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COLLISIONS IN THREE-DIMENSIONAL FLUID STRUCTURE INTERACTION PROBLEMS*

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Abstract. This paper deals with a system composed of a rigid ball moving into a viscous incompressible fluid over a fixed horizontal plane. The equations of motion for the fluid are the Navier–Stokes equations, and the equations for the motion of the rigid ball are obtained by applying Newton's laws. We show that for any weak solution of the corresponding system satisfying the energy inequality, the rigid ball never touches the plane. This result is the extension of that obtained in [M. Hillairet, *Comm. Partial Differential Equations*, 32 (2007), pp. 1345–1371] in the two-dimensional setting.

Key words. fluid structure interactions, Cauchy theory, qualitative properties, collisions

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1. Introduction.

In the last decade, several studies showed that collisions between rigid bodies in a fluid would lead to great difficulties in the mathematical treatment of fluid-structure interaction models. For example, in [4, p. 287], Feireisl constructs a solution in which a sphere remains stuck to the ceiling of the cavity regardless of the intensity of the gravity. This example emphasizes that collisions would lead to unphysical solutions to standard mathematical systems. Indeed, Starovoitov proves also that several solutions exist when contact occurs [11, 12]. Therefore, at least one of these solutions does not represent a physical configuration.

Before handling the description of these collisions, several studies proved they do not occur in fluid structure systems. In [15], Vázquez and Zuazua prove no collision can occur between particles for a one-dimensional toy model. Then Starovoitov obtains a criterion for the velocity field of solutions [12] in the multidimensional setting. Namely, he proves no collision can occur if the gradient of the velocity field is sufficiently integrable. Finally, two parallel studies [8, 9] prove a no-collision result when there is only one body in a bounded (or partially bounded) two-dimensional cavity. In the first case, the author considers a rigid disk inside a bigger disk. In the second case, the author considers a rigid disk above a ramp. The aim of the present study is to extend these two-dimensional results to three-dimensional comparable configurations, i.e., for a rigid sphere above a ramp in $\mathbb{R}^3$.

1.1. Mathematical model.

We consider a homogeneous rigid sphere $\mathcal{B}$ with radius 1 and density $\rho_{\mathcal{B}}$. We denote by $\mathbf{G}$ its center (of mass), by $\mathbf{V}$ (resp., $\boldsymbol{\omega}$) its translation (resp., angular) velocity, and by m (resp., $\mathbb{J}$) its mass (resp., inertia). Notice that $\boldsymbol{\omega}$ is a vector in $\mathbb{R}^3$ and $\mathbb{J} = J\mathrm{I}_3$, where $J \in (0,\infty)$, and I_3 is the 3×3

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identity matrix. The velocity field of $\mathcal{B}$ reads $\mathbf{V}+\boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G})$ for all $\mathbf{x}\in\mathcal{B}$. The sphere evolves over a ramp $\mathcal{P}$. The remainder of the cavity $\mathbb{R}^3_+$ is denoted by $\mathcal{F}$. It contains an incompressible viscous and Newtonian fluid which does not slip on boundaries and has constant density $\rho_{\mathcal{F}}=1$ and viscosity μ. The whole system evolves only through the interactions between solid and fluid without any external force field.

The evolution of the fluid is described by $(\mathbf{u},p)$, a velocity/pressure field satisfying the incompressible Navier–Stokes equations:

$$\begin{cases}\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}=\operatorname{div}\mathbb{T}(\mathbf{u},p)\\ \operatorname{div}\mathbf{u}=0\end{cases}\quad\text{in }\mathcal{F},\tag{1.1}$$

where $\mathbb{T}(\mathbf{u},p)$ is the Newtonian stress tensor:

$$\mathbb{T}(\mathbf{u},p)=2\mu D(\mathbf{u})-p\mathrm{I}_3.$$

Here $D(\mathbf{u})$ stands for the symmetric part of the gradient of $\mathbf{u}$.

To describe the evolution of $\mathcal{B}$, we apply Newton's laws assuming continuity of the stress-tensor of the fluid on $\partial\mathcal{B}$. It yields

$$\begin{cases}-\displaystyle\int_{\partial\mathcal{B}}\mathbb{T}(\mathbf{u},p)\mathbf{n}\;\mathrm{d}\sigma=m\dot{\mathbf{V}},\\ -\displaystyle\int_{\partial\mathcal{B}}\mathbb{T}(\mathbf{u},p)\mathbf{n}\times(\mathbf{x}-\mathbf{G})\;\mathrm{d}\sigma=J\dot{\boldsymbol{\omega}}.\end{cases}\tag{1.2}$$

Here $\mathbf{n}$ stands for the normal to $\partial\mathcal{B}$ directed towards $\mathcal{B}$. As $\mathbb{J}$ is a scalar matrix, the inertial term $\mathbb{J}\boldsymbol{\omega}\times\boldsymbol{\omega}$ vanishes in the conservation of momentum.

This system is complemented with the boundary conditions

$$\mathbf{u}_{|\partial\mathcal{B}}=\mathbf{V}+\boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}),\quad\mathbf{u}_{|\mathcal{P}}=0,\quad\mathbf{u}_{|\infty}=0\tag{1.3}$$

and initial conditions

$$\mathbf{u}(0,\cdot)=\mathbf{u}^0,\quad\mathbf{V}(0)=\mathbf{V}^0,\quad\boldsymbol{\omega}(0)=\boldsymbol{\omega}^0,\quad\mathbf{G}(0)=\mathbf{G}^0.\tag{1.4}$$

For short, we shall refer to the whole system (1.1)–(1.3) as (FSIS) for fluid-solid interaction system. We emphasize that this system is strongly coupled. On the one hand, the position of the sphere $\mathcal{B}$ fixes the domain $\mathcal{F}$ where the incompressible Navier–Stokes equations (1.1) have to be solved and the movement of $\mathcal{B}$ fixes the boundary conditions for (1.1) on $\partial\mathcal{B}$. In particular, we lay stress upon these time-dependences, denoting by $\mathcal{F}(t)$ (resp., $\mathcal{B}(t)$) the domain occupied by the fluid (resp., the solid body) at time t in the following. We shall reserve the notation $\mathcal{B}$ (resp., $\mathcal{F}$) for the sphere (resp., the fluid) as "actors" in the scenarios provided by our solutions to (FSIS). On the other hand, the solution $(\mathbf{u},p)$ prescribes the displacement of $\mathcal{B}$ via the computation of the forces and torques applied to $\mathcal{B}$:

$$-\int_{\partial\mathcal{B}}\mathbb{T}(\mathbf{u},p)\mathbf{n}\;\mathrm{d}\sigma,\quad-\int_{\partial\mathcal{B}}\mathbb{T}(\mathbf{u},p)\mathbf{n}\times(\mathbf{x}-\mathbf{G})\;\mathrm{d}\sigma.$$

Our main result is that no collision can occur between $\mathcal{B}$ and $\mathcal{P}$ in finite time in solutions to (FSIS). This reads as follows.

THEOREM 1.1. *Given $T>0$, let $(\mathbf{u},\mathbf{G})$ be a weak solution to (FSIS) over $(0,T)$ with initial data $(\mathbf{u}^0,\mathbf{G}^0)$. Then there exists a decreasing function $h_{min}\in$*

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$\mathcal{C}([0,T];(0,\infty))$ *depending only on initial data* $(\mathbf{u}^0, \mathbf{G}^0)$ *such that* $h(t) := \mathrm{dist}(\mathcal{B}(t), \mathcal{P})$ *satisfies*

$$h(t) \geqslant h_{min}(t) \quad \forall\, t \in (0,T).$$

This result has been expected ever since the computations of Cooley and O'Neill [1] in the slow motion regime. However, until now no rigorous mathematical result has been available in the full nonlinear case. We emphasize that this result still holds true when one adds a reasonable external force field $\mathbf{f}$, for example, the gravity or $\mathbf{f} \in L^2((0,T)\times\mathbb{R}^3_+)$. In section 3, we provide an interpretation for the weak formulation of (FSIS) explaining how the distance can be estimated from below with a suitable test function. Then we construct a test function explicitly. In section 4, the interpretation of the weak formulation is applied to the constructed test function. Technical details are postponed to the appendix.

As mentioned in Theorem 1.1, our result applies to any weak solution to (FSIS). However, the Cauchy theory of weak solutions has been developed only in bounded and in exterior domains (to our knowledge) so that we do not know whether one solution exists to (FSIS). For the sake of completeness, we extend classical results for the Cauchy theory of (FSIS) to a half-space in the next section. Eventually, we obtain that, for a sufficiently large class of initial data $(\mathbf{u}^0, \mathbf{G}^0)$, the system (FSIS) predicts that no collision can occur between $\mathcal{B}$ and $\mathcal{P}$.

1.2. Notation. Throughout, bold symbols stand for vectors. Given $\mathbf{a} \in \mathbb{R}^3$ we denote by $\mathbf{a}\otimes\mathbf{a}$ the symmetric matrix with entries $a_i a_j$. Coordinates $\mathbf{x} = (x_1, x_2, x_3)$ are centered on $\mathcal{P}$. For example, we have $\mathcal{P} := \{(x_1,x_2,0),\ (x_1,x_2) \in \mathbb{R}^2\}$. The half-space above $\mathcal{P}$ is $\mathbb{R}^3_+$ and $\mathbb{R}^3_{++}$ stands for $\{(x_1,x_2,x_3)$ with $x_3 > 1\}$. This is the domain where the center of mass $\mathbf{G}$ can evolve as long as no collision between $\mathcal{B}$ and $\mathcal{P}$ occurs.

Given $\mathbf{G} \in \mathbb{R}^3$ and $\delta > 0$, we denote by $\mathcal{B}(\mathbf{G},\delta)$ the sphere with center $\mathbf{G}$ and radius δ. For short, we also set $\mathcal{B}_{\mathbf{G}} = \mathcal{B}(\mathbf{G},1)$. This is the domain occupied by $\mathcal{B}$ when its center of mass meets $\mathbf{G}$. In this case, the fluid domain $\mathcal{F}_{\mathbf{G}}$ is the complementary of $\mathcal{B}_{\mathbf{G}}$ in $\mathbb{R}^3_+$. If the orthogonal projection of $\mathbf{G}$ on $\mathcal{P}$ is the center of coordinates, we have $\mathbf{G} = \mathbf{G}_h = (0,0,1+h)$ with $h = \mathrm{dist}(\mathcal{B}_{\mathbf{G}}, \mathcal{P})$. In this case the suitable parameter is h and not $\mathbf{G}$. Thus, when using notation with h as subscript instead of $\mathbf{G}$, we implicitly mean that the subscript should be $\mathbf{G}_h$. For example, $\mathcal{B}_h := \mathcal{B}_{\mathbf{G}_h}$.

In the whole paper, we denote by $\eta : [0,\infty) \to [0,1]$ a smooth function such that

$$\eta(s) = \begin{cases} 1 & \text{if } s < \frac{1}{2}, \\ 0 & \text{if } s > 1, \end{cases}$$

and we set $\eta_\alpha = \eta(\cdot/\alpha)$ for all parameters $\alpha > 0$.

We use the classical Lebesgue and Sobolev spaces $L^\alpha(A)$, $W^{\beta,\alpha}(A)$, $H^\beta(A)$ with A an open set, $\alpha \geqslant 1$, and $\beta \geqslant 0$. We define

$$\mathcal{H} = \{\boldsymbol{\phi} \in L^2(\mathbb{R}^3_+)\ ;\ \mathrm{div}\,\boldsymbol{\phi} = 0,\ \boldsymbol{\phi}\cdot\mathbf{n} = 0 \text{ on } \mathcal{P}\},$$
$$\mathcal{V} = \{\boldsymbol{\phi} \in H^1(\mathbb{R}^3_+)\ ;\ \mathrm{div}\,\boldsymbol{\phi} = 0,\ \boldsymbol{\phi} = 0 \text{ on } \mathcal{P}\}.$$

We recall that $\mathcal{H}$ and $\mathcal{V}$ are closed subspaces of $L^2(\mathbb{R}^3_+)$ and $H^1_0(\mathbb{R}^3_+)$, respectively. Thus, they form Hilbert spaces with respect to the induced inner products. For an open subset $A \subset \mathbb{R}^3_+$, we also consider the following Hilbert spaces:

$$\mathbb{H}(A) = \{\boldsymbol{\phi} \in \mathcal{H}\ ;\ D(\boldsymbol{\phi}) = 0 \text{ in } A\},$$
$$\mathbb{V}(A) = \{\boldsymbol{\phi} \in \mathcal{V}\ ;\ D(\boldsymbol{\phi}) = 0 \text{ in } A\}.$$

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To simplify, if $\mathbf{G} \in \mathbb{R}^3_{++}$, we set

$$\mathbb{H}(\mathbf{G}) = \mathbb{H}(\mathcal{B}_\mathbf{G}), \quad \mathbb{V}(\mathbf{G}) = \mathbb{V}(\mathcal{B}_\mathbf{G}).$$

For all $\mathbf{G} \in \mathbb{R}^3_{++}$, we also denote by $\rho_\mathbf{G}$ the function

$$\rho_\mathbf{G}(\mathbf{x}) = \begin{cases} \rho_\mathcal{B} & \text{if } \mathbf{x} \in \mathcal{B}_\mathbf{G}, \\ 1 & \text{if } \mathbf{x} \in \mathcal{F}_\mathbf{G}. \end{cases}$$

If $\mathbf{v} \in \mathbb{H}(\mathbf{G})$, from [14, p. 18], there exists a unique pair $(\mathbf{V}[\mathbf{v}], \boldsymbol{\omega}[\mathbf{v}]) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\mathbf{v}|_{\mathcal{B}_\mathbf{G}} = \mathbf{V}[\mathbf{v}] + \boldsymbol{\omega}[\mathbf{v}] \times (\mathbf{x} - \mathbf{G}).$$

In particular, if $(\mathbf{u}, \mathbf{v}) \in \mathbb{H}(\mathbf{G})^2$,

$$\int_{\mathbb{R}^3_+} \rho_\mathbf{G} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3_+ \setminus \mathcal{B}_\mathbf{G}} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + m\mathbf{V}[\mathbf{u}] \cdot \mathbf{V}[\mathbf{v}] + J\boldsymbol{\omega}[\mathbf{u}] \cdot \boldsymbol{\omega}[\mathbf{v}].$$

2. Cauchy theory.

First, we give the definition of weak solution to (FSIS).

DEFINITION 2.1. *Given* $\mathbf{G}^0 \in \mathbb{R}^3_{++}$ *and* $\mathbf{u}^0 \in \mathbb{H}(\mathbf{G}^0)$, *a pair* $(\mathbf{u}, \mathbf{G})$ *is called a weak solution to (FSIS) on* $(0,T)$ *with initial data* $(\mathbf{u}^0, \mathbf{G}^0)$ *if*

$$\mathbf{G} \in W^{1,\infty}(0,T; \mathbb{R}^3_{++}), \quad \textit{with} \quad \mathbf{G}(0) = \mathbf{G}^0, \tag{2.1}$$

$$\mathbf{u} \in L^\infty(0,T; \mathcal{H}) \cap L^2(0,T; \mathcal{V}), \tag{2.2}$$

$$\mathbf{u} = \mathbf{V} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}) \quad \textit{in } \mathcal{B}_\mathbf{G}, \quad \textit{with} \quad \mathbf{V} = \dot{\mathbf{G}}; \tag{2.3}$$

if for all $\mathbf{v} \in \mathcal{C}([0,T]; H^1_0(\mathbb{R}^3_+)) \cap H^1(0,T; L^2(\mathbb{R}^3_+))$ *with compact support in* $(0,T) \times \mathbb{R}^3_+$ *and such that* $\mathbf{v} \in \mathbb{V}(\mathbf{G}(t))$ *for all* $t \in [0,T]$,

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^3_+} \rho_\mathbf{G} \mathbf{u} \cdot \partial_t \mathbf{v} \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + 2\mu \int_0^T \int_{\mathbb{R}^3_+} D(\mathbf{u}) : D(\mathbf{v}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \\ - \int_0^T \int_{\mathbb{R}^3_+} \mathbf{u} \otimes \mathbf{u} : D(\mathbf{v}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t = 0; \end{aligned} \tag{2.4}$$

if for all $\mathbf{v} \in \mathcal{C}([0,T]; L^2(\mathbb{R}^3_+))$ *with compact support in* $[0,T) \times \mathbb{R}^3_+$ *and such that* $\mathbf{v} \in \mathbb{H}(\mathbf{G}(t))$ *for all* $t \in [0,T]$ *we have*

$$W : t \mapsto \int_{\mathbb{R}^3_+} \rho_\mathbf{G} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \in \mathcal{C}([0,T]) \quad \textit{with} \quad W(0) = \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0} \mathbf{u}^0 \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}; \tag{2.5}$$

if the energy estimate holds true:

$$\frac{1}{2} \int_{\mathbb{R}^3_+} \rho_\mathbf{G} |\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + 2\mu \int_0^t \int_{\mathbb{R}^3_+} |D(\mathbf{u})|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}s \leqslant \frac{1}{2} \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0} |\mathbf{u}^0|^2 \, \mathrm{d}\mathbf{x} \quad \textit{for a.a. } t \in (0,T).$$

Before going to our existence result for such weak solutions, let us recall some of their straightforward properties. First, combining (2.2) and (2.3) yields that $\mathbf{u}(t,\cdot) \in$

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$\mathbb{V}(\mathbf{G}(t))$ for almost all $t \in (0,T)$. Moreover, it follows from standard arguments that the pair $(\mathbf{V},\boldsymbol{\omega})$ such that (2.3) holds satisfies

$$|\mathbf{V}|^2 + |\boldsymbol{\omega}|^2 \leqslant C \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} |\mathbf{u}|^2 \, \mathrm{d}\mathbf{x}, \tag{2.6}$$

where C depends only on ρ_B. In particular, the pair $(\mathbf{V},\boldsymbol{\omega})$ associated to a weak solution $(\mathbf{u},\mathbf{G})$ belongs to $L^\infty(0,T)$.

The result of well-posedness we obtain for (FSIS) can be stated as follows.

THEOREM 2.2. *Assuming* $\mathbf{G}^0 \in \mathbb{R}^3_{++}$ *and* $\mathbf{u}^0 \in \mathbb{H}(\mathbf{G}^0)$, *there exists at least one maximal weak solution* $(T_0,(\mathbf{U},\mathbf{G}))$ *to (FSIS) with initial data* $(\mathbf{U}^0,\mathbf{G}^0)$. *Moreover, we have the alternative:*

- $T_0 = \infty$,
- $T_0 < \infty$ *and* $G_3(t) \to 1$ *as* $t \to T_0$.

The proof of Theorem 2.2 given in what follows is inspired by methods developed in other papers, and since we use many similar arguments, we choose to present here only the main ideas and refer to the appropriate references to avoid repeating technical calculations which are not the main interest of this paper. From now on $(\mathbf{G}^0,\mathbf{u}^0) \in \mathbb{R}^3_{++} \times \mathbb{H}(\mathbf{G}^0)$ is a fixed initial condition. We denote $(\mathbf{V}^0,\boldsymbol{\omega}^0) = (\mathbf{V}[\mathbf{u}^0],\boldsymbol{\omega}[\mathbf{u}^0])$ and $d^0 := \mathrm{dist}(\mathcal{B}_{\mathbf{G}^0},\mathcal{P})$. Due to our assumption, there holds $d^0 > 0$. The remainder of the section is devoted to the proof of Theorem 2.2.

2.1. Strong solutions for an approximate system. As in [7], we prove the existence of weak solutions by first obtaining strong solutions for an approximate problem of (FSIS). More precisely, we consider an even nonnegative function $\kappa \in C^\infty_0(\mathbb{R})$ such that $\kappa(s) = 0$ if $|s| \geqslant 1$. We define for all $\varepsilon > 0$

$$K_\varepsilon(\mathbf{x}) = \frac{c}{\varepsilon^3}\kappa\left(\left|\frac{\mathbf{x}}{\varepsilon}\right|^2\right) \quad (\mathbf{x} \in \mathbb{R}^3). \tag{2.7}$$

The constant c is chosen so that

$$\int_{\mathbb{R}^3} K_\varepsilon(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 1 \quad \forall \varepsilon > 0.$$

Then, for all $\mathbf{u} \in L^2((0,T)\times\mathbb{R}^3_+)$ and for all $\varepsilon > 0$, we set

$$\mathbf{u}_\varepsilon(t,\mathbf{x}) = \int_{\mathbb{R}^3_+} K_\varepsilon(\mathbf{x}-\mathbf{y})\mathbf{u}(t,\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Let us consider the following problem, which approximates (FSIS):

$$\partial_t \mathbf{u} - \mu\Delta\mathbf{u} + (\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u} + \nabla p = 0 \quad \text{in } \mathcal{F}_{\mathbf{G}(t)},\ t \in (0,T), \tag{2.8}$$

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \mathcal{F}_{\mathbf{G}(t)},\ t \in (0,T), \tag{2.9}$$

$$\mathbf{u} = 0 \quad \text{on } \mathcal{P},\ t \in (0,T), \tag{2.10}$$

$$\mathbf{u}_{|\infty} = 0, \quad t \in (0,T), \tag{2.11}$$

$$\mathbf{u} = \dot{\mathbf{G}} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}) \quad \text{on } \partial\mathcal{B}_{\mathbf{G}},\ t \in (0,T), \tag{2.12}$$

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$$m\ddot{\mathbf{G}} = -\int_{\partial\mathcal{B}_{\mathbf{G}}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\ \mathrm{d}\sigma + \frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}}} ((\mathbf{u}_\varepsilon - \mathbf{u})\cdot\mathbf{n})\mathbf{u}\ \mathrm{d}\sigma \quad \text{in } (0,T), \tag{2.13}$$

$$\begin{aligned} J\dot{\boldsymbol{\omega}} = &-\int_{\partial\mathcal{B}_{\mathbf{G}}} (\mathbf{x}-\mathbf{G})\times\mathbb{T}(\mathbf{u},p)\mathbf{n}\ \mathrm{d}\sigma \\ &+ \frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}}} ((\mathbf{u}_\varepsilon - \mathbf{u})\cdot\mathbf{n})(\mathbf{x}-\mathbf{G})\times\mathbf{u}\ \mathrm{d}\sigma \quad \text{in } (0,T). \end{aligned} \tag{2.14}$$

We complete the system with the initial conditions

$$\mathbf{u}(0,\cdot) = \mathbf{u}^0, \quad \dot{\mathbf{G}}(0) = \mathbf{V}^0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}^0, \quad \mathbf{G}(0) = \mathbf{G}^0. \tag{2.15}$$

We define the space $\widehat{H^1}(A)$ by

$$\widehat{H^1}(A) = \{q \in L^2_{loc}(\overline{A})\ ;\ \nabla q \in L^2(A)\}.$$

We denote

$$\mathcal{F}_T = \{(t,\mathbf{x}) \in [0,T]\times\mathbb{R}^3\ ;\ \mathbf{x}\in\mathcal{F}_{\mathbf{G}(t)}\}.$$

Consider a smooth mapping $\mathbf{X} : \mathbb{R}^3_{++}\times\mathcal{F}_{\mathbf{G}^0} \to \mathbb{R}^3$ such that for all $\mathbf{G}\in\mathbb{R}^3_{++}$, $\mathbf{X}(\mathbf{G},\cdot)$ is a C^∞-diffeomorphism from $\mathcal{F}_{\mathbf{G}^0}$ onto $\mathcal{F}_{\mathbf{G}}$. Moreover, suppose that the mappings

$$(\mathbf{G},\mathbf{y}) \mapsto D_{\mathbf{G}} D_y^\alpha \mathbf{X}(\mathbf{G},\mathbf{y}), \quad \alpha\in\mathbb{N}^3,$$

exist and are continuous and compactly supported in $\mathcal{F}_{\mathbf{G}^0}$. For any $\mathbf{g} : \mathcal{F}_T \to \mathbb{R}^3$, we denote by $\mathbf{g}_{\mathbf{X}} : [0,T]\times\mathcal{F}_{\mathbf{G}^0} \to \mathbb{R}^3$ the mapping $\mathbf{g}_{\mathbf{X}}(t,\mathbf{y}) = \mathbf{g}(t,\mathbf{X}(\mathbf{G}(t),\mathbf{y}))$ for all $t \geqslant 0$ for all $\mathbf{y}\in\mathcal{F}_{\mathbf{G}^0}$. We use similar notation for $g : \mathcal{F}_T \to \mathbb{R}$.

We introduce the following function spaces in a variable domain:

$$\begin{aligned} L^2(0,T;H^2(\mathcal{F}(t))) &= \{\mathbf{u}\ ;\ \mathbf{u}_{\mathbf{X}} \in L^2(0,T;H^2(\mathcal{F}_{\mathbf{G}^0}))\}, \\ H^1(0,T;L^2(\mathcal{F}(t))) &= \{\mathbf{u}\ ;\ \mathbf{u}_{\mathbf{X}} \in H^1(0,T;L^2(\mathcal{F}_{\mathbf{G}^0}))\}, \\ C([0,T],H^1(\mathcal{F}(t))) &= \{\mathbf{u}\ ;\ \mathbf{u}_{\mathbf{X}} \in C([0,T],H^1(\mathcal{F}_{\mathbf{G}^0}))\}, \\ L^2(0,T;\widehat{H}^1(\mathcal{F}(t))) &= \{p\ ;\ p_{\mathbf{X}} \in L^2(0,T;\widehat{H}^1(\mathcal{F}_{\mathbf{G}^0}))\}. \end{aligned}$$

THEOREM 2.3. *Assume the initial conditions satisfy*

$$\operatorname{dist}(\mathcal{B}_{\mathbf{G}^0},\mathcal{P}) = d^0 > 0, \quad \mathbf{u}^0 \in \mathbb{V}(\mathbf{G}^0).$$

Then, given $\varepsilon > 0$, *there exist a time* $T > 0$ *depending only on* $\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)}$ *and a* 4*-uplet* $(\mathbf{u},\mathbf{G},\omega,p)$ *satisfying*

$$\mathbf{G}\in H^2(0,T), \quad \operatorname{dist}(\mathcal{B}_{\mathbf{G}(t)},\mathcal{P}) \geqslant \frac{d^0}{2} > 0 \quad \forall\, t\in[0,T], \tag{2.16}$$

$$\mathbf{u}\in L^2(0,T;H^2(\mathcal{F}(t)))\cap C([0,T];H^1(\mathcal{F}(t)))\cap H^1(0,T;L^2(\mathcal{F}(t))), \tag{2.17}$$

$$p\in L^2(0,T;\widehat{H^1}(\mathcal{F}(t))), \quad \boldsymbol{\omega}\in H^1(0,T), \tag{2.18}$$

and satisfying (2.8)–(2.14) *almost everywhere on* $[0,T]$ *or in the trace sense.*

Proof. One can obtain local existence of strong solutions to (2.8)–(2.14) via arguments similar to those in the proof of Theorem 1.1 in [2] (see also [3, 13, 10] for

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results obtained applying similar techniques). For the sake of brevity, we compute only energy estimates here in order to prove that these local strong solutions can be continued up to collision between $\mathcal{B}$ and $\mathcal{P}$. We refer the reader to the mentioned articles for technical details.

First, we multiply (2.8) by $\mathbf{u}$, (2.13) by $\dot{\mathbf{G}}$, and (2.14) by $\boldsymbol{\omega}$. We deduce the energy estimate

$$\frac{1}{2}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}(t)}|\mathbf{u}|^2(t)\ \mathrm{d}\mathbf{x} + 2\mu\int_0^t\int_{\mathbb{R}^3_+} |D(\mathbf{u})|^2\ \mathrm{d}\mathbf{x}\ \mathrm{d}s = \frac{1}{2}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0}|\mathbf{u}^0|^2\ \mathrm{d}\mathbf{x}. \tag{2.19}$$

From the above estimate and (2.6) we obtain a time $T > 0$ depending on $\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}^0}$ $|\mathbf{u}^0|^2\ \mathrm{d}\mathbf{x}$ such that (2.16) holds for all ε.

Then we introduce a smooth function Υ with compact support in $\mathcal{B}(\mathbf{G}^0, 1+d_0/4)$ and such that $\Upsilon = 1$ on $\mathcal{B}_{\mathbf{G}^0}$. We also set

$$\mathbf{w}_R = \frac{1}{2}\left(\dot{\mathbf{G}}\times(\mathbf{x}-\mathbf{G}) + |\mathbf{x}-\mathbf{G}|^2\boldsymbol{\omega}\right),$$
$$\mathbf{u}_R(t,\mathbf{x}) = \mathrm{curl}\left[\Upsilon\left(\mathbf{x}-\mathbf{G}(t)+\mathbf{G}^0\right)\mathbf{w}_R(t,\mathbf{x})\right].$$

The function $\mathbf{u}_R$ has a compact support in $\mathcal{B}(\mathbf{G}^0, 1+d_0/4)$ and satisfies

$$\mathrm{div}\,\mathbf{u}_R = 0, \quad \mathbf{u}_R = \dot{\mathbf{G}} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}) \text{ on } \mathcal{B}(t).$$

We multiply (2.8) by

$$\boldsymbol{\varphi} = \partial_t\mathbf{u} + (\mathbf{u}_R\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u}_R,$$

which yields

$$\int_{\mathcal{F}(t)} (\partial_t\mathbf{u} + \mathbf{u}_\varepsilon\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\ \mathrm{d}\mathbf{x} = \int_{\mathcal{F}(t)} \mathrm{div}\,\mathbb{T}(\mathbf{u},p)\cdot\boldsymbol{\varphi}\ \mathrm{d}\mathbf{x}. \tag{2.20}$$

Using the regularization of the nonlinear term, we obtain the existence of a constant C_0 depending only on $\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)}$ such that the left-hand side of (2.20) can be written as

$$LHS = \int_{\mathcal{F}(t)} |\partial_t\mathbf{u}|^2\ \mathrm{d}\mathbf{x} + R_l \quad \text{with} \quad |R_l| \leqslant C_0\int_{\mathcal{F}(t)}|\nabla\mathbf{u}|^2\ \mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathcal{F}(t)}|\partial_t\mathbf{u}|^2\ \mathrm{d}\mathbf{x}.$$

Concerning the right-hand side of (2.20), we apply the same arguments as in the proof of Lemma 4.3 in [3] and obtain (see [3, (4.24)])

$$RHS = -\mu\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{F}(t)}|D(\mathbf{u})|^2\ \mathrm{d}\mathbf{x} + I_r + R_r,$$

where

$$I_r = \int_{\partial\mathcal{B}(t)} \mathbb{T}(\mathbf{u},p)\mathbf{n}\cdot\left[\ddot{\mathbf{G}} + \dot{\boldsymbol{\omega}}\times(\mathbf{x}-\mathbf{G}) - \boldsymbol{\omega}\times\dot{\mathbf{G}}\right]\ \mathrm{d}\sigma,$$

and, with the same convention as previously for C_0, we obtain

$$|R_r| \leqslant C_0\left[1 + \int_{\mathcal{F}(t)}|\nabla\mathbf{u}|^2\ \mathrm{d}\mathbf{x}\right].$$

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Finally, using (2.13)–(2.14), we deduce

$$I_r = -\left[m|\ddot{\mathbf{G}}|^2 + J|\dot{\boldsymbol{\omega}}|^2 - m\ddot{\mathbf{G}}\cdot(\boldsymbol{\omega}\times\dot{\mathbf{G}})\right] + \tilde{R}_r,$$

with

$$\tilde{R}_r = \frac{1}{2}\int_{\partial\mathcal{B}(t)} (\mathbf{u}-\mathbf{u}_\varepsilon)\cdot\mathbf{n}\left(\dot{\mathbf{G}} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G})\right)\cdot\left(\ddot{\mathbf{G}} - \boldsymbol{\omega}\times\dot{\mathbf{G}} + \dot{\boldsymbol{\omega}}\times(\mathbf{x}-\mathbf{G})\right)\,\mathrm{d}\sigma.$$

In this last integral, extending the rigid velocity fields to $\mathcal{B}(\mathbf{G}, 1+d_0/4)$ in a similar fashion to that in $\mathbf{u}_R$, we obtain, after integration by parts,

$$|\tilde{R}_r| \leqslant C_0 + \frac{m}{2}|\ddot{\mathbf{G}}|^2 + \frac{J}{2}|\dot{\boldsymbol{\omega}}|^2.$$

Combining all these estimates, we finally deduce

$$\mu\frac{\mathrm{d}}{\mathrm{dt}}\left[\int_{\mathcal{F}(t)}|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\right] + \frac{1}{2}\left[m|\ddot{\mathbf{G}}|^2 + J|\dot{\boldsymbol{\omega}}|^2 + \int_{\mathcal{F}(t)}|\partial_t\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\right]$$
$$\leqslant C_0\left[1 + \int_{\mathbb{R}^3_+}|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\right].$$

As a consequence, we have obtained that the mapping $t\mapsto\|\mathbf{u}\|_{H^1(\mathcal{F}(t))}$ is bounded on $[0,T]$. ◻

2.2. Convergences. As in the assumptions of our theorem, we assume that the initial condition $\mathbf{u}^0$ belongs to $\mathbb{H}(\mathbf{G}^0)$; we introduce a sequence $\mathbf{u}^{0k}\in\mathbb{V}(\mathbf{G}^0)$ such that

$$\mathbf{u}^{0k}\to\mathbf{u}^0 \quad \text{in } \mathbb{H}(\mathbf{G}^0).$$

We also take a sequence $\varepsilon^k\to 0$. Then, applying Theorem 2.3, we can consider a (uniform) time T such that, for all k, the corresponding solutions $(\mathbf{u}^k, \mathbf{G}^k, \omega^k, p^k)$ exist on $[0,T]$ and satisfy

$$\mathrm{dist}(\mathcal{B}_{\mathbf{G}^k(t)}, \mathcal{P}) > \frac{d^0}{2} \quad \forall\, t\in[0,T],\ \forall\, k.$$

In the following, we extend $\mathbf{u}^k$ with the value of the rigid velocity field $\mathbf{V}^k + \boldsymbol{\omega}^k\times(\mathbf{x}-\mathbf{G}^k)$ on the solid domain. From (2.19), the following holds:

$$\mathbf{u}^k \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{R}^3_+))\cap L^2(0,T;H^1_0(\mathbb{R}^3_+)),$$

so that, up to a subsequence (which we do not relabel), we can assume

$$\mathbf{u}^k\rightharpoonup\mathbf{u} \quad \text{in } L^2(0,T;H^1(\mathbb{R}^3_+))\text{-weak and } L^\infty(0,T;L^2(\mathbb{R}^3_+))\text{-weak}^*, \tag{2.21}$$

$$\mathbf{G}^k\to\mathbf{G} \quad \text{in } C([0,T];\mathbb{R}^3_{++}). \tag{2.22}$$

Taking any $\mathbf{U}\in\mathcal{D}((0,T)\times\mathbb{R}^3_+)$ such that $D(\mathbf{U})=0$ in a neighborhood of $\mathcal{B}_{\mathbf{G}(t)}$ for all $t\in(0,T)$, we can multiply (2.8) by $\mathbf{U}$ for k sufficiently large. Integrating by

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parts and using Reynolds' transport theorem yields

$$
\begin{aligned}
(2.23)\quad & -\int_0^T \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^k} \mathbf{u}^k \cdot \partial_t \mathbf{U} \, \mathrm{d}\mathbf{x} \, \mathrm{d}s \\
& + \int_0^T \int_{\mathbb{R}^3_+} (\mathbf{u}^k \otimes \mathbf{u}^k) : D(\mathbf{U}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}s + 2\mu \int_0^T \int_{\mathbb{R}^3_+} D(\mathbf{u}^k) : D(\mathbf{U}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}s \\
= & \int_0^T \int_{\mathcal{F}_{\mathbf{G}^k(t)}} \left[((\mathbf{u}^k_{\varepsilon^k} - \mathbf{u}^k) \cdot \nabla)\mathbf{U}\right] \cdot \mathbf{u}^k \, \mathrm{d}\mathbf{x} - \frac{1}{2} \int_0^T \int_{\partial \mathcal{B}_{\mathbf{G}^k(t)}} (\mathbf{U} \cdot \mathbf{u}^k)((\mathbf{u}^k_{\varepsilon^k} - \mathbf{u}^k) \cdot \mathbf{n}) \, \mathrm{d}\sigma.
\end{aligned}
$$

In order to pass to the limit in this weak formulation, we need to prove L^2-compactness of the $\mathbf{u}^k$. As usual, this is the main difficulty of the proof. The procedure to prove this compactness property follows closely the method developed in [7]. The main difference here is that our cavity is unbounded. Therefore, we do not look for a compactness property of the sequence $\mathbf{u}^k$ on the whole domain but only locally in space. Indeed, with the help of Friedrichs' lemma (see [6, Lemma II.4.2]) we have the following: for all relatively compact $\mathcal{O} \subset \mathbb{R}^3_+$, for any $\gamma > 0$ there exist $I = I(\gamma, \mathcal{O}) \in \mathbb{N}$ and functions $\boldsymbol{\psi}_j \in L^\infty(\mathcal{O})$, $j = 1, \ldots, I$, such that

$$
\begin{aligned}
(2.24)\quad & \left\|\mathbf{u}^k - \mathbf{u}\right\|^2_{L^2(0,T;L^2(\mathcal{O}))} \\
& \leqslant \sum_{j=1}^{I} \int_0^T \left(\int_{\mathcal{O}} \rho_{\mathbf{G}} \left(\mathbf{u}^k(t) - \mathbf{u}(t)\right) \cdot \boldsymbol{\psi}_j \, \mathrm{d}\mathbf{y} \right)^2 \mathrm{d}t + \gamma \left\|\nabla \mathbf{u}^k - \nabla \mathbf{u}\right\|^2_{L^2(0,T;L^2(\mathcal{O}))}.
\end{aligned}
$$

Due to the uniform bound on $\mathbf{u}^k$ in $L^2(0,T;H^1_0(\mathbb{R}^3_+))$, our remaining task is to prove that, for any $\boldsymbol{\psi} \in L^\infty(\mathcal{O})$, there exists a subsequence (which we do not relabel) for which there holds

$$
(2.25)\qquad \lim_{k \to \infty} \int_0^T \left(\int_{\mathcal{O}} \rho_{\mathbf{G}} \left(\mathbf{u}^k(t) - \mathbf{u}(t)\right) \cdot \boldsymbol{\psi} \, \mathrm{d}\mathbf{y} \right)^2 \mathrm{d}t = 0.
$$

As a first step, we obtain results similar to those of (2.25) for another family of test functions (not included in $L^\infty(\mathcal{O})$. To this end, we divide the segment $[0,T]$ into N segments $[t_{i-1}, t_i]$, with $\Delta t = t_i - t_{i-1} = T/N$, $i = 1, \ldots, N$. For all i and for $\delta < \frac{d^0}{2}$, we consider an orthonormal basis $(\mathbf{e}^{i,\delta}_j)$ of $\mathbb{V}(\mathcal{B}(\mathbf{G}(t_i), 1+\delta))$. Without further restrictions, we assume all the $\mathbf{e}^{i,\delta}_j$ with compact support. We also consider the set of piecewise linear functions in t:

$$
(2.26)\qquad \mathbf{U}^\delta(t, \mathbf{x}) = \mathbf{e}^{i-1,\delta}_j(\mathbf{x}) + \frac{t - t_i}{\Delta t} \left(\mathbf{e}^{i,\delta}_l(\mathbf{x}) - \mathbf{e}^{i-1,\delta}_j(\mathbf{x}) \right)
$$

for $t \in [t_{i-1}, t_i]$, $j, l \in \mathbb{N}$, $i \in \{1, \ldots, N\}$. There exists a countable set of functions satisfying (2.26).

It is worth noting that, since $\mathbf{G}$ is uniformly continuous, for N big enough, the functions $\mathbf{U}^\delta$ of the above form satisfy $D(\mathbf{U}^\delta(t)) = 0$ in $\mathcal{B}(\mathbf{G}(t), 1 + \delta/2)$ for all t. In particular, these functions are in $C([0,T]; \mathbb{V}(\mathbf{G}(t)))$. These functions are important to approximate continuous functions in time with value in $\mathbb{H}(\mathbf{G}(t))$ and thus functions in $L^2(0,T;\mathbb{H}(\mathbf{G}(t)))$ (see Lemmas 4.1 and 4.2 in [7]). From (2.22), there exists $k_0 = k_0(\delta)$ such that for $k \geqslant k_0$,

$$
D(\mathbf{U}^\delta) = 0 \quad \text{in } \mathcal{B}_{\mathbf{G}^k(t)} \quad \forall\, t \in [0,T]
$$

for all functions satisfying (2.26).

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We multiply (2.8) by $\mathbf{U}^\delta$ satisfying (2.26). After similar computations to those leading to (2.23), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}^\delta \,\mathrm{d}\mathbf{x} = -2\mu\int_{\mathcal{F}^k(t)} D(\mathbf{u}^k) : D(\mathbf{U}^\delta)\,\mathrm{d}\mathbf{x} + \int_{\mathcal{F}^k(t)} \left(\partial_t\mathbf{U}^\delta + (\mathbf{u}^k_\varepsilon\cdot\nabla)\mathbf{U}^\delta\right)\cdot\mathbf{u}^k\,\mathrm{d}\mathbf{x} - \frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}} (\mathbf{U}^\delta\cdot\mathbf{u}^k)((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\mathbf{n})\,\mathrm{d}\sigma. \tag{2.27}$$

Following the estimates of [7], using Arzelà and Ascoli and the diagonal Cantor procedure, we obtain that for all $\mathbf{U}^\delta$ satisfying (2.26), we have

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}^\delta\,\mathrm{d}\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}^\delta\,\mathrm{d}\mathbf{x} \quad \text{in } C([0,T]). \tag{2.28}$$

However, as $\mathbf{G}^k \to \mathbf{G}$ in $\mathcal{C}([0,T])$, and $\mathbf{u}^k$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^3_+))\cap L^2(0,T; H^1_0(\mathbb{R}^3_+))$, this also leads to

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}^k\cdot\mathbf{U}^\delta\,\mathrm{d}\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}^\delta\,\mathrm{d}\mathbf{x} \quad \text{in } C([0,T]). \tag{2.29}$$

Via a density argument (see Lemmas 4.1 and 4.2 in [7] for details), we might extend this convergence result to all $\mathbf{U}\in C([0,T];L^2(\mathbb{R}^3_+))$, $\operatorname{div}\mathbf{U}=0$, $D(\mathbf{U})=0$ in $\mathcal{B}(\mathbf{G}(t),1)$,

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}^k\cdot\mathbf{U}\,\mathrm{d}\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}\mathbf{x} \quad \text{in } C([0,T]), \tag{2.30}$$

and to all $\mathbf{U}\in L^2(0,T;L^2(\mathbb{R}^3_+))$, $\operatorname{div}\mathbf{U}=0$, $D(\mathbf{U})=0$ in $\mathcal{B}(\mathbf{G}(t),1)$,

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}^k\cdot\mathbf{U}\,\mathrm{d}\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}\mathbf{x} \quad \text{in } L^2(0,T). \tag{2.31}$$

Relation (2.30) implies in particular that $\mathbf{u}$ satisfies the initial conditions.

However, in (2.25), no assumption is made on the velocity field ψ over $\mathcal{B}_{\mathbf{G}}$. In order to reduce (2.25) to the above convergence result, we need to project such $\psi\in L^\infty(\mathcal{O})$ on velocity fields which are rigid over $\mathcal{B}_{\mathbf{G}}$. Similarly, $\mathbf{u}^k$ is rigid on $\mathcal{B}_{\mathbf{G}^k}$, so we also modify $\mathbf{u}^k$ slightly to obtain a function with the same properties but which is rigid in $\mathcal{B}_{\mathbf{G}}$. To this end, we extend $\mathbf{u}^k$ by 0 in $\mathbb{R}^3\setminus\mathbb{R}^3_+$, and since $\mathbf{u}^k=0$ on $\mathcal{P}$, we have $\mathbf{u}^k\in L^2(0,T;H^1(\mathbb{R}^3))$. We set

$$\widehat{\mathbf{u}}^k(t,\mathbf{y}) = \mathbf{u}^k(t,\mathbf{y}+\mathbf{G}^k(t)-\mathbf{G}(t)).$$

We have

$$\operatorname{div}\widehat{\mathbf{u}}^k = 0,\quad \widehat{\mathbf{u}}^k = 0 \quad \text{on } \{\mathbf{y}\in\mathbb{R}^3\ ;\ y_3 = G_3 - G_3^k\},\quad D(\mathbf{u}^k)=0 \quad \text{in } \mathcal{B}_{\mathbf{G}}$$

and

$$\|\widehat{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2((0,T)\times\mathbb{R}^3)} \leqslant \|\mathbf{G}^k-\mathbf{G}\|_{L^\infty(0,T)}\|\nabla\mathbf{u}^k\|_{L^2((0,T)\times\mathbb{R}^3)} \to 0. \tag{2.32}$$

We define

$$\mathcal{L}^2(\mathbb{R}^3) = \left\{\mathbf{v}\in L^2(\mathbb{R}^3)\ ;\ \operatorname{div}\mathbf{v}=0 \quad \text{in } \mathbb{R}^3\right\}.$$

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Following [7] (see (4.37) in [7]), there exists a function $\Lambda : \mathcal{L}^2(\mathbb{R}^3) \to \mathcal{L}^2(\mathbb{R}^3)$ such that for all $\mathbf{v} \in \mathcal{L}^2(\mathbb{R}^3)$, $\mathbf{u} = \Lambda(\mathbf{v})$ satisfies

$$\mathbf{u} = \mathbf{v} \quad \text{in } \mathcal{B}_{\mathbf{G}^0}, \quad \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{B}\left(\mathbf{G}^0, 1 + \frac{d^0}{4}\right)},$$

$$\|\mathbf{u}\|_{L^2(\mathbb{R}^3)} \leqslant c\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}.$$

Moreover,

$$\text{if} \quad \mathbf{v} \in C([0,T]; L^2(\mathbb{R}^3)), \quad \text{then} \quad \Lambda\mathbf{v} \in C([0,T]; L^2(\mathbb{R}^3)),$$

and

$$\text{if} \quad \mathbf{v} \in L^2(0,T; L^2(\mathbb{R}^3)), \quad \text{then} \quad \Lambda\mathbf{v} \in L^2(0,T; L^2(\mathbb{R}^3)).$$

We also define $\check{\Lambda}^t : \mathcal{L}^2(\mathbb{R}^3) \to \mathcal{L}^2(\mathbb{R}^3)$ as follows:

$$\check{\Lambda}^t \check{\mathbf{v}}(\mathbf{y}) = [\Lambda \check{\mathbf{v}}]\,(\mathbf{y} + \mathbf{G}^0 - \mathbf{G}(t)),$$

where

$$\check{\mathbf{v}}(\mathbf{x}) = \mathbf{v}(\mathbf{x} + \mathbf{G}(t) - \mathbf{G}^0).$$

Then, as in [7], we consider

$$\overline{\mathbf{u}}^k(\mathbf{x},t) = \mathbf{u}^k + \check{\Lambda}^t(\widehat{\mathbf{u}}^k - \mathbf{u}^k).$$

This function is rigid in $\mathcal{B}_{\mathbf{G}(t)}$ and

$$\|\overline{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2((0,T)\times\mathbb{R}^3)} \to 0. \tag{2.33}$$

We are now in a position to prove (2.25) by using (2.31). Assume that $\boldsymbol{\psi} \in L^\infty(\mathcal{O})$. We set

$$I_k := \int_0^T \left(\int_{\mathcal{O}} \rho_{\mathbf{G}} \left(\mathbf{u}^k(t) - \mathbf{u}(t)\right) \cdot \boldsymbol{\psi} \; \mathrm{d}\mathbf{y} \right)^2 \, \mathrm{d}t$$

and

$$\overline{I}_k = \int_0^T \left(\int_{\mathcal{O}} \rho_{\mathbf{G}} \left(\overline{\mathbf{u}}^k(t) - \mathbf{u}(t)\right) \cdot \boldsymbol{\psi} \; \mathrm{d}\mathbf{y} \right)^2 \, \mathrm{d}t.$$

From (2.33), to prove that I_k goes to 0, it is sufficient to prove that $\overline{I}_k$ goes to 0. Then, given $\mathbf{G} \in \mathbb{R}^3_+$ such that $\operatorname{dist}(\mathcal{B}_{\mathbf{G}}, \mathcal{P}) > 0$, we denote by $P(\mathbf{G})$ the orthogonal projection from $L^2(\mathbb{R}^3_+, \rho_{\mathbf{G}}\mathrm{d}\mathbf{x})$ onto $\mathbb{H}(\mathbf{G})$ and introduce

$$\widetilde{\boldsymbol{\psi}}(t,\mathbf{x}) = P(\mathbf{G}(t))\boldsymbol{\psi}(\mathbf{x}).$$

We notice that $\widetilde{\boldsymbol{\psi}} \in L^\infty(0,T; L^2(\mathbb{R}^3_+))$, $\operatorname{div} \widetilde{\boldsymbol{\psi}} = 0$, $D(\widetilde{\boldsymbol{\psi}}) = 0$ in $\mathcal{B}(\mathbf{G}(t), 1)$ and

$$\overline{I}_k := \int_0^T \left(\int_{\mathcal{O}} \rho_{\mathbf{G}} \left(\overline{\mathbf{u}}^k(t) - \mathbf{u}(t)\right) \cdot \widetilde{\boldsymbol{\psi}} \; \mathrm{d}\mathbf{y} \right)^2 \, \mathrm{d}t.$$

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We set

$$\widetilde{I}_k := \int_0^T \left(\int_{\mathcal{O}} \rho_{\mathbf{G}} \left(\mathbf{u}^k(t) - \mathbf{u}(t) \right) \cdot \widetilde{\boldsymbol{\psi}} \, \mathrm{d}\mathbf{y} \right)^2 \, \mathrm{d}t$$

and notice that, as before, if $\widetilde{I}_k$ goes to 0, then $\overline{I}_k$ goes to 0. However, since $\widetilde{\boldsymbol{\psi}} \in L^\infty(0,T;L^2(\mathbb{R}^3_+))$ we can take $\mathbf{U} = \widetilde{\boldsymbol{\psi}}$ in (2.31). Consequently, we obtain that there exists a subsequence we do not relabel such that $\overline{I}_k \to 0$. This ends the proof of L^2-compactness of the sequence $\mathbf{u}^k$. In particular we conclude from (2.24) that

$$\left\| \mathbf{u}^k - \mathbf{u} \right\|_{L^2(0,T;L^2(\mathcal{O}))} \to 0.$$

Combining (2.33) and the above relation, we obtain that

$$\left\| \overline{\mathbf{u}}^k - \mathbf{u} \right\|_{L^2(0,T;L^2(\mathcal{B}_{\mathbf{G}}))} \to 0.$$

This implies that

$$\dot{\mathbf{G}}^k \to \dot{\mathbf{G}} \quad \text{and} \quad \boldsymbol{\omega}^k \to \boldsymbol{\omega} \quad \text{in } L^2(0,T).$$

Using the above compactness, we can pass to the limit in (2.23) in the first three terms. For the two terms of the last line of (2.23), we proceed as follows. First, we can notice that

$$\int_{\mathcal{F}_{\mathbf{G}^k(t)}} \left| \mathbf{u}^k_{\varepsilon^k} - \mathbf{u}^k \right|^2 \, \mathrm{d}\mathbf{x} \leqslant C\varepsilon^k \|\mathbf{u}^k\|^2_{H^1(\mathcal{F}_{\mathbf{G}^k(t)})}$$

and thus

$$\int_0^T \int_{\mathcal{F}_{\mathbf{G}^k(t)}} \left[((\mathbf{u}^k_{\varepsilon^k} - \mathbf{u}^k) \cdot \nabla)\mathbf{U} \right] \cdot \mathbf{u}^k \, \mathrm{d}\mathbf{x} \to 0,$$

as $k \to \infty$.

Second, from the choice of κ and K_ε (see (2.7)), we can easily check that

$$\mathbf{u}^k_{\varepsilon^k} = \dot{\mathbf{G}}^k + \boldsymbol{\omega}^k \times (\mathbf{x} - \mathbf{G}^k) \quad \text{in } \mathcal{B}(\mathbf{G}^k(t), 1 - \varepsilon^k).$$

As a consequence, using Lemma 4.10 in [5], we conclude that

$$\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}} \left| \mathbf{u}^k_{\varepsilon^k} - \mathbf{u}^k \right|^2 \, \mathrm{d}\sigma \leqslant C\varepsilon^k \int_{\mathcal{B}_{\mathbf{G}^k(t)}} \left| \nabla \mathbf{u}^k_{\varepsilon^k} - \nabla \mathbf{u}^k \right|^2 \, \mathrm{d}\mathbf{x}.$$

The above inequality implies that

$$-\frac{1}{2} \int_0^T \int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}} (\mathbf{U} \cdot \mathbf{u}^k)((\mathbf{u}^k_{\varepsilon^k} - \mathbf{u}^k) \cdot \mathbf{n}) \, \mathrm{d}\sigma \to 0,$$

as $k \to \infty$.

Finally, we can pass to the limit in (2.23) and obtain the weak formulation (2.4) for smooth test functions. We can pass from smooth test functions to the required regularity for $\mathbf{v}$ by applying the same approximation technique as when we obtained (2.29).

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3. Constructing test functions. Let us begin with some notation. We introduce (r,θ,z), the cylindrical coordinates associated to (x_1,x_2,x_3):

$$x_1 = r\cos(\theta), \quad x_2 = r\sin(\theta), \quad x_3 = z.$$

Given $h>0$ and $l>0$, we denote by $\Omega_{h,l}$ the cylindric domain under $\mathcal{B}_h$ with radius l:

$$\Omega_{h,l} := \{(r,\theta,z) \in \mathcal{F}_h \text{ such that } r \in [0,l), z \in (0,1+h)\}. \tag{3.1}$$

We notice that whenever $l<1$, the upper boundary of $\Omega_{h,\delta}$ is parametrized by (r,θ):

$$(r,\theta,z) \in \partial\Omega_{h,l} \cap \partial\mathcal{B}_h \Leftrightarrow \{r \in [0,\delta) \text{ and } z = \delta_h(r)\},$$

where, for arbitrary nonnegative h,

$$\delta_h(s) := 1 + h - \sqrt{1-s^2} \quad \forall\, s \in [0,1). \tag{3.2}$$

As in [9], we estimate the distance between $\mathcal{B}$ and $\mathcal{P}$ from below with a suitable choice of test function in the weak formulation. To this end, we introduce an approximation of the Stokes solution for a given position of $\mathcal{B}$ in $\mathbb{R}^3_+$ (namely, $\mathcal{B}_h$). We call these approximations "static functions" and denote them by $(\mathbf{w}_h)_{h>0}$. Given a weak solution $(\mathbf{u},\mathbf{G})$ to (FSIS) in $(0,T)$, we construct admissible test functions by setting

$$\begin{array}{rcll} \widetilde{\mathbf{w}}: & (0,T)\times\mathbb{R}^3_+ & \longrightarrow & \mathbb{R}^3, \\ & (t,\mathbf{x}) & \longmapsto & \zeta(t)\mathbf{w}_{h(t)}(x_1 - G_1(t), x_2 - G_2(t), x_3) \end{array} \tag{3.3}$$

for arbitrary $\zeta \in \mathcal{D}(0,T)$. In this definition $h(t)$ stands for the distance between the sphere and the ramp at time t.

Applying the weak formulation, we obtain

$$\int_0^T \int_{\mathbb{R}^3_+} [\rho_{\mathbf{G}}\mathbf{u}\cdot\widetilde{\mathbf{w}}_t + (\mathbf{u}\otimes\mathbf{u} - 2\mu\mathrm{D}(\mathbf{u})) : \mathrm{D}(\widetilde{\mathbf{w}})] \quad \mathrm{d}\mathbf{x}\ \mathrm{d}t = 0.$$

In this equation, the key ingredient is

$$\int_{\mathbb{R}^3_+} \mathrm{D}(\mathbf{u}) : \mathrm{D}(\mathbf{w}_h)\ \mathrm{d}\mathbf{x}.$$

It shall behave like $\dot{h}/h^{\alpha}$ with an exponent α to be made precise. The other terms appear as remainders. We shall bound them by an integrable (in time) function. This relies on the following lemma.

Lemma 3.1. *Given $h>0$, $r_0>0$, and $(\mathbf{u},\mathbf{w}) \in H^1_0(\mathbb{R}^3_+) \times (\mathbb{H}(\mathbf{G}_h) \cap \mathcal{C}^\infty(\mathcal{F}_h))$, we assume $\mathbf{w}$ is with compact support. Then there exists C depending only on the size of the support of $\mathbf{w}$ such that*

$$\left| \int_{\mathbb{R}^3_+} \mathbf{u}\cdot\mathbf{w} \quad \mathrm{d}\mathbf{x} \right| \leqslant C\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^3_+)} \left[\|\mathbf{w}\|_{2,2} + \|\mathbf{w}\|_{L^2\left(\mathbb{R}^3_+\setminus\Omega_{h,r_0}\right)} \right], \tag{3.4}$$

where

$$\|\mathbf{w}\|^2_{2,2} = \int_0^{r_0} \left(\delta_h(r)^2 \left[\int_0^{\delta_h(r)} \sup_{\theta\in(0,2\pi)} \{|\mathbf{w}(r,\theta,z)|^2)\}\ \mathrm{d}z \right] \right) r\ \mathrm{d}r.$$

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If, moreover, $\mathbf{w} \in \mathbb{V}(\mathbf{G}_h)$*, we have*

$$\left| \int_{\mathbb{R}^3_+} \mathbf{u} \otimes \mathbf{u} : D(\mathbf{w}) \ \ \mathrm{d}\mathbf{x} \right| \leqslant C \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{R}^3_+)} \left[\|\mathbf{w}\|_{\infty,2} + \|D(\mathbf{w})\|_{L^\infty\left(\mathcal{F}_h \setminus \Omega_{h,r_0}\right)} \right], \tag{3.5}$$

where

$$\|\mathbf{w}\|_{\infty,2} = \sup_{r \in (0,r_0)} \left\{ \delta_h(r)^{\frac{3}{2}} \left[\int_0^{\delta_h(r)} \sup_{\theta \in (0,2\pi)} \{ |\nabla \mathbf{w}(r,\theta,z)|^2 \} \ \mathrm{d}z \right]^{\frac{1}{2}} \right\}.$$

Proof. We denote by I_1 and I_2 the two integrals we want to estimate in (3.5) and (3.4).

We first deal with I_1. As $D(\mathbf{w}) = 0$ in $\mathcal{B}_h$, we might restrict the integration domain to $\mathcal{F}_h$. We split the integral into an integral in $\mathcal{F}_h \setminus \Omega_{h,r_0}$ and an integral in Ω_{h,r_0}: $I_1 = I_1^{in} + I_1^{out}$ with

$$|I_1^{out}| = \left| \int_{\mathcal{F}_h \setminus \Omega_{h,r_0}} \mathbf{u} \otimes \mathbf{u} : D(\mathbf{w}) \ \mathrm{d}\mathbf{x} \right| \leqslant \|\mathbf{u}\|^2_{L^2(\mathrm{Supp}(\mathbf{w}))} \|D(\mathbf{w})\|_{L^\infty\left(\mathcal{F}_h \setminus \Omega_{h,r_0}\right)}.$$

Because $\mathrm{Supp}(\mathbf{w})$ is bounded and $\mathbf{u} \in H^1_0(\mathbb{R}^3_+)$, we can use the Poincaré inequality. Concerning the integral in Ω_{h,r_0}, we have

$$I_1^{in} = \int_0^{2\pi} \int_0^{r_0} \int_0^{\delta_h(r)} [\mathbf{u}(r,\theta,z) \otimes \mathbf{u}(r,\theta,z) : D(\mathbf{w})(r,\theta,z)] \, r \ \mathrm{d}z \ \mathrm{d}r \ \mathrm{d}\theta.$$

Using a Hölder inequality with respect to the z-variable, we deduce

$$|I_1^{in}| \leqslant C \int_0^{2\pi} \int_0^{r_0} \left[\int_0^{\delta_h(r)} |\mathbf{u}(r,\theta,z)|^4 \mathrm{d}z \right]^{\frac{1}{2}} \left[\int_0^{\delta_h(r)} |D(\mathbf{w})|^2 \ \mathrm{d}z \right]^{\frac{1}{2}} r \ \mathrm{d}r \ \mathrm{d}\theta.$$

Then a direct generalization of the Poincaré inequality (see Lemma 12 in [9]) implies

$$\left[\int_0^{\delta_h(r)} |\mathbf{u}(r,\theta,z)|^4 \ \mathrm{d}z \right]^{\frac{1}{2}} \leqslant C \delta_h(r)^{\frac{3}{2}} \left[\int_0^{\delta_h(r)} |\nabla \mathbf{u}(r,\theta,z)|^2 \ \mathrm{d}z \right].$$

Substituting this in I_1^{in} and using again a Hölder inequality, we then obtain (3.5).

To estimate I_2, we decompose it in the same manner as I_1, and with the same proof, we deduce that there exists $C = C(\mathrm{Supp}(\mathbf{w}))$ such that

$$|I_2^{out}| \leqslant C \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3_+)} \|\mathbf{w}\|_{L^2\left(\mathbb{R}^3_+ \setminus \Omega_{h,r_0}\right)}.$$

It remains to estimate the integral in Ω_{h,r_0}:

$$I_2^{in} = \int_0^{2\pi} \int_0^{r_0} \int_0^{\delta_h(r)} [\mathbf{u}(r,\theta,z) \cdot \mathbf{w}(r,\theta,z)] \, r \ \mathrm{d}z \ \mathrm{d}r \ \mathrm{d}\theta.$$

As above, a Hölder inequality in the z-variable associated to the Poincaré inequality implies

$$|I_2^{in}| \leqslant C \int_0^{2\pi} \int_0^{r_0} \left[\int_0^{\delta_h(r)} |\nabla \mathbf{u}(r,\theta,z)|^2 \ \mathrm{d}z \right]^{\frac{1}{2}} \delta_h(r) \left[\int_0^{\delta_h(r)} |\mathbf{w}|^2 \ \mathrm{d}z \right]^{\frac{1}{2}} r \ \mathrm{d}r \ \mathrm{d}\theta.$$

We conclude by using a Cauchy–Schwarz inequality. ∎

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3.1. Explicit formula. From now on h is a fixed positive parameter. As in [9], we introduce a velocity field which is a good approximation (in a sense to be made precise) to the solution to the Stokes problem

$$\left\{\begin{array}{r} \operatorname{div}\mathbb{T}(\mathbf{w},q)=0 \\ \operatorname{div}\mathbf{w}=0 \\ \mathbf{w}_{|\mathcal{P}}=0, \\ \mathbf{w}_{|\partial\mathcal{B}_h}=\mathbf{e}_3. \end{array}\right. \quad \text{in } \mathcal{F}_h, \tag{3.6}$$

At first, we focus on the divergence-free and boundary conditions. So we introduce a potential vector field $\mathbf{a}_h$ and set $\mathbf{w}_h=\operatorname{curl}\mathbf{a}_h$. One choice for $\mathbf{a}_h$ could be

$$\mathbf{a}_h^s(\mathbf{x}):=\frac{\eta_{h_0}(|\mathbf{x}-\mathbf{G}_h|-1)}{2}(\mathbf{e}_3\times(\mathbf{x}-\mathbf{G}_h)) \quad \forall\,\mathbf{x}\in\mathcal{F}_h, \quad \text{with } h_0>0.$$

The field $\mathbf{w}_h^s:=\operatorname{curl}\mathbf{a}_h^s$ satisfies the divergence-free and boundary conditions regardless of the value of $h_0<h$. However, when h goes to 0, this particular velocity field does not take advantage of the particular shape of the aperture between $\mathcal{B}$ and $\mathcal{P}$. Thus, we need to find another velocity field, especially in this aperture, in $\Omega_{h,1/2}$.

As we want to obtain an approximation of the Stokes problem, we construct a velocity field in which the fluid escapes from under the sphere with the most efficiency. Consequently, we want the velocity field to be planar and radial in each plane. Thus, our potential vector field reads, in cylindrical coordinates,

$$\mathbf{a}_h^d(r,\theta,z)=\left(-\phi_h^d(r,z)\sin(\theta),\phi_h^d(r,z)\cos(\theta),0\right)^\top \quad \forall\,(r,\theta,z)\in\Omega_{h,1/2},$$

so that, for all $(r,\theta,z)\in\Omega_{h,1/2}$,

$$\mathbf{w}_h^d(r,\theta,z)=\left(-\partial_z\phi_h^d(r,z)\cos(\theta),-\partial_z\phi_h^d(r,z)\sin(\theta),\partial_r\phi_h^d(r,z)+\frac{\phi_h^d(r,z)}{r}\right)^\top.$$

We set, in order to fit boundary conditions (this shall be critical in Lemma 3.2),

$$\phi_h^d(r,z)=r\chi_o\left(\frac{z}{\delta_h(r)}\right), \text{ with } \chi_o(s)=\frac{s^2(3-2s)}{2} \quad \forall\,s\in(0,1).$$

From now on, we set $h_0=(\sqrt{17/16}-1)/2$. It remains to interpolate $\mathbf{w}_h^s$ and $\mathbf{w}_h^d$ so that we obtain

$$\mathbf{a}_h(\mathbf{x})=\left\{\begin{array}{rl} \eta_{1/2}(r)\mathbf{a}_h^d(r,\theta,z)+(1-\eta_{1/2}(r))\mathbf{a}_h^s(\mathbf{x}) & \text{in } \Omega_{h,1/2}, \\ \mathbf{a}_h^s(\mathbf{x}) & \text{in } \mathbb{R}_+^3\setminus\Omega_{h,1/2} \end{array}\right.$$

and $\mathbf{w}_h=\operatorname{curl}\mathbf{a}_h$. Explicitly, in $\Omega_{h,1/2}$, we have

$$\mathbf{w}_h(r,\theta,z)=\eta_{1/2}(r)\mathbf{w}_h^d(r,\theta,z)+(1-\eta_{1/2}(r))\mathbf{w}_h^s(\mathbf{x})+\mathbf{rem}_0(\mathbf{x}), \tag{3.7}$$

where, denoting by $\boldsymbol{n\pi}_3(\mathbf{x})=(x_1,x_2,0)^\top/\sqrt{x_1^2+x_2^2}$, we have

$$\mathbf{rem}_0(\mathbf{x})=\eta_{1/2}'(r)\boldsymbol{n\pi}_3(\mathbf{x})\times\left(\mathbf{a}_h^d(r,\theta,z)-\mathbf{a}_h^s(\mathbf{x})\right) \quad \text{in } \Omega_{h,1/2}.$$

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3.2. From static to moving test function. The main point in this subsection is to prove that, given a weak solution to (FSIS) $(\mathbf{u}, \mathbf{G})$ and $\zeta \in \mathcal{D}(0,T)$, the function $\widetilde{\mathbf{w}}$ constructed in (3.3) is a suitable test function. To this end, we need to extend $\mathbf{w}_h$ on $\mathbb{R}^3_+$ first. This is possible thanks to the following technical result.

LEMMA 3.2. *Given $h > 0$, we have*

$$\begin{aligned} &\mathbf{w}_h(\mathbf{x}) = \mathbf{e}_3, \quad \mathbf{a}_h(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2 \quad \forall \mathbf{x} \in \partial\mathcal{B}_h, \\ &\mathbf{w}_h(\mathbf{x}) = 0, \quad \mathbf{a}_h(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathcal{P}. \end{aligned}$$

Proof. We set $\lambda = z/\delta_h(r)$ and differentiations of λ by subscripts. We have

$$\partial_z \phi_h^d(r,z) = r\lambda_z \chi_o'(\lambda), \quad \partial_r \phi_h^d(r,z) = \chi_o(\lambda) + r\lambda_r \chi_o'(\lambda). \tag{3.8}$$

Computing with the value of λ yields

$$\lambda_z = \frac{1}{\delta_h(r)}, \quad \lambda_r = -\frac{z\delta_h'(r)}{(\delta_h(r))^2}.$$

Our choice for χ_o implies that

$$\chi_o(0) = \chi_o'(0) = 0, \quad \chi_o(1) = \frac{1}{2}, \quad \chi_o'(1) = 0.$$

Replacing λ by 0 in (3.8) yields

$$\phi_h^d(r,z) = \partial_z \phi_h^d(r,z) = \partial_r \phi_h^d(r,z) = 0 \quad \text{on } \mathcal{P}.$$

Consequently, $\mathbf{a}_h^d = \mathbf{w}_h^d = 0$ on $\mathcal{P}$. Replacing λ by 1,

$$\partial_z \phi_h^d(r,z) = 0, \quad \phi_h^d(r,z) = \frac{r}{2}, \quad \partial_r \phi_h^d(r,z) = \frac{1}{2} \quad \text{on } \partial\mathcal{B}_h.$$

Consequently, $\mathbf{a}_h^d(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2$ and $\mathbf{w}_h^d = \mathbf{e}_3$ on $\mathcal{B}_h$.

Concerning the smooth part, a straightforward computation leads to

$$\mathbf{w}_h^s(\mathbf{x}) = \eta_{h_0}(|\mathbf{x} - \mathbf{G}_h| - 1)\mathbf{e}_3 + \eta_{h_0}'(|\mathbf{x} - \mathbf{G}_h| - 1)\frac{\mathbf{x} - \mathbf{G}_h}{|\mathbf{x} - \mathbf{G}_h|} \times \frac{(\mathbf{e}_3 \times (\mathbf{x} - \mathbf{G}_h))}{2}.$$

Due to our choice, we have

$$\eta_{h_0}(|\mathbf{x} - \mathbf{G}_h| - 1) = 1, \quad \eta_{h_0}'(|\mathbf{x} - \mathbf{G}_h| - 1) = 0 \quad \text{on } \partial\mathcal{B}_h.$$

Consequently $\mathbf{a}_h^s(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2$ and $\mathbf{w}_h^s = \mathbf{e}_3$ on $\partial\mathcal{B}_h$. Then

$$\eta_{h_0}(|\mathbf{x} - \mathbf{G}_h| - 1) = 0, \quad \eta_{h_0}'(|\mathbf{x} - \mathbf{G}_h| - 1) = 0 \quad \text{if } |\mathbf{x} - \mathbf{G}_h| \geqslant 1 + 2h_0 = \sqrt{17/16}.$$

Moreover, if $\mathbf{x} \in \mathcal{P} \setminus \overline{\Omega_{h,1/4}}$, we have $r > 1/4$ and, as $\mathbf{G}_h = (0,0,1+h)$,

$$|\mathbf{x} - \mathbf{G}_h|^2 > (1+h)^2 + (1/4)^2 > 17/16.$$

Consequently, $\mathbf{a}_h^s(\mathbf{x}) = \mathbf{w}_h^s(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathcal{P} \setminus \overline{\Omega_{h,1/4}}$.

It remains to check that boundary conditions are satisfied in the transition region, i.e., when $\mathbf{x} \in \overline{\Omega_{h,1/2}} \setminus \Omega_{h,1/4}$. On $\mathcal{P}$, we remark that $\mathbf{w}_h^d(\mathbf{x}) = \mathbf{w}_h^s(\mathbf{x}) = 0 = \mathbf{a}_h^s(\mathbf{x}) =$

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$\mathbf{a}_h^d(\mathbf{x}) = 0$. Interpolating the potential vector fields, we obtain $\mathbf{w}_h = 0$ on $\mathcal{P}$. Finally, on $\overline{\mathcal{B}}_h \cap \overline{\Omega_{h,1/4}}$ we have already computed

$$\mathbf{w}_h^d(\mathbf{x}) = \mathbf{w}_h^s(\mathbf{x}) = \mathbf{e}_3 \quad \text{and} \quad \mathbf{a}_h^s(\mathbf{x}) = \mathbf{a}_h^d(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2.$$

Interpolating the potential vector fields, we deduce $\mathbf{w}_h(\mathbf{x}) = \mathbf{e}_3$. This concludes the proof. □

Remark 3.1. According to this lemma, we extend $\mathbf{a}_h$ (resp., $\mathbf{w}_h$) to $\mathbb{R}^3_+$ with the value $(\mathbf{e}_3 \times \mathbf{x})/2$ (resp., $\mathbf{e}_3$) in $\mathcal{B}_h$. In what follows, we consider the functions $\mathbf{a} : (h, \mathbf{x}) \to \mathbf{a}_h(\mathbf{x})$ and $\mathbf{w} : (h, \mathbf{x}) \to \mathbf{w}_h(\mathbf{x})$. Denoting by $\mathcal{Q}_c = \{(h, \mathbf{x}) \in (0, 1) \times \mathbb{R}^3 \; ; \; \mathbf{x} \in \mathcal{B}_h\}$, standard analytic arguments imply $\mathbf{a} \in \mathcal{C}^\infty(\mathcal{Q}_c) \cap \mathcal{C}^\infty(((0,1) \times \mathbb{R}^3_+) \setminus \overline{\mathcal{Q}}_c)$. We note that $\mathbf{w}_h$ vanishes as soon as $|\mathbf{x} - \mathbf{G}_h| > (\sqrt{17/16} - 1)/2$ and $|\mathbf{x}| > 1/2$. Consequently, the above lemma implies $\mathbf{w} \in H^1((\bar{h}, 1) \times \mathbb{R}^3_+)$ for any $\bar{h} > 0$ and, after standard composition arguments, this yields

$$\widetilde{\mathbf{w}} \in \mathcal{C}([0, T]; \mathrm{H}^1(\mathbb{R}^3_+)) \cap H^1(0, T; \mathrm{L}^2(\mathbb{R}^3_+))$$

as long as $h(t) \in (\bar{h}, 1]$ for all $t \in (0, T)$. So, $\widetilde{\mathbf{w}}$ is a suitable test function for the weak formulation as long as $h(0, T) \subset (\bar{h}, 1)$.

3.3. Estimate of remainder terms. In order to exploit the weak formulation with our test function, we need to dominate remainder terms according to Lemma 3.1. We begin with estimates on Sobolev norms of $\mathbf{w}_h$.

By construction, our test functions behave differently under the sphere (in $\Omega_{h,\delta}$) and above the sphere (in $\mathcal{F}_h \setminus \Omega_{h,\delta}$ for arbitrary fixed $\delta > 0$). Above the sphere we have the following.

LEMMA 3.3. *Given $\alpha \geqslant 0$ and $\delta > 0$ there exists $C(\alpha, \delta) < \infty$ such that*

$$\|\mathbf{a}_h\|_{H^\alpha(\mathcal{F}_h \setminus \Omega_{h,\delta})} \leqslant C(\alpha, \delta) \quad \forall \, h \in (0, 1).$$

Proof. By construction the restriction $\mathbf{a} : \mathcal{Q}_{c,\delta} \to \mathbb{R}^3$, with

$$\mathcal{Q}_{c,\delta} := \{(h, \mathbf{x}) \in [0, 1] \times \overline{\mathbb{R}^3_+} \quad \text{with } \mathbf{x} \notin \Omega_{h,\delta}\},$$

is smooth and with compact support. □

Inside $\Omega_{h,1/4}$, estimates rely essentially on dominations of integrals:

$$\int_0^{\frac{1}{4}} \frac{r^\alpha \, \mathrm{d}r}{[\delta_h(r)]^\beta}.$$

We refer the reader to the appendix for such computations.

LEMMA 3.4. *The family $(\mathbf{w}_h)_{0<h<1}$ is uniformly bounded in $L^2(\mathbb{R}^3_+)$.*

Proof. Because of the previous lemma, we focus on $\mathbf{w}_h^d$ inside $\Omega_{h,1/4}$.

In this region, we have

$$(\mathbf{w}_h^d(r, \theta, z))_1 = -\partial_z \phi_h^d(r, z) \cos(\theta), \quad (\mathbf{w}_h^d(r, \theta, z))_2 = -\partial_z \phi_h^d(r, z) \sin(\theta),$$

and

$$(\mathbf{w}_h^d(r, \theta, z))_3 = \partial_r \phi_h^d(r, z) + \frac{\phi_h^d(r, z)}{r}.$$

Thus

$$|\mathbf{w}_h^d(r, \theta, z)| \leqslant |\partial_z \phi_h^d(r, z)| + |\partial_r \phi_h^d(r, z)| + \frac{|\phi_h^d(r, z)|}{r}.$$

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Applying Lemma A.3, this leads to

$$|\mathbf{w}_h^d(r,\theta,z)| \leqslant C\left(1+\frac{r}{\delta_h(r)}\right) \quad \forall\,(r,\theta,z)\in\Omega_{h,1/4}, \quad \forall\, h\in(0,1).$$

The result then follows from Lemma A.1 with $(\alpha,\beta)=(3,1)$. ∎

As a technical device for applying Lemma 3.1, we have the following.

LEMMA 3.5. *Let us define*

$$w_h(r,\theta,z) = |\partial_r \mathbf{w}_h^d(r,\theta,z)| + \frac{|\partial_\theta \mathbf{w}_h^d(r,\theta,z)|}{r} + |\partial_h \mathbf{w}_h^d(r,\theta,z)| \quad \forall\,(r,\theta,z)\in\Omega_{h,1/4}.$$

Then

$$\int_0^{\frac{1}{4}} \left(\delta_h(r)^2\left[\int_0^{\delta_h(r)} \sup_{\theta\in(0,2\pi)} |w_h(r,\theta,z)|^2 \,\mathrm{d}z\right]\right) r\,\mathrm{d}r \tag{3.9}$$

is uniformly bounded for $h\in(0,1)$.

Proof. A straightforward computation yields, for all $(r,\theta,z)\in\Omega_{h,1/4}$,

$$|\partial_r \mathbf{w}_h^d(r,\theta,z)| \leqslant C\left(|\partial_{rz}\phi_h^d(r,z)| + |\partial_{rr}\phi_h^d(r,z)| + \left|\frac{\partial_r\phi_h^d(r,z)}{r} - \frac{\phi_h^d(r,z)}{r^2}\right|\right)$$

and

$$\frac{|\partial_\theta \mathbf{w}_h^d(r,\theta,z)|}{r} \leqslant \frac{|\partial_z \phi_h^d(r,z)|}{r},$$
$$|\partial_h \mathbf{w}_h^d(r,\theta,z)| \leqslant \frac{|\partial_h \phi_h^d(r,z)|}{r} + |\partial_{hr}\phi_h^d(r,z)| + |\partial_{hz}\phi_h^d(r,z)|.$$

Combining the above inequalities with Lemma A.3, we deduce there exists a constant C independent of h such that

$$|w_h(r,\theta,z)| \leqslant C\left(\frac{1}{\delta_h(r)} + \frac{r}{\delta_h(r)^2}\right)$$

for all $(r,\theta,z)\in\Omega_{h,1/4}$. Consequently,

$$\int_0^{1/4} \left(\delta_h(r)^2\left[\int_0^{\delta_h(r)} \sup_{\theta\in(0,2\pi)} |w_h(r,\theta,z)|^2 \,\mathrm{d}z\right]\right) r\,\mathrm{d}r \leqslant C\int_0^{1/4} (\delta_h(r)+1) r\,\mathrm{d}r.$$

As the last integral remains bounded when h goes to 0, the same holds for the integral of w_h. ∎

Then, to dominate the trilinear form, we need the following result.

LEMMA 3.6. *We set*

$$dw_h(r,\theta,z) = |\partial_r \mathbf{w}_h^d(r,\theta,z)| + \frac{|\partial_\theta \mathbf{w}_h^d(r,\theta,z)|}{r} + |\partial_z \mathbf{w}_h^d(r,\theta,z)| \quad \forall\,(r,\theta,z)\in\Omega_{h,1/4}.$$

Then the quantity

$$\sup_{r\in(0,1/4)} \left\{\delta_h(r)^{\frac{3}{2}}\left[\int_0^{\delta_h(r)} \sup_{\theta\in(0,2\pi)} \{|dw_h(r,\theta,z)|^2\}\,\mathrm{d}z\right]^{\frac{1}{2}}\right\} \tag{3.10}$$

is uniformly bounded for $h\in(0,1)$.

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Proof. As in the previous proof, there exists a constant C independent of h such that

$$|dw_h(r,\theta,z)| \leqslant C\left(\frac{1}{\delta_h(r)} + \frac{r}{\delta_h(r)^2}\right)$$

for all $(r,\theta,z) \in \Omega_{h,1/4}$. Therefore

$$\int_0^{\delta_h(r)} |dw_h(r,z)|^2 \, \mathrm{d}z \leqslant C\left(\frac{1}{\delta_h(r)} + \frac{r^2}{\delta_h(r)^3}\right)$$

and

$$\sup_{r\in(0,1/4)}\left\{\delta_h(r)^{\frac{3}{2}}\left[\int_0^{\delta_h(r)} \sup_{\theta\in(0,2\pi)}\{|dw_h(r,\theta,z)|^2\}\, \mathrm{d}z\right]^{\frac{1}{2}}\right\} \leqslant C \sup_{r\in(0,1/4)} (\delta_h(r)+r),$$

which is uniformly bounded when $h \in (0,1)$. ∎

Finally, there holds the following lemma, which is reminiscent of works by Starovoitov.

LEMMA 3.7. *There exists a constant $C > 0$ such that*

$$|\nabla \mathbf{w}_h|_2^2 \geqslant \frac{C}{h} \quad \forall\, h \in (0,1).$$

Proof. Given $h > 0$ we already noticed that

$$\mathbf{w}_h(\mathbf{x}) = \mathbf{w}_h^d(\mathbf{x}) \quad \forall\, \mathbf{x} \in \Omega_{h,1/4}.$$

Consequently,

$$|\nabla \mathbf{w}_h(\mathbf{x})| \geqslant |\partial_z \mathbf{w}_h^d(\mathbf{x})| \quad \forall\, \mathbf{x} \in \Omega_{h,1/4}.$$

With the explicit formula for $\mathbf{w}_h^d$, we have $|\partial_z \mathbf{w}_h^d| \geqslant |\partial_{zz}\phi_h^d|$, where $|\partial_{zz}\phi_h^d(r,z)| = \dfrac{r}{\delta_h^2(r)}\chi_o''(\lambda)$. Consequently,

$$|\nabla \mathbf{w}_h|_2^2 \geqslant 2\pi \int_0^{\frac{1}{4}} \frac{r^3 \, \mathrm{d}r}{\delta_h(r)^3} \int_0^1 |\chi_o''(s)|^2 \, \mathrm{d}s.$$

As χ is a polynomial with degree 3, its second derivative does not vanish, and neither does the s-integral. Then we obtain the result by applying Lemma A.1 with $\alpha = \beta = 3$. ∎

3.4. Our test function and the Stokes problem. First, we prove that our choice is a good one because it is a good approximation of the solution to the Stokes problem.

LEMMA 3.8. *There exist $q_h \in \mathcal{C}^\infty(\overline{\mathcal{F}_h})$ and $\mathbf{f}_h \in \mathcal{C}_c^\infty(\overline{\mathcal{F}_h})$ such that*

$$\begin{cases} \mu\Delta \mathbf{w}_h - \nabla q_h = \mathbf{f}_h \\ \operatorname{div} \mathbf{w}_h = 0 \end{cases} \quad \textit{in } \mathcal{F}_h, \tag{3.11}$$

where there exists an absolute constant C for which

$$\int_0^{2\pi}\int_0^{\frac{1}{4}}\left(\delta_h(r)^2\left[\int_0^{\delta_h(r)} |\mathbf{f}_h|^2 \, \mathrm{d}z\right]\right) r \, \mathrm{d}r \, \mathrm{d}\theta + \|\mathbf{f}_h\|^2_{L^2(\mathcal{F}_h\setminus\Omega_{h,1/4})} \leqslant C.$$

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Proof. By construction, we have $\mathbf{w}_h = \operatorname{curl} \widetilde{\mathbf{a}}_h^d + \operatorname{curl} \widetilde{\mathbf{a}}_h^s$, where

$$\widetilde{\mathbf{a}}_h^d(\mathbf{x}) = \begin{cases} \eta_{1/2}(r)\mathbf{a}_h^d(\mathbf{x}), & \mathbf{x} \in \Omega_{h,1/2}, \\ 0 & \text{else}, \end{cases} \quad \widetilde{\mathbf{a}}_h^s(\mathbf{x}) = \begin{cases} (1-\eta_{1/2}(r))\mathbf{a}_h^s(\mathbf{x}), & \mathbf{x} \in \Omega_{h,1/2}, \\ \mathbf{a}_h^s(\mathbf{x}) & \text{else}. \end{cases}$$

Then, according to Lemma 3.3, the smooth part $\widetilde{\mathbf{a}}_h^s$ is bounded in any Sobolev space uniformly in h. Consequently, $\widetilde{\mathbf{f}}_h = \mu\Delta \operatorname{curl} \widetilde{\mathbf{a}}_h^s$ is bounded in all Sobolev spaces. We have

$$\mu\Delta\mathbf{w}_h = \mu\Delta \operatorname{curl} \widetilde{\mathbf{a}}_h^d + \widetilde{\mathbf{f}}_h.$$

In the following we write $\widetilde{\phi}_h^d(r,z) = \eta_{1/2}(r)\phi_h^d(r,z)$ for all $(r,\theta,z) \in \Omega_{h,1/2}$. Let us recall that in cylindrical coordinates we have

$$\Delta = \frac{\partial_r[r\partial_r]}{r} + \frac{\partial_{\theta\theta}}{r^2} + \partial_{zz}.$$

Consequently,

$$\left[\Delta \operatorname{curl} \widetilde{\mathbf{a}}_h^d\right]_1 = -\left[\partial_{rrz}\widetilde{\phi}_h^d + \frac{\partial_{rz}\widetilde{\phi}_h^d}{r} - \frac{\partial_z\widetilde{\phi}_h^d}{r^2} + \partial_{zzz}\widetilde{\phi}_h^d\right]\cos(\theta)$$

and

$$\left[\Delta \operatorname{curl} \widetilde{\mathbf{a}}_h^d\right]_2 = -\left[\partial_{rrz}\widetilde{\phi}_h^d + \frac{\partial_{rz}\widetilde{\phi}_h^d}{r} - \frac{\partial_z\widetilde{\phi}_h^d}{r^2} + \partial_{zzz}\widetilde{\phi}_h^d\right]\sin(\theta),$$

$$\left[\Delta \operatorname{curl} \widetilde{\mathbf{a}}_h^d\right]_3 = \partial_{rrr}\widetilde{\phi}_h^d + 2\frac{\partial_{rr}\widetilde{\phi}_h^d}{r} - \frac{\partial_r\widetilde{\phi}_h^d}{r^2} + \frac{\widetilde{\phi}_h^d}{r^3} + \partial_{zz}\left[\partial_r\widetilde{\phi}_h^d + \frac{\widetilde{\phi}_h^d}{r}\right].$$

We remark here that

$$\partial_{zzz}\widetilde{\phi}_h^d(r,z) = -6\frac{r\,\eta_{1/2}(r)}{\delta_h^3(r)}.$$

Consequently, denoting by Φ a primitive of $s \mapsto -6s\eta_{1/2}(s)/(\delta_h(s))^3$, we have

$$\nabla\Phi(r) = (\partial_{zzz}\phi_h^d\cos(\theta), \partial_{zzz}\phi_h^d\sin(\theta), 0)^\top.$$

We set

$$q_h(\mathbf{x}) = \mu\Phi(r) + \mu\partial_z\left[\partial_r\widetilde{\phi}_h^d + \frac{\widetilde{\phi}_h^d}{r}\right], \quad \check{\mathbf{f}}_h = \mu\Delta \operatorname{curl} \widetilde{\mathbf{a}}_h^d - \nabla q_h.$$

In particular $\mu\Delta\mathbf{w}_h - \nabla q_h = \widetilde{\mathbf{f}}_h + \check{\mathbf{f}}_h$, so that our result follows from the same result for $\check{\mathbf{f}}_h$. Denoting by $\check{f}_1, \check{f}_2, \check{f}_3$ the Cartesian components of $\check{\mathbf{f}}_h$, straightforward computations show that

$$|\check{f}_1|^2 + |\check{f}_2|^2 \leqslant 4\left[\partial_{rrz}\widetilde{\phi}_h^d + \frac{\partial_{rz}\widetilde{\phi}_h^d}{r} - \frac{\partial_z\widetilde{\phi}_h^d}{r^2}\right]^2.$$

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As $\eta'_{1/2}$ vanishes uniformly in $\Omega_{h,1/4}$, Lemma 3.3 implies there exists a universal constant C such that

$$|\check{f}_1|^2 + |\check{f}_2|^2 \leqslant C\left[1 + \left|\partial_{rrz}\phi_h^d + \frac{\partial_{rz}\phi_h^d}{r} - \frac{\partial_z\phi_h^d}{r^2}\right|\right]^2.$$

Then, for the same reasons, we have

$$|\check{f}_3|^2 \leqslant C\left[1 + \left|\partial_{rrr}\phi_h^d\right| + \left|\frac{2\partial_{rr}\phi_h^d}{r}\right| + \left|\frac{\phi_h^d}{r^3} - \frac{\partial_r\phi_h^d}{r^2}\right|\right]^2.$$

Replacing with the size computed in Lemma A.3, we obtain

$$|\check{\mathbf{f}}_h(\mathbf{x})|^2 \leqslant C\left(\frac{r}{\delta_h^2(r)} + \frac{1}{\delta_h(r)}\right)^2.$$

Consequently, for arbitrary $r \in (0,1/2)$

$$\int_0^{\delta_h(r)} |\check{\mathbf{f}}_h|^2 \,\mathrm{d}z \leqslant C\left(\frac{r^2}{\delta_h(r)^3} + \frac{1}{\delta_h(r)}\right)$$

and

$$\int_0^{2\pi}\int_0^{1/2} \delta_h(r)^2 \int_0^{\delta_h(r)} |\check{\mathbf{f}}_h|^2 \,\mathrm{d}z r \,\mathrm{d}r \,\mathrm{d}\theta \leqslant C\int_0^{1/2}\left(\frac{r^3}{\delta_h(r)} + r\delta_h(r)\right) \mathrm{d}r,$$

which is uniformly bounded for $h \in (0,1)$. This concludes the proof. ∎

As a direct corollary, we get the following lemma.

LEMMA 3.9. *There exist $K_m < \infty$ and a function $\widetilde{n}_3 : [0,1] \to \mathbb{R}_+$ such that, for any $h < 1$ and $\mathbf{w} \in \mathbb{V}(\mathbf{G}_h)$ such that $\mathbf{w} = \mathbf{V}_\mathbf{w} + \mathbf{R}_\mathbf{w} \times (\mathbf{x} - \mathbf{G}_h)$ in $\mathcal{B}_h$, we have*

$$\left|2\mu\int_{\mathbb{R}^3_+} D(\mathbf{w}_h) : D(\mathbf{w}) \quad \mathrm{d}\mathbf{x} - \widetilde{n}_3(h)\mathbf{V}_\mathbf{w}\cdot\mathbf{e}_3\right| \leqslant K_m\|\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3_+)}. \tag{3.12}$$

Moreover, there exist $h_m > 0$ and a constant $c > 0$ such that $\widetilde{n}_3(h) \geqslant c/h$ for all $h < h_m$.

Proof. Given $h > 0$ and $\mathbf{w} \in \mathbb{V}(\mathbf{G}_h)$, we apply the Stokes identity with (3.11) and obtain

$$2\mu\int_{\mathbb{R}^3_+} \mathrm{D}(\mathbf{w}_h) : \mathrm{D}(\mathbf{w}) \,\mathrm{d}\mathbf{x} = \int_{\partial\mathcal{B}_h} \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n}\cdot\mathbf{w} \,\mathrm{d}\sigma - \int_{\mathcal{F}_h} \mathbf{f}_h\cdot\mathbf{w} \,\mathrm{d}\mathbf{x}. \tag{3.13}$$

For symmetry reasons, there exists $\widetilde{n}_3 : (0,1) \to \mathbb{R}$ such that

$$\int_{\partial\mathcal{B}_h} \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n} \,\mathrm{d}\sigma = \widetilde{n}_3(h)\mathbf{e}_3$$

and

$$\int_{\partial\mathcal{B}_h} (\mathbf{x} - \mathbf{G}_h) \times \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n} \,\mathrm{d}\sigma = 0.$$

On the other hand, applying (3.4) and Lemma 3.8, we also deduce

$$\left|\int_{\mathcal{F}_h} \mathbf{f}_h\cdot\mathbf{w} \,\mathrm{d}\mathbf{x}\right| \leqslant C\|\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3_+)} \quad \forall\, h \in (0,1),$$

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where C is a positive constant. Finally, we have obtained the existence of a constant K such that, for arbitrary $h \in (0,1)$ and $\mathbf{w} \in \mathbb{V}(\mathbf{G})$,

$$\left| 2\mu \int_{\mathbb{R}^3_+} \mathrm{D}(\mathbf{w}_h) : \mathrm{D}(\mathbf{w}) \; \mathrm{d}\mathbf{x} - \widetilde{n}_3(h)\mathbf{V}_{\mathbf{w}} \cdot \mathbf{e}_3 \right| \leqslant K \|\nabla \mathbf{w}\|_{L^2(\mathbb{R}^3_+)}.$$

In order to estimate $\widetilde{n}_3$, we take $\mathbf{w} = \mathbf{w}_h$ in (3.13) and obtain

$$\widetilde{n}_3 = \int_{\partial \mathcal{B}_h} \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n} \cdot \mathbf{e}_3 \; \mathrm{d}\sigma = 2\mu \int_{\mathbb{R}^3_+} |D(\mathbf{w}_h)|^2 \; \mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h} \mathbf{f}_h \cdot \mathbf{w}_h \; \mathrm{d}\mathbf{x}. \tag{3.14}$$

Dealing as previously with the last integral, we deduce that

$$\left| \int_{\mathcal{F}_h} \mathbf{f}_h \cdot \mathbf{w}_h \; \mathrm{d}\mathbf{x} \right| \leqslant K \|\nabla \mathbf{w}_h\|_{L^2(\mathbb{R}^3_+)} \quad \forall\, h \in (0,1).$$

But, applying Lemma 3.7, we have that

$$2\mu \int_{\mathbb{R}^3_+} |D(\mathbf{w}_h)|^2 \; \mathrm{d}\mathbf{x} = \mu \int_{\mathbb{R}^3_+} |\nabla \mathbf{w}_h|^2 \; \mathrm{d}\mathbf{x} \geqslant \frac{C}{h} \quad \forall\, h \in (0,1).$$

Consequently, the asymptotic behavior of the right-hand side in (3.14) when h goes to 0 is prescribed by the first integral. Hence, there exist $h_m > 0$ and constants $\widetilde{c}, c > 0$ such that

$$\widetilde{n}_3(h) \geqslant \widetilde{c} \int_{\mathbb{R}^3_+} |D(\mathbf{w}_h)|^2 \; \mathrm{d}\mathbf{x} \geqslant \frac{c}{h} \quad \forall\, h \in (0, h_m). \qquad \square$$

4. Proof of Theorem 1.1. We let the reader convince himself that Theorem 1.1 is a direct consequence of the following theorem.

THEOREM 4.1. *Given* $(\mathbf{u}, \mathbf{G})$ *a weak solution to (FSIS) on* $(0,T)$ *with initial data* $(\mathbf{u}^0, \mathbf{G}^0)$, *we assume there exists* $0 \leqslant \tau_0 < \tau_1 \leqslant T$ *for which*

$$h(t) := \mathrm{dist}(\mathcal{B}(t), \mathcal{P}) \leqslant 1 \quad \forall\, t \in [\tau_0, \tau_1].$$

Then there exists $C(\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)}) < \infty$ *depending only on the* L^2*-norm of initial data such that*

$$h(t) \geqslant h(\tau_0) \exp\left[-C(\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)})(1 + \sqrt{T}) \right] \quad \forall\, t \in (\tau_0, \tau_1).$$

The remainder of this paper is devoted to the proof of this result. From now on $(\mathbf{u}, \mathbf{G})$ is a given weak solution to (FSIS) with initial data $(\mathbf{u}^0, \mathbf{G}^0)$. For simplicity, we assume that $h(t) \leqslant 1$ for all $t \in (0,T)$. This means that $\tau_0 = 0$ and $\tau_1 = T$ in the assumptions of our theorem.

As mentioned before, we estimate the distance h from below with our approximation of the Stokes problem. So, from now on, $(\mathbf{w}_h)_{h \in (0,1)}$ are the approximations constructed in section 3.1. Given $0 < t_0 < t_1 < 1$, we set

$$\zeta_\varepsilon(t) = \eta_\varepsilon(\mathrm{dist}(t, [t_0, t_1])).$$

Then $\zeta_\varepsilon \in \mathcal{D}(0,T)$ whenever ε is sufficiently small. Consequently, according to Remark 3.1, for ε sufficiently small

$$\begin{array}{rccl} \widetilde{\mathbf{w}}_\varepsilon : & (0,T) \times \mathbb{R}^3_+ & \longrightarrow & \mathbb{R}^3, \\ & (t, \mathbf{x}) & \longmapsto & \zeta_\varepsilon(t)\mathbf{w}_{h(t)}(x_1 - G_1(t), x_2 - G_2(t), x_3) \end{array}$$

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can be taken as a test function in (2.4):

$$\int_{(0,T)\times\mathbb{R}^3_+} [\rho_{\mathbf{G}}\mathbf{u}\cdot\partial_t\widetilde{\mathbf{w}}_\varepsilon + (\mathbf{u}\otimes\mathbf{u} - 2\mu\mathrm{D}(\mathbf{u})) : \mathrm{D}(\widetilde{\mathbf{w}}_\varepsilon)] \quad \mathrm{d}\mathbf{x} \quad \mathrm{d}t = 0. \tag{4.1}$$

In the following, we set

$$I_1 := \int_{(0,T)\times\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}\cdot\partial_t\widetilde{\mathbf{w}}_\varepsilon \ \mathrm{d}t\ \mathrm{d}\mathbf{x},$$
$$I_2 := \int_{(0,T)\times\mathbb{R}^3_+} \mathbf{u}\otimes\mathbf{u} : D(\widetilde{\mathbf{w}}_\varepsilon)\ \mathrm{d}t\ \mathrm{d}\mathbf{x},$$
$$I_3 := \int_{(0,T)\times\mathbb{R}^3_+} D(\mathbf{u}) : D(\widetilde{\mathbf{w}}_\varepsilon)\ \mathrm{d}t\ \mathrm{d}\mathbf{x}.$$

After a change of variables, we have for almost all $t\in(0,T)$

$$\int_{\mathbb{R}^3_+} D(\mathbf{u})(t,\cdot) : D(\widetilde{\mathbf{w}}_\varepsilon)(t,\cdot)\ \mathrm{d}\mathbf{x} = \zeta_\varepsilon(t)\int_{\mathbb{R}^3_+} D(\mathbf{u})(t,x_1+G_1,x_2+G_2,x_3) : D(\mathbf{w}_{h(t)})\ \mathrm{d}\mathbf{x}.$$

Thus, applying Lemma 3.9,

$$\int_{\mathbb{R}^3_+} D(\mathbf{u})(t,x_1+G_1,x_2+G_2,x_3) : D(\mathbf{w}_{h(t)})\ \mathrm{d}\mathbf{x} = \dot{h}\widetilde{n}_3(h) + E(t),$$

where $|E(t)| = K_M|\nabla\mathbf{u}(t,\cdot)|_2$. Consequently,

$$I_3 = \int_0^T \zeta_\varepsilon(t)\dot{h}(t)\widetilde{n}_3(h(t))\ \mathrm{d}t + \widetilde{E}, \tag{4.2}$$

where

$$|\widetilde{E}| \leqslant K_M\sqrt{T}\|\mathbf{u}_0\|_2. \tag{4.3}$$

Similarly, for almost all $t\in(0,T)$,

$$\int_{\mathbb{R}^3_+}[\mathbf{u}\otimes\mathbf{u}](t,\cdot) : D(\widetilde{\mathbf{w}}_\varepsilon)(t,\cdot)\ \mathrm{d}\mathbf{x}$$
$$= \zeta_\varepsilon(t)\int_{\mathbb{R}^3_+}[\rho_{\mathbf{G}_h}\mathbf{u}\otimes\mathbf{u}](t,x_1+G_1,x_2+G_2,x_3) : D(\mathbf{w}_{h(t)})(\mathbf{x})\ \mathrm{d}\mathbf{x}.$$

Consequently, applying Lemma 3.1 together with Lemmas 3.6 and 3.3, we obtain

$$\left|\int_{\mathbb{R}^3_+}[\mathbf{u}\otimes\mathbf{u}](t,\cdot) : D(\widetilde{\mathbf{w}}_\varepsilon)(t,\cdot)\ \mathrm{d}\mathbf{x}\right| \leqslant K_m\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^3_+)}.$$

Thus,

$$|I_2| \leqslant K_m\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}. \tag{4.4}$$

Finally, computing $\partial_t\widetilde{\mathbf{w}}_\varepsilon$ we have, after our change of variable,

$$I_1 = I_1^\chi + I_1^w,$$

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where

$$I_1^\chi := \int_0^T \int_{\mathbb{R}^3_+} [\rho_{\mathbf{G}_h}\mathbf{u}](t, x_1+G_1, x_2+G_2, x_3)] \cdot \zeta_\varepsilon'(t)\mathbf{w}_{h(t)}(\mathbf{x})\ \mathrm{d}\mathbf{x}\ \mathrm{d}t$$

and

$$I_1^w := \int_0^T \zeta_\varepsilon(t) \int_{\mathbb{R}^3_+} [\rho_{\mathbf{G}_h}\mathbf{u}](x_1+G_1, x_2+G_2, x_3) \cdot \left[\dot{h}\partial_h \mathbf{w}_h - V_1\partial_{x_1}\mathbf{w}_h - V_2\partial_{x_2}\mathbf{w}_h\right](\mathbf{x})\ \mathrm{d}\mathbf{x}\ \mathrm{d}t.$$

Applying the Cauchy–Schwarz inequality and Lemma 3.4 on $\mathbf{w}_h$, we deduce that

$$|I_1^\chi| \leqslant C\left[\int_{t_0-\varepsilon}^{t_0} |\zeta_\varepsilon'(t)|\ \ \|\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3_+)}\ \mathrm{d}t + \int_{t_1}^{t_1+\varepsilon} |\zeta_\varepsilon'(t)|\ \ \|\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3_+)}\ \mathrm{d}t\right],$$

and therefore, using the uniform L^2-bound on $\mathbf{u}$, we obtain

$$|I_1^\chi| \leqslant K\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)}. \tag{4.5}$$

Finally, applying (3.4) in Lemma 3.1 together with (3.9) in Lemma 3.5, we conclude that

$$|I_1^w| \leqslant K_m \int_0^T [|\dot{h}|^2 + |\mathbf{V}_1|^2 + |\mathbf{V}_2|^2]^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^3_+)}\ \ \mathrm{d}t,$$

so that, with standard energy estimate,

$$|I_1^w| \leqslant K_m\sqrt{T}\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}. \tag{4.6}$$

Gathering (4.2), (4.4), (4.5), (4.6) with (4.1) yields

$$\left|\int_0^T \zeta_\varepsilon(t)\dot{h}(t)\widetilde{n}_3(h(t))\ \ \mathrm{d}t\right| \leqslant K_m(1+\sqrt{T})\left\{\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}\right\},$$

where we emphasize that K_m depend only on our choice for the approximation of the solution to the Stokes problem, but not on ε. Thus, letting ε go to 0, as h and $\widetilde{n}_3$ are continuous functions, we obtain

$$|N_3(h(t_1)) - N_3(h(t_0))| \leqslant K_m(1+\sqrt{T})\left\{\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}\right\},$$

where N_3 is a primitive of $\widetilde{n}_3$ which vanishes in $h = 1$, for example. Applying Lemma 3.9, we have $\widetilde{n}_3(h) \geqslant c/h$ when $0 < h < h_m$ for some $c > 0$ and $h_m > 0$, and we finally deduce

$$|\ln(h(t)/h(t_0))| \leqslant K_m(1+\sqrt{T})\left\{\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}\right\}.$$

Because h is continuous, letting t_0 tend to 0, we finally obtain

$$h(t) \geqslant h(0)\exp\left[-K_m(1+\sqrt{T})\left\{\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}\right\}\right].$$

This is the expected result.

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Appendix. Detailed description of ϕ_h^d. In this section we estimate the size of ϕ_h^d and its derivatives. We recall that

$$\phi_h^d(r,\theta,z) = r\chi_o(z/\delta_h(r)), \quad \text{with} \quad \chi_o(s) = \frac{s^2(3-2s)}{2}.$$

In order to compare functions in the following, we introduce the following conventions. Given families $(f_h : \Omega_{h,1/4} \to \mathbb{R})_{h\in(0,1)}$ and $(g_h : \Omega_{h,1/4} \to \mathbb{R})_{h\in(0,1)}$ we denote $f_h \prec g_h$ if there exists an absolute constant such that

$$|f_h(\mathbf{x})| \leqslant C g_h(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{h,1/4} \text{ and } h < 1.$$

Given nonnegative functions $f : (0,1) \to \mathbb{R}^+$ and $g : (0,1) \to \mathbb{R}^+$, we also denote

$$f(s) \sim g(s) \quad \forall s \in (0,1)$$

if there exist two positive constants c and C such that

$$cf(s) \leqslant g(s) \leqslant Cf(s) \quad \forall s \in (0,1).$$

First, we compute typical $L^1(0,1/4)$-sizes of functions $r \mapsto r^\alpha/(\delta_h(r))^\beta$.

LEMMA A.1. *Given $(\alpha,\beta) \in (\mathbb{R}_+)^2$, we have the following estimations for all $h \in (0,1)$:*

$$\int_0^{\frac14} \frac{r^\alpha}{\delta_h(r)^\beta}\,\mathrm{d}r \sim \begin{cases} 1 & \text{if } \alpha > 2\beta - 1, \\ h^{\frac{(\alpha+1)-2\beta}{2}} & \text{if } \alpha < 2\beta - 1. \end{cases}$$

Proof. As in [9], we remark that, for all $h \in (0,1)$, we have

$$h + \frac{s^2}{2} \leqslant \delta_h(s) \leqslant h + s^2 \quad \forall s \in (0,1/4).$$

Consequently, we can replace $\delta_h(r)$ by $h + \gamma r^2$ with some generic parameter $\gamma > 0$, and we are bound to calculate

$$I_{\alpha,\beta} := \int_0^{1/4} \frac{r^\alpha}{(h+\gamma r^2)^\beta}\,\mathrm{d}r,$$

in which we set $r = \sqrt{h}s$. This yields

$$I_{\alpha,\beta} := h^{\frac{(\alpha+1)-2\beta}{2}} \int_0^{\frac{1}{4\sqrt{h}}} \frac{s^\alpha}{(1+\gamma s^2)^\beta}\,\mathrm{d}s.$$

Consequently, if $\alpha > 2\beta - 1$, the integral behaves like $Ch^{-\frac{(\alpha+1)-2\beta}{2}}$, and we obtain the first case, while if $\alpha < 2\beta - 1$, the integral goes to a finite positive value as $h \to \infty$, and we obtain the second case. ◻

We now compare $\lambda(r,z,h) = z/\delta_h(r)$ to member functions $(r,\theta,z) \mapsto r^\alpha/(\delta_h(r))^\beta$ in $\Omega_{h,1/4}$.

LEMMA A.2. *We have the following sizes:*

$$\begin{array}{llll} \lambda \prec 1, & \lambda_r \prec r/\delta_h, & \lambda_z \prec 1/\delta_h, & \lambda_h \prec 1/\delta_h, \\ \lambda_{rh} \prec r/\delta_h^2, & \lambda_{zh} \prec 1/\delta_h^2, & \lambda_{rr} \prec 1/\delta_h, & \lambda_{rz} \prec r/\delta_h^2, \\ \lambda_{rrz} \prec 1/\delta_h^2, & & \lambda_{rrr} \prec r/\delta_h^2. & \end{array}$$

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Proof. The reader may rapidly check that all the derivatives of δ_h are independent of h and that all the odd ones are bounded by r over $(0,1/4)$. Then in $\Omega_{h,1/4}$ we have $z \in (0, \delta_h(r))$, and consequently $\lambda \prec 1$. Then

$$\lambda_r = -\frac{\lambda \delta_h'}{\delta_h}, \quad \lambda_z = \frac{1}{\delta_h}, \quad \lambda_h = -\frac{\lambda}{\delta_h}.$$

As δ' is bounded by r necessarily independent of h, we get $\lambda_r \prec r/\delta_h$ and $\lambda_z \prec 1/\delta_h$, $\lambda_h \prec 1/\delta_h$. To the next order, we get

$$\lambda_{rz} = -\frac{\delta_h'}{\delta_h^2}, \quad \lambda_{rr} = \lambda \left(2\frac{(\delta_h')^2}{\delta_h^2} - \frac{\delta_h''}{\delta_h} \right), \quad \lambda_{rh} = -2\frac{\lambda \delta_h'}{\delta_h^2}, \quad \lambda_{zh} = -\frac{1}{\delta_h^2}.$$

As δ'' is bounded independently of h and $r^2 \leqslant h + r^2$, we obtain

$$\lambda_{rz} \prec \frac{r}{\delta_h^2}, \quad \lambda_{rr} \prec \frac{1}{\delta_h}, \quad \lambda_{rh} \prec \frac{r}{\delta_h}, \quad \lambda_{zh} \prec \frac{1}{\delta_h^2}.$$

Finally,

$$\lambda_{rrz} = \frac{1}{\delta_h} \left(2\frac{(\delta_h')^2}{\delta_h^2} - \frac{\delta_h''}{\delta_h} \right), \quad \lambda_{rrr} = \lambda \left(6\frac{\delta_h''\delta_h'}{\delta_h^2} - 6\frac{(\delta_h')^3}{\delta_h^3} - \frac{\delta_h^{(3)}}{\delta_h} \right).$$

And, as $\delta_h^{(3)}$ is bounded by r and $r^2 \leqslant \delta_h$,

$$\lambda_{rrz} \prec \frac{1}{\delta_h^2}, \quad \lambda_{rrr} \prec \frac{r}{\delta_h^2}. \qquad \square$$

Then we obtain the following lemma.

LEMMA A.3. *We have the following sizes:*

$$\begin{array}{llll}
\phi_h^d \prec r, & \partial_r \phi_h^d \prec 1, & \partial_z \phi_h^d \prec r/\delta_h, & \partial_r \phi_h^d / r - \phi_h^d / r^2 \prec r/\delta_h, \\
\partial_h \phi_h^d \prec r/\delta_h, & \partial_{rh} \phi_h^d \prec 1/\delta_h, & \partial_{zh} \phi_h^d \prec r/\delta_h^2, & \partial_{rz} \phi_h^d / r - \partial_z \phi_h^d / r^2 \prec r/\delta_h^2, \\
\partial_{rr} \phi_h^d \prec r/\delta_h, & \partial_{rz} \phi_h^d \prec 1/\delta_h, & \partial_{zz} \phi_h^d \prec r/\delta_h^2, & \\
\partial_{rrr} \phi_h^d \prec 1/\delta_h, & \partial_{rzz} \phi_h^d \prec 1/\delta_h^2, & \partial_{rrz} \phi_h^d \prec r/\delta_h^2, & \partial_{zzz} \phi_h^d \prec r/\delta_h^3.
\end{array}$$

Proof. By definition, we have $\phi_h^d(r,z) = r\chi_o(\lambda)$, where χ_o is a fixed polynomial and, according to the previous lemma, λ is bounded. Consequently, we obtain $\phi_h^d \prec r$.

In the following, we shall drop all dependencies of χ_o in λ. Due to the same argument as for χ_o, all those quantities depending only on χ_o are bounded independently of (h,r,z) in $\Omega_{h,1/4}$.

So, we compute

$$\partial_r \phi_h^d = \chi_o + r\lambda_r \chi_o', \quad \partial_z \phi_h^d = r\lambda_z \chi_o', \quad \partial_h \phi_h^d = r\lambda_h \chi_o'.$$

Applying the previous lemma and $r^2 \leqslant \delta_h(r)$, we get

$$\partial_r \phi_h^d \prec 1, \quad \partial_z \phi_h^d \prec r/\delta_h, \quad \partial_h \phi_h^d \prec r/\delta_h, \quad \partial_r \phi_h^d / r - \phi_h^d / r^2 = \lambda_r \chi_o' \prec r/\delta_h.$$

To the next order, we obtain, as λ_z is independent of z,

$$\partial_{rr} \phi_h^d = (2\lambda_r + r\lambda_{rr})\chi_o' + r(\lambda_r)^2 \chi_o'',$$

$$\partial_{rz} \phi_h^d = (\lambda_z + r\lambda_{rz})\chi_o' + r\lambda_r \lambda_z \chi_o'' \quad \partial_{zz} \phi_h^d = r(\lambda_z)^2 \chi_o'',$$

$$\partial_{zh} \phi_h^d = r\lambda_z \lambda_h \chi_o'' + r\lambda_{hz} \chi_o', \quad \text{and} \quad \partial_{rh} \phi_h^d = \lambda_h \chi_o' + r\lambda_{rh} \chi_o' + r\lambda_r \lambda_h \chi_o''.$$

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As above,

$$\partial_{rr}\phi_h^d \prec r/\delta_h, \quad \partial_{rz}\phi_h^d \prec 1/\delta_h, \quad \partial_{zz}\phi_h^d \prec r/\delta_h^2, \quad \partial_{rh}\phi_h^d \prec 1/\delta_h, \quad \partial_{zh}\phi_h^d \prec r/\delta_h^2$$

and

$$\frac{\partial_{rz}\phi_h^d}{r} - \frac{\partial_z\phi_h^d}{r^2} = \lambda_{rz}\chi_o' + \lambda_r\lambda_z\chi_o'' \prec r/\delta_h^2.$$

To the next order, we obtain

$$\partial_{rrr}\phi_h^d = (3\lambda_{rr} + r\lambda_{rrr})\chi_o' + (3\lambda_r^2 + 3r\lambda_{rr}\lambda_r))\chi_o'' + r(\lambda_r)^3\chi_o^{(3)},$$

and thus $\partial_{rrr}\phi_h^d \prec 1/\delta_h$; and

$$\partial_{rzz}\phi_h^d = (\lambda_z^2 + 2r\lambda_{rz}\lambda_z)\chi_o'' + r(\lambda_z)^2\lambda_r\chi_o^{(3)},$$

so $\partial_{rzz}\phi_h^d \prec 1/\delta_h^2$; and

$$\partial_{rrz}\phi_h^d = (2\lambda_{rz} + r\lambda_{rrz})\chi_o' + (2\lambda_r\lambda_z + r(\lambda_{rr}\lambda_z + 2\lambda_{rz}\lambda_r)\chi_o'' + r(\lambda_r)^2\lambda_z\chi_o^{(3)},$$

so $\partial_{rrz}\phi_h^d \prec r/\delta_h^2$. Finally, $\partial_{zzz}\phi_h^d = r(\lambda_z)^3\chi_o^{(3)}$, so that $\partial_{zzz}\phi_h^d \prec r/{\delta_h}^3$. ∎

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