \in \{\partial\mathcal{S}\}_\varepsilon. \tag{3.2}$$

We can extend  $\psi$  to  $\mathcal{S}$  by putting  $\psi(y) = 0$  for all  $y \in \mathcal{S}$ .

Let us consider a fixed number  $m > 4$  and let us denote by  $\|\cdot\|_\infty$  the  $L^\infty(\Omega)$ -norm. Then for  $\lambda > 0$  and  $s > 0$ , we define the following functions defined in  $\mathcal{C}_T$ :

$$\varphi(y, t) = \frac{e^{\lambda(\psi(y)+m\|\psi\|_\infty)}}{(t(T-t))^4}, \tag{3.3}$$

$$\tilde{\varphi}(t) = \min_{y \in \overline{\Omega}} \varphi(y, t) = \frac{e^{\lambda m \|\psi\|_\infty}}{(t(T-t))^4}, \tag{3.4}$$

$$\hat{\varphi}(t) = \max_{y \in \overline{\Omega}} \varphi(y, t) = \frac{e^{\lambda(m+1)\|\psi\|_\infty}}{(t(T-t))^4}, \tag{3.5}$$

$$\alpha(y, t) = \frac{e^{\lambda(\psi(y)+m\|\psi\|_\infty)} - e^{\frac{5}{4}m\lambda\|\psi\|_\infty}}{(t(T-t))^4}, \tag{3.6}$$

$$\tilde{\alpha}(t) = \min_{y \in \overline{\Omega}} \alpha(y, t) = \frac{e^{\lambda m \|\psi\|_\infty} - e^{\frac{5}{4}m\lambda\|\psi\|_\infty}}{(t(T-t))^4}, \tag{3.7}$$

$$\hat{\alpha}(t) = \max_{y \in \overline{\Omega}} \alpha(y, t) = \frac{e^{\lambda(m+1)\|\psi\|_\infty} - e^{\frac{5}{4}m\lambda\|\psi\|_\infty}}{(t(T-t))^4}, \tag{3.8}$$

$$\xi_1(t) = s\lambda\hat{\varphi}e^{s\hat{\alpha}}, \quad \xi_2(t) = s^{15/4}\hat{\varphi}^{15/4}e^{2s\hat{\alpha}-s\tilde{\alpha}}, \tag{3.9}$$

$$\xi_3 = (\xi_1\xi_2^{-1})^2 = \lambda^2s^{-11/2}\hat{\varphi}^{-11/2}e^{2s(\tilde{\alpha}-\hat{\alpha})}, \tag{3.10}$$

where  $(y, t) \in \mathcal{C}_T$ . We can notice that  $\tilde{\alpha} - \hat{\alpha} < 0$  (see (3.7) and (3.8)) and thus  $\xi_3 \leq C\lambda^2$ , with a constant  $C$  independent of  $s$  and  $\lambda$ .

At the first step we send  $\nabla\tilde{q}$  into the right-hand side of Eq. (1.10):

$$\frac{\partial\tilde{w}}{\partial t} - \nu\Delta\tilde{w} = \hat{f} = \tilde{f} + \nabla\tilde{q} \quad \text{in } Q. \tag{3.11}$$

We are going to obtain Carleman estimate of the system (1.10)–(1.16) by considering the above equation as a heat equation and by following the proof of [17]. Here, however, we have to deal with the boundary conditions (1.13).

For sake of simplicity, in the sequel of this section, we take  $\nu = 1$ .

We first make the change of variables:

$$\begin{aligned} w(y, t) &= \tilde{w}(y, t)e^{s\alpha}, & q(y, t) &= \tilde{q}(y, t)e^{s\alpha}, \\ g(t) &= \tilde{g}(t)e^{s\tilde{\alpha}}, & \omega(t) &= \tilde{\omega}(t)e^{s\tilde{\alpha}}. \end{aligned}$$

Eqs. (1.10)–(1.15) are transformed into the following system:

$$\begin{aligned} & \frac{\partial w}{\partial t} - \Delta w + 2s\lambda\varphi(\nabla w)(\nabla\psi) + s\lambda^2\varphi|\nabla\psi|^2w - s^2\lambda^2\varphi^2|\nabla\psi|^2w \\ & + s\lambda\varphi w\Delta\psi - s\frac{\partial\alpha}{\partial t}w = \hat{f}e^{s\alpha}, \quad \text{in } \Omega \times [0, T], \end{aligned} \tag{3.12}$$

$$w(y, t) = 0, \quad y \in \partial\mathcal{S}, \quad t \in [0, T], \tag{3.13}$$

$$w(y, t) = g(t) + \omega(t)y^\perp, \quad y \in \partial\mathcal{S}, \quad t \in [0, T], \tag{3.14}$$

$$Mg'(t) - Ms\tilde{\alpha}'(t)g(t) = - \int_{\partial\mathcal{S}} \sigma(\tilde{w}, \tilde{q})e^{s\tilde{\alpha}}n \, d\Gamma + \tilde{l}e^{s\tilde{\alpha}}, \quad t \in [0, T], \tag{3.15}$$

$$J\omega'(t) - Js\tilde{\alpha}'(t)\omega(t) = - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(\tilde{w}, \tilde{q})e^{s\tilde{\alpha}}n \, d\Gamma + \tilde{k}e^{s\tilde{\alpha}}, \quad t \in [0, T]. \tag{3.16}$$

We introduce the operators  $L_1, L_2$ , as follows:

$$L_1w = -\Delta w - \lambda^2s^2\varphi^2|\nabla\psi|^2w - s\frac{\partial\alpha}{\partial t}w, \tag{3.17}$$

$$L_2w = \frac{\partial w}{\partial t} + 2s\lambda\varphi(\nabla w)(\nabla\psi) + 2s\lambda^2\varphi|\nabla\psi|^2w. \tag{3.18}$$

It follows from (3.12), (3.17) and (3.18) that

$$L_1w + L_2w = f_s \quad \text{in } Q, \tag{3.19}$$

where

$$f_s = \hat{f}e^{s\alpha} - s\lambda\varphi w\Delta\psi + s\lambda^2\varphi|\nabla\psi|^2w. \tag{3.20}$$

Taking  $L^2$ -norm of both sides of (3.19), we obtain:

$$\|f_s\|_{L^2(Q)}^2 = \|L_1w\|_{L^2(Q)}^2 + \|L_2w\|_{L^2(Q)}^2 + 2(L_1w, L_2w)_{L^2(Q)}. \tag{3.21}$$

By (3.17) and (3.18), we have the following equality:

$$\begin{aligned} (L_1w, L_2w)_{L^2(Q)} &= \left( -\Delta w - \lambda^2s^2\varphi^2|\nabla\psi|^2w - s\frac{\partial\alpha}{\partial t}w, \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2w \right)_{L^2(Q)} \\ &\quad - \int_Q \left[ 2\lambda^3s^3\varphi^3|\nabla\psi|^2w + 2s^2\lambda\varphi\frac{\partial\alpha}{\partial t}w \right] \cdot [(\nabla w)(\nabla\psi)] \, dy \, dt \\ &\quad - \int_Q 2s\lambda\varphi(\Delta w) \cdot [(\nabla w)(\nabla\psi)] \, dy \, dt. \end{aligned} \tag{3.22}$$

Denote by  $A_0$  the first term in the right-hand side of (3.22), i.e.,

$$A_0 = \left( -\Delta w - \lambda^2s^2\varphi^2|\nabla\psi|^2w - s\frac{\partial\alpha}{\partial t}w, \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2w \right)_{L^2(Q)}.$$

Then, by integrating by parts, we obtain that

$$\begin{aligned} A_0 &= \int_Q \left( \frac{1}{2} \frac{\partial}{\partial t} |\nabla w|^2 - \frac{\lambda^2s^2\varphi^2}{2} |\nabla\psi|^2 \frac{\partial}{\partial t} |w|^2 - \frac{s}{2} \frac{\partial\alpha}{\partial t} \frac{\partial}{\partial t} |w|^2 \right. \\ &\quad \left. - 2s^3\varphi^3\lambda^4 |\nabla\psi|^4 |w|^2 - 2s^2\lambda^2\varphi \frac{\partial\alpha}{\partial t} |\nabla\psi|^2 |w|^2 \right) dy \, dt \\ &\quad + \int_Q 2s\lambda^2\varphi |\nabla\psi|^2 |\nabla w|^2 - s\lambda^2 |w|^2 \Delta(\varphi |\nabla\psi|^2) \, dy \, dt \\ &\quad + \int_\Sigma s\lambda^2 |w|^2 \frac{\partial}{\partial n} (\varphi |\nabla\psi|^2) \, d\Sigma - \int_\Sigma \left( \frac{\partial w}{\partial t} + 2s\lambda^2\varphi |\nabla\psi|^2w \right) \cdot \frac{\partial w}{\partial n} \, d\Sigma \end{aligned}$$

and therefore,

$$\begin{aligned}
 A_0 = & \int_Q \left( \frac{\lambda^2 s^2 |\nabla \psi|^2}{2} \frac{\partial}{\partial t} (\varphi^2) |w|^2 + \frac{s|w|^2}{2} \frac{\partial^2 \alpha}{\partial t^2} \right. \\
 & \left. - 2s^3 \varphi^3 \lambda^4 |\nabla \psi|^4 |w|^2 - 2s^2 \lambda^2 \frac{\partial \alpha}{\partial t} \varphi |\nabla \psi|^2 |w|^2 \right) dy dt \\
 & + \int_Q (2s \lambda^2 \varphi |\nabla \psi|^2 |\nabla w|^2 - s \lambda^2 |w|^2 \Delta (\varphi |\nabla \psi|^2)) dy dt \\
 & + \int_{\Sigma} s \lambda^2 |w|^2 \frac{\partial}{\partial n} (\varphi |\nabla \psi|^2) d\Sigma - \int_{\Sigma} \left( \frac{\partial w}{\partial t} + 2s \lambda^2 \varphi |\nabla \psi|^2 w \right) \cdot \frac{\partial w}{\partial n} d\Sigma. \tag{3.23}
 \end{aligned}$$

Integrating by parts in the second term of the right-hand side of (3.22), we have:

$$\begin{aligned}
 A_1 = & - \int_Q \left[ 2\lambda^3 s^3 \varphi^3 |\nabla \psi|^2 w \cdot [(\nabla w)(\nabla \psi)] + 2s^2 \lambda \frac{\partial \alpha}{\partial t} \varphi w \cdot [(\nabla w)(\nabla \psi)] \right] dy dt \\
 = & - \int_Q \left( \lambda^3 s^3 \varphi^3 |\nabla \psi|^2 \nabla \psi \cdot \nabla |w|^2 + s^2 \varphi \lambda \frac{\partial \alpha}{\partial t} \nabla \psi \cdot \nabla |w|^2 \right) dy dt \\
 = & \int_Q \left[ 3\lambda^4 s^3 \varphi^3 |\nabla \psi|^4 |w|^2 + |w|^2 \varphi^3 \lambda^3 s^3 \operatorname{div}(|\nabla \psi|^2 \nabla \psi) + \operatorname{div} \left( s^2 \lambda \varphi \frac{\partial \alpha}{\partial t} \nabla \psi \right) |w|^2 \right] dy dt \\
 & - \int_{\Sigma} \left( \lambda^3 s^3 \varphi^3 |\nabla \psi|^2 + s^2 \frac{\partial \alpha}{\partial t} \varphi \lambda \right) \frac{\partial \psi}{\partial n} |w|^2 d\Sigma. \tag{3.24}
 \end{aligned}$$

Finally, integrating by parts the third term of the right-hand side of (3.22), we have:

$$\begin{aligned}
 A_2 = & - \int_Q 2s \lambda \varphi (\Delta w) \cdot [(\nabla w)(\nabla \psi)] dy dt \\
 = & \int_Q \left( 2s \lambda^2 \varphi |(\nabla w)(\nabla \psi)|^2 + 2s \lambda \varphi \sum_{i,j,k=1}^2 \frac{\partial w_j}{\partial y_i} \frac{\partial w_j}{\partial y_k} \frac{\partial^2 \psi}{\partial y_i \partial y_k} - s \lambda^2 \varphi |\nabla \psi|^2 |\nabla w|^2 - |\nabla w|^2 s \lambda \varphi \Delta \psi \right) dy dt \\
 & + \int_{\Sigma} \left( 2s \lambda \varphi |\nabla \psi| \left| \frac{\partial w}{\partial n} \right|^2 - s \lambda \varphi |\nabla \psi| |\nabla w|^2 \right) d\Sigma. \tag{3.25}
 \end{aligned}$$

From (3.23)–(3.25), we obtain:

$$\begin{aligned}
 (L_1 w, L_2 w)_{L^2(Q)} = & \int_Q (\lambda^4 s^3 \varphi^3 |\nabla \psi|^4 |w|^2 + s \lambda^2 \varphi |\nabla \psi|^2 |\nabla w|^2 + 2s \lambda^2 \varphi |(\nabla w)(\nabla \psi)|^2) dy dt \\
 & + \int_{\Sigma} \left( 2s \lambda \varphi |\nabla \psi| \left| \frac{\partial w}{\partial n} \right|^2 - s \lambda \varphi |\nabla \psi| |\nabla w|^2 \right) d\Sigma \\
 & - \int_{\Sigma} \left( \lambda^3 s^3 \varphi^3 |\nabla \psi|^2 + s^2 \frac{\partial \alpha}{\partial t} \varphi \lambda \right) \frac{\partial \psi}{\partial n} |w|^2 d\Sigma \\
 & - \int_{\Sigma} \left( \frac{\partial w}{\partial t} + 2s \lambda^2 \varphi |\nabla \psi|^2 w \right) \cdot \frac{\partial w}{\partial n} d\Sigma + \int_{\Sigma} s \lambda^2 |w|^2 \frac{\partial}{\partial n} (\varphi |\nabla \psi|^2) d\Sigma + X_1, \tag{3.26}
 \end{aligned}$$

where we put,

$$\begin{aligned}
 X_1 = & \int_Q (-s\lambda^2|w|^2\Delta(\varphi|\nabla\psi|^2)) + \frac{\lambda^2s^2|\nabla\psi|^2}{2} \frac{\partial}{\partial t}(\varphi^2)|w|^2 + \frac{s|w|^2}{2} \frac{\partial^2\alpha}{\partial t^2} \\
 & + 2s\lambda\varphi \sum_{i,j,k=1}^2 \frac{\partial w_j}{\partial y_i} \frac{\partial w_j}{\partial y_k} \frac{\partial^2\psi}{\partial y_i\partial y_k} - s\lambda\varphi(\Delta\psi)|\nabla w|^2 - 2s^2\lambda^2 \frac{\partial\alpha}{\partial t} \varphi|\nabla\psi|^2|w|^2 \\
 & + |w|^2\varphi^3\lambda^3s^3 \operatorname{div}(|\nabla\psi|^2\nabla\psi) + \operatorname{div}\left(s^2\lambda\varphi \frac{\partial\alpha}{\partial t} \nabla\psi\right)|w|^2 \, dy \, dt.
 \end{aligned}$$

One can estimate  $X_1$  as

$$|X_1| \leq C_0 \int_Q ((s^3\lambda^3\varphi^3 + (T^{16} + T^{14})s\lambda^4\varphi^3 + T^7s^2\lambda^3\varphi^3)|w|^2 + s\lambda\varphi|\nabla w|^2) \, dy \, dt, \quad s > 0, \lambda \geq 1, \quad (3.27)$$

where the constant  $C_0$  is independent on  $s, \lambda$  and  $T$ .

Therefore, by using (3.26) and (3.27), we obtain the existence of two constants:

$$s_0(\Omega, \mathcal{O}_0) > 0 \quad \text{and} \quad \lambda_0(\Omega, \mathcal{O}_0) \geq 1,$$

such that

$$\begin{aligned}
 2(L_1w, L_2w)_{L^2(Q)} \geq & C_1 \int_Q (\lambda^4s^3\varphi^3|w|^2 + s\lambda^2\varphi|\nabla w|^2) \, dy \, dt \\
 & - C_2 \int_{\mathcal{O}_0 \times (0, T)} (\lambda^4s^3\varphi^3|w|^2 + s\lambda^2\varphi|\nabla w|^2) \, dy \, dt \\
 & + 2 \int_{\Sigma} \left( 2s\lambda\varphi|\nabla\psi| \left| \frac{\partial w}{\partial n} \right|^2 - s\lambda\varphi|\nabla\psi||\nabla w|^2 \right) \, d\Sigma \\
 & - 2 \int_{\Sigma} \left( \lambda^3s^3\varphi^3|\nabla\psi|^2 + s^2 \frac{\partial\alpha}{\partial t} \varphi\lambda \right) \frac{\partial\psi}{\partial n} |w|^2 \, d\Sigma \\
 & - 2 \int_{\Sigma} \left( \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2w \right) \cdot \frac{\partial w}{\partial n} \, d\Sigma \\
 & + 2 \int_{\Sigma} s\lambda^2|w|^2 \frac{\partial}{\partial n} (\varphi|\nabla\psi|^2) \, d\Sigma, \quad \forall s \geq s_0(T^7 + T^8), \forall \lambda \geq \lambda_0. \quad (3.28)
 \end{aligned}$$

Next we have to investigate the boundary integrals in (3.28) note that  $\Sigma = \Sigma_0 \cup \Sigma_1$ , where we have  $\Sigma_0 = [0, T] \times \partial\mathcal{C}$  and  $\Sigma_1 = [0, T] \times \partial\mathcal{S}$ . Thanks to the zero Dirichlet boundary conditions of the function  $w$  on  $\Sigma_0$ , we obtain:

$$\begin{aligned}
 & \int_{\Sigma_0} \left( 2s\lambda\varphi|\nabla\psi| \left| \frac{\partial w}{\partial n} \right|^2 - s\lambda\varphi|\nabla\psi||\nabla w|^2 \right) \, d\Sigma - \int_{\Sigma_0} \left( \lambda^3s^3\varphi^3|\nabla\psi|^2 + s^2 \frac{\partial\alpha}{\partial t} \varphi\lambda \right) \frac{\partial\psi}{\partial n} |w|^2 \, d\Sigma \\
 & - \int_{\Sigma_0} \left( \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2w \right) \cdot \frac{\partial w}{\partial n} \, d\Sigma + \int_{\Sigma_0} s\lambda^2|w|^2 \frac{\partial}{\partial n} (\varphi|\nabla\psi|^2) \, d\Sigma \\
 & = \int_{\Sigma_0} s\lambda\varphi|\nabla\psi| \left| \frac{\partial w}{\partial n} \right|^2 \, d\Sigma. \quad (3.29)
 \end{aligned}$$

Next we estimate the boundary integrals over  $\Sigma_1$ :

$$\begin{aligned}
 I_1 &= \int_{\Sigma_1} \left( 2s\lambda\varphi|\nabla\psi| \left| \frac{\partial w}{\partial n} \right|^2 - s\lambda\varphi|\nabla\psi||\nabla w|^2 \right) d\Sigma \\
 &\quad - \int_{\Sigma_1} \left( \lambda^3 s^3 \varphi^3 |\nabla\psi|^2 + s^2 \frac{\partial\alpha}{\partial t} \varphi\lambda \right) \frac{\partial\psi}{\partial n} |w|^2 d\Sigma \\
 &\quad - \int_{\Sigma_1} \left( \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2 w \right) \cdot \frac{\partial w}{\partial n} d\Sigma + \int_{\Sigma_1} s\lambda^2|w|^2 \frac{\partial}{\partial n} (\varphi|\nabla\psi|^2) d\Sigma.
 \end{aligned}$$

By using (3.14) and the fact that  $|\nabla\psi| \equiv 2$  on  $\partial\mathcal{S}$  we get that

$$\begin{aligned}
 I_1 &= \int_{\Sigma_1} \left( 4s\lambda\varphi \left| \frac{\partial w}{\partial n} \right|^2 - 2s\lambda\varphi |\nabla w|^2 \right) d\Sigma + 2 \int_{\Sigma_1} \left( 4\lambda^3 s^3 \varphi^3 + s^2 \frac{\partial\alpha}{\partial t} \varphi\lambda \right) |w|^2 d\Sigma \\
 &\quad - \int_{\Sigma_1} \left( g'(t) + \omega'(t)y^\perp + 8s\lambda^2\varphi w \right) \cdot \frac{\partial w}{\partial n} d\Sigma + \int_{\Sigma_1} s\lambda^2|w|^2 \frac{\partial}{\partial n} (\varphi|\nabla\psi|^2) d\Sigma.
 \end{aligned}$$

The above equation and (3.15)–(3.16) yield that for  $s \geq s_0(T^7 + T^8)$  and  $\lambda \geq \lambda_0$ ,

$$\begin{aligned}
 I_1 &\geq 2 \int_{\Sigma_1} \left( s\lambda\tilde{\varphi} \left| \frac{\partial w}{\partial n} \right|^2 - s\lambda\tilde{\varphi} \left| \frac{\partial w}{\partial \tau} \right|^2 \right) d\Sigma + 2 \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma \\
 &\quad - \int_{\Sigma_1} \left( s\tilde{\alpha}'g - \frac{1}{M} \int_{\partial\mathcal{S}} \sigma(\tilde{w}, \tilde{q}) e^{s\tilde{\alpha}} n d\Gamma + \frac{1}{M} \tilde{l} e^{s\tilde{\alpha}} + s\tilde{\alpha}'\omega y^\perp \right. \\
 &\quad \left. - J^{-1} \left( \int_{\partial\mathcal{S}} x^\perp \cdot \sigma(\tilde{w}, \tilde{q}) e^{s\tilde{\alpha}} n d\Gamma \right) y^\perp + J^{-1} \tilde{k} y^\perp e^{s\tilde{\alpha}} + 8s\lambda^2 \tilde{\varphi} w \right) \cdot \frac{\partial w}{\partial n} d\Sigma.
 \end{aligned}$$

Taking into account that  $\frac{\partial w}{\partial \tau}(y, t) = -\omega(t)y$  for  $y \in \partial\mathcal{S}$ , we obtain that

$$\begin{aligned}
 I_1 &\geq 2 \int_{\Sigma_1} s\lambda\tilde{\varphi} \left| \frac{\partial w}{\partial n} \right|^2 d\Sigma + 2 \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma \\
 &\quad - \int_{\Sigma_1} \left( s\tilde{\alpha}'g - \frac{1}{M} \int_{\partial\mathcal{S}} \sigma(\tilde{w}, \tilde{q}) e^{s\tilde{\alpha}} n d\Gamma + \frac{1}{M} \tilde{l} e^{s\tilde{\alpha}} + s\tilde{\alpha}'\omega y^\perp \right. \\
 &\quad \left. - J^{-1} \left( \int_{\partial\mathcal{S}} x^\perp \cdot \sigma(\tilde{w}, \tilde{q}) e^{s\tilde{\alpha}} n d\Gamma \right) y^\perp + J^{-1} \tilde{k} y^\perp e^{s\tilde{\alpha}} + 8s\lambda^2 \tilde{\varphi} w \right) \cdot \frac{\partial w}{\partial n} d\Sigma - 2 \int_{\Sigma_1} s\lambda\tilde{\varphi} |\omega(t)|^2 d\Sigma,
 \end{aligned}$$

and therefore,

$$\begin{aligned}
 I_1 &\geq \frac{3}{2} \int_{\Sigma_1} s\lambda\tilde{\varphi} \left| \frac{\partial w}{\partial n} \right|^2 d\Sigma + 2 \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma \\
 &\quad - C \int_{\Sigma_1} \left( sT|\tilde{\varphi}|^{5/4}|g| + \int_{\partial\mathcal{S}} (|\nabla w| + s\tilde{\varphi}\lambda|w|) d\Gamma + sT|\tilde{\varphi}|^{5/4}|w| + 8s\lambda^2\tilde{\varphi}|w| \right) \left| \frac{\partial w}{\partial n} \right| d\Sigma \\
 &\quad - C \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt - C \int_{\Sigma_1} |q|^2 d\Sigma - 2 \int_{\Sigma_1} s\lambda\tilde{\varphi} |\omega(t)|^2 d\Sigma.
 \end{aligned} \tag{3.30}$$

We have:

$$\int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma = \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 (g(t) + \omega(t)y^\perp)^2 d\Sigma = 2\pi \int_0^T \lambda^3 s^3 \tilde{\varphi}^3 (|g(t)|^2 + \omega^2(t)) dt. \tag{3.31}$$

Combining (3.30) and (3.31) we obtain that for  $\lambda \geq \lambda_0$  and  $s \geq s_0(T^7 + T^8)$ ,

$$I_1 \geq \int_{\Sigma_1} s\lambda\tilde{\varphi} \left| \frac{\partial w}{\partial n} \right|^2 d\Sigma + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma - C \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt - C \int_{\Sigma_1} |q|^2 d\Sigma. \tag{3.32}$$

Relations (3.28), (3.29) and (3.32) imply that for  $\lambda \geq \lambda_0$  and  $s \geq s_0(T^7 + T^8)$ ,

$$\begin{aligned} 2(L_1 w, L_2 w)_{L^2(Q)} &\geq C_1 \int_Q (\lambda^4 s^3 \varphi^3 |w|^2 + s\lambda^2 \varphi |\nabla w|^2) dy dt \\ &\quad - C_2 \int_{\mathcal{O}_0 \times (0, T)} (\lambda^4 s^3 \varphi^3 |w|^2 + s\lambda^2 \varphi |\nabla w|^2) dy dt \\ &\quad + C_3 \int_{\Sigma} s\lambda\varphi |\nabla \psi| \left| \frac{\partial w}{\partial n} \right|^2 d\Sigma + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma \\ &\quad - C \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt - C \int_{\Sigma_1} |q|^2 d\Sigma. \end{aligned}$$

The above relation and (3.21) yield that for  $\lambda \geq \lambda_0$  and  $s \geq s_0(T^7 + T^8)$ ,

$$\begin{aligned} &C_1 \int_Q (\lambda^4 s^3 \varphi^3 |w|^2 + s\lambda^2 \varphi |\nabla w|^2) dy dt + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma + \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q)}^2 \\ &\leq C_2 \int_{\mathcal{O}_0 \times (0, T)} (\lambda^4 s^3 \varphi^3 |w|^2 + s\lambda^2 \varphi |\nabla w|^2) dy dt + C \int_{\Sigma_1} |q|^2 d\Sigma + \|f_s\|_{L^2(Q)}^2 + C \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt. \end{aligned}$$

Taking into account the definitions (3.17), (3.18) and (3.20) of  $L_1 w$ ,  $L_2 w$  and of  $f_s$ , we obtain that for  $\lambda \geq \lambda_0$  and  $s \geq s_0(T^7 + T^8)$ ,

$$\begin{aligned} &\int_Q s^{-1} \varphi^{-1} \left( |\Delta w|^2 + \left| \frac{\partial w}{\partial t} \right|^2 \right) dy dt + \int_Q (\lambda^4 s^3 \varphi^3 |w|^2 + s\lambda^2 \varphi |\nabla w|^2) dy dt + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 d\Sigma \\ &\leq C \left( \int_{\mathcal{O}_0 \times (0, T)} (\lambda^4 s^3 \varphi^3 |w|^2 + s\lambda^2 \varphi |\nabla w|^2) dy dt + \|\tilde{f} e^{s\alpha}\|_{L^2(Q)}^2 \right. \\ &\quad \left. + \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt + \int_{\Sigma_1} |q|^2 d\Sigma + \|\nabla \tilde{q} e^{s\alpha}\|_{L^2(Q)}^2 \right). \tag{3.33} \end{aligned}$$

Now, we eliminate the integral on  $|\nabla w|^2$  of the right-hand side of the above equation. Let  $\mathcal{O}_1$  be a subdomain such that  $\mathcal{O}_0 \Subset \mathcal{O}_1 \Subset \mathcal{O}$  and let  $\chi$  be a function such that

$$\chi \in C_c^2(\mathcal{O}_1), \quad \chi \equiv 1 \text{ in } \mathcal{O}_0, \quad 0 \leq \chi \leq 1.$$

Then, some calculations give that

$$\begin{aligned} \int_{\mathcal{O}_0 \times (0, T)} s \lambda^2 \varphi |\nabla w|^2 \, dy \, dt &\leq \int_{\mathcal{O}_1 \times (0, T)} s \lambda^2 \chi \varphi |\nabla w|^2 \, dy \, dt \\ &\leq \varepsilon \int_Q s^{-1} \varphi^{-1} |\Delta w|^2 \, dy \, dt + C \int_{\mathcal{O}_1 \times (0, T)} \lambda^4 s^3 \varphi^3 |w|^2 \, dy \, dt, \end{aligned}$$

for a small enough constant  $\varepsilon > 0$  and for all  $\lambda \geq \lambda_0, s \geq s_0(T^7 + T^8)$ . The above equation and (3.33) imply that

$$\begin{aligned} &\int_Q s^{-1} \varphi^{-1} \left( |\Delta w|^2 + \left| \frac{\partial w}{\partial t} \right|^2 \right) \, dy \, dt + \int_Q (\lambda^4 s^3 \varphi^3 |w|^2 + s \lambda^2 \varphi |\nabla w|^2) \, dy \, dt + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |w|^2 \, d\Sigma \\ &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} \lambda^4 s^3 \varphi^3 |w|^2 \, dy \, dt + \|\tilde{f} e^{s\alpha}\|_{L^2(Q)}^2 + \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} \, dt + \int_{\Sigma_1} |q|^2 \, d\Sigma + \|\nabla \tilde{q} e^{s\alpha}\|_{L^2(Q)}^2 \right), \end{aligned}$$

for all  $\lambda \geq \lambda_0$  and for all  $s \geq s_0(T^7 + T^8)$ .

By using the definition of  $w$ , the above inequality implies (after some calculations),

$$\begin{aligned} &\int_Q s^{-1} \varphi^{-1} \left( |\Delta \tilde{w}|^2 + \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \right) e^{2s\alpha} \, dy \, dt + \int_Q (\lambda^4 s^3 \varphi^3 |\tilde{w}|^2 + s \lambda^2 \varphi |\nabla \tilde{w}|^2) e^{2s\alpha} \, dy \, dt + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |\tilde{w}|^2 e^{2s\tilde{\alpha}} \, d\Sigma \\ &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} \, dy \, dt + \|\tilde{f} e^{s\alpha}\|_{L^2(Q)}^2 \right. \\ &\quad \left. + \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} \, dt + \int_{\Sigma_1} |q|^2 \, d\Sigma + \|\nabla \tilde{q} e^{s\alpha}\|_{L^2(Q)}^2 \right), \end{aligned} \tag{3.34}$$

for all  $\lambda \geq \lambda_0$  and for all  $s \geq s_0(T^7 + T^8)$ .

In the sequel, we use the notation,

$$\begin{aligned} I(s, \lambda, \varphi) &= \int_Q s^{-1} \varphi^{-1} \left( |\Delta \tilde{w}|^2 + \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \right) e^{2s\alpha} \, dy \, dt \\ &\quad + \int_Q (\lambda^4 s^3 \varphi^3 |\tilde{w}|^2 + s \lambda^2 \varphi |\nabla \tilde{w}|^2) e^{2s\alpha} \, dy \, dt + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |\tilde{w}|^2 e^{2s\tilde{\alpha}} \, d\Sigma, \end{aligned} \tag{3.35}$$

so that (3.34) can be written under the following form:

$$\begin{aligned} I(s, \lambda, \varphi) &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} \, dy \, dt + \|\tilde{f} e^{s\alpha}\|_{L^2(Q)}^2 \right. \\ &\quad \left. + \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} \, dt + \int_{\Sigma_1} |\tilde{q}|^2 e^{2s\tilde{\alpha}} \, d\Sigma + \|\nabla \tilde{q} e^{s\alpha}\|_{L^2(Q)}^2 \right), \end{aligned} \tag{3.36}$$

for all  $\lambda \geq \lambda_0$  and for all  $s \geq s_0(T^7 + T^8)$ .

Now in order to get rid of the terms with the pressure in the right-hand side of (3.36) we need the following result proved in [18]. Consider the elliptic equation:

$$\Delta u = \operatorname{div} f \quad \text{in } \Omega, \tag{3.37}$$

$$u|_{\partial\Omega} = h. \tag{3.38}$$

Then, we have that

**Theorem 3.1.** *Suppose that  $f \in L^2(\Omega)^2$ ,  $h \in H^{\frac{1}{2}}(\partial\Omega)$ . Let  $\psi$  be a function as above and  $\eta = e^{\lambda\psi}$ . Then there exists a constant  $C$  independent of  $s$  and  $\lambda$  and parameters  $\hat{\lambda} > 1$  and  $\hat{s} > 1$  such that for all  $s > \hat{s}$  and  $\lambda > \hat{\lambda}$ ,*

$$\int_{\Omega} (|\nabla u|^2 + s^2\lambda^2\eta^2|u|^2)e^{2s\eta} \, dy \leq C \left( \int_{\mathcal{O}_1} (|\nabla u|^2 + s^2\lambda^2\eta^2|u|^2)e^{2s\eta} \, dy + s^{\frac{1}{2}}e^{2s} \|h\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + s \int_{\Omega} |f|^2\eta e^{2s\eta} \, dy \right). \tag{3.39}$$

Applying to Eq. (1.10) the operator  $\text{div}$ , we obtain:

$$\Delta \tilde{q} = \text{div} \tilde{f} \quad \text{for almost all } t \in [0, T], \, y \in \Omega.$$

Consequently, by using Theorem 3.1, we obtain that for almost all  $t \in (0, T)$ ,

$$\int_{\Omega} (|\nabla \tilde{q}|^2 + s^2\lambda^2\eta^2|\tilde{q}|^2)e^{2s\eta} \, dy \leq C \left( \int_{\mathcal{O}_1} (|\nabla \tilde{q}|^2 + s^2\lambda^2\eta^2|\tilde{q}|^2)e^{2s\eta} \, dy + s^{\frac{1}{2}}e^{2s} \|\tilde{q}\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + s \int_{\Omega} |\tilde{f}|^2\eta e^{2s\eta} \, dy \right). \tag{3.40}$$

Next in (3.40) we make the change of  $s \rightarrow \frac{se^{\frac{1}{2}\lambda m \|\psi\|_{\infty}}}{t^4(T-t)^4}$  and multiply both sides of the inequality by  $\exp(-2s\frac{e^{\frac{5}{4}\lambda m \|\psi\|_{\infty}}}{t(T-t)^4})$ . We then obtain that for  $s \geq s_0(T^7 + T^8)$ ,

$$\int_{\Omega} (|\nabla \tilde{q}|^2 + s^2\lambda^2\varphi^2|\tilde{q}|^2)e^{2s\alpha} \, dy \leq C \left( \frac{s^{\frac{1}{2}}e^{\frac{1}{2}\lambda m \|\psi\|_{\infty}}}{t^2(T-t)^2} e^{2s\tilde{\alpha}} \|\tilde{q}\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + s \int_{\Omega} |\tilde{f}|^2\varphi e^{2s\alpha} \, dy + \int_{\mathcal{O}_1} (|\nabla \tilde{q}|^2 + s^2\lambda^2\varphi^2|\tilde{q}|^2)e^{2s\alpha} \, dy \right). \tag{3.41}$$

Combining the estimates (3.36), (3.41) and the fact that

$$\int_{\Sigma_1} |\tilde{q}|^2 e^{2s\tilde{\alpha}} \, d\Sigma \leq \int_0^T \frac{s^{\frac{1}{2}}e^{\frac{1}{2}\lambda m \|\psi\|_{\infty}}}{t^2(T-t)^2} e^{2s\tilde{\alpha}} \|\tilde{q}\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \, dt$$

for  $s \geq s_0T^8$ , we obtain that

$$\begin{aligned} I(s, \lambda, \varphi) &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} \, dy \, dt + \|\tilde{f}e^{s\alpha}\|_{L^2(Q)}^2 + \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} \, dt \right. \\ &\quad + \int_{\mathcal{O}_1 \times (0, T)} (|\nabla \tilde{q}|^2 + s^2\lambda^2\varphi^2|\tilde{q}|^2) e^{2s\alpha} \, dy \, dt \\ &\quad \left. + \int_0^T \frac{s^{\frac{1}{2}}e^{\frac{1}{2}\lambda m \|\psi\|_{\infty}}}{t^2(T-t)^2} e^{2s\tilde{\alpha}} \|\tilde{q}\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \, dt + s \int_Q |\tilde{f}|^2 \varphi e^{2s\alpha} \, dy \, dt \right), \tag{3.42} \end{aligned}$$

for all  $\lambda \geq \lambda_0$  and for all  $s \geq s_0(T^7 + T^8)$ .

We set:

$$\mu(t) = \frac{s^{\frac{1}{4}}e^{\frac{1}{4}\lambda m \|\psi\|_{\infty}}}{t(T-t)} e^{s\tilde{\alpha}(t)},$$

and

$$w^*(y, t) = \mu(t)\tilde{w}(y, t), \quad q^*(y, t) = \mu(t)\tilde{q}(y, t), \quad g^*(t) = \mu(t)\tilde{g}(t), \quad \omega^*(t) = \mu(t)\tilde{\omega}(t).$$

Then,  $(w^*, q^*, g^*, \omega^*)$  verifies the system (1.10)–(1.16) with  $f^* = \mu\tilde{f} + \mu'\tilde{w}$ ,  $l^* = \mu\tilde{l} + M\mu'\tilde{g}$ ,  $k^* = \mu\tilde{k} + J\mu'\tilde{\omega}$ . By applying the estimate (2.1) to  $(w^*, q^*, g^*, \omega^*)$  and by using the trace theorem, we obtain that

$$\begin{aligned} & \left( \int_0^T \frac{s^{\frac{1}{2}} e^{\frac{1}{2}\lambda m \|\psi\|_\infty}}{t^2(T-t)^2} e^{2s\tilde{\alpha}} \|\tilde{q}\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 dt \right)^{\frac{1}{2}} \leq C \|(w^*, q^*)\|_{H^{1,2}(Q) \times L^2(0,T; H^1(\Omega))} \\ & \leq C(\|\mu\tilde{f}\|_{L^2(Q)} + \|\mu'\tilde{w}\|_{L^2(Q)} + \|\mu'\tilde{g}\|_{L^2(0,T)} + \|\mu'\tilde{\omega}\|_{L^2(0,T)} + \|\mu(|\tilde{l}| + |\tilde{k}|)\|_{L^2(0,T)}) \\ & \leq C(\|s^{\frac{1}{4}}\varphi^{\frac{1}{4}}\tilde{f}e^{s\alpha}\|_{L^2(Q)} + Ts^{\frac{5}{4}}\|\varphi^{\frac{3}{2}}\tilde{w}e^{s\alpha}\|_{L^2(Q)} + Ts^{\frac{5}{4}}\|\tilde{\varphi}^{\frac{3}{2}}\tilde{g}e^{s\tilde{\alpha}}\|_{L^2(0,T)} \\ & \quad + Ts^{\frac{5}{4}}\|\tilde{\varphi}^{\frac{3}{2}}\tilde{\omega}e^{s\tilde{\alpha}}\|_{L^2(0,T)} + \|s^{\frac{1}{4}}\tilde{\varphi}^{\frac{1}{4}}(|\tilde{l}| + |\tilde{k}|)e^{s\tilde{\alpha}}\|_{L^2(0,T)}), \quad \forall s \geq s_0(T^7 + T^8). \end{aligned} \tag{3.43}$$

Combining the above estimate and (3.42), we get:

$$\begin{aligned} I(s, \lambda, \varphi) \leq C & \left( \int_{\mathcal{O}_1 \times (0,T)} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} dy dt + \|\tilde{f}e^{s\alpha}\|_{L^2(Q)}^2 + \int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt \right. \\ & + \int_{\mathcal{O}_1 \times (0,T)} (|\nabla\tilde{q}|^2 + s^2\lambda^2\varphi^2|\tilde{q}|^2) e^{2s\alpha} dy dt \\ & + s \int_Q |\tilde{f}|^2 \varphi e^{2s\alpha} dy dt + \|s^{\frac{1}{4}}\varphi^{\frac{1}{4}}\tilde{f}e^{s\alpha}\|_{L^2(Q)}^2 + \|s^{\frac{1}{4}}\tilde{\varphi}^{\frac{1}{4}}(|\tilde{l}| + |\tilde{k}|)e^{s\tilde{\alpha}}\|_{L^2(0,T)}^2 \\ & \left. + T^2s^{\frac{5}{2}}\|\varphi^{\frac{3}{2}}\tilde{w}e^{s\alpha}\|_{L^2(Q)}^2 + T^2s^{\frac{5}{2}}\|\tilde{\varphi}^{\frac{3}{2}}\tilde{g}e^{s\tilde{\alpha}}\|_{L^2(0,T)}^2 + T^2s^{\frac{5}{2}}\|\tilde{\varphi}^{\frac{3}{2}}\tilde{\omega}e^{s\tilde{\alpha}}\|_{L^2(0,T)}^2 \right), \end{aligned} \tag{3.44}$$

for all  $\lambda \geq \lambda_0$  and for all  $s \geq s_0(T^7 + T^8)$ .

We can now absorb the last three terms in (3.44) by taking  $s \geq s_1T^4$ :

$$\begin{aligned} & T^2s^{\frac{5}{2}}\|\varphi^{\frac{3}{2}}\tilde{w}e^{s\alpha}\|_{L^2(Q)}^2 + T^2s^{\frac{5}{2}}\|\tilde{\varphi}^{\frac{3}{2}}\tilde{g}e^{s\tilde{\alpha}}\|_{L^2(0,T)}^2 + T^2s^{\frac{5}{2}}\|\tilde{\varphi}^{\frac{3}{2}}\tilde{\omega}e^{s\tilde{\alpha}}\|_{L^2(0,T)}^2 \\ & \leq \frac{1}{2} \left( \int_Q (\lambda^4 s^3 \varphi^3 |w|^2 dy dt + \int_{\Sigma_1} \lambda^3 s^3 \varphi^3 |w|^2 d\Sigma) \right). \end{aligned} \tag{3.45}$$

On the other hand, for all  $s \geq s_2T^8$ , we have that

$$\|\tilde{f}e^{s\alpha}\|_{L^2(Q)}^2 + \|s^{\frac{1}{4}}\varphi^{\frac{1}{4}}\tilde{f}e^{s\alpha}\|_{L^2(Q)}^2 \leq \frac{1}{2}s \int_Q |\tilde{f}|^2 \varphi e^{2s\alpha} dy dt \tag{3.46}$$

and

$$\int_0^T (|\tilde{l}|^2 + |\tilde{k}|^2) e^{2s\tilde{\alpha}} dt \leq \|s^{\frac{1}{4}}\tilde{\varphi}^{\frac{1}{4}}(|\tilde{l}| + |\tilde{k}|)e^{s\tilde{\alpha}}\|_{L^2(0,T)}^2 \leq \frac{1}{2}s \int_{\mathcal{C}_T} |\tilde{f}|^2 \varphi e^{2s\alpha} dy dt. \tag{3.47}$$

(Recall that we have extended  $\tilde{f}$  by the formula (2.6).)

Consequently, from (3.44)–(3.47), we get that

$$\begin{aligned} I(s, \lambda, \varphi) \leq C & \left( \int_{\mathcal{O}_1 \times (0,T)} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} dy dt + \int_{\mathcal{O}_1 \times (0,T)} (|\nabla\tilde{q}|^2 + s^2\lambda^2\varphi^2|\tilde{q}|^2) e^{2s\alpha} dy dt \right. \\ & \left. + s \int_{\mathcal{C}_T} |\tilde{f}|^2 \varphi e^{2s\alpha} dy dt \right), \end{aligned} \tag{3.48}$$

for all  $\lambda \geq \lambda_3$  and for all  $s \geq s_3(T^4 + T^8)$ .

Next we estimate the term containing the pressure in the right-hand side of (3.48). Since the pressure is defined up to a constant we may choose it in such a way that

$$\int_{\mathcal{O}_1} \tilde{q}(y, t) \, dy = 0 \quad \forall t \in [0, T].$$

Then by the Poincaré inequality,

$$\int_{\mathcal{O}_1 \times (0, T)} s^2 \lambda^2 \varphi^2 |\tilde{q}|^2 e^{2s\alpha} \, dy \, dt \leq C \int_{\mathcal{O}_1 \times (0, T)} s^2 \lambda^2 \hat{\varphi}^2 |\nabla \tilde{q}|^2 e^{2s\hat{\alpha}} \, dy \, dt.$$

Consequently, for all  $\lambda \geq \lambda_3$  and for all  $s \geq s_3(T^4 + T^8)$ , we have that

$$\int_{\mathcal{O}_1 \times (0, T)} (|\nabla \tilde{q}|^2 + s^2 \lambda^2 \varphi^2 |\tilde{q}|^2) e^{2s\alpha} \, dy \, dt \leq C \int_{\mathcal{O}_1 \times (0, T)} s^2 \lambda^2 \hat{\varphi}^2 |\nabla \tilde{q}|^2 e^{2s\hat{\alpha}} \, dy \, dt$$

and by using Eq. (1.10), we obtain that

$$\begin{aligned} \int_{\mathcal{O}_1 \times (0, T)} (|\nabla \tilde{q}|^2 + s^2 \lambda^2 \varphi^2 |\tilde{q}|^2) e^{2s\alpha} \, dy \, dt &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} s^2 \lambda^2 \hat{\varphi}^2 |\tilde{f}|^2 e^{2s\hat{\alpha}} \, dy \, dt \right. \\ &\left. + \int_{\mathcal{O}_1 \times (0, T)} s^2 \lambda^2 \hat{\varphi}^2 |\Delta \tilde{w}|^2 e^{2s\hat{\alpha}} \, dy \, dt + \int_{\mathcal{O}_1 \times (0, T)} s^2 \lambda^2 \hat{\varphi}^2 \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 e^{2s\hat{\alpha}} \, dy \, dt \right). \end{aligned} \tag{3.49}$$

Combining (3.48) and (3.49) and using the notation (3.9), we get that

$$\begin{aligned} I(s, \lambda, \varphi) &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} \, dy \, dt + \int_{\mathcal{O}_1 \times (0, T)} \xi_1^2 |\tilde{f}|^2 \, dy \, dt \right. \\ &\left. + \int_{\mathcal{O}_1 \times (0, T)} \xi_1^2 |\Delta \tilde{w}|^2 \, dy \, dt + \int_{\mathcal{O}_1 \times (0, T)} \xi_1^2 \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \, dy \, dt + s \int_{\tilde{\mathcal{C}}_T} |\tilde{f}|^2 \varphi e^{2s\alpha} \, dy \, dt \right), \end{aligned} \tag{3.50}$$

for all  $\lambda \geq \lambda_3$  and for all  $s \geq s_3(T^4 + T^8)$ .

The sequel of this section is devoted to get rid of the local integral of  $|\Delta \tilde{w}|^2$  and  $|\frac{\partial \tilde{w}}{\partial t}|^2$ . First, we are going to estimate the local integral of  $|\Delta \tilde{w}|^2$ . We introduce an open set  $\mathcal{O}_2$  such that

$$\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}.$$

Then, following [10], we have that

$$\int_{\mathcal{O}_1 \times (0, T)} \xi_1^2 |\Delta \tilde{w}|^2 \, dy \, dt \leq C \left( \int_{\mathcal{O}_2 \times (0, T)} |\xi_1'|^2 |\tilde{w}|^2 \, dy \, dt + \int_{\mathcal{O}_2 \times (0, T)} \xi_2^2 (|\tilde{f}|^2 + |\tilde{w}|^2) \, dy \, dt \right). \tag{3.51}$$

Now, to estimate the local integral of  $|\frac{\partial \tilde{w}}{\partial t}|^2$  we proceed as in [10]: let us introduce the pairs  $(w^1, q^1, g^1, \omega^1)$  and  $(w^2, q^2, g^2, \omega^2)$  solutions to the following systems:

$$\left\{ \begin{array}{ll} \frac{\partial w^1}{\partial t} - \Delta w^1 + \nabla q^1 = \xi_2 \tilde{f} & \text{in } \Omega \times (0, T), \\ \operatorname{div} w^1 = 0 & \text{in } \Omega \times (0, T), \\ w^1 = 0 & \text{on } \partial\mathcal{C} \times [0, T], \\ w^1 = g^1 + \omega^1 y^\perp & \text{on } \partial\mathcal{S} \times [0, T], \\ M(g^1)' = - \int_{\partial\mathcal{S}} \sigma(w^1, q^1) n \, d\Gamma + \xi_2 \tilde{l} & \text{in } (0, T), \\ J(\omega^1)' = - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(w^1, q^1) n \, d\Gamma + \xi_2 \tilde{k} & \text{in } (0, T), \\ (w^1(0), g^1(0), \omega^1(0)) = 0, & \end{array} \right.$$

and

$$\left\{ \begin{array}{ll} \frac{\partial w^2}{\partial t} - \Delta w^2 + \nabla q^2 = \xi_2' \tilde{w} & \text{in } \Omega \times (0, T), \\ \operatorname{div} w^2 = 0 & \text{in } \Omega \times (0, T), \\ w^2 = 0 & \text{on } \partial\mathcal{C} \times [0, T], \\ w^2 = g^2 + \omega^2 y^\perp & \text{on } \partial\mathcal{S} \times [0, T], \\ M(g^2)' = - \int_{\partial\mathcal{S}} \sigma(w^2, q^2) n \, d\Gamma + M \xi_2' \tilde{g} & \text{in } (0, T), \\ J(\omega^2)' = - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(w^2, q^2) n \, d\Gamma + J \xi_2' \tilde{\omega} & \text{in } (0, T), \\ (w^2(0), g^2(0), \omega^2(0)) = 0. & \end{array} \right.$$

By uniqueness of solution to the system (1.10)–(1.16), we have:

$$\xi_2 \tilde{w} = w^1 + w^2, \quad \xi_2 \tilde{q} = q^1 + q^2, \quad \xi_2 \tilde{g} = g^1 + g^2, \quad \xi_2 \tilde{\omega} = \omega^1 + \omega^2.$$

Therefore, we get the following relations:

$$\begin{aligned} \xi_2 \frac{\partial \tilde{w}}{\partial t} &= -\xi_2' \tilde{w} + \frac{\partial w^1}{\partial t} + \frac{\partial w^2}{\partial t}, \\ \xi_1 \frac{\partial \tilde{w}}{\partial t} &= \xi_1 \xi_2^{-1} \left( -\xi_2' \tilde{w} + \frac{\partial w^1}{\partial t} + \frac{\partial w^2}{\partial t} \right), \end{aligned}$$

$$\left\| \xi_1 \frac{\partial \tilde{w}}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))} \leq \left\| \xi_1 \xi_2^{-1} \xi_2' \tilde{w} \right\|_{L^2(\mathcal{O}_1 \times (0, T))} + \left\| \xi_1 \xi_2^{-1} \frac{\partial w^1}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))} + \left\| \xi_1 \xi_2^{-1} \frac{\partial w^2}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))}. \tag{3.52}$$

Now, we are going to estimate the right-hand side of (3.52). First, we have that

$$\int_{\mathcal{O}_1 \times (0, T)} (\xi_1 \xi_2^{-1})^2 (\xi_2')^2 |\tilde{w}|^2 \, dy \, dt \leq C \int_{\mathcal{O}_1 \times (0, T)} \lambda^2 s^{\frac{9}{2}} T^2 \hat{\varphi}^{\frac{9}{2}} e^{2s\hat{\alpha}} |\tilde{w}|^2 \, dy \, dt, \tag{3.53}$$

for all  $\lambda \geq \lambda_4$  and for all  $s \geq s_4(T^7 + T^8)$ .

By using Lemma 2.1, we have that

$$\left\| \frac{\partial w^1}{\partial t} \right\|_{L^2(\mathcal{Q})} + \|w^1\|_{L^2(\mathcal{Q})} + \|\omega^1\|_{H^1(0, T)} + \|g^1\|_{H^1(0, T)} \leq C \left( \int_{\mathcal{C}_T} |\xi_2 \tilde{f}|^2 \, dy \, dt \right)^{\frac{1}{2}}. \tag{3.54}$$

Consequently, we can easily estimate the second term in the right-hand side of (3.52):

$$\left\| \xi_1 \xi_2^{-1} \frac{\partial w^1}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 \leq C \lambda^2 \int_{\mathcal{C}_T} |\xi_2 \tilde{f}|^2 \, dy \, dt, \tag{3.55}$$

for all  $\lambda \geq \lambda_4$  and for all  $s \geq s_4(T^7 + T^8)$ .

In order to treat the third term of (3.52), we first integrate by parts twice with respect to  $t$  and we obtain that

$$\left\| \xi_1 \xi_2^{-1} \frac{\partial w^2}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 = \frac{1}{2} \int_{\mathcal{O}_1 \times (0, T)} \xi_3'' |w^2|^2 \, dy \, dt - \int_{\mathcal{O}_1 \times (0, T)} \xi_3 \frac{\partial^2 w^2}{\partial t^2} \cdot w^2 \, dy \, dt. \tag{3.56}$$

By (3.52)–(3.54) we have that for all  $\lambda \geq \lambda_4$  and for all  $s \geq s_4(T^4 + T^8)$ ,

$$\left| \frac{1}{2} \int_{\mathcal{O}_1 \times (0, T)} \xi_3'' |w^2|^2 \, dy \, dt \right| \leq C \lambda^2 \left( \|\xi_2 \tilde{w}\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 + \int_{\mathcal{C}_T} |\xi_2 \tilde{f}|^2 \, dy \, dt \right), \tag{3.57}$$

and

$$\left| \int_{\mathcal{O}_1 \times (0, T)} \xi_3 \frac{\partial^2 w^2}{\partial t^2} w^2 \, dy \, dt \right| \leq C \left\| \xi_3 \frac{\partial^2 w^2}{\partial t^2} \right\|_{L^2(\mathcal{O}_1 \times (0, T))} \|w^2\|_{L^2(\mathcal{O}_1 \times (0, T))}. \tag{3.58}$$

We then set  $w^3 = \xi_3 \frac{\partial w^2}{\partial t}$ ,  $q^3 = \xi_3 \frac{\partial q^2}{\partial t}$ ,  $g^3 = \xi_3 \frac{\partial g^2}{\partial t}$ ,  $\omega^3 = \xi_3 \frac{\partial \omega^2}{\partial t}$ . The functions  $w^3, q^3, g^3, \omega^3$  satisfy the system:

$$\begin{cases} \frac{\partial w^3}{\partial t} - \Delta w^3 + \nabla q^3 = \xi_3 \frac{\partial(\xi_2' \tilde{w})}{\partial t} + \xi_3' \frac{\partial w^2}{\partial t} & \text{in } \Omega \times (0, T), \\ \operatorname{div} w^3 = 0 & \text{in } \Omega \times (0, T), \\ w^3 = 0 & \text{on } \partial \mathcal{C} \times [0, T], \\ w^3 = g^3 + \omega^3 y^\perp & \text{on } \partial \mathcal{S} \times [0, T], \\ M(g^3)' = - \int_{\partial \mathcal{S}} \sigma(w^3, q^3) n \, d\Gamma + M \xi_3'(g^2)' + M \xi_3(\xi_2' \tilde{g})' & \text{in } (0, T), \\ J(\omega^3)' = - \int_{\partial \mathcal{S}} y^\perp \cdot \sigma(w^3, q^3) n \, d\Gamma + J \xi_3'(\omega^2)' + J \xi_3(\xi_2' \tilde{\omega})' & \text{in } (0, T), \\ (w^3(0), g^3(0), \omega^3(0)) = 0. \end{cases}$$

Using the a priori estimate for the system (1.10)–(1.16), we obtain:

$$\begin{aligned} \left\| \frac{\partial w^3}{\partial t} \right\|_{L^2(Q)} &\leq C \left( \left\| \xi_3 \frac{\partial(\xi_2' \tilde{w})}{\partial t} + \xi_3' \frac{\partial w^2}{\partial t} \right\|_{L^2(Q)} + \|\xi_3'(g^2)'\| + \xi_3(\xi_2' \tilde{g})' \right\|_{L^2(0, T)} \\ &\quad + \|\xi_3'(\omega^2)'\| + \xi_3(\xi_2' \tilde{\omega})' \Big\|_{L^2(0, T)} \end{aligned}$$

and thus that

$$\begin{aligned} \left\| \xi_3 \frac{\partial^2 w^2}{\partial t^2} \right\|_{L^2(Q)} &\leq C \left( \left\| \xi_3 \frac{\partial(\xi_2' \tilde{w})}{\partial t} \right\|_{L^2(Q)} + \left\| \xi_3' \frac{\partial w^2}{\partial t} \right\|_{L^2(Q)} + \|\xi_3'(g^2)'\| + \xi_3(\xi_2' \tilde{g})' \right\|_{L^2(0, T)} \\ &\quad + \|\xi_3'(\omega^2)'\| + \xi_3(\xi_2' \tilde{\omega})' \Big\|_{L^2(0, T)}. \end{aligned} \tag{3.59}$$

Combining the above relation with (3.54) and with,

$$|\xi_3' \xi_2'| + |\xi_3 \xi_2''| + |\xi_3'' \xi_2| \leq C s^{\frac{3}{4}} \tilde{\varphi}^{\frac{3}{4}} \lambda^2 e^{s\tilde{\alpha}}, \tag{3.60}$$

$$|\xi_3 \xi_2'| + |\xi_3' \xi_2| \leq C s^{-\frac{1}{2}} \tilde{\varphi}^{-\frac{1}{2}} \lambda^2 e^{s\tilde{\alpha}}, \tag{3.61}$$

we get that for all  $\lambda \geq \lambda_6$  and for all  $s \geq s_6(T^4 + T^8)$ ,

$$\begin{aligned} \left\| \xi_3 \frac{\partial^2 w^2}{\partial t^2} \right\|_{L^2(Q)}^2 &\leq C \left( \int_0^T s^{\frac{3}{2}} \tilde{\varphi}^{\frac{3}{2}} \lambda^4 e^{2s\tilde{\alpha}} \left( \int_{\Omega} |\tilde{w}|^2 \, dy + |\tilde{\omega}|^2 + |\tilde{g}|^2 \right) dt \right. \\ &\quad \left. + \int_0^T s^{-1} \hat{\varphi}^{-1} \lambda^4 e^{2s\tilde{\alpha}} \left( \int_{\Omega} \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \, dy + |\tilde{\omega}'|^2 + |\tilde{g}'|^2 \right) dt + \lambda^4 \int_{C_T} |\xi_2 \tilde{f}|^2 \, dy \, dt \right). \end{aligned}$$

Combining the above relation and the estimates (3.56)–(3.58), we obtain:

$$\begin{aligned} \left\| \xi_1 \xi_2^{-1} \frac{\partial w^2}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 &\leq C \left( \lambda^5 \|\xi_2 \tilde{w}\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 + \lambda^5 \int_{C_T} |\xi_2 \tilde{f}|^2 \, dy \, dt \right. \\ &\quad \left. + \int_0^T \lambda^{-1} s^{\frac{3}{2}} \tilde{\varphi}^{\frac{3}{2}} e^{2s\tilde{\alpha}} \left( \int_{\Omega} |\tilde{w}|^2 \, dy + |\tilde{\omega}|^2 + |\tilde{g}|^2 \right) dt \right. \\ &\quad \left. + \int_0^T \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\tilde{\alpha}} \left( \int_{\Omega} \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \, dy + |\tilde{\omega}'|^2 + |\tilde{g}'|^2 \right) dt \right), \end{aligned}$$

for all  $\lambda \geq \lambda_7$  and for all  $s \geq s_7(T^4 + T^8)$ .

By using (3.52), (3.53), (3.55) and the above equation, we obtain that

$$\begin{aligned} \left\| \xi_1 \frac{\partial \tilde{w}}{\partial t} \right\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 &\leq C \left( \int_{\mathcal{O}_1 \times (0, T)} \lambda^2 s^{\frac{9}{2}} \tilde{\varphi}^{\frac{9}{2}} e^{2s\tilde{\alpha}} |\tilde{w}|^2 \, dy \, dt + \lambda^5 \|\xi_2 \tilde{w}\|_{L^2(\mathcal{O}_1 \times (0, T))}^2 \right. \\ &\quad \left. + \lambda^5 \int_{C_T} |\xi_2 \tilde{f}|^2 \, dy \, dt + \int_0^T \lambda^{-1} s^{\frac{3}{2}} \tilde{\varphi}^{\frac{3}{2}} e^{2s\tilde{\alpha}} \left( \int_{\Omega} |\tilde{w}|^2 \, dy + |\tilde{\omega}|^2 + |\tilde{g}|^2 \right) dt \right. \\ &\quad \left. + \int_0^T \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\tilde{\alpha}} \left( \int_{\Omega} \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 + |\tilde{\omega}'|^2 + |\tilde{g}'|^2 \right) dt \right), \end{aligned} \tag{3.62}$$

for all  $\lambda \geq \lambda_8$  and for all  $s \geq s_8(T^4 + T^8)$ .

Combining (3.50), (3.51) and (3.62), we have:

$$\begin{aligned} I(s, \lambda, \varphi) &\leq C \left( \int_{\mathcal{O}_2 \times (0, T)} \lambda^5 s^{\frac{15}{2}} \tilde{\varphi}^{\frac{15}{2}} e^{4s\tilde{\alpha} - 2s\tilde{\alpha}} |\tilde{w}|^2 \, dy \, dt + \int_{C_T} \lambda^5 s^{\frac{15}{2}} \tilde{\varphi}^{\frac{15}{2}} |\tilde{f}|^2 e^{4s\tilde{\alpha} - 2s\tilde{\alpha}} \, dy \, dt \right. \\ &\quad \left. + \int_0^T \lambda^{-1} s^{\frac{3}{2}} \tilde{\varphi}^{\frac{3}{2}} e^{2s\tilde{\alpha}} \left( \int_{\Omega} |\tilde{w}|^2 \, dy + |\tilde{\omega}|^2 + |\tilde{g}|^2 \right) dt \right. \\ &\quad \left. + \int_0^T \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\tilde{\alpha}} \left( \int_{\Omega} \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 \, dy + |\tilde{\omega}'|^2 + |\tilde{g}'|^2 \right) dt \right), \end{aligned} \tag{3.63}$$

for all  $\lambda \geq \lambda_9$  and for all  $s \geq s_9(T^4 + T^8)$ .

By using Eqs. (1.14) and (1.15), we easily obtain that

$$\int_0^T \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\hat{\alpha}} (|\tilde{\omega}'|^2 + |\tilde{g}'|^2) dt \leq C \left( \int_{\Sigma_1} \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\hat{\alpha}} (|\tilde{q}|^2 + |\nabla \tilde{w}|^2) dt d\Gamma + \int_0^T \lambda^{-1} e^{2s\hat{\alpha}} (|\tilde{l}|^2 + |\tilde{k}|^2) dt \right),$$

for all  $\lambda \geq \lambda_9$  and for all  $s \geq s_9(T^4 + T^8)$ .

The above relation and Lemma 2.1 yield that

$$\int_0^T \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\hat{\alpha}} (|\tilde{\omega}'|^2 + |\tilde{g}'|^2) dt \leq C \lambda^{-1} \left( \int_Q s^{\frac{1}{2}} \varphi^{\frac{1}{2}} |\tilde{f}|^2 e^{2s\alpha} dy dt + \int_0^T s^{\frac{1}{2}} \tilde{\varphi}^{\frac{1}{2}} e^{2s\hat{\alpha}} (|\tilde{l}|^2 + |\tilde{k}|^2) dt + \int_0^T s^3 \left( \int_{\Omega} \varphi^3 |\tilde{w}|^2 e^{2s\alpha} dy + \tilde{\varphi}^3 |\tilde{g}|^2 e^{2s\hat{\alpha}} + \tilde{\varphi}^3 |\tilde{\omega}|^2 e^{2s\hat{\alpha}} \right) dt \right),$$

for all  $\lambda \geq \lambda_9$  and for all  $s \geq s_9(T^4 + T^8)$ .

Gathering the above relation and (3.63), we get that

$$\begin{aligned} I(s, \lambda, \varphi) \leq & C \left( \int_{\mathcal{O}_2 \times (0, T)} \lambda^5 s^{\frac{15}{2}} \hat{\varphi}^{\frac{15}{2}} |\tilde{w}|^2 e^{4s\hat{\alpha} - 2s\hat{\alpha}} dy dt + \int_{\mathcal{C}_T} \lambda^5 s^{\frac{15}{2}} \hat{\varphi}^{\frac{15}{2}} |\tilde{f}|^2 e^{4s\hat{\alpha} - 2s\hat{\alpha}} dy dt \right. \\ & + \int_0^T \lambda^{-1} s^{\frac{3}{2}} \tilde{\varphi}^{\frac{3}{2}} e^{2s\hat{\alpha}} \left( \int_{\Omega} |\tilde{w}|^2 dy + |\tilde{\omega}|^2 + |\tilde{g}|^2 \right) dt \\ & + \int_0^T s^3 \left( \int_{\Omega} \varphi^3 |\tilde{w}|^2 e^{2s\alpha} dy + \tilde{\varphi}^3 |\tilde{g}|^2 e^{2s\hat{\alpha}} + \tilde{\varphi}^3 |\tilde{\omega}|^2 e^{2s\hat{\alpha}} \right) dt \\ & \left. + \int_Q \lambda^{-1} s^{-1} \hat{\varphi}^{-1} e^{2s\hat{\alpha}} \left| \frac{\partial \tilde{w}}{\partial t} \right|^2 dy dt \right), \end{aligned} \tag{3.64}$$

for all  $\lambda \geq \lambda_9$  and for all  $s \geq s_9(T^4 + T^8)$ .

Consequently, by using the definition (3.35) of  $I(s, \lambda, \varphi)$  and the definitions (3.4)–(3.8), we obtain that

$$I(s, \lambda, \varphi) \leq C \left( \int_{\mathcal{O}_2 \times (0, T)} \lambda^5 s^{\frac{15}{2}} \hat{\varphi}^{\frac{15}{2}} |\tilde{w}|^2 e^{4s\hat{\alpha} - 2s\hat{\alpha}} dy dt + \int_{\mathcal{C}_T} \lambda^5 s^{\frac{15}{2}} \hat{\varphi}^{\frac{15}{2}} |\tilde{f}|^2 e^{4s\hat{\alpha} - 2s\hat{\alpha}} dy dt \right),$$

for all  $\lambda \geq \lambda_{10}$  and for all  $s \geq s_{10}(T^4 + T^8)$ , and thus that

$$\begin{aligned} & \int_Q \lambda^4 s^3 \varphi^3 |\tilde{w}|^2 e^{2s\alpha} dy dt + \int_{\Sigma_1} \lambda^3 s^3 \tilde{\varphi}^3 |\tilde{w}|^2 e^{2s\hat{\alpha}} d\Sigma \\ & \leq C \left( \int_{\mathcal{O}_2 \times (0, T)} \lambda^5 s^{\frac{15}{2}} \hat{\varphi}^{\frac{15}{2}} |\tilde{w}|^2 e^{4s\hat{\alpha} - 2s\hat{\alpha}} dy dt + \int_{\mathcal{C}_T} \lambda^5 s^{\frac{15}{2}} \hat{\varphi}^{\frac{15}{2}} |\tilde{f}|^2 e^{4s\hat{\alpha} - 2s\hat{\alpha}} dy dt \right). \end{aligned}$$

### 4. Some controllability results

#### 4.1. Null controllability of nonhomogeneous systems

In this subsection, we give some equivalence results between controllability properties and Carleman’s type inequalities. Let  $\mathbb{U}, \mathbb{H}, \mathbb{X}$  be Hilbert spaces and consider the initial value problem:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) + f(t), \\ \dot{a}(t) = Cz(t), \\ z(0) = z^0 \in \mathbb{H}, \\ a(0) = a^0 \in \mathbb{X}, \end{cases} \tag{4.1}$$

where  $A$  is the generator of a strongly continuous semi-group  $\mathbb{S}(t)$  on the space  $\mathbb{H}$ ,  $B \in \mathcal{L}(\mathbb{U}, \mathbb{H})$  is a control operator and  $C \in \mathcal{L}(\mathbb{H}, \mathbb{X})$  is a bounded operator. For all  $a_T \in \mathbb{X}$ , we want to find a control  $u$  such that  $z(T) = 0$  and  $a(T) = a_T$ . By a change of variable, we can assume that  $a_T = 0$ , so that we will only provide null controllability results for the system (4.1).

We also introduce the adjoint system of (4.1):

$$\begin{cases} -\dot{\zeta}(t) = A^*\zeta(t) + \gamma^1(t) + C^*\gamma^2, \\ \zeta(T) = 0. \end{cases} \tag{4.2}$$

Let us consider  $\rho_i : [0, T] \rightarrow \mathbb{R}$  ( $i \in \{1, 2, 3\}$ ) some continuous functions such that for all  $i \in \{1, 2, 3\}$ ,

$$\rho_i(T) = 0, \quad \rho_i > 0 \quad \text{in } [0, T).$$

The idea of the sequel is that if  $\frac{z}{\rho_2} \in C([0, T]; \mathbb{V})$  (where  $\mathbb{V}$  is a Hilbert space), then the above condition implies that  $z(T) = 0$ .

We introduce the following spaces:

$$\begin{aligned} \mathcal{F} &= \left\{ f \in L^2(0, T; \mathbb{H}); \frac{1}{\rho_1} f \in L^2(0, T; \mathbb{H}) \right\}, \\ \mathcal{Z} &= \left\{ z \in L^2(0, T; \mathbb{H}); \frac{1}{\rho_2} z \in L^2(0, T; \mathbb{H}) \right\}, \\ \mathcal{U} &= \left\{ u \in L^2(0, T; \mathbb{U}); \frac{1}{\rho_3} u \in L^2(0, T; \mathbb{U}) \right\}. \end{aligned}$$

**Theorem 4.1.** *Under the above assumptions, we have that the following statements are equivalent*

(i) *For all  $\gamma = (\gamma^1, \gamma^2) \in L^2(0, T; \mathbb{H}) \times \mathbb{X}$ , the solution of (4.2) satisfies:*

$$\|\gamma^2\|_{\mathbb{X}}^2 + \|\zeta(0)\|_{\mathbb{H}}^2 + \int_0^T \|\rho_1 \zeta\|_{\mathbb{H}}^2 dt \leq C \left( \int_0^T \|\rho_2 \gamma^1\|_{\mathbb{H}}^2 dt + \int_0^T \|\rho_3 B^* \zeta\|_{\mathbb{U}}^2 dt \right). \tag{4.3}$$

(ii) *For all  $(a^0, z^0, f) \in \mathbb{X} \times \mathbb{H} \times \mathcal{F}$ , there exists  $u \in \mathcal{U}$  such that  $z \in \mathcal{Z}$  and  $a(T) = 0$ .*

**Remark 4.2.** In particular, assume that (i) holds true; then for all  $(a^0, z^0, f) \in \mathbb{X}^2 \times \mathbb{H} \times \mathcal{F}$  there exist  $u \in L^2(0, T; \mathbb{U})$  such that  $z(T) = 0$  and  $a(T) = 0$ . But we have a stronger property:

**Corollary 4.3.** *Under the assumptions of Theorem 4.1, suppose that condition (i) of Theorem 4.1 holds true. Then there exists a linear bounded operator:*

$$E_T : \mathbb{X} \times \mathbb{H} \times \mathcal{F} \rightarrow \mathcal{U},$$

such that for any  $(a^0, z^0, f) \in \mathbb{X} \times \mathbb{H} \times \mathcal{F}$ , the control  $u = E_T((a^0, z^0, f))$  is such that  $z \in \mathcal{Z}$  and  $a(T) = 0$ .

Moreover assume that  $z^0 \in D((-A)^{\frac{1}{2}})$  and assume that there exists a function  $\rho \in C^1([0, T], \mathbb{R})$  such that

$$\rho \geq 0, \quad \rho(t) = 0 \iff (t = T),$$

$$\frac{\rho' \rho_2}{(\rho)^2} \in L^\infty(0, T) \quad \text{and} \quad \frac{\rho_i}{\rho} \in L^\infty(0, T) \quad (i = 1, 3).$$

Then we have that

$$\frac{z}{\rho} \in L^2(0, T; D(A)) \cap C([0, T]; D((-A)^{\frac{1}{2}})) \cap H^1(0, T; \mathbb{H}),$$

and

$$\left\| \frac{z}{\rho} \right\|_{L^2(0, T; D(A)) \cap C([0, T]; D((-A)^{\frac{1}{2}})) \cap H^1(0, T; \mathbb{H})} \leq C(\|f\|_{\mathcal{F}} + \|a^0\|_{\mathbb{X}} + \|z^0\|_{D((-A)^{\frac{1}{2}})}).$$

The proof of the corollary is quite classical so we omit its proof.

**Proof of Theorem 4.1.** First let us notice that for any  $(a^0, z^0, f) \in \mathbb{X} \times \mathbb{H} \times \mathcal{F}$  and for any  $u \in \mathcal{U}$ , the solution  $z$  and  $a$  of (4.1) can be written under the form:

$$\begin{cases} z(t) = \mathbb{S}(t)z^0 + \int_0^t \mathbb{S}(t-s)f(s) \, ds + \int_0^t \mathbb{S}(t-s)Bu(s) \, ds, \\ a(t) = C \int_0^t \left[ \mathbb{S}(s)z^0 + \int_0^s \mathbb{S}(s-\tau)f(\tau) \, d\tau + \int_0^s \mathbb{S}(s-\tau)Bu(\tau) \, d\tau \right] dt + a^0. \end{cases}$$

Therefore, if we consider the following applications:

$$L_T : \mathbb{X} \times \mathbb{H} \times \mathcal{F} \rightarrow L^2(0, T; \mathbb{H}) \times \mathbb{X},$$

$$\begin{pmatrix} a^0 \\ z^0 \\ f \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{S}(t)z^0 + \int_0^t \mathbb{S}(t-s)f(s) \, ds \\ C \int_0^T \left[ \mathbb{S}(t)z^0 + \int_0^t \mathbb{S}(t-s)f(s) \, ds \right] dt + a^0 \end{pmatrix},$$

$$M_T : \mathcal{Z} \times \mathcal{U} \rightarrow L^2(0, T; \mathbb{H}) \times \mathbb{X},$$

$$\begin{pmatrix} z \\ u \end{pmatrix} \mapsto \begin{pmatrix} z(t) - \int_0^t \mathbb{S}(t-s)Bu(s) \, ds \\ -C \int_0^T \int_0^t \mathbb{S}(t-s)Bu(s) \, ds \, dt \end{pmatrix}$$

then we see that condition (ii) is equivalent to

$$Range(L_T) \subset Range(M_T).$$

From [6], the above condition is equivalent to the existence of a constant  $C > 0$  such that

$$\|L_T^* \gamma\|_{\mathbb{X} \times \mathbb{H} \times \mathcal{F}'} \leq C \|M_T^* \gamma\|_{\mathcal{Z}' \times \mathcal{U}'} \quad \forall \gamma \in L^2(0, T; \mathbb{H}) \times \mathbb{X}, \tag{4.4}$$

where  $\mathcal{F}', \mathcal{Z}', \mathcal{U}'$  are the duals of  $\mathcal{F}, \mathcal{Z}, \mathcal{U}$  with respect to the pivot space  $L^2(0, T, \mathbb{H})$ . Thus we have obtained that (ii) is equivalent to (4.4) and is suffices to prove the equivalence between (4.4) and (i).

In order to establish this, we first compute  $L_T^*$  and  $M_T^*$ :

$$\begin{aligned} \langle L_T(a^0, z^0, f), \gamma \rangle_{L^2(0, T; \mathbb{H}) \times \mathbb{X}} &= \int_0^T \left( \mathbb{S}(t)z^0 + \int_0^t \mathbb{S}(t-s)f(s) ds, \gamma^1(t) \right)_{\mathbb{H}} dt \\ &\quad + \left( C \int_0^T \left[ \mathbb{S}(t)z^0 + \int_0^t \mathbb{S}(t-s)f(s) ds \right] dt + a^0, \gamma^2 \right)_{\mathbb{X}} \end{aligned}$$

thus,

$$L_T^*(\gamma) = \begin{pmatrix} \gamma^2 \\ \int_0^T \mathbb{S}^*(t)\gamma^1(t) dt + \int_0^T \mathbb{S}^*(t)C^*\gamma^2 dt \\ \int_s^T \mathbb{S}^*(t-s)\gamma^1(t) dt + \int_s^T \mathbb{S}^*(t-s)C^*\gamma^2 dt \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \langle M_T(z, u), \gamma \rangle_{L^2(0, T; \mathbb{H}) \times \mathbb{X}} &= \int_0^T \left( z(t) - \int_0^t \mathbb{S}(t-s)Bu(s) ds, \gamma^1(t) \right)_{\mathbb{H}} dt \\ &\quad - \left( C \left[ \int_0^T \int_0^t \mathbb{S}(t-s)Bu(s) ds dt \right], \gamma^2 \right)_{\mathbb{X}}, \end{aligned}$$

and therefore

$$M_T^*(\gamma) = \begin{pmatrix} \gamma^1 \\ \int_s^T B^*\mathbb{S}^*(t-s)\gamma^1(t) dt + \int_s^T B^*\mathbb{S}^*(t-s)C^*\gamma^2 dt \end{pmatrix}.$$

Let us note that

$$\zeta(s) = \int_s^T \mathbb{S}^*(t-s)(\gamma^1(t) + C^*\gamma^2) dt$$

is the solution of (4.2). Thus (4.4) can be written as (4.3) and the proof of Theorem 4.1 is complete.

#### 4.2. Controllability of an auxiliary linear system

Let us consider the following linear system:

$$\frac{\partial V}{\partial t} - \nu \Delta V + \nabla P + 1_{\mathcal{O}}u = F, \quad t \in (0, T), \quad y \in \Omega, \tag{4.5}$$

$$\operatorname{div} V = 0, \quad t \in (0, T), \quad y \in \Omega, \tag{4.6}$$

$$V(y, t) = 0, \quad y \in \partial\mathcal{C}, \quad t \in [0, T], \tag{4.7}$$

$$V(y, t) = h'(t) + \theta'(t)y^\perp, \quad y \in \partial\mathcal{S}, \quad t \in [0, T], \tag{4.8}$$

$$Mh''(t) = - \int_{\partial\mathcal{S}} \sigma(V, P)n d\Gamma, \quad t \in (0, T), \tag{4.9}$$

$$J\theta''(t) = - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(V, P)n d\Gamma, \quad t \in (0, T), \tag{4.10}$$

$$V(y, 0) = V^0(y), \quad y \in \Omega, \tag{4.11}$$

$$h(0) = h^0, \quad h'(0) = h^1, \quad \theta(0) = \theta^0, \quad \theta'(0) = \theta^1. \tag{4.12}$$

In order to prove the controllability result for the system (1.1)–(1.8), we are going to prove that the above system is controllable and then use a fixed point procedure.

First we fix the parameters  $s$  and  $\lambda$  in such a way that  $\lambda > \lambda^*$ ,  $s \geq s^*(T^4 + T^8)$  (where  $\lambda^*$  and  $s^*$  are as in Theorem 1.3) and we introduce the following notation:

$$\rho_1(t) = \begin{cases} \lambda^2 s^{\frac{3}{2}} \tilde{\varphi}(T/2)^{\frac{3}{2}} e^{s\tilde{\alpha}(T/2)} & \text{if } t \in (0, T/2), \\ \lambda^2 s^{\frac{3}{2}} \tilde{\varphi}^{\frac{3}{2}} e^{s\tilde{\alpha}} & \text{if } t \in (T/2, T); \end{cases} \tag{4.13}$$

$$\rho_2(t) = \begin{cases} \lambda^{\frac{5}{2}} s^{\frac{15}{4}} \hat{\varphi}(T/2)^{\frac{15}{4}} e^{2s\hat{\alpha}(T/2) - s\tilde{\alpha}(T/2)} & \text{if } t \in (0, T/2), \\ \lambda^{\frac{5}{2}} s^{\frac{15}{4}} \hat{\varphi}^{\frac{15}{4}} e^{2s\hat{\alpha} - s\tilde{\alpha}} & \text{if } t \in (T/2, T), \end{cases} \tag{4.14}$$

$$\rho_3 = \rho_2, \tag{4.15}$$

and

$$\rho(t) = \begin{cases} e^{\frac{7}{4}s\hat{\alpha}(T/2) - s\tilde{\alpha}(T/2)} & \text{if } t \in (0, T/2), \\ e^{\frac{7}{4}s\hat{\alpha} - s\tilde{\alpha}} & \text{if } t \in (T/2, T). \end{cases} \tag{4.16}$$

We also set

$$\mathcal{K} = \left\{ F \in L^2(Q); \int_Q \left( \frac{1}{\rho_1} \right)^2 |F|^2 \, dy \, dt < \infty \right\}. \tag{4.17}$$

**Theorem 4.4.** *Assume that  $V^0 \in H^1(\Omega)$  and that*

$$\begin{cases} \operatorname{div} V^0 = 0 & \text{in } \Omega, \\ V^0(y) = h^1 + \theta^1 y^\perp & (y \in \partial\mathcal{S}), \\ V^0(y) = 0 & (y \in \partial\mathcal{C}). \end{cases}$$

*There exists a linear bounded operator;*

$$E_T : \mathbb{R}^3 \times \mathbb{H} \times \mathcal{K} \rightarrow L^2(0, T; L^2(\mathcal{O})),$$

*such that for all  $F \in \mathcal{K}$ , if we put  $u = E_T(h^0, \theta^0, V^0, F)$  in (4.5)–(4.12) then the solution  $(V, h, \theta)$  of (4.5)–(4.12) is such that*

$$\begin{aligned} \frac{V}{\rho} &\in H^{1,2}(Q), & \frac{\nabla P}{\rho} &\in L^2(Q), \\ \frac{h'}{\rho}, \frac{\theta'}{\rho} &\in H^1(0, T) \quad \text{and} \quad h(T) = 0, \quad \theta(T) = 0. \end{aligned}$$

**Remark 4.5.** Notice that in the above theorem, we have extended  $V^0$  by the formula,

$$V^0(y) = h^1 + \theta^1 y^\perp \quad (y \in \mathcal{S}),$$

so that  $V^0 \in \mathbb{H}$  (see (2.2) and (2.5)) and so that the control  $u$  depends on  $h^1$  and  $\theta^1$ .

**Proof.** We first notice that the system (4.5)–(4.12) can be written under the form (4.1), where

- the space  $\mathbb{H}$  is defined by (2.2), and the spaces  $\mathbb{X}$  and  $\mathbb{U}$  are respectively  $\mathbb{R}^3$  and  $L^2(\mathcal{O})$ ,
- the unknowns  $z$  and  $a$  are respectively  $V$  and  $(h, \theta)$ , where  $V$  has been extended to  $\mathcal{S}$  by:

$$V(y, t) = h'(t) + \theta'(t)y^\perp \quad (y \in \mathcal{S}, t \in [0, T]),$$

- the operator  $A$  is defined by (2.7)–(2.9),

- the operator  $B$  is defined by:

$$Bu = \mathbb{P}(\mathbb{E}(u)),$$

where  $\mathbb{E}$  is the extension by 0 to  $\mathcal{C} \setminus \mathcal{O}$  and where  $\mathbb{P}$  is the orthogonal projection of  $L^2(\mathcal{C})$  onto  $\mathbb{H}$ ,

- the operator  $C$  is defined by:

$$Cv = (l_v, k_v), \quad \text{if } v = l_v + k_v y^\perp \text{ in } \mathcal{S}.$$

In particular, the equation  $a' = Cz$  corresponds to the equations  $h' = l_V$  and  $\theta' = k_V$  where

$$V(y, t) = l_V(t) + k_V(t)y^\perp \quad (y \in \mathcal{S}, t \in [0, T]),$$

and where the equation  $z' = Az + Bu + f$  corresponds to Eqs. (4.5)–(4.8) and to the equations

$$Ml'_V(t) = - \int_{\partial\mathcal{S}} \sigma(V, P)n \, d\Gamma, \quad t \in (0, T),$$

and

$$Jk'_V(t) = - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(V, P)n \, d\Gamma, \quad t \in (0, T).$$

From Section 4.1, in order to prove the controllability of (4.5)–(4.12) in the sense that it satisfies the assumption (ii) of Theorem 4.1, we have to show that the solutions of system (4.2) satisfy the estimate (4.3).

Assume that  $\zeta$  is the solution of (4.2) corresponding to  $\gamma$ . Since  $\zeta(t) \in \mathbb{H}$ , then there exist  $(l_\zeta, k_\zeta) \in \mathbb{R}^3$  such that

$$\zeta(t, y) = l_\zeta(t) + k_\zeta(t)y^\perp, \quad y \in \mathcal{S}, t \in [0, T].$$

According to Proposition 2.2, the operator  $A$  is self-adjoint and if  $\mathbb{R}^3$  is endowed with the scalar product

$$((l, k), (g, \omega))_{\mathbb{R}^3} = Ml \cdot g + Jk \omega, \quad (l, k), (g, \omega) \in \mathbb{R}^3,$$

then  $C^*$  is given by:

$$C^*(l, k) = \mathbb{P}(1_{\mathcal{S}}(l + ky^\perp)),$$

where  $\mathbb{P}$  is the orthogonal projection of  $L^2(\mathcal{C})$  onto  $\mathbb{H}$ .

Consequently, (4.2) can be written under the form:

$$\begin{aligned} -\frac{\partial \zeta}{\partial t} - \nu \Delta \zeta + \nabla \pi &= \gamma^1, \quad \text{in } \Omega \times (0, T), \\ \operatorname{div} \zeta &= 0, \quad \text{in } \Omega \times (0, T), \\ \zeta(y, t) &= 0, \quad y \in \partial\mathcal{C}, t \in [0, T], \\ \zeta(y, t) &= l_\zeta(t) + k_\zeta(t)y^\perp, \quad y \in \partial\mathcal{S}, t \in [0, T], \\ -Ml'_\zeta(t) &= - \int_{\partial\mathcal{S}} \sigma(\zeta, \pi)n \, d\Gamma + l_{\gamma^1} + l, \quad t \in (0, T), \\ -Jk'_\zeta(t) &= - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(\zeta, \pi)n \, d\Gamma + k_{\gamma^1} + k, \quad t \in (0, T), \\ \zeta(y, T) &= 0, \quad y \in \Omega, \quad l_\zeta(T) = 0, \quad k_\zeta(T) = 0, \end{aligned}$$

where  $\gamma^2 = (l, k) \in \mathbb{R}^3$  and where the function  $\gamma^1$  is defined in  $\mathcal{S}$  by,

$$\gamma^1(y, t) = l_{\gamma^1}(t) + k_{\gamma^1}(t)y^\perp \quad (y \in \mathcal{S}, t \in [0, T]). \tag{4.18}$$

Then, by denoting  $\tilde{w}(t) = \zeta(T - t)$ ,  $\tilde{g}(t) = l_\zeta(T - t)$ ,  $\tilde{\omega}(t) = k_\zeta(T - t)$ ,  $\tilde{f}(t) = \gamma^1(T - t)$ ,  $\tilde{l}(t) = l_{\gamma^1}(T - t) + l$  and  $\tilde{k}(t) = k_{\gamma^1}(T - t) + k$ , it is easy to check that  $(\tilde{w}, \tilde{g}, \tilde{\omega})$  and  $(\tilde{f}, \tilde{l}, \tilde{k})$  satisfy (1.10)–(1.16) for pressure  $\tilde{q} = \pi(T - t)$ .

Let us also denote  $\bar{\rho}_i(t) := \rho_i(T - t)$ , where  $\rho_i$  are defined by (4.13)–(4.15), and  $\bar{\gamma}^1(t) := \gamma^1(T - t)$ . Then, condition (4.3) is clearly satisfied if there exists a constant  $C > 0$  such that

$$|\gamma^2|^2 + \|\tilde{w}(T)\|_{\mathbb{H}}^2 + \int_0^T \|\bar{\rho}_1 \tilde{w}\|_{\mathbb{H}}^2 dt \leq C \left( \int_0^T \|\bar{\rho}_2 \bar{\gamma}^1\|_{\mathbb{H}}^2 dt + \int_0^T \int_{\mathcal{O}} |\bar{\rho}_3 \tilde{w}|^2 dy dt \right). \tag{4.19}$$

(Recall that in the above inequality, we extend  $\tilde{w}$  to  $\mathcal{S}$  by (2.5) and that  $\bar{\gamma}^1$  is defined in  $\mathcal{S}$  by (4.18).)

On the other hand, inequality (1.17) implies that

$$\int_0^T \|\rho_1^* \tilde{w}\|_{\mathbb{H}}^2 dt \leq C \left( \int_0^T \int_{\mathcal{O}} |\rho_3^* \tilde{w}|^2 dy dt + \int_0^T \|\rho_2^* (\bar{\gamma}^1 + C^* \gamma^2)\|_{\mathbb{H}}^2 dt \right), \tag{4.20}$$

where  $\rho_i^*$  are smooth functions related to  $\bar{\rho}_i$  by the relations

$$\rho_i^*(t) = \bar{\rho}_i(t) \quad \text{for } t \in \left[0, \frac{T}{2}\right] \quad \text{and} \quad \rho_i^*(t) = \bar{\rho}_i(T - t) \quad \text{for } t \in \left[\frac{T}{2}, T\right].$$

Then, acting as in [10], we can deduce from (4.20) that

$$\|\tilde{w}(T)\|_{\mathbb{H}}^2 + \int_0^T \|\bar{\rho}_1 \tilde{w}\|_{\mathbb{H}}^2 dt \leq C \left( \int_0^T \int_{\mathcal{O}} |\bar{\rho}_3 \tilde{w}|^2 dy dt + \int_0^T \|\bar{\rho}_2 (\bar{\gamma}^1 + C^* \gamma^2)\|_{\mathbb{H}}^2 dt \right). \tag{4.21}$$

Consequently, to prove that (4.19), it suffices to prove:

$$|\gamma^2|^2 \leq C \left( \int_0^T \int_{\mathcal{O}} |\bar{\rho}_3 \tilde{w}|^2 dy dt + \int_0^T \|\bar{\rho}_2 \bar{\gamma}^1\|_{\mathbb{H}}^2 dt \right). \tag{4.22}$$

To show the above inequality, we can use a contradiction argument: assume that (4.22) is false. Then there exists a sequence  $(\gamma_n^1, \gamma_n^2)$  such that

$$\int_0^T \int_{\mathcal{O}} |\bar{\rho}_3 \tilde{w}_n|^2 dy dt + \int_0^T \|\bar{\rho}_2 \bar{\gamma}_n^1\|_{\mathbb{H}}^2 dt \rightarrow 0 \tag{4.23}$$

and

$$|\gamma_n^2|^2 = 1. \tag{4.24}$$

From the above inequality, we can assume that for any  $\varepsilon > 0$ ,

$$\tilde{w}_n \rightharpoonup \tilde{w} \quad \text{weakly in } L^2(\varepsilon, T; D(A)) \cap H^1(\varepsilon, T; \mathbb{H}),$$

$$\bar{\gamma}_n^1 \rightharpoonup 0 \quad \text{weakly in } L^2(\varepsilon, T; \mathbb{H}),$$

and

$$\gamma_n^2 \rightarrow \gamma^2 \quad \text{in } \mathbb{X}$$

for some functions  $\tilde{w} \in L^2(\varepsilon, T; D(A))$  and  $\gamma^2 \in \mathbb{X}$  such that

$$|\gamma^2| = 1. \tag{4.25}$$

Therefore the function  $\tilde{w}$  satisfies for some pressure  $\tilde{q}$  the system:

$$\frac{\partial \tilde{w}}{\partial t} - \nu \Delta \tilde{w} + \nabla \tilde{q} = 0, \quad \text{in } \Omega \times (\varepsilon, T), \tag{4.26}$$

$$\operatorname{div} \tilde{w} = 0, \quad \text{in } \Omega \times (\varepsilon, T), \tag{4.27}$$

$$\tilde{w}(y, t) = 0, \quad y \in \partial\mathcal{C}, \quad t \in [\varepsilon, T], \tag{4.28}$$

$$\tilde{w}(y, t) = \tilde{g}(t) + \tilde{\omega}(t)y^\perp, \quad y \in \partial\mathcal{S}, \quad t \in [\varepsilon, T], \tag{4.29}$$

$$M\tilde{g}'(t) = - \int_{\partial\mathcal{S}} \sigma(\tilde{w}, \tilde{q})n \, d\Gamma + l, \quad t \in (\varepsilon, T), \tag{4.30}$$

$$J\tilde{\omega}'(t) = - \int_{\partial\mathcal{S}} y^\perp \cdot \sigma(\tilde{w}, \tilde{q})n \, d\Gamma + k, \quad t \in (\varepsilon, T), \tag{4.31}$$

where  $(l, k) = \gamma^2 \in \mathbb{R}^3$ .

From (4.23) we have moreover that

$$\tilde{w} = 0 \quad \text{in } \mathcal{O} \times (\varepsilon, T).$$

Thus, using [7], we get that

$$\tilde{w} = \nabla\tilde{q} = 0 \quad \text{in } \Omega \times (\varepsilon, T) \quad \text{and} \quad (\tilde{g}, \tilde{\omega}) = 0 \quad \text{in } (\varepsilon, T).$$

In particular, from (4.30) and (4.31), we have that  $\gamma^2 = (l, k) = 0$  which contradicts (4.25). This establishes the inequality (4.22) which, combined with (4.21), implies (4.19). We can thus apply Theorem 4.1 and Corollary 4.3. In particular, since  $\frac{\rho_i}{\rho}$  are continuous functions of  $[0, T]$  for  $i = 1, 3$  and since there exists two positive constants  $C, b$  such that

$$\left| \frac{\rho' \rho_2}{(\rho)^2} \right| \leq C \hat{\varphi}^b e^{\frac{1}{4}s\hat{\alpha}} \quad (t \in [0, T]),$$

we can apply the second part of Corollary 4.3 and we get that

$$\frac{V}{\rho} \in L^2(0, T; D(A)) \cap C([0, T]; D((-A)^{\frac{1}{2}})) \cap H^1(0, T; \mathbb{H}).$$

The above relation and the definitions (2.2) of  $\mathbb{H}$  and (2.7) of  $D(A)$  completes the proof of the theorem.  $\square$

### 5. Proof of the main result

#### 5.1. The change of variables

Since

$$\overline{\mathcal{S}(0)} = \overline{B(h^0)} \subset \mathcal{C} \setminus \overline{\mathcal{O}},$$

and since  $\mathcal{C}$  in an open subset, there exists  $\bar{\varepsilon} > 0$  such that

$$\overline{B(h^0, 1 + \bar{\varepsilon})} \subset \mathcal{C} \setminus \overline{\mathcal{O}}.$$

Thus, if we have that

$$|h(t) - h^0| < \frac{\bar{\varepsilon}}{3} \quad (t \in [0, T]), \tag{5.1}$$

then for all  $t \in [0, T]$ ,

$$B(h(t)) \subset \mathcal{C} \setminus \overline{\mathcal{O}}.$$

Relation (5.1) holds true in particular if we have:

$$\|h'\|_{L^\infty(0, T)} < \frac{\bar{\varepsilon}}{3T}. \tag{5.2}$$

With the above assumptions, we can construct, as in [27], a change of variables with the following properties.

**Lemma 5.1.** *There exist  $C^1$  applications  $\mathcal{X}$  and  $\mathcal{Y}$  from  $\mathcal{C}_T$  into  $\mathcal{C}$  such that*

- for any  $t \in [0, T]$ ,  $\mathcal{X}(t)$  and  $\mathcal{Y}(t)$  are diffeomorphisms from  $\mathcal{C}$  onto itself,
- $\mathcal{X}(\mathcal{Y}(x, t), t) = x$  for all  $(x, t) \in \mathcal{C}_T$  and  $\mathcal{Y}(\mathcal{X}(y, t), t) = y$  for all  $(y, t) \in \mathcal{C}_T$ ,
- for all  $y \in B(h^0, 1 + \frac{\bar{\epsilon}}{3})$ , we have that  $\mathcal{X}(y, t) = y + h(t)$ ; in particular,  $\mathcal{X}(t)(\mathcal{S}(T)) = \mathcal{S}(t)$ ,
- for all  $(y, t) \in \mathcal{C}_T$  such that  $|y - h^0| > 1 + \bar{\epsilon}$ , we have that  $\mathcal{X}(y, t) = y$ ,
- the function  $\mathcal{X}(t)$  fulfils,

$$\det \nabla \mathcal{X}(y, t) = 1, \quad \forall y \in \mathcal{C},$$

$$\mathcal{X}(y, T) = y, \quad \forall y \in \mathcal{C}.$$

Now let us denote:

$$V(y, t) = (\nabla \mathcal{Y})(\mathcal{X}(y, t), t) \cdot v(\mathcal{X}(y, t), t), \quad P(y, t) = p(\mathcal{X}(y, t), t), \tag{5.3}$$

and

$$V^0(y) = v^0(y + h^0) \quad (y \in \mathcal{C}).$$

Then, after some calculation (cf. [19]), we have that

$$\frac{\partial V}{\partial t} + [\mathcal{M}V] + [\mathcal{N}V] - v[\mathcal{L}V] + [\mathcal{G}P] + 1_{\mathcal{O}}u = 0, \quad t \in (0, T), \quad y \in \Omega, \tag{5.4}$$

$$\operatorname{div} V = 0, \quad t \in (0, T), \quad y \in \Omega, \tag{5.5}$$

$$V(y, t) = 0, \quad y \in \partial \mathcal{C}, \quad t \in [0, T], \tag{5.6}$$

$$V(y, t) = h'(t) + \theta'(t)y^\perp, \quad y \in \partial \mathcal{S}, \quad t \in [0, T], \tag{5.7}$$

$$Mh''(t) = - \int_{\partial \mathcal{S}} \sigma(V, P)n \, d\Gamma, \quad t \in (0, T), \tag{5.8}$$

$$J\theta''(t) = - \int_{\partial \mathcal{S}} y^\perp \cdot \sigma(V, P)n \, d\Gamma, \quad t \in (0, T), \tag{5.9}$$

$$V(y, 0) = V^0(y), \quad y \in \Omega(T), \tag{5.10}$$

$$h(0) = h^0, \quad h'(0) = h^1, \quad \theta(0) = \theta^0, \quad \theta'(0) = \theta^1, \tag{5.11}$$

with

$$[\mathcal{L}V]_i = \sum_{j,k} \frac{\partial}{\partial y_j} \left( g^{jk} \frac{\partial V_i}{\partial y_k} \right) + 2 \sum_{j,k,l} g^{kl} \Gamma_{jk}^i \frac{\partial V_j}{\partial y_l} + \sum_{j,k,l} \left\{ \frac{\partial}{\partial y_k} (g^{kl} \Gamma_{jl}^i) + \sum_m g^{kl} \Gamma_{jl}^m \Gamma_{km}^i \right\} V_j, \tag{5.12}$$

$$[\mathcal{N}V]_i = \sum_j V_j \frac{\partial V_i}{\partial y_j} + \sum_{j,k} \Gamma_{jk}^i V_j V_k, \tag{5.13}$$

$$[\mathcal{M}V]_i = \sum_j \frac{\partial \mathcal{Y}_j}{\partial t} \frac{\partial V_i}{\partial y_j} + \sum_{j,k} \left\{ \Gamma_{jk}^i \frac{\partial \mathcal{Y}_k}{\partial t} + \frac{\partial \mathcal{Y}_i}{\partial x_k} \frac{\partial^2 \mathcal{X}_k}{\partial t \partial y_j} \right\} V_j, \tag{5.14}$$

$$[\mathcal{G}P]_i = \sum_{j=1}^2 g^{ij} \frac{\partial P}{\partial y_j}, \tag{5.15}$$

where we have denoted:

$$g^{ij} = \sum_k \frac{\partial \mathcal{Y}_i}{\partial x_k} \frac{\partial \mathcal{Y}_j}{\partial x_k}, \tag{5.16}$$

$$g_{ij} = \sum_k \frac{\partial \mathcal{X}_i}{\partial y_k} \frac{\partial \mathcal{X}_j}{\partial y_k}, \tag{5.17}$$

and

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left\{ \frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} + \frac{\partial g_{ij}}{\partial y_l} \right\}. \tag{5.18}$$

Consequently, we have to control the above system and we have to prove that, with our control, relation (5.2) holds true. In order to achieve this goal, we use the results of the previous section combined with some estimates on the coefficients appearing in the above system.

In fact, we have proved in [27] that

**Lemma 5.2.** *There exist  $\mathcal{X}$  and  $\mathcal{Y}$  satisfying the properties of Lemma 5.1 and such that there exists  $d \in \mathbb{N}$  for which, for all  $(V, P) \in H^2(\Omega) \times H^1(\Omega)$  and for all  $t \in [0, T]$ , the following relations holds true:*

$$\begin{aligned} \|(\mathcal{L} - \Delta)V\|_{L^2(\Omega)} &\leq C(\|h'\|_{L^\infty(0,T)} + 1)^d \rho(t) \left\| \frac{h'}{\rho} \right\|_{L^2(0,T)} \|V\|_{H^2(\Omega)}, \\ \|\mathcal{N}V\|_{L^2(\Omega)} &\leq C(\|h'\|_{L^\infty(0,T)} + 1)^d \|V\|_{L^2(\Omega)} \|V\|_{H^1(\Omega)}, \\ \|\mathcal{M}V\|_{L^2(\Omega)} &\leq C(\|h'\|_{L^\infty(0,T)} + 1)^d \rho(t) \left\| \frac{h'}{\rho} \right\|_{L^2(0,T)} \|V\|_{H^1(\Omega)}, \\ \|(\mathcal{G} - \nabla)P\|_{L^2(\Omega)} &\leq C(\|h'\|_{L^\infty(0,T)} + 1)^d \rho(t) \left\| \frac{h'}{\rho} \right\|_{L^2(0,T)} \|\nabla P\|_{L^2(\Omega)}. \end{aligned}$$

5.2. Proof of Theorem 1.1

We are now in position to prove Theorem 1.1. Let  $r \in \mathbb{R}$ ,  $r > 0$ . We denote by  $\mathcal{K}_r$  the set:

$$\mathcal{K}_r = \left\{ F \in L^2(Q); \int_Q \left(\frac{1}{\rho_1}\right)^2 |F|^2 \, dy \, dt < r \right\},$$

where  $\mathcal{K}$  is defined by (4.17).

Assume that  $F \in \mathcal{K}_r$ . Then by taking  $u = E_T(h^0, \theta^0, V^0, F)$ , we have that the solution  $(V, h, \theta)$  of (4.5)–(4.12) is such that

$$\begin{aligned} \frac{V}{\rho} &\in H^{1,2}(Q), & \frac{\nabla P}{\rho} &\in L^2(Q), \\ \frac{h'}{\rho}, \frac{\theta'}{\rho} &\in H^1(0, T) & \text{and } h(T) = 0, \quad \theta(T) = 0. \end{aligned}$$

(See Theorem 4.4.)

Moreover there exist a positive constant  $C$  such that

$$\left\| \frac{V}{\rho} \right\|_{H^{1,2}(Q)} \leq C\lambda \left( \left\| \frac{F}{\rho_1} \right\|_{L^2(Q)} + C_0 \right), \tag{5.19}$$

$$\left\| \frac{\nabla P}{\rho} \right\|_{L^2(Q)} \leq C \left( \left\| \frac{F}{\rho_1} \right\|_{L^2(Q)} + C_0 \right), \tag{5.20}$$

$$\left\| \frac{(h', \theta')}{\rho} \right\|_{H^1(0,T)} \leq C \left( \left\| \frac{F}{\rho_1} \right\|_{L^2(Q)} + C_0 \right), \tag{5.21}$$

where  $C_0 = \|v^0\|_{H^1(\Omega)} + |h^1| + |\theta^1| + |h^0| + |\theta^0|$ .

In particular, by using Sobolev inequalities, we have that

$$\|h'\|_{L^\infty(0,T)} \leq C \left( \left\| \frac{F}{\rho_1} \right\|_{L^2(Q)} + C_0 \right).$$

Thus, since  $F \in \mathcal{K}_r$ , we have that

$$\|h'\|_{L^\infty(0,T)} \leq C(r + C_0).$$

Consequently, if

$$\|v^0\|_{H^1(\Omega)} + |h^1| + |\theta^1| + |h^0| + |\theta^0| < r, \quad (5.22)$$

and  $r$  is small enough, we have (5.1) and (5.2) hold true. We can thus consider  $\mathcal{X}$  and  $\mathcal{Y}$  as in Lemmas 5.1 and 5.2 and the operators  $\mathcal{L}$ ,  $\mathcal{M}$ ,  $\mathcal{N}$ ,  $\mathcal{G}$  defined by (5.12)–(5.18).

We can thus consider the following application:

$$\begin{aligned} \mathcal{Y} : \mathcal{K}_r &\rightarrow \mathcal{K}, \\ F &\mapsto -[\mathcal{M}V] - [\mathcal{N}V] + \nu[(\mathcal{L} - \Delta)V] + [(\nabla - \mathcal{G})P]. \end{aligned}$$

In fact, combining Lemma 5.2, relations (5.19)–(5.21) and (5.22), and

$$\frac{(\rho)^2}{\rho_1} \in L^\infty(0, T), \quad (5.23)$$

we get that

$$\left\| \frac{\mathcal{Y}(F)}{\rho_1} \right\|_{L^2(Q)} \leq C(1+r)^{d+1}r^2.$$

Thus, for  $r$  small enough, we see that  $\mathcal{Y}(\mathcal{K}_r) \subset \mathcal{K}_r$ .

If we combine again Lemma 5.2, relations (5.19)–(5.21) and (5.23), we also get that for  $F^1, F^2 \in \mathcal{K}_r$ , we have:

$$\left\| \frac{\mathcal{Y}(F^1) - \mathcal{Y}(F^2)}{\rho_1} \right\|_{L^2(Q)} \leq C(1+r)^{d+1}r \left\| \frac{\mathcal{Y}(F^1) - \mathcal{Y}(F^2)}{\rho_1} \right\|_{L^2(Q)}.$$

Thus for  $r$  small enough, we also see that  $\mathcal{Y}|_{\mathcal{K}_r}$  is a contraction and thus has a fixed point.

We have consequently obtained that there exists a control  $u \in L^2(0, T; L^2(\mathcal{O}))$  such that the solution of (5.4)–(5.11) satisfies:

$$V(\cdot, T) = 0, \quad h'(T) = 0, \quad \theta'(T) = 0, \quad h(T) = 0, \quad \theta(T) = 0.$$

Moreover, from (5.3), we have also proved that there exists a control  $u \in L^2(0, T; L^2(\mathcal{O}))$  such that the solution of (1.1)–(1.8) satisfies:

$$v(\cdot, T) = 0, \quad h'(T) = 0, \quad \theta'(T) = 0, \quad h(T) = 0, \quad \theta(T) = 0,$$

which ends the proof of Theorem 1.1.

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