Homogenization and singular limits for the complete Navier-Stokes-Fourier system

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Abstract

In this paper we apply the methods of homogenization to the full Navier-Stokes-Fourier system describing the motion of a general viscous, compressible, and heat conducting fluid. We study the asymptotic behavior of solutions in perforated domains with tiny holes, where the diameter of the holes is proportional to their mutual distance. As a limit system, we identify a porous medium type equation with a nonlinear Darcy's law.

Abstract

Dans cet article, nous appliquons les méthodes d'homogénéisation au système complet des équations de Navier-Stokes-Fourier décrivant le mouvement d'un fluide général, visqueux, compressible et conducteur de chaleur. Nous étudions le comportement asymptotique des solutions dans un domaine perforé par de petits trous dont le diamètre est proportionnel à leur distance mutuelle. Nous obtenons à la limite une équation de type milieux poreux avec une loi de Darcy non linéaire.

Keywords: compressible Navier–Stokes–Fourier equations, homogenization, porous medium equation

Mots-clés : équations de Navier–Stokes–Fourier, homogénéisation, équation des milieux poreux

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1 Introduction

Many equations and systems in continuum fluid mechanics can be identified with a singular limit of the full *Navier-Stokes-Fourier* system governing the motion of a general viscous, compressible, and heat conducting fluid. The aim of *homogenization* theory is to describe the macroscopic behavior of microscopically heterogeneous systems. In the present paper, we study the asymptotic behavior of solutions to the Navier-Stokes-Fourier system in perforated domains with tiny holes. We focus on the case, where the diameter of the holes is proportional to their mutual distance. This problem is nowadays well understood in the case of the Stokes system. As shown by Tartar [28] (see also Keller [19], J.-L.Lions [20], Sanchez-Palencia [27], among others), the asymptotic limit gives rise to the classical Darcy's law. These results were later generalized to the incompressible Navier-Stokes system by Allaire [1], [2], [3], [4], Mikelič [25]) (cf. also Berlyand and Khruslov [6], Ciorănescu et al. [8]). For extensions to random distribution of the holes see Belyaev and Kozlov [5]. Recently, Masmoudi [22], [24] adapted these methods to models of compressible barotropic fluids, with low Strouhal number. Our goal is to extend these results to general viscous, compressible and *heat conducting* fluids.

Following the philosophy of Allaire [2], Ciorănescu and Murat [10], [9] we introduce an abstract framework through hypotheses [H1] - [H5] in Section 2, imposed on a family of perforated domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$. Specifically, we consider a bounded regular domain $\Omega \subset \mathbb{R}^3$, together with a family of "holes" (solid objects) $\{B^s_{\varepsilon,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^3}$ such that $B^s_{\varepsilon,\mathbf{k}} \subset \mathbb{R}^3$ are regular domains, and

$$B^s_{\varepsilon,\mathbf{k}} \subset \overline{B^s_{\varepsilon,\mathbf{k}}} \subset \varepsilon C_{\mathbf{k}}, \text{ with } C_{\mathbf{k}} \equiv (0,1)^3 + \mathbf{k},\ \mathbf{k} \in \mathbb{Z}^3. \tag{1.1}$$

Finally, we set

$$\Omega_\varepsilon = \Omega \setminus \bigcup \{B^s_{\varepsilon,\mathbf{k}} \mid \varepsilon C_{\mathbf{k}} \subset \Omega\}, \text{ and } B^f_{\varepsilon,\mathbf{k}} = \varepsilon C_{\mathbf{k}} \setminus B^s_{\varepsilon,\mathbf{k}}. \tag{1.2}$$

1.1 Primitive system

We consider a scaled *Navier-Stokes-Fourier system* in the form

$$\varepsilon^2 \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.3}$$

$$\varepsilon^2 \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}, \tag{1.4}$$

$$\varepsilon^2 \partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \Sigma \tag{1.5}$$

in the space-time cylinder $(0,T) \times \Omega_\varepsilon$, where ϱ is the fluid density, $\mathbf{u}$ is the velocity field, ϑ stands for the absolute temperature, p is the pressure, $\mathbb{S}$ denotes the viscous stress determined through *Newton's rheological law*

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{1.6}$$

$\mathbf{q}$ denotes the heat flux given by *Fourier's law*

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta, \tag{1.7}$$

$s = s(\varrho, \vartheta)$ is the specific entropy, and Σ is the entropy production rate satisfying

$$\Sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{1.8}$$

The inequality sign in (1.8) is a peculiar feature of the concept of weak solutions introduced in [12] and adopted in the present paper. However, it is easy to check that (1.8) reduces to the classical relation

$$\Sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

as soon as the system is energetically insulated (cf. the boundary conditions (1.9), (1.10) below) and the solution is smooth (see [12]).

Finally, we impose the no-slip boundary conditions for the velocity field

$$\mathbf{u}|_{\partial\Omega_\varepsilon} = 0 \tag{1.9}$$

and suppose the boundary is thermally insulated, meaning,

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = -\kappa(\vartheta)\nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{1.10}$$

where $\mathbf{n}$ denotes the unit normal vector to $\partial\Omega_\varepsilon$.

1.2 Target system

The parameter ε^2 in system (1.3 - 1.5) is termed *Strouhal number* that is proportional to the characteristic length of the physical system and inversely proportional to the product of the characteristic velocity with time. When the Strouhal number is small, the fluid approaches an equilibrium state. In the presence of small "holes" in the underlying physical space, however, a suitable change of scale gives rise to a porous medium like equation, where the spatially homogeneous temperature field is interrelated to the density through a total energy balance equation.

To be more specific, we introduce an *extension operator*

$$h \mapsto \langle h\rangle(x) = \left\{ \begin{array}{l} h(x) \text{ for } x\in\Omega_\varepsilon, \\ \\ \frac{1}{|B^f_{\varepsilon,\mathbf{k}}|}\int_{B^f_{\varepsilon,\mathbf{k}}} h(x)\ \mathrm{d}x \text{ for } x\in\Omega\cap B^s_{\varepsilon,\mathbf{k}} \end{array}\right. \tag{1.11}$$

for any $h\in L^1(\Omega_\varepsilon)$, where the sets $B^s_{\varepsilon,\mathbf{k}}$, $B^f_{\varepsilon,\mathbf{k}}$ have been introduced in (1.1).

Let $\{\varrho_\varepsilon,\mathbf{u}_\varepsilon,\vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.3 - 1.10). Our aim is to show that

$$\left\{ \begin{array}{c} \langle\varrho_\varepsilon\rangle \to \varrho \text{ in } L^2((0,T)\times\Omega), \\ \\ \langle\vartheta_\varepsilon\rangle \to \Theta \text{ in } L^2((0,T)\times\Omega), \\ \\ \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \to \mathbf{U} \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3), \end{array}\right\}$$

where $\Theta = \Theta(t) > 0$ is a spatially homogeneous function. Moreover,

$$\partial_t\varrho + \mathrm{div}_x\mathbf{J} = 0, \tag{1.12}$$

$$2\mu(\Theta)\mathcal{A}[\mathbf{J}] = -\nabla_x\mathcal{P}(\varrho,\Theta), \tag{1.13}$$

$$\mathcal{E}(t) \equiv \int_\Omega \varrho e(\varrho,\Theta)(t,\cdot)\ \mathrm{d}x = \mathcal{E}_0 \text{ for all } t, \tag{1.14}$$

where $\mathcal{P}$ satisfies

$$\frac{\partial \mathcal{P}(\varrho,\vartheta)}{\partial \varrho} = \varrho \frac{\partial p(\varrho,\vartheta)}{\partial \varrho},$$

$e = e(\varrho,\vartheta)$ is the (specific) internal energy interrelated to p and s through *Gibbs' equation*

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right), \tag{1.15}$$

and $\mathcal{A}$ denotes the so-called *permeability matrix*, the form of which is determined by the specific shape of the holes (cf. hypothesis [H2] below).

A similar problem for the *isentropic Navier-Stokes system*, where $p = p(\varrho) = a\varrho^\gamma$, and μ, η are positive constants, was studied by Masmoudi [22]. Comparable results for the full Navier-Stokes-Fourier system were obtained in [15] on a fixed spatial domain, where the effect of "holes" was replaced by a singular friction term in the momentum equation (1.4).

The present problem requires a non-trivial extension of the methods developed in [15], [22]. To begin, the assumptions imposed on the pressure in [22], namely $p = a\varrho^\gamma$, with $\gamma \geq 3$, are quite strong and rather inconvenient from the physical point of view. Moreover, the influence of the temperature on the pressure gives rise to an additional difficulty when showing strong convergence of the densities. Indeed, in sharp contrast with the situation that appears in the existence theory, the temperature field is known to converge only weakly at the moment when the strong convergence of the densities is to be established. To overcome this stumbling block, we introduce a concept of *oscillations defect measure*, reminiscent of that used in [15]. Last but not least, we introduce a "weighted" analogue of Nečas' lemma in order to handle the temperature dependent viscosity coefficients.

The paper is organized as follows. In Section 2, we introduce the abstract framework of the theory, recall the definitions of the weak solutions for both the primitive and target system, and state our main result. Section 3 summarizes some preliminary considerations concerning functionals defined on perforated domains. In Section 4, we derive uniform estimates on the family of solutions to the primitive system independent of ε. Sections 5, 6 represent the heart of the paper. We introduce the oscillations defect measure and prove strong convergence of the density and temperature. The proof of the main result is completed in Section 7.

2 Main result

To begin, we introduce the abstract framework of the theory through the following list of hypotheses imposed on the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ defined in (1.1), (1.2).

[H1] *Uniform volume fraction.* There exists $\sigma \in (0,1)$ such

$$\lim_{\varepsilon\to 0} \frac{|B^f_{\varepsilon,\mathbf{k}}|}{|\varepsilon C_{\mathbf{k}}|} = \lim_{\varepsilon\to 0} \frac{|B^f_{\varepsilon,\mathbf{k}}|}{\varepsilon^3} = \sigma \text{ uniformly for } \mathbf{k} \in \mathbb{Z}^3. \tag{2.1}$$

[H2] *Boundary layer test functions.* There exist families of functions $\{\mathbf{w}^n_\varepsilon\}_{\varepsilon>0}$, $\{\mathbf{v}^n_\varepsilon\}_{\varepsilon>0}$, $\{q^n_\varepsilon\}_{\varepsilon>0}$, and a matrix $\mathcal{A} \in L^\infty(\Omega;\mathbb{R}^3)$ (permeability matrix), such that

$$\mathbf{v}^n_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^3),\ q^n_\varepsilon \in W^{1,\infty}(\Omega_\varepsilon); \tag{2.2}$$

$$\mathrm{div}_x \mathbf{v}^n_\varepsilon = 0 \text{ in } \Omega,\ \|q^n_\varepsilon\|_{L^\infty(\Omega_\varepsilon;\mathbb{R}^3)} + \varepsilon\Big(\|\nabla_x \mathbf{v}^n_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})} + \|\nabla_x q^n_\varepsilon\|_{L^\infty(\Omega_\varepsilon;\mathbb{R}^3)}\Big) \leq c; \tag{2.3}$$

$$-\varepsilon^2 \Delta \mathbf{v}^n_\varepsilon + \varepsilon \nabla_x q^n_\varepsilon = \mathbf{w}^n_\varepsilon \text{ in } \mathcal{D}'(\Omega_\varepsilon; R^3), \tag{2.4}$$

with

$$\mathbf{w}^n_\varepsilon \in L^2(\Omega_\varepsilon; R^3), \ \| \mathbf{w}^n_\varepsilon - \mathcal{A}[\mathbf{e}^n] \|_{L^2(\Omega_\varepsilon; R^3)} \to 0 \text{ as } \varepsilon \to 0, \tag{2.5}$$

$$\mathbf{v}^n_\varepsilon \to \mathbf{e}^n \text{ weakly-(*) in } L^\infty(\Omega; R^3) \text{ as } \varepsilon \to 0, \tag{2.6}$$

where $\mathbf{e}^n$, $n = 1, 2, 3$, is the canonical basis of R^3.

[H3] *Restriction operator.* There exists an operator $\mathcal{R}_\varepsilon$ with the following properties:

$$\mathcal{R}_\varepsilon \text{ is a bounded linear operator on } W^{1,2}(\Omega; R^3) \text{ ranging in } W^{1,2}_0(\Omega_\varepsilon; R^3),$$

$$\mathcal{R}_\varepsilon[\mathbf{w}] = \mathbf{w}|_{\Omega_\varepsilon} \text{ provided } \mathbf{w} = 0 \text{ in } \Omega \setminus \Omega_\varepsilon;$$

$$\mathrm{div}_x \mathbf{w} = 0 \text{ in } \Omega \text{ implies } \mathrm{div}_x \mathcal{R}_\varepsilon[\mathbf{w}] = 0 \text{ in } \Omega_\varepsilon;$$

$$\| \mathcal{R}_\varepsilon[\mathbf{w}] \|_{L^2(\Omega_\varepsilon; R^3)} + \varepsilon \| \nabla_x \mathcal{R}_\varepsilon[\mathbf{w}] \|_{L^2(\Omega_\varepsilon; R^{3\times 3})} \leq c \Big(\| \mathbf{w} \|_{L^2(\Omega_\varepsilon; R^3)} + \varepsilon \| \nabla_x \mathbf{w} \|_{L^2(\Omega_\varepsilon; R^{3\times 3})} \Big). \tag{2.7}$$

In addition, we assume the restriction operator $\mathcal{R}_\varepsilon$ satisfies a *compatibility relation* with the extension operator introduced in (1.11), specifically,

$$< \nabla_x \langle h \rangle \ ; \varphi > \equiv \int_\Omega \langle h \rangle \, \mathrm{div}_x \varphi \ \mathrm{d}x = \int_{\Omega_\varepsilon} h \mathrm{div}_x (\mathcal{R}_\varepsilon[\varphi]) \ \mathrm{d}x \text{ for any } h \in L^1(\Omega_\varepsilon), \ \varphi \in C^\infty_c(\Omega). \tag{2.8}$$

[H4] *Bogovskii's operator.* There exists a linear operator $\mathcal{B}_\varepsilon$ with the following properties: If $\mathbf{f} \in L^p(\Omega_\varepsilon)$, then $\mathbf{w} = \mathcal{B}_\varepsilon[f]$ satisfies

$$\mathbf{w} \in W^{1,p}(\Omega_\varepsilon), \ \mathrm{div}_x \mathbf{w} = f - \frac{1}{|\Omega_\varepsilon|} \int_\Omega f \ \mathrm{d}x \text{ in } \Omega_\varepsilon, \ \mathbf{w}|_{\partial \Omega_\varepsilon} = 0.$$

Moreover,

$$\varepsilon \| \mathcal{B}_\varepsilon[f] \|_{W^{1,p}(\Omega_\varepsilon; R^3)} \leq c \| f \|_{L^p(\Omega_\varepsilon)}, \ 1 < p < \infty. \tag{2.9}$$

If, in addition $f = \mathrm{div}_x \mathbf{g}$, $\mathbf{g} \in L^q(\Omega_\varepsilon; R^3)$, $\mathbf{g} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0$, then

$$\| \mathcal{B}_\varepsilon[f] \|_{L^q(\Omega_\varepsilon; R^3)} \leq c \| \mathbf{g} \|_{L^q(\Omega_\varepsilon; R^3)}, \ 1 < q < \infty, \tag{2.10}$$

where the constants in (2.9), (2.10) are independent of ε.

[H5] *Uniform bounds for Stokes' problem.* Let $1 < p < \infty$. For any $\mathbf{f} \in L^p(\Omega_\varepsilon; R^3)$, the Stokes problem

$$-\Delta \mathbf{v} + \nabla_x q = \mathbf{f}, \ \mathrm{div}_x \mathbf{v} = 0 \text{ in } \Omega_\varepsilon, \ \mathbf{v}|_{\partial \Omega_\varepsilon} = 0, \ \int_{\Omega_\varepsilon} q \ \mathrm{d}x = 0 \tag{2.11}$$

admits a unique solution $\mathbf{v} \in W^{2,p}(\Omega_\varepsilon; R^3)$, $q \in W^{1,p}(\Omega_\varepsilon)$ satisfying

$$\| \Delta \mathbf{v} \|_{L^p(\Omega_\varepsilon)} + \| \nabla_x q \|_{L^p(\Omega_\varepsilon; R^3)} \leq c \| \mathbf{f} \|_{L^p(\Omega_\varepsilon; R^3)}, \tag{2.12}$$

Similarly, if $\mathbf{f} \in W^{-1,p}(\Omega_\varepsilon; R^3)$, problem (2.11) admits a unique solution $\mathbf{v} \in W^{1,p}_0(\Omega_\varepsilon; R^3)$, $q \in L^p(\Omega_\varepsilon)$, and

$$\| \nabla_x \mathbf{v} \|_{L^p_0(\Omega_\varepsilon; R^{3\times 3})} + \| \nabla_x q \|_{W^{-1,p}(\Omega_\varepsilon; R^3)} \leq c \| \mathbf{f} \|_{W^{-1,p}(\Omega_\varepsilon; R^3)}, \tag{2.13}$$

where the constant c is independent of ε.

The above hypotheses deserve some comments. To begin, let us point out that **[H1]** - **[H5]** are satisfied at least in the simplest case of periodically distributed holes sharing the same shape after scaling. More precisely,

$$\mathcal{B}^s_{\varepsilon,\mathbf{k}} = \varepsilon(\mathcal{B}^s + \mathbf{k}) \text{ for a smooth domain } \mathcal{B}^s \subset \overline{\mathcal{B}^s} \subset (0,1)^3. \tag{2.14}$$

The meaning of **[H1]** is obvious. The existence of the special test functions introduced in **[H2]** was discussed by Allaire [2], Ciorănescu and Murat [10], [9], among others. Note that in the simple case of periodically distributed holes described through (2.14), the functions $\mathbf{v}^n_\varepsilon$, q^n_ε can be obtained by means of a simple scaling $\mathbf{v}^n_\varepsilon = \mathbf{v}(x/\varepsilon)$, $q^n_\varepsilon = q(x/\varepsilon)$, where $\mathbf{v}^n$, q^n are solutions of a non-homogeneous Stokes problem, supplemented with periodic boundary conditions on the unit cell $(0,1)^3$ (cf. Masmoudi [22], Mikelič [25]).

The restriction $\mathcal{R}_\varepsilon$ operator was introduced by Tartar [28]. His construction was later used by Avellaneda et al. [21] in order to establish the compatibility relation (2.8) in the purely periodic framework described in (2.14). Tartar's construction can be easily adapted to a more general distribution and shapes of the holes, however, some uniformity of their mutual distance and smoothness of the boundary are still necessary in order to keep the constant in (2.7) independent of ε (cf. Allaire [2]).

There are many ways how to construct the operator $\mathcal{B}_\varepsilon$ appearing in **[H4]**. An explicit formula that applies on any Lipschitz domain was proposed by Bogovskii [7]. The uniform bounds required in (2.9), (2.10) can be derived by the methods of Galdi [17]. If the holes are given by (2.14), the relevant estimates were deduced by Masmoudi [22], [23].

The uniform bounds (2.12), (2.13) concerning the solutions of the Stokes problem (2.11) represent a non-trivial issue. They were established by Masmoudi [23] in the purely periodic case (2.14), however, any extension to a more general class of perforated domains seems far from being straightforward. A weaker version of (2.11), (2.12), with $c \approx \varepsilon^{-\alpha}$, $\alpha = |3/2 - 3/p|$ can be proved by means of local regularity for the Stokes problem (see Farwig et al. [14], Masmoudi [22]). As a matter of fact, these weaker estimates are still sufficient for the proof of our main result stated in Theorem 2.1 provided we assume $\gamma > 2$ in hypothesis (2.16).

2.1 Structural hypotheses on constitutive equations

Motivated by certain models in astrophysics (see Van Wylen and Sonntag [29]) we consider a state equation for the pressure in the form

$$p(\varrho,\vartheta) = p_e(\varrho) + p_\vartheta(\varrho)\vartheta + \frac{\varepsilon}{3}\vartheta^4, \tag{2.15}$$

where

$$p_e \in C^1[0,\infty),\ p_e(0) = 0,\ p_e'(\varrho) \geq a\varrho^{\gamma-1},\ p_e(\varrho) \leq c(\varrho^\gamma + 1),\ \text{with } a > 0,\ \gamma \geq 2,\ c > 0 \tag{2.16}$$

$$p_\vartheta \in C^1[0,\infty),\ p_\vartheta(0) = 0,\ p_\vartheta' \geq 0,\ p_\vartheta(\varrho) \leq c(\varrho^\beta + 1),\ \text{for } 0 \leq \beta < \frac{2}{3}\gamma,\ c > 0. \tag{2.17}$$

We point out that the presence of the (small) radiation component $(\varepsilon/3)\vartheta^4$ is essential in the *existence* theory (cf. [12]) but entirely irrelevant in the analysis of the singular limit. The reader may consult [12] for more details concerning the physical background of the state equation (2.15).

Accordingly, the (specific) internal energy can be taken in the form

$$e_\varepsilon(\varrho,\vartheta) = b\vartheta^{3/2} + P_e(\varrho) + \varepsilon\frac{\vartheta^4}{\varrho},\ b > 0, \tag{2.18}$$

where

$$P_e(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2}\ \mathrm{d}z, \tag{2.19}$$

while the (specific) entropy reads

$$s_\varepsilon(\varrho,\vartheta) = 3b\sqrt{\vartheta} - P_\vartheta(\varrho) + \varepsilon\frac{4}{3}\frac{\vartheta^3}{\varrho}, \tag{2.20}$$

with

$$P_\vartheta(\varrho) = \int_1^\varrho \frac{p_\vartheta(z)}{z^2}\ \mathrm{d}z. \tag{2.21}$$

In (2.18), the function $c_v(\vartheta) : \vartheta \to (3/2)b\sqrt{\vartheta}$ represents the specific heat at constant volume. The specific form of this quantity simplifies considerably future considerations but the main result of this paper can be extended to more general constitutive relations including the standard choice $c_v(\vartheta) = \mathrm{const} > 0$.

Finally, we assume that the transport coefficients μ and κ are continuously differentiable functions of the absolute temperature $\vartheta \in [0,\infty)$ such that

$$\mu \in W^{1,\infty}[0,\infty),\ 0 < \underline{\mu}(1+\vartheta) \le \mu(\vartheta), \tag{2.22}$$

$$0 < \underline{\kappa}(1+\vartheta^3) \le \kappa(\vartheta) \le \overline{\kappa}(1+\vartheta^3) \tag{2.23}$$

for all $\vartheta \ge 0$, where $\underline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants. For the sake of simplicity, we take the bulk viscosity coefficient $\eta \equiv 0$, however, our method applies with minor modifications provided η is a continuously differentiable function of ϑ satisfying

$$0 \le \eta(\vartheta) \le \overline{\eta}(1+\vartheta),$$

see [12] concerning the physical background of (2.22), (2.23).

2.2 Weak solutions

We shall say that a trio $\{\varrho,\vartheta,\mathbf{u}\}$ is a *weak solution* to the Navier-Stokes-Fourier system (1.3 - 1.8), supplemented with the boundary conditions (1.9), (1.10), and the initial data

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0, \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{2.24}$$

if

- $\varrho \in L^\infty(0,T;L^\gamma(\Omega_\varepsilon))$, $\varrho \ge 0$, $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$, and the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon^2\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = -\int_{\Omega_\varepsilon} \varepsilon^2\varrho_0\varphi(0,\cdot)\ \mathrm{d}x \tag{2.25}$$

  holds for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}_\varepsilon)$;

- $p(\varrho,\vartheta) \in L^q((0,T)\times\Omega_\varepsilon)$, $\mathbb{S} \in L^q((0,T)\times\Omega_\varepsilon;L^q(\mathbb{R}^{3\times3}))$, for a certain $q > 1$, and

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon^2\varrho\mathbf{u}\cdot\partial_t\varphi + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.26}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \mathbb{S} : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_{\Omega_\varepsilon} \varepsilon^2\varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

  for any $\varphi \in C_c^\infty([0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$;

- $\vartheta \in L^\infty(0,T;L^4(\Omega_\varepsilon)) \cap L^2(0,T;W^{1,2}(\Omega_\varepsilon))$, $\vartheta > 0$ a.a. in $(0,T)\times\Omega_\varepsilon$, and the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon^2 \varrho s_\varepsilon(\varrho,\vartheta)\partial_t\varphi + \varrho s_\varepsilon(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta\cdot\nabla_x\varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.27}$$

$$\leq \int_0^T \int_{\Omega_\varepsilon} \frac{1}{\vartheta}\Big(\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla_x\vartheta|^2 \Big)\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_{\Omega_\varepsilon} \varepsilon^2 \varrho_0 s_\varepsilon(\varrho_0,\vartheta_0)\varphi \ \mathrm{d}x$$

  holds for any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega}_\varepsilon)$, $\varphi \geq 0$;

- the energy equality

$$E(t) \equiv \int_{\Omega_\varepsilon} \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho P_e(\varrho) + b\varrho\vartheta^{3/2} + \varepsilon\vartheta^4 \Big)(t,\cdot) \ \mathrm{d}x \tag{2.28}$$

$$= E_0 \equiv \int_{\Omega_\varepsilon} \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 P_e(\varrho_0) + b\varrho_0\vartheta_0^{3/2} + \varepsilon\vartheta_0^4 \Big) \ \mathrm{d}x$$

  holds for a.a. $t \in (0,T)$.

Similarly, we shall say that $\{\varrho,\Theta\}$ is a solution of the target problem (1.12 - 1.14), supplemented with the boundary condition

$$\mathbf{J}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.29}$$

and the initial condition

$$\varrho(0,\cdot) = \varrho_0, \tag{2.30}$$

if

- $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$, $\Theta = \Theta(t) \in L^\infty(0,T)$, $\mathbf{J} \in L^q((0,T)\times\Omega;\mathbb{R}^3))$ for a certain $q > 1$, and the integral identity

$$\int_0^T \int_\Omega \Big(\varrho\partial_t\varphi + \mathbf{J}\cdot\nabla_x\varphi \Big) \ \mathrm{d}x = - \int_\Omega \varrho_0\varphi(0,\cdot) \ \mathrm{d}x \tag{2.31}$$

  is satisfied for any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$;

- $\mathcal{P}(\varrho,\vartheta) \in L^q(0,T;W^{1,q}(\Omega))$ for a certain $q > 1$, and

$$2\mu(\Theta)\mathcal{A}[\mathbf{J}] = -\nabla_x\mathcal{P}(\varrho,\Theta) \text{ a.a. in } (0,T)\times\Omega; \tag{2.32}$$

- the energy equality

$$\mathcal{E}(t) \equiv \int_\Omega \varrho P_e(\varrho)(t,\cdot) \ \mathrm{d}x + Mb\Theta^{3/2}(t) = \mathcal{E}_0 \text{ holds for a.a. } t \in (0,T), \tag{2.33}$$

  where $M = \int_\Omega \varrho \ \mathrm{d}x$.

2.3 Main result

The remaining part of the paper is devoted to the proof of the following result.

Theorem 2.1 *Let* Ω, $\{\Omega_\varepsilon\}_{\varepsilon>0}$, *be a family of smooth domains in* $\mathbb{R}^3$ *given by (1.1), (1.2) and satisfying hypotheses* **[H1]** - **[H5]**. *Assume that the constitutive relations comply with the structural hypotheses (2.15 - 2.23). Let* $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions of the Navier-Stokes-Fourier system on* $(0,T)\times\Omega_\varepsilon$ *in the sense of (2.24 - 2.28), with*

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon}, \ \textit{where } \varrho_{0,\varepsilon} \to \varrho_0 \ \textit{weakly in } L^\gamma(\Omega), \tag{2.34}$$

$$E_\varepsilon(0) = E_{0,\varepsilon} \to E_0, \ S_\varepsilon(0) = \int_{\Omega_\varepsilon} \varrho_{0\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \ \mathrm{d}x > S_0, \ \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \ \mathrm{d}x = M_\varepsilon \geq M > 0, \tag{2.35}$$

where $\varrho_{0,\varepsilon}$ *were extended to be zero outside* Ω_ε.

Then

$$\langle \varrho_\varepsilon \rangle \to \varrho \ \textit{in } L^2((0,T)\times\Omega), \tag{2.36}$$

$$\langle \vartheta_\varepsilon \rangle \to \Theta \ \textit{in } L^2((0,T)\times\Omega), \tag{2.37}$$

$$\frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \to \mathbf{U} \ \textit{weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3)), \tag{2.38}$$

where ϱ, Θ *solve the target problem (2.29 - 2.33), with* $\mathbf{J} = \varrho\mathbf{U}$, $\sigma\mathcal{E}_0 = E_0$, *and the permeability matrix* $\mathcal{A}$ *specified in* **[H2]**.

3 Preliminaries

In this section, we list several elementary results that will be used in the proof of Theorem 2.1.

- We start with a variant of *Poincaré's inequality*

$$\|v\|_{L^p(\Omega_\varepsilon)} \leq \varepsilon c \|\nabla_x v\|_{L^p(\Omega_\varepsilon;\mathbb{R}^3)} \ \text{for any } v \in W_0^{1,p}(\Omega_\varepsilon). \tag{3.1}$$

- *Continuity of the extension operator.* The following assertion is a direct consequence of Jensen's inequality:

$$\| \langle h \rangle \|^p_{L^p(\Omega)} \leq \frac{1}{\sigma_\varepsilon} \|h\|^p_{L^p(\Omega_\varepsilon)} \ \text{for all } 1 \leq p \leq \infty, \tag{3.2}$$

where

$$\sigma_\varepsilon = \inf_{\mathbf{k}\in\mathbb{Z}^3} \frac{|B^f_{\varepsilon,\mathbf{k}}|}{\varepsilon^3}$$

(cf. hypothesis **[H1]**).

- *Extension operator and strong convergence*

Lemma 3.1 *Let* $g \in C(\mathbb{R}^N;\mathbb{R})$ *be a uniformly bounded function. Assume that* $\{h_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega_\varepsilon;\mathbb{R}^N)$ *is a sequence of functions such that*

$$\langle h_\varepsilon \rangle \to h \ \textit{in } L^1(\Omega;\mathbb{R}^N). \tag{3.3}$$

Then

$$g(\langle h_\varepsilon \rangle) \to g(h) \ \textit{in } L^1(\Omega).$$

Remark 3.1 *The conclusion of Lemma 3.1 remains valid provided the sequence* $\{h_\varepsilon\}_{\varepsilon>0}$ *is bounded in* $L^p(\Omega_\varepsilon;\mathbb{R}^N)$ *and* g *complies with a growth restriction*

$$g(h) \leq c(1+|h|^q) \text{ for a certain } 1 \leq q < p.$$

Proof:

We write

$$\int_\Omega \Big| \langle g(h_\varepsilon)\rangle - g(h)\Big| \, \mathrm{d}x$$

$$\leq \int_{\Omega_\varepsilon} \Big| \langle g(h_\varepsilon)\rangle - p(h)\Big| \, \mathrm{d}x + \int_{\Omega\setminus\Omega_\varepsilon} \Big| \langle g(h_\varepsilon)\rangle - \langle 1_{\Omega_\varepsilon} g(h)\rangle \Big| \mathrm{d}x + \int_{\Omega\setminus\Omega_\varepsilon} \Big| \langle 1_{\Omega_\varepsilon} g(h)\rangle - g(h)\Big| \, \mathrm{d}x,$$

where, in accordance with hypothesis (3.3),

$$\int_{\Omega_\varepsilon} \Big| \langle g(h_\varepsilon)\rangle - g(h)\Big| \, \mathrm{d}x \leq \int_\Omega \Big| g(\langle h_\varepsilon\rangle) - g(h)\Big| \, \mathrm{d}x. \to 0 \tag{3.4}$$

Moreover, by virtue of (3.2),

$$\int_{\Omega\setminus\Omega_\varepsilon} \Big| \langle g(h_\varepsilon)\rangle - \langle 1_{\Omega_\varepsilon} g(h)\rangle \Big| \mathrm{d}x$$

$$\leq \frac{1}{\sigma_\varepsilon} \int_{\Omega_\varepsilon} \Big| g(h_\varepsilon) - g(h)\Big| \mathrm{d}x \leq \frac{1}{\sigma_\varepsilon} \int_\Omega \Big| g(h_\varepsilon) - g(h)\Big| \mathrm{d}x \to 0.$$

Finally, we observe that

$$\int_{\Omega\setminus\Omega_\varepsilon} \Big| \langle 1_{\Omega_\varepsilon} g(h)\rangle - g(h)\Big| \, \mathrm{d}x \ \mathrm{d}t \to 0. \tag{3.5}$$

To see that, we introduce a family of linear operators

$$\xi \mapsto \langle 1_{\Omega_\varepsilon} \xi\rangle \ \text{ for } \xi \in L^1(\Omega).$$

By virtue of (3.2), these operators are bounded in $L^1(\Omega)$ uniformly for $\varepsilon \to 0$. Consequently, (3.5) follows as soon as we can show that

$$\langle 1_{\Omega_\varepsilon} \xi\rangle \to \xi \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0$$

for any *smooth* function ξ. However, this is obvious as

$$\sup_{x\in\Omega} |(\langle 1_{\Omega_\varepsilon}\xi\rangle - \xi)(x)| \leq c\varepsilon \sup_{x\in\Omega} |\xi'(x)|.$$

Q.E.D.

- *Sobolev embedding theorem.*

Proposition 3.1 *Under hypotheses* **[H1]**, **[H4]**, *we have*

$$\|v\|_{L^6(\Omega_\varepsilon)} \leq c\Big(\Big| \int_{\Omega_\varepsilon} v \, \mathrm{d}x \Big| + \Big\| \frac{\nabla_x v}{\varepsilon} \Big\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \Big)$$

for any $v \in W^{1,2}(\Omega_\varepsilon)$, *with* c *independent of* ε.

Proof: Obviously, it is enough to show

$$\|v\|_{L^6(\Omega_\varepsilon)} \leq c \left\| \frac{\nabla_x v}{\varepsilon} \right\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \quad \text{for all } v \in W^{1,2}(\Omega_\varepsilon), \ \int_{\Omega_\varepsilon} v \ {\rm d}x = 0.$$

We have

$$\|v\|_{L^6(\Omega_\varepsilon)} = \sup_{\|\varphi\|_{L^{6/5}(\Omega_\varepsilon)} \leq 1, \int_{\Omega_\varepsilon} \varphi \ {\rm d}x = 0} \int_{\Omega_\varepsilon} v\varphi \ {\rm d}x$$

$$= \sup_{\|\varphi\|_{L^{6/5}(\Omega_\varepsilon)} \leq 1, \int_{\Omega_\varepsilon} \varphi \ {\rm d}x = 0} \int_{\Omega_\varepsilon} v {\rm div}_x \mathcal{B}_\varepsilon[\varphi] \ {\rm d}x = \sup_{\|\varphi\|_{L^{6/5}(\Omega_\varepsilon)} \leq 1, \int_{\Omega_\varepsilon} \varphi \ {\rm d}x = 0} \int_{\Omega_\varepsilon} \frac{\nabla_x v}{\varepsilon} \cdot \varepsilon \mathcal{B}_\varepsilon[\varphi] \ {\rm d}x,$$

where, by virtue of (2.9) and the standard embedding relation $W^{1,6/5}_0(\Omega_\varepsilon) \hookrightarrow L^2(\Omega_\varepsilon)$,

$$\sup_{\|\varphi\|_{L^{6/5}(\Omega_\varepsilon)} \leq 1, \int_{\Omega_\varepsilon} \varphi \ {\rm d}x = 0} \int_{\Omega_\varepsilon} \frac{\nabla_x v}{\varepsilon} \cdot \varepsilon \mathcal{B}_\varepsilon[\varphi] \ {\rm d}x \leq c \left\| \frac{\nabla_x v}{\varepsilon} \right\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)}.$$

Here, we have used the fact that functions in $W^{1,6/5}_0(\Omega_\varepsilon)$, extended by zero outside Ω_ε, belong to $W^{1,6/5}(\mathbb{R}^3)$.

Q.E.D.

4 Uniform bounds

In this section, we collect all available uniform bounds on the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$.

4.1 Energy estimates

The total energy of the system is a constant of motion, in particular,

$$E_\varepsilon(t) \equiv \int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon P_e(\varrho_\varepsilon) + b \varrho_\varepsilon \vartheta^{3/2} + \varepsilon \vartheta^4_\varepsilon \right)(t,\cdot) \ {\rm d}x = E_{0,\varepsilon} \to E_0. \tag{4.1}$$

Accordingly, we deduce the uniform estimates listed below:

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.2}$$

$$\{\varrho_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^\gamma(\Omega)), \tag{4.3}$$

$$\{\varrho_\varepsilon \vartheta_\varepsilon^{3/2}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega)), \tag{4.4}$$

$$\{\varepsilon \vartheta_\varepsilon^4\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega)), \tag{4.5}$$

where we have used the structural hypotheses (2.16), (2.17). Here and hereafter, all functions defined *a priori* on Ω_ε are extended to be zero outside Ω_ε, if not stated otherwise.

4.2 Bounds based on mechanical energy dissipation

Integrating the entropy balance (2.27) we get

$$S_\varepsilon(\tau) \geq S_0 \tag{4.6}$$

$$+ \int_0^\tau \int_{\Omega_\varepsilon} \frac{1}{\vartheta_\varepsilon} \left(\frac{\mu(\vartheta_\varepsilon)}{2\varepsilon^2} \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \right) \mathrm{d}x \ \mathrm{d}t \text{ for a.a. } \tau \in (0,T),$$

where

$$S_\varepsilon(\tau) \equiv \int_{\Omega_\varepsilon} \left(3b\varrho_\varepsilon \sqrt{\vartheta_\varepsilon} - \varrho_\varepsilon P_\vartheta(\varrho_\varepsilon) + \varepsilon \frac{4}{3} \vartheta_\varepsilon^3 \right) (\tau, \cdot) \ \mathrm{d}x. \tag{4.7}$$

As a consequence of the uniform bounds established in (4.3 - 4.5), and by virtue of hypotheses (2.16), (2.17), the quantity S_ε is bounded uniformly with respect to ε. Accordingly, using hypotheses (2.22 - 2.23), we may infer that

$$\int_0^T \int_{\Omega_\varepsilon} \left| \frac{\nabla_x \vartheta_\varepsilon^\alpha}{\varepsilon} \right|^2 \mathrm{d}x \ \mathrm{d}t \leq c \text{ for any } 0 \leq \alpha \leq 3/2 \tag{4.8}$$

uniformly for $\varepsilon \to 0$.

Next, extending $\mathbf{u}_\varepsilon$ to be zero outside Ω_ε we are allowed to use Korn's inequality to obtain

$$\left\{ \frac{\nabla_x \mathbf{u}_\varepsilon}{\varepsilon} \right\}_{\varepsilon > 0} \text{ bounded in } L^2((0,T) \times \Omega; \mathbb{R}^{3\times 3}), \tag{4.9}$$

and, as a direct consequence of (3.1),

$$\left\{ \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \right\}_{\varepsilon > 0} \text{ bounded in } L^2((0,T) \times \Omega; \mathbb{R}^3). \tag{4.10}$$

4.3 Refined temperature estimates

Our next goal is to apply Proposition 3.1, together with (4.8), to deduce uniform bounds on the temperature. However, this cannot be done directly as a suitable bound on the mean of ϑ_ε is still missing. To fill this gap, we introduce the following version of Nečas' lemma (cf. Nečas [26]):

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let M, K be two positive real numbers and ϱ a non-negative function such that*

$$0 < M \leq M_\varrho = \int_\Omega \varrho \ \mathrm{d}x, \ \int_\Omega \varrho^\gamma \ \mathrm{d}x \leq K \textit{ for a certain } \gamma \geq 2.$$

Then there exists a constant $c = c(M,K)$ such that

$$\left\| w - \frac{1}{M_\varrho} \int_\Omega \varrho w \ \mathrm{d}x \right\|_{L^2(\Omega)} \leq c(M,K) \|\nabla_x w\|_{W^{-1,2}(\Omega;\mathbb{R}^3)}$$

for any $w \in L^2(\Omega)$.

Proof: Assuming the contrary we construct a sequence $\{w_n\}_{n=1}^{\infty} \subset L^2(\Omega)$ and $\{\varrho_n\}_{n=1}^{\infty}$ such that

$$\varrho_n \to \varrho \text{ weakly in } L^{\gamma}(\Omega), \ \int_{\Omega} \varrho \ {\rm d}x = M_{\varrho} \geq M,$$

$$\left\| w_n - \frac{1}{M_{\varrho_n}} \int_{\Omega} \varrho_n w_n \ {\rm d}x \right\|_{L^2(\Omega)} \geq n \|\nabla_x w_n\|_{W^{-1,2}(\Omega;\mathbb{R}^3)} > 0.$$

Setting

$$v_n = w_n - \frac{1}{M_{\varrho_n}} \int_{\Omega} \varrho_n w_n \ {\rm d}x, \ z_n = \frac{v_n}{\|v_n\|_{L^2(\Omega)}},$$

we readily get

$$\nabla_x z_n \to 0 \text{ in } W^{-1,2}(\Omega), \ \|z_n\|_{L^2(\Omega)} = 1.$$

By virtue of the standard Nečas' lemma (see Nečas [26]), we have

$$z_n \to \overline{z} \text{ (strongly) in } L^2(\Omega), \text{ where } \overline{z} \neq 0 \text{ is a constant.}$$

On the other hand, however,

$$0 = \int_{\Omega} \varrho_n z_n \ {\rm d}x \to \int_{\Omega} \varrho \overline{z} \ {\rm d}x = \overline{z} M_{\varrho}$$

- a contradiction.

Q.E.D.

In accordance with (2.8), we may write

$$< \nabla_x \langle \vartheta_\varepsilon \rangle ; \varphi >= - \int_{\Omega} \langle \vartheta_\varepsilon \rangle \,{\rm div}_x \varphi \ {\rm d}x = - \int_{\Omega_\varepsilon} \vartheta_\varepsilon {\rm div}_x \mathcal{R}_\varepsilon[\varphi] \ {\rm d}x = \int_{\Omega_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \mathcal{R}_\varepsilon[\varphi] \ {\rm d}x;$$

whence, by virtue of (2.7),

$$| < \nabla_x \langle \vartheta_\varepsilon \rangle ; \varphi > | \leq c \|\nabla_x \vartheta_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \Big(\|\varphi\|_{L^2(\Omega;\mathbb{R}^3)} + \varepsilon \|\nabla_x \varphi\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \Big) \text{ for all } \varphi \in C^\infty_c(\Omega;\mathbb{R}^3).$$

Thus, in agreement with (4.8), we conclude that

$$\|\nabla_x \langle \vartheta_\varepsilon \rangle \|_{L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))} \leq \varepsilon c. \tag{4.11}$$

Consequently, we have

$$\langle \vartheta_\varepsilon \rangle = \langle \vartheta_\varepsilon \rangle - \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \vartheta_\varepsilon \ {\rm d}x + \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \vartheta_\varepsilon \ {\rm d}x, \quad M_\varepsilon \equiv \int_{\Omega_\varepsilon} \varrho_\varepsilon \, {\rm d}x, \tag{4.12}$$

where, in accordance with Lemma 4.1, and (4.11),

$$\left\| \langle \vartheta_\varepsilon \rangle - \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \vartheta_\varepsilon \ {\rm d}x \right\|_{L^2((0,T)\times\Omega)} \leq \varepsilon c, \tag{4.13}$$

while

$$\left\{ \Theta_\varepsilon \equiv \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \vartheta_\varepsilon \ {\rm d}x \right\}_{\varepsilon > 0} \text{ is bounded in } L^\infty(0,T) \tag{4.14}$$

as a direct consequence of (4.4).

Finally, combining Proposition 3.1, with estimates (4.8), (4.13), (4.14), we may apply the same argument to $\vartheta_\varepsilon^{3/2}$ to obtain

$$\int_0^T \|\vartheta_\varepsilon\|^3_{L^9(\Omega_\varepsilon)} \ {\rm d}t \le c \tag{4.15}$$

uniformly for $\varepsilon \to 0$.

4.4 Pressure estimates

In order to deduce uniform bounds on the pressure, we use the quantities

$$\mathbf{v} = \psi(t)\mathcal{B}_\varepsilon \left[b(\varrho) \right], \ \psi \in C^\infty_c(0,T), \ \text{with a suitable function } b(\varrho) = \varrho^\nu \text{ for } \varrho \ge 1,$$

as test functions in the momentum equation (2.26), where the symbol $\mathcal{B}_\varepsilon$ denotes Bogovskii's operator introduced in hypothesis **[H4]**. This step is rather technical but nowadays well-understood. In particular, we use the fact that $b(\varrho_\varepsilon)$ satisfies the renormalized continuity equation

$$\int_0^T \int_\Omega \Big(\varepsilon^2 b(\varrho_\varepsilon)\partial_t \varphi + b(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \varphi + (b'(\varrho_\varepsilon)\varrho_\varepsilon - b(\varrho_\varepsilon)){\rm div}_x \mathbf{u}_\varepsilon \varphi \Big) \ {\rm d}x \ {\rm d}t = 0 \tag{4.16}$$

for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$ provided ϱ_ε, $\mathbf{u}_\varepsilon$ were extended by zero outside Ω_ε. Indeed as $\gamma \ge 2$, (4.16) can be deduced from (2.25) by means of a regularization procedure introduced by DiPerna and Lions [11].

After a bit tedious but rather straightforward manipulation we arrive at the following identity:

$$\int_0^T \psi \int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) b(\varrho_\varepsilon) \ {\rm d}x \ {\rm d}t = \sum_{i=1}^6 I_{i,\varepsilon}, \tag{4.17}$$

where

$$I_{1,\varepsilon} = \frac{1}{|\Omega_\varepsilon|} \int_0^T \psi \left(\int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) \ {\rm d}x \right) \left(\int_{\Omega_\varepsilon} b(\varrho_\varepsilon) \ {\rm d}x \right) \ {\rm d}t,$$

$$I_{2,\varepsilon} = \int_0^T \psi \int_{\Omega_\varepsilon} \mu(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) : \nabla_x \mathcal{B}_\varepsilon[b(\varrho_\varepsilon)] \ {\rm d}x \ {\rm d}t$$
$$+ \int_0^T \psi \int_{\Omega_\varepsilon} \eta(\vartheta_\varepsilon){\rm div}_x \mathbf{u}_\varepsilon \left(b(\varrho_\varepsilon) - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} b(\varrho_\varepsilon) dx \right) \ {\rm d}x \, {\rm d}t,$$

$$I_{3,\varepsilon} = \int_0^T \psi \int_{\Omega_\varepsilon} \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathcal{B}_\varepsilon[b(\varrho_\varepsilon)] \ {\rm d}x \ {\rm d}t,$$

$$I_{4,\varepsilon} = \varepsilon^2 \int_0^T \partial_t \psi \int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}_\varepsilon[b(\varrho_\varepsilon)] \ {\rm d}x \ {\rm d}t,$$

$$I_{5,\varepsilon} = \int_0^T \psi \int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}_\varepsilon \Big[(b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon) \mathrm{div}_x \mathbf{u}_\varepsilon \Big] \ \mathrm{d}x \ \mathrm{d}t,$$

and

$$I_{6,\varepsilon} = - \int_0^T \psi \int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}_\varepsilon \Big[\mathrm{div}_x (b(\varrho_\varepsilon)\mathbf{u}_\varepsilon) \Big] \ \mathrm{d}x \ \mathrm{d}t.$$

Now, in view of the uniform bounds established in (4.2), (4.3), (4.5), (4.9), (4.10), (4.15), and in accordance with (2.9), we deduce that

$$|I_{1,\varepsilon}| + |I_{2,\varepsilon}| \leq c \text{ uniformly with respect to } \varepsilon$$

provided $\nu > 0$ is small enough. Similarly, making use of the properties of the operator $\mathcal{B}_\varepsilon$ stated in **[H4]**, we deduce that

$$\sum_{i=3}^{6} |I_{i,\varepsilon}| \leq \varepsilon c, \tag{4.18}$$

with c independent of ε. We remark that, strictly speaking, the quantity $b(\varrho_\varepsilon)\mathbf{u}_\varepsilon$ appearing in $I_{6,\varepsilon}$ does not belong to the class required in (2.10). This technical problem may be overcome by using suitable regularization of ϱ_ε exactly as in [16], or by extending (2.10) to $\mathbf{f}$ belonging to the dual space $[W^{1,p'}(\Omega_\varepsilon; \mathbb{R}^3)]^*$ as in Geißert et al. [18]. Thus we conclude that

$$\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon^{\gamma+\nu} \ \mathrm{d}x \ \mathrm{d}t \leq c \text{ for a certain } \nu > 0, \tag{4.19}$$

uniformly for $\varepsilon \to 0$.

5 Strong convergence of the density

In future considerations, we will systematically use the following properties of the extension operator introduced in (1.11) :

$$G(\langle h \rangle) \leq \langle G(h) \rangle \ \text{ for any convex function } G, \tag{5.1}$$

and

$$\left\{ \begin{array}{c} \langle h_\varepsilon \rangle \to \chi \text{ weakly in } L^1((0,T)\times\Omega) \\ \\ \text{only if} \\ \\ h_\varepsilon \to \sigma\chi \text{ weakly in } L^1((0,T)\times\Omega), \end{array} \right\} \tag{5.2}$$

where $\sigma \in (0,1]$ is the constant appearing in (2.1). Note that (5.1) is a direct consequence of Jensen's inequality, while (5.2) can be shown exactly as in Masmoudi [22, Lemma 1.3].

In view of (3.2) and the uniform bounds (4.3) we may assume that

$$\langle \varrho_\varepsilon \rangle \to \varrho \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\Omega)).$$

Our goal is to show that $\langle \varrho_\varepsilon \rangle$ converge to ϱ pointwise (a.a.) in $(0,T)\times\Omega$.

5.1 Oscillations defect measure

The following quantity will play a crucial role in the analysis of density oscillations:

$$D_k = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(p_e(\varrho_\varepsilon) T_k(\varrho_\varepsilon) - \overline{\langle p_e(\varrho) \rangle} \ \overline{T_k(\varrho)} \Big) \ \mathrm{d}x \ \mathrm{d}t, \ \ k \geq 1,$$

where we have introduced the cut-off functions

$$T_k(\varrho) = \left\{ \begin{array}{l} \min\{\varrho, k\} \text{ for } \varrho \geq 0, \\ \\ -T_k(-\varrho) \text{ if } \varrho < 0, \end{array} \right. \tag{5.3}$$

and where the symbol $\overline{G(v)}$ denotes a weak L^1-limit of a sequence $\{v_n\}_{n=1}^\infty$. We recall that ϱ_ε as well as other ε-dependent quantities defined solely on Ω_ε are set to be zero outside Ω_ε.

In accordance with (5.2), we have

$$D_k = \sigma \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(\langle p_e(\varrho_\varepsilon) T_k(\varrho_\varepsilon) \rangle - \overline{\langle p_e(\varrho) \rangle} \ \overline{\langle T_k(\varrho) \rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t.$$

Next, we introduce a function Q_k,

$$Q_k(r) = \left\{ \begin{array}{l} \int_0^r p_e(s) \ \mathrm{d}s \text{ for } r \in [0, k], \\ \\ \infty \text{ for } r > k. \end{array} \right.$$

Since p_e is increasing, Q_k is convex and lower semi-continuous on the interval $[0, \infty)$. Moreover, we have

$$p_e(\varrho_\varepsilon) T_k(\varrho_\varepsilon) = Q_k(T_k(\varrho_\varepsilon)) + Q_k^*(p_e(\varrho_\varepsilon)), \tag{5.4}$$

where the symbol Q_k^* stands for the conjugate function, see Ekeland, Temam [13].

On the other hand,

$$\overline{\langle p_e(\varrho) \rangle} \ \overline{\langle T_k(\varrho) \rangle} \leq Q_k(\overline{\langle T_k(\varrho) \rangle}) + Q_k^*(\overline{\langle p_e(\varrho) \rangle}), \tag{5.5}$$

which, combined with (5.4), gives rise to

$$D_k \geq \sigma \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(\ \langle Q_k(T_k(\varrho_\varepsilon) \rangle + \langle Q_k^*(p_e(\varrho_\varepsilon)) \rangle - Q_k(\overline{\langle T_k(\varrho) \rangle}) - Q_k^*(\overline{\langle p_e(\varrho) \rangle}) \Big) \ \mathrm{d}x \ \mathrm{d}t. \tag{5.6}$$

Thus, using (5.1), together with lower semi-continuity of convex functions, we may infer that

$$D_k \geq \sigma \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(Q_k(\langle T_k(\varrho_\varepsilon) \rangle) - Q_k(\overline{\langle T_k(\varrho) \rangle}) \Big) \ \mathrm{d}x \ \mathrm{d}t. \tag{5.7}$$

Relation (5.7) can be rewritten as

$$D_k \geq \sigma \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(Y(\langle T_k(\varrho_\varepsilon) \rangle) \, \langle T_k(\varrho_\varepsilon) \rangle - Y(\overline{\langle T_k(\varrho) \rangle}) \overline{\langle T_k(\varrho) \rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t, \tag{5.8}$$

where

$$Y(z) = \frac{Q(z)}{z} \text{ for all } 0 \leq z \leq k, \ \text{ with } Q(z) = \int_0^z p_e(s) \ \mathrm{d}s.$$

On the other hand, it follows from hypothesis (2.16) that

$$p_e(\varrho) = \Gamma \varrho^\gamma + q(\varrho), \ \text{with } \Gamma > 0, \ q \in C^1[0,\infty), \ q(0) = 0, \ q' \geq 0.$$

Consequently, relation (5.8) gives rise to

$$D_k \geq \frac{\Gamma\sigma}{\gamma+1} \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Big(\langle T_k(\varrho_\varepsilon)\rangle^{\gamma+1} - \overline{\langle T_k(\varrho_\varepsilon)\rangle^{\gamma+1}} \Big) \ \mathrm{d}x \ \mathrm{d}t, \tag{5.9}$$

where we have used the fact that the function $z \mapsto \frac{1}{z}\int_0^z q(s) \ \mathrm{d}s$ is non-decreasing.

Finally, (5.9) yields

$$D_k \geq \frac{\Gamma\sigma}{\gamma+1} \lim_{\varepsilon\to 0} \int_0^T \int_\Omega |\, \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}|^{\gamma+1} \ \mathrm{d}x \ \mathrm{d}t, \tag{5.10}$$

where we have used

$$\overline{\langle T_k(\varrho)\rangle^{\gamma+1}} \leq \overline{\langle T_k(\varrho)\rangle^{\gamma}} \ \overline{\langle T_k(\varrho)\rangle}.$$

The quantity

$$\mathbf{osc}_k[\langle\varrho_\varepsilon\rangle \to \varrho] \equiv \lim_{\varepsilon\to 0} \int_0^T \int_\Omega |\, \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}|^{\gamma+1} \ \mathrm{d}x \ \mathrm{d}t \tag{5.11}$$

will be termed *oscillation defect measure* associated to the sequence $\{\langle\varrho_\varepsilon\rangle\}_{\varepsilon>0}$.

5.2 Thermal pressure

Our goal is to show that $\mathbf{osc}_k[\langle\varrho_\varepsilon\rangle \to \varrho] \to 0$ for $k \to \infty$ that implies strong convergence of $\{\langle\varrho_\varepsilon\rangle\}_{\varepsilon>0}$. To this end, we examine the limit of the "thermal part" of the pressure, specifically,

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Big(p_\vartheta(\varrho_\varepsilon)\vartheta_\varepsilon T_k(\varrho_\varepsilon) - \overline{\langle p_\vartheta(\varrho)\vartheta\rangle} \ \overline{T_k(\varrho)} \Big) \ \mathrm{d}x \ \mathrm{d}t = \tag{5.12}$$

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Big(p_\vartheta(\varrho_\varepsilon)\vartheta_\varepsilon T_k(\varrho_\varepsilon) - \overline{p_\vartheta(\varrho)\vartheta} \ \overline{\langle T_k(\varrho)\rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t = \lim_{\varepsilon\to 0} \int_0^T \int_\Omega p_\vartheta(\varrho_\varepsilon)\vartheta_\varepsilon \Big(T_k(\varrho_\varepsilon) - \overline{\langle T_k(\varrho)\rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \vartheta_\varepsilon \Big(p_\vartheta(\varrho_\varepsilon) - p_\vartheta(\overline{\langle T_k(\varrho)\rangle}) \Big) \Big(T_k(\varrho_\varepsilon) - \overline{\langle T_k(\varrho)\rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \lim_{\varepsilon\to 0} \int_0^T \int_\Omega p_\vartheta(\overline{\langle T_k(\varrho)\rangle}) \ \vartheta_\varepsilon \Big(T_k(\varrho_\varepsilon) - \overline{\langle T_k(\varrho)\rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\geq \lim_{\varepsilon\to 0} \int_0^T \int_\Omega p_\vartheta(\overline{\langle T_k(\varrho)\rangle}) \ \vartheta_\varepsilon \Big(T_k(\varrho_\varepsilon) - \overline{\langle T_k(\varrho)\rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega p_\vartheta(\overline{\langle T_k(\varrho)\rangle}) \ \vartheta_\varepsilon \Big(\langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle} \Big) \ \mathrm{d}x \ \mathrm{d}t,$$

where we have used monotonicity of p_ϑ, specifically the inequality

$$\vartheta_\varepsilon \Big(p_\vartheta(\varrho_\varepsilon) - p_\vartheta(\overline{\langle T_k(\varrho)\rangle}) \Big) \Big(T_k(\varrho_\varepsilon) - \overline{\langle T_k(\varrho)\rangle} \Big) \geq 0.$$

Note that, in accordance with hypotheses (2.16), (2.17), and the uniform bound (4.3),

$$p_\vartheta(\overline{\langle T_k(\varrho)\rangle}) \in L^\infty(0,T;L^\beta(\Omega)) \ \text{for a certain} \ \beta > 3/2. \tag{5.13}$$

The last integral in (5.12) can be handled as follows:

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\ \vartheta_\varepsilon\Big(\langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\Big)\ \mathrm{d}x\ \mathrm{d}t =$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \vartheta_\varepsilon\Big(\langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\lim_{\varepsilon\to 0}\int_0^T\int_\Omega (\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \vartheta_\varepsilon\Big(\langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\Big)\ \mathrm{d}x\ \mathrm{d}t,$$

where

$$\left|\lim_{\varepsilon\to 0}\int_0^T\int_\Omega (\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \vartheta_\varepsilon\Big(\langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\Big)\ \mathrm{d}x\ \mathrm{d}t\right| \leq$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left|(\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\right|\ \langle\vartheta_\varepsilon\rangle\ \left| \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\right|\ \mathrm{d}x\ \mathrm{d}t$$

$$= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left|(\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\right|\ \Big(\langle\vartheta_\varepsilon\rangle - \Theta_\varepsilon\Big) \left| \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\right|\ \mathrm{d}x\ \mathrm{d}t$$

$$+\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Theta_\varepsilon \left|(\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\right|\ \left| \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\right|\ \mathrm{d}x\ \mathrm{d}t,$$

where the quantities Θ_ε were introduced in (4.14).

Consequently, by virtue of (4.13),

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \left|(\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\right|\ \Big(\langle\vartheta_\varepsilon\rangle - \Theta_\varepsilon\Big) \left| \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\right|\ \mathrm{d}x\ \mathrm{d}t = 0, \tag{5.14}$$

while

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \Theta_\varepsilon \left|(\mathrm{Id} - T_M)\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\right|\ \left| \langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\right|\ \mathrm{d}x\ \mathrm{d}t \leq h(M)\Big(\mathbf{osc}_k[\langle\varrho_\varepsilon\rangle - \varrho]\Big)^{1/3}, \tag{5.15}$$

where, in accordance with (4.14), (5.13),

$$h(M) \to 0 \ \text{as} \ M \to \infty.$$

Finally,

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \vartheta_\varepsilon\Big(\langle T_k(\varrho_\varepsilon)\rangle - \overline{\langle T_k(\varrho)\rangle}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \langle\vartheta_\varepsilon\rangle\ \Big(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)}\Big)\ \mathrm{d}x\ \mathrm{d}t,$$

where, by means of the same arguments as above,

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \langle\vartheta_\varepsilon\rangle\left(T_k(\varrho_\varepsilon)-\overline{T_k(\varrho)}\right)\,\mathrm{d}x\ \mathrm{d}t$$

$$=\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left(T_k(\varrho_\varepsilon)-\overline{T_k(\varrho)}\right)\,\mathrm{d}x\ \mathrm{d}t.$$

Furthermore,

$$\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left(T_k(\varrho_\varepsilon)-\overline{T_k(\varrho)}\right)\,\mathrm{d}x\ \mathrm{d}t$$

$$=\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left(\varrho_\varepsilon-\overline{\varrho}\right)\,\mathrm{d}x\ \mathrm{d}t$$

$$+\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left(T_k(\varrho_\varepsilon)-\varrho_\varepsilon\right)\,\mathrm{d}x\ \mathrm{d}t$$

$$+\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left(\overline{\varrho}-\overline{T_k(\varrho)}\right)\,\mathrm{d}x\ \mathrm{d}t,$$

where, in accordance with (2.25), (4.3), (4.10), and (4.19),

$$\varrho_\varepsilon\to\overline{\varrho}\ \text{in}\ C([0,T];W^{-1,2}(\Omega)). \tag{5.16}$$

Thus we have

$$\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left(\varrho_\varepsilon-\overline{\varrho}\right)\,\mathrm{d}x\ \mathrm{d}t=0,$$

while,

$$\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left|T_k(\varrho_\varepsilon)-\varrho_\varepsilon\right|\,\mathrm{d}x\ \mathrm{d}t$$

$$+\lim_{\varepsilon\to 0}\int_0^T\Theta_\varepsilon\int_\Omega T_M\left(p_\vartheta(\overline{\langle T_k(\varrho)\rangle})\right)\ \left|\overline{\varrho}-\overline{T_k(\varrho)}\right|\,\mathrm{d}x\ \mathrm{d}t\le c\frac{M}{k}.$$

Combining the estimates obtained in the preceding two sections, we may infer that

$$\frac{\Gamma\sigma}{\Gamma+1}\mathbf{osc}_k[\langle\varrho_\varepsilon\rangle-\varrho]\equiv\frac{\Gamma\sigma}{\Gamma+1}\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\left|\langle T_k(\varrho_\varepsilon)\rangle-\overline{\langle T_k(\varrho)\rangle}\right|^{\gamma+1}\,\mathrm{d}x\ \mathrm{d}t \tag{5.17}$$

$$\le\lim_{\varepsilon\to 0}\int_0^T\int_\Omega\left(p(\varrho_\varepsilon,\vartheta_\varepsilon)T_k(\varrho_\varepsilon)-\overline{\langle p(\varrho,\vartheta)\rangle}\ \overline{T_k(\varrho)}\right)\,\mathrm{d}x\ \mathrm{d}t+c\frac{M}{k}+h(M)\Big(\mathbf{osc}_k[\langle\varrho_\varepsilon\rangle-\varrho]\Big)^{1/3}.$$

Here, we have used a simple interpolation inequality

$$\|\varepsilon^{1/4}\vartheta_\varepsilon\|^{17/3}_{L^{17/3}(\Omega_\varepsilon)}\le c\varepsilon^{3/4}\|\vartheta_\varepsilon\|^3_{L^9(\Omega_\varepsilon)}\|\varepsilon^{1/4}\vartheta_\varepsilon\|^{8/3}_{L^4(\Omega_\varepsilon)},$$

together with the uniform bounds established in (4.5), (4.15) to eliminate the radiation component of the pressure, specifically,

$$\varepsilon\vartheta_\varepsilon^4\to 0\ \text{in}\ L^{17/12}((0,T)\times\Omega). \tag{5.18}$$

5.3 Refined pressure estimates

In view of (5.17), we have strong convergence of the densities $\{\langle \varrho_\varepsilon \rangle\}_{\varepsilon > 0}$ as soon as we can show that

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(p(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon) - \overline{\langle p(\varrho, \vartheta) \rangle}\ \overline{T_k(\varrho)} \Big) \ {\rm d}x \ {\rm d}t \leq h(k), \tag{5.19}$$

with $h(k) \to 0$ for $k \to \infty$. In order to see (5.19), we follow the arguments presented in Section 4.4. However, in contrast with the method developed in that section, the pressure must be split into several parts that are treated separately.

Following Masmoudi [22], we write

$$p(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) \ {\rm d}x = \sum_{i=1}^4 p_{i,\varepsilon},$$

where $p_{i,\varepsilon}$ are the (unique) solutions to the Stokes problems:

$$\Delta \mathbf{w}_{1,\varepsilon} + \nabla_x p_{1,\varepsilon} = {\rm div}_x \Big(\mu(\Theta_\varepsilon) \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) \Big) \text{ in } (0,T) \times \Omega_\varepsilon, \tag{5.20}$$

$$\Delta \mathbf{w}_{2,\varepsilon} + \nabla_x p_{2,\varepsilon} = {\rm div}_x \Big(\Big(\mu(\vartheta_\varepsilon) - \mu(\Theta_\varepsilon) \Big) \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) \Big) \text{ in } (0,T) \times \Omega_\varepsilon, \tag{5.21}$$

$$\Delta \mathbf{w}_{3,\varepsilon} + \nabla_x p_{3,\varepsilon} = -{\rm div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \text{ in } (0,T) \times \Omega_\varepsilon, \tag{5.22}$$

$$\Delta \mathbf{w}_{4,\varepsilon} + \nabla_x p_{4,\varepsilon} = -\varepsilon^2 \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) \text{ in } (0,T) \times \Omega_\varepsilon, \tag{5.23}$$

$${\rm div}_x \mathbf{w}_{i,\varepsilon} = 0 \text{ in } (0,T) \times \Omega_\varepsilon, \ \mathbf{w}_{i,\varepsilon}|_{\partial \Omega_\varepsilon} = 0, \ \int_{\Omega_\varepsilon} p_{i,\varepsilon} \ {\rm d}x = 0 \text{ for } i = 1 \ldots 4.$$

In accordance with hypothesis **[H5]** and the uniform bounds (4.9), (4.14), we have

$$\nabla_x p_{1,\varepsilon} = {\rm div}_x \mathbb{H}_\varepsilon \text{ in } (0,T) \times \Omega_\varepsilon, \tag{5.24}$$

where

$$\| \mathbb{H}_\varepsilon \|_{L^2((0,T) \times \Omega_\varepsilon; \mathbb{R}^{3 \times 3}))} \leq \varepsilon c. \tag{5.25}$$

On the other hand, it follows from (2.8) that

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} p_{1,\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x \ {\rm d}t = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \langle p_{1,\varepsilon} \rangle \left(T_k(\varrho_\varepsilon) - \frac{1}{|\Omega|} \int_{\Omega_\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x \right) \ {\rm d}x \ {\rm d}t \tag{5.26}$$

$$= \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \langle p_{1,\varepsilon} \rangle \, {\rm div}_x \mathcal{B}[T_k(\varrho_\varepsilon)] \ {\rm d}x \ {\rm d}t = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} p_{1,\varepsilon} {\rm div}_x \mathcal{R}_\varepsilon \left(\mathcal{B}[T_k(\varrho_\varepsilon)] \right) \ {\rm d}x \ {\rm d}t,$$

where we have used that $\int_\Omega < h > \ {\rm d}x = \frac{1}{\sigma} \int_\Omega h \, {\rm d}x$ and where $\mathcal{B}$ is the Bogovskii operator defined on the *whole* domain Ω. Namely, the operator $\mathcal{B}$ satisfies $\mathcal{B}(f) = f - \frac{1}{|\Omega|} \int_\Omega f \, {\rm d}x$ on Ω and estimates (2.9), (2.10), where Ω_ε is replaced by Ω and ε is set to be 1.

Moreover, by the same token,

$$\int_0^T \int_\Omega \overline{\langle p_1 \rangle}\ \overline{T_k(\varrho)} \ {\rm d}x \ {\rm d}t = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \langle p_{1,\varepsilon} \rangle \, \overline{T_k(\varrho)} \ {\rm d}x \ {\rm d}t$$

$$= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} p_{1,\varepsilon} {\rm div}_x \mathcal{R}_\varepsilon \left(\mathcal{B}[\overline{T_k(\varrho)}] \right) \ {\rm d}x \ {\rm d}t.$$

Consequently, we get

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} p_{1,\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x \ {\rm d}t - \int_0^T \int_\Omega \overline{\langle p_1 \rangle} \ \overline{T_k(\varrho)} \ {\rm d}x \ {\rm d}t$$

$$= \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} \mathbb{H}_\varepsilon : \nabla_x \left(\mathcal{R}_\varepsilon \left(\mathcal{B} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] \right) \right) \ {\rm d}x \ {\rm d}t$$

$$= \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} \mathbb{H}_\varepsilon : \nabla_x \left(\mathcal{R}_\varepsilon \left(\mathcal{B} \left[\varrho_\varepsilon - \overline{\varrho} \right] \right) \right) \ {\rm d}x \ {\rm d}t$$

$$+ \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} \mathbb{H}_\varepsilon : \nabla_x \left(\mathcal{R}_\varepsilon \left(\mathcal{B} \left[T_k(\varrho_\varepsilon) - \varrho_\varepsilon \right] \right) \right) \ {\rm d}x \ {\rm d}t$$

$$+ \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} \mathbb{H}_\varepsilon : \nabla_x \left(\mathcal{R}_\varepsilon \left(\mathcal{B} \left[\overline{\varrho} - \overline{T_k(\varrho)} \right] \right) \right) \ {\rm d}x \ {\rm d}t,$$

where, in accordance with (4.19),

$$\| T_k(\varrho_\varepsilon) - \varrho_\varepsilon \|_{L^2((0,T)\times\Omega)} + \| \overline{T_k(\varrho)} - \overline{\varrho} \|_{L^2((0,T)\times\Omega)} \leq h(k) \text{ with } h(k) \to 0 \text{ as } k \to \infty. \tag{5.27}$$

In addition, we deduce from (5.16) that

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } C_{\rm weak}([0,T]; L^2(\Omega)), \ \overline{\varrho} = \sigma \varrho;$$

whence

$$\mathcal{B}[\varrho_\varepsilon - \overline{\varrho}] \to 0 \text{ (strongly) in } L^2((0,T)\times\Omega). \tag{5.28}$$

Combining (5.25 - 5.28) with (2.7) we conclude that

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} p_{1,\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x \ {\rm d}t - \int_0^T \int_\Omega \overline{\langle p_1 \rangle} \ \overline{T_k(\varrho)} \ {\rm d}x \ {\rm d}t \leq h(k), \ h(k) \to 0 \text{ as } k \to \infty. \tag{5.29}$$

Moreover, going back to (5.24), (5.25), and making use of (2.7), (2.8), we readily obtain that

$$\nabla_x \overline{\langle p_1 \rangle} \in L^2(0,T; L^2(\Omega)). \tag{5.30}$$

Now, we apply a similar treatment to $p_{2,\varepsilon}, \dots, p_{4,\varepsilon}$ replacing (5.26) by the identity

$$\int_{\Omega_\varepsilon} p_{i,\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x = \int_{\Omega_\varepsilon} p_{i,\varepsilon} \left(T_k(\varrho_\varepsilon) - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x \right) \ {\rm d}x$$

$$= \int_{\Omega_\varepsilon} p_{i,\varepsilon} {\rm div}_x \mathcal{B}_\varepsilon [T_k(\varrho_\varepsilon)] \ {\rm d}x, \ i = 2,3,4.$$

Thus, making use of (4.9), (4.13 - 4.15), we deduce that

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} p_{2,\varepsilon} T_k(\varrho_\varepsilon) \ {\rm d}x = 0, \ \overline{\langle p_2 \rangle} = 0, \tag{5.31}$$

and, by virtue of (4.2 - 4.5), (4.9), (4.10),

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} p_{3,\varepsilon} T_k(\varrho_\varepsilon) \ \mathrm{d}x = 0, \ \overline{\langle p_3 \rangle} = 0. \tag{5.32}$$

Furthermore, it follows from (5.23), combined with (4.3), (4.9–4.10) and hypothesis **[H5]**, that

$$\nabla_x p_{4,\varepsilon} = -\varepsilon^2 \partial_t \mathbf{h}_\varepsilon, \ \text{with} \ \|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega_\varepsilon))} \leq \varepsilon c. \tag{5.33}$$

Now, similarly to Section (4.4), we can use the renormalized continuity equation (4.16) to justify that

$$\int_0^T \psi \int_{\Omega_\varepsilon} p_{4,\varepsilon} T_k(\varrho_\varepsilon) \, \mathrm{d}x\mathrm{d}t = \int_0^T \psi \int_{\Omega_\varepsilon} p_{4,\varepsilon} \mathrm{div}_x (\mathcal{B}_\varepsilon \left(T_k(\varrho_\varepsilon)\right) \ \mathrm{d}x\mathrm{d}t =$$

$$-\varepsilon^2 \int_0^T \psi' \int_{\Omega_\varepsilon} \mathbf{h}_\varepsilon \cdot \mathcal{B}_\varepsilon \left(T_k(\varrho_\varepsilon)\right) \ \mathrm{d}x\mathrm{d}t + \int_0^T \psi \int_{\Omega_\varepsilon} \mathbf{h}_\varepsilon \cdot \mathcal{B}_\varepsilon \left(\mathrm{div}_x (T_k(\varrho_\varepsilon)\mathbf{u}_\varepsilon)\right) \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T \psi \int_{\Omega_\varepsilon} \mathbf{h}_\varepsilon \cdot \mathcal{B}_\varepsilon \left((\varrho_\varepsilon T_k'(\varrho_\varepsilon) - T_k(\varrho_\varepsilon))\mathrm{div}_x \mathbf{u}_\varepsilon\right) \ \mathrm{d}x\mathrm{d}t;$$

whence, in view of uniform bounds (4.3), (4.9), (4.10), we obtain

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} p_{4,\varepsilon} T_k(\varrho_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t = 0, \ \overline{\langle p_4 \rangle} = 0. \tag{5.34}$$

Finally, we observe that

$$\Big(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \Big) \int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) \ \mathrm{d}x = (\varrho_\varepsilon - \overline{\varrho}) \int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) \ \mathrm{d}x$$

$$+ \Big(T_k(\varrho_\varepsilon) - \varrho_\varepsilon \Big) \int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) \ \mathrm{d}x + \Big(\overline{\varrho} - \overline{T_k(\varrho)} \Big) \int_{\Omega_\varepsilon} p(\varrho_\varepsilon, \vartheta_\varepsilon) \ \mathrm{d}x.$$

Summing up the results achieved in this section, we are allowed to conclude that (5.19) holds. This fact, combined with (5.17), implies the desirable conclusion

$$\langle \varrho_\varepsilon \rangle \to \varrho \text{ in } L^2((0,T) \times \Omega). \tag{5.35}$$

In addition, using (5.30), (5.31), (5.32), and (5.34), we conclude that

$$\nabla_x \overline{\langle p(\varrho, \vartheta) \rangle} \in L^2((0,T) \times \Omega). \tag{5.36}$$

6 Strong convergence of the temperature

In order to establish strong (pointwise a.a.) convergence of the temperature field, we use the entropy balance equation (2.27). Similarly to Section 4.3, we write

$$\left\langle \sqrt{\vartheta_\varepsilon} \right\rangle = \left\langle \sqrt{\vartheta_\varepsilon} \right\rangle - \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \sqrt{\vartheta_\varepsilon} \ \mathrm{d}x + \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \sqrt{\vartheta_\varepsilon} \ \mathrm{d}x,$$

where

$$\left\langle \sqrt{\vartheta_\varepsilon} \right\rangle - \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \sqrt{\vartheta_\varepsilon} \ {\rm d}x \to 0 \text{ in } L^2((0,T)\times\Omega).$$

On the other hand, it follows from the entropy balance equation (2.27) that

$$\partial_t \int_{\Omega_\varepsilon} \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) \, {\rm d}x = \partial_t \int_{\Omega_\varepsilon} \left(\varrho_\varepsilon \sqrt{\vartheta_\varepsilon} - \frac{1}{3b} \varrho_\varepsilon P_\vartheta(\varrho_\varepsilon) + \varepsilon \frac{4}{9b} \vartheta_\varepsilon^3 \right) \ {\rm d}x$$

is a sequence of positive measures bounded in the space $(C[0,T])^*$, and therefore precompact, in particular, in $W^{-1,2}\ (0,T)$.

In accordance with (5.18), the mapping

$$t \mapsto \int_{\Omega_\varepsilon} \varrho_\varepsilon \sqrt{\vartheta_\varepsilon} \ {\rm d}x \text{ is precompact in } L^2(0,T)$$

as soon as we observe that

$$t \mapsto \int_{\Omega_\varepsilon} \varrho_\varepsilon P_\vartheta(\varrho_\varepsilon) \ {\rm d}x \text{ is precompact in } L^1(0,T). \tag{6.1}$$

Indeed, by virtue of (5.35), we may assume

$$\langle \varrho_\varepsilon \rangle \, P_\vartheta(\langle \varrho_\varepsilon \rangle)(t,\cdot) \to \varrho P_\vartheta(\varrho)(t,\cdot) \text{ (strongly) in } L^1(\Omega) \text{ for a.a. } t \in (0,T);$$

whence (6.1) follows.

Thus we have shown that $\left\langle \sqrt{\vartheta_\varepsilon} \right\rangle$ converges a.a to a spatially homogeneous function. Using Lemma 3.1 (cf. also Remark 3.1) we therefore conclude that

$$\langle \vartheta_\varepsilon \rangle \to \Theta \text{ (strongly) in } L^2((0,T)\times\Omega), \tag{6.2}$$

where $\Theta = \Theta(t)$ is given as

$$\Theta(t) = \lim_{\varepsilon\to 0} \Theta_\varepsilon(t) \equiv \lim_{\varepsilon \to 0} \frac{1}{M_\varepsilon} \int_{\Omega_\varepsilon} \varrho_\varepsilon \vartheta_\varepsilon(t,\cdot) \ {\rm d}x \text{ for a.a. } t \in (0,T). \tag{6.3}$$

7 Conclusion - proof of the main result

For a function h defined on $(0,T)\times\Omega$ let $[h]_\delta$, $\delta > 0$ denote a suitable (time-space) regularization of h, e.g. the standard mollification of h extended by zero outside $(0,T)\times\Omega$.

We use the quantities

$$[T_k(\langle \varrho_\varepsilon \rangle)]_\delta \mathbf{v}_\varepsilon^n \varphi, \ \varphi \in C_c^\infty((0,T)\times\Omega)$$

as test functions in equation (2.26), where $\mathbf{v}_\varepsilon^n$ are the test functions introduced in hypothesis **[H2]**. After a simple manipulation, we get

$$\int_0^T \int_{\Omega_\varepsilon} \mu(\vartheta_\varepsilon) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \right) : \nabla_x \left([T_k(\langle \varrho_\varepsilon \rangle)]_\delta \mathbf{v}_\varepsilon^n \varphi \right) \ {\rm d}x \ {\rm d}t$$

$$= \int_0^T \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \left([T_k(\langle \varrho_\varepsilon \rangle)]_\delta \mathbf{v}_\varepsilon^n \varphi \right) \ {\rm d}x \ {\rm d}t$$

$$+\int_0^T\int_\Omega \varepsilon^2 \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \left(\partial_t([T_k(\langle\varrho_\varepsilon\rangle)]_\delta\varphi)\mathbf{v}^n_\varepsilon\right)\ {\rm d}x\ {\rm d}t + \int_0^T\int_\Omega p(\varrho_\varepsilon,\vartheta_\varepsilon){\rm div}_x\left([T_k(\langle\varrho_\varepsilon\rangle)]_\delta \mathbf{v}^n_\varepsilon\varphi\right)\ {\rm d}x\ {\rm d}t.$$

Exploiting (2.3), (2.6), (4.2), (4.3), (4.9), (4.10), we show that the first and second terms at the right hand side tend to zero as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ at the left hand side, we obtain, by the same token

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon} \mu(\vartheta_\varepsilon)\left(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x\mathbf{u}_\varepsilon\mathbb{I}\right) : \nabla_x\left([T_k(\langle\varrho_\varepsilon\rangle)]_\delta\mathbf{v}^n_\varepsilon\varphi\right)\ {\rm d}x\ {\rm d}t$$

$$= \lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon}\left(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x\mathbf{u}_\varepsilon\mathbb{I}\right) : \nabla_x\left([T_k(\langle\varrho_\varepsilon\rangle)]_\delta\mathbf{v}^n_\varepsilon\varphi\right)\ {\rm d}x\ {\rm d}t =$$

$$2\lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon}\nabla_x\mathbf{v}^n_\varepsilon : \nabla_x([T_k(\langle\varrho_\varepsilon\rangle)]_\delta\mathbf{u}_\varepsilon\varphi)\ {\rm d}x\ {\rm d}t;$$

whence

$$2\lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon}\nabla_x\mathbf{v}^n_\varepsilon : \nabla_x([T_k(\langle\varrho_\varepsilon\rangle)]_\delta\mathbf{u}_\varepsilon\varphi)\ {\rm d}x\ {\rm d}t \tag{7.1}$$

$$= \int_0^T\int_\Omega p(\varrho,\Theta)\mathbf{e}^n\cdot\nabla_x([T_k(\varrho)]_\delta\varphi])\ {\rm d}x\ {\rm d}t,$$

where we have used again (2.6).

On the other hand, it follows from (2.4), (2.5) that

$$\lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon}\nabla_x\mathbf{v}^n_\varepsilon : \nabla_x([T_k(\langle\varrho_\varepsilon\rangle)]_\delta\mathbf{u}_\varepsilon\varphi)\ {\rm d}x\ {\rm d}t$$

$$= \lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon} -\frac{1}{\varepsilon}\nabla_x q^n_\varepsilon\cdot\mathbf{u}_\varepsilon[T_k(\langle\varrho_\varepsilon\rangle)]_\delta\varphi\ {\rm d}x\ {\rm d}t + \lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon}\mathbf{w}^n_\varepsilon\cdot\frac{\mathbf{u}_\varepsilon}{\varepsilon^2}[T_k(\langle\varrho_\varepsilon\rangle)]_\delta\varphi\ {\rm d}x\ {\rm d}t,$$

where

$$\lim_{\varepsilon\to 0}\int_0^T \mu(\Theta)\int_{\Omega_\varepsilon}\mathbf{w}^n_\varepsilon\cdot\frac{\mathbf{u}_\varepsilon}{\varepsilon^2}[T_k(\langle\varrho_\varepsilon\rangle)]_\delta\varphi\ {\rm d}x\ {\rm d}t = \int_0^T \mu(\Theta)\int_\Omega \mathcal{A}[\mathbf{e}^n]\cdot\mathbf{U}[T_k(\varrho)]_\delta\varphi\ {\rm d}x\ {\rm d}t. \tag{7.2}$$

Furthermore,

$$\int_0^T \mu(\Theta)\int_\Omega \frac{1}{\varepsilon}\nabla_x q^n_\varepsilon\cdot\mathbf{u}_\varepsilon[T_k(\langle\varrho_\varepsilon\rangle)]_\delta\varphi\ {\rm d}x\ {\rm d}t$$

$$= \int_0^T \mu(\Theta)\int_\Omega \varepsilon\nabla_x q^n_\varepsilon\cdot\frac{\mathbf{u}_\varepsilon}{\varepsilon^2}\Big([T_k(\langle\varrho_\varepsilon\rangle)]_\delta - T_k(\langle\varrho_\varepsilon\rangle)\Big)\varphi\ {\rm d}x\ {\rm d}t$$

$$+ \int_0^T [\mu(\Theta)]_\omega\int_\Omega \varepsilon\nabla_x q^n_\varepsilon\cdot\frac{\mathbf{u}_\varepsilon}{\varepsilon^2}T_k(\varrho_\varepsilon)\varphi\ {\rm d}x\ {\rm d}t$$

$$+ \int_0^T \Big(\mu(\Theta) - [\mu(\Theta)]_\omega\Big)\int_\Omega \varepsilon\nabla_x q^n_\varepsilon\cdot\frac{\mathbf{u}_\varepsilon}{\varepsilon^2}T_k(\varrho_\varepsilon)\varphi\ {\rm d}x\ {\rm d}t,$$

where, by virtue of (4.10), (4.19), (5.35), and (2.3),

$$\left| \int_0^T \mu(\Theta) \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \Big([T_k(\langle \varrho_\varepsilon \rangle)]_\delta - T_k(\langle \varrho_\varepsilon \rangle) \Big) \varphi \ \mathrm{d}x \ \mathrm{d}t \right| \leq h_1(k) + h_2(\delta), \tag{7.3}$$

$$h_1(k) \to 0 \text{ for } k \to \infty, \ \ h_2(\delta) \to 0 \text{ for } \delta \to 0.$$

Similarly,

$$\left| \int_0^T \Big(\mu(\Theta) - [\mu(\Theta)]_\omega \Big) \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} T_k(\varrho_\varepsilon) \varphi \ \mathrm{d}x \ \mathrm{d}t \right| \leq h_2(\omega), \tag{7.4}$$

and, finally,

$$\int_0^T [\mu(\Theta)]_\omega \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} T_k(\varrho_\varepsilon) \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^T [\mu(\Theta)]_\omega \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \Big(T_k(\varrho_\varepsilon) - \varrho_\varepsilon \Big) \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T [\mu(\Theta)]_\omega \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \varrho_\varepsilon \varphi \ \mathrm{d}x \ \mathrm{d}t.$$

Here, exactly as above

$$\left| \int_0^T [\mu(\Theta)]_\omega \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \Big(T_k(\varrho_\varepsilon) - \varrho_\varepsilon \Big) \varphi \ \mathrm{d}x \ \mathrm{d}t \right| \leq h_1(k),$$

while, by virtue of equation (2.25) and the hypothesis **[H2]**,

$$\int_0^T [\mu(\Theta)]_\omega \int_\Omega \varepsilon \nabla_x q_\varepsilon^n \cdot \frac{\mathbf{u}_\varepsilon}{\varepsilon^2} \varrho_\varepsilon \varphi \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0.$$

Summing up the previous results, relation (7.1) reduces to

$$2 \int_0^T \mu(\Theta) \int_\Omega \mathcal{A}[\mathbf{e}^n] \cdot \mathbf{U} [T_k(\varrho)]_\delta \varphi \ \mathrm{d}x \ \mathrm{d}t = \tag{7.5}$$

$$\int_0^T \int_\Omega p(\varrho, \Theta) \mathbf{e}^n \cdot \nabla_x ([T_k(\varrho)]_\delta \varphi]) \ \mathrm{d}x \ \mathrm{d}t + h_1(k) + h_2(\delta) + h_2(\omega).$$

Letting $\omega \to 0$, $k \to \infty$ we obtain

$$2 \int_0^T \mu(\Theta) \int_\Omega [\varrho]_\delta \mathbf{U} \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{7.6}$$

$$= \int_0^T \int_\Omega p(\varrho, \Theta) \nabla_x [\varrho]_\delta \cdot \mathcal{A} \varphi \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega p(\varrho, \Theta) \mathcal{A} : [\nabla_x \varphi] [\varrho]_\delta \ \mathrm{d}x \ \mathrm{d}t + h_2(\delta)$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^3)$.

At this stage, we use Lemma 3.1 to deduce that

$$p(\varrho,\Theta) = \overline{\langle p(\varrho,\vartheta)\rangle}, \tag{7.7}$$

in particular, by virtue of (5.36),

$$\nabla_x p(\varrho,\Theta) \in L^2((0,T)\times\Omega).$$

Accordingly, we can let $\delta \to 0$ in (7.6) to obtain the desired conclusion (2.32).

Strict positivity of the temperature as well as the energy balance (2.33) can be shown exactly as in [15]. We have proved Theorem 2.1.

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