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SOLVING INVERSE SOURCE PROBLEMS USING OBSERVABILITY. APPLICATIONS TO THE EULER–BERNOULLI PLATE EQUATION*

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Abstract. The aim of this paper is to provide a general framework for solving a class of inverse source problems by using exact observability of infinite dimensional systems. More precisely, we show that if a system is exactly observable, then a source term in this system can be identified by knowing its intensity and appropriate observations which often correspond to measurements of some boundary traces. This abstract theory is then applied to obtain new identifiability results for a system governed by the Euler–Bernoulli plate equation. Using a different methodology, we show that exact observability can be used to identify both the locations and the intensities of combinations of point sources in the plate equation.

Key words. inverse source problem, exact observability, plate equation

AMS subject classifications. 93C25, 93B07, 93C20, 35R30

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1. Introduction.

In this paper we consider the problem of determining sources in infinite dimensional systems by using appropriate observation operators. In the case of systems governed by PDEs these observation operators often correspond to measurements of some boundary traces. As it has been remarked in Puel and Yamamoto [19] this inverse problem is closely related to exact controllability properties (see also El Badia and Ha-Duong in [11]).

One of our aims is to give a general framework, in terms of functional analysis, of the connection between exact observability (which is dual to exact controllability) and identifiability (possibly stable) of sources. In the case of sources of the form $\lambda(t)f$ with $\lambda : [0,\infty) \to \mathbb{C}$ given and f unknown, this approach can be used to derive in a systematic manner identifiability of sources for various PDEs. To illustrate the versatility of the proposed method, we apply the abstract results to several PDE examples, in particular for the Euler–Bernoulli plate equation. As far as we know, the results obtained in this way either are new (as for the two dimensional plate equation with pointwise source) or they improve the existing ones (as for the plate equation with regular source).

In order to describe this abstract framework and our general results, let X, Y be Hilbert spaces and $\mathcal{D}(A)$, $\mathcal{D}(C)$ two subspaces of X. Let $A : \mathcal{D}(A) \to X$ be a generator of a strongly continuous group in X and let $C : \mathcal{D}(C) \to Y$ be an observation operator. In this paper we study inverse source problems for the differential equation

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$$\dot{z}(t) = Az(t) + g(t) \qquad (t \in (0,\tau)), \tag{1.1}$$

$$z(0) = z_0, \tag{1.2}$$

$$y(t) = Cz(t) \qquad (t \in (0,\tau)), \tag{1.3}$$

where $\tau > 0$, $z_0 \in X$ are given and $g : [0,\tau] \to Z'$ is (partially) unknown. The space $Z' \supset X$ is the dual space of a space $Z \subset X$ containing $\mathcal{D}(A)$, as will be made precise in section 2. More precisely, we consider the problem of determining g from appropriate measurements y and the following classical questions associated with it:

- *Identifiability:* Is the mapping $g \mapsto y$ one-to-one?
- *Stability:* Assume that we have two sources $g^{(1)}$ and $g^{(2)}$ and let $y^{(1)}$ and $y^{(2)}$ be the corresponding observations. Can we find a positive constant K such that

  $$\|g^{(1)} - g^{(2)}\| \leqslant K\|y^{(1)} - y^{(2)}\|,$$

  with appropriate norms?
- *Reconstruction:* Is it possible to "reconstruct," in some sense, g from the observation y?

We will focus just on the first two topics and study them using a method that relies on the exact observability of the system

$$\dot{z}(t) = Az(t), \quad z(0) = z_0, \tag{1.4}$$

$$y(t) = Cz(t). \tag{1.5}$$

Recall that system (1.4)–(1.5) is exactly observable in time $\tau > 0$ if there exists $k_\tau > 0$ such that

$$\int_0^\tau \|y(t)\|_Y^2 \mathrm{d}t \geqslant k_\tau^2 \|z_0\|_X^2 \quad \forall z_0 \in \mathcal{D}(A). \tag{1.6}$$

Our first main result is Theorem 4.3, which states that the exact observability in time $\tau > 0$ of (1.4)–(1.5) implies the stability (and identifiability) for the inverse source problem of (1.1)–(1.3) in the case where $g(t) = \lambda(t)f$, with $\lambda \in H^1((0,\tau))$ known and $f \in Z'$ to be determined.

Combined with various exact observability results, we use Theorem 4.3 to obtain *new identifiability results for the Euler–Bernoulli plate equation.* More precisely, consider the following initial value problem for the Euler–Bernoulli plate equation:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \Delta^2 w = \lambda(t)\delta_\xi & \text{in } (0,\tau) \times \Omega, \\ w = \Delta w = 0 & \text{on } (0,\tau) \times \partial\Omega, \\ w(0,x) = w_0(x), \ \dfrac{\partial w}{\partial t}(0,x) = w_1(x), & x \in \Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\tau > 0$, $\lambda \in H^1((0,\tau))$ with $\lambda(0) \neq 0$, $\xi \in \Omega$, and δ_ξ is the Dirac mass concentrated in ξ. Assume Γ is a nonempty open subset of $\partial\Omega$. We will show in Theorem 5.5 that the mapping

$$\Omega \to L^2((0,\tau); L^2(\Gamma)), \quad \xi \mapsto y = \left.\frac{\partial w}{\partial \nu}\right|_{(0,\tau)\times\Gamma} \tag{1.8}$$

is well defined and that

(H1) if Ω is smooth and Γ satisfies the geometric optics condition of Bardos, Lebeau, and Rauch [4],
or
(H2) if Ω is a rectangle $(0,a)\times(0,b)$ and Γ contains both a horizontal and a vertical segment of nonzero length,

then, for all $\xi^{(1)},\xi^{(2)}\in\Omega$, there exists $K>0$ such that

$$|\xi^{(1)}-\xi^{(2)}|\leqslant K\left\|\frac{\partial w^{(1)}}{\partial\nu}-\frac{\partial w^{(2)}}{\partial\nu}\right\|_{L^2((0,\tau);L^2(\Gamma))}.$$

In this inequality $w^{(1)}$ and $w^{(2)}$ are the solutions of (1.7) corresponding to $\xi^{(1)}$ and $\xi^{(2)}$, respectively, with suitable assumptions on the initial data.

The use of the Bardos–Lebeau–Rauch condition for the control of the wave equation was originally limited to domains with a C^∞ boundary. The extension to domains with less regular boundaries (including rectangles and boundaries of class C^3), which we use in this paper, is due to Burq [6, 7]. Note that assumptions (H1) and (H2) are quite different. More precisely, Ω and Γ can clearly satisfy (H2) without satisfying the geometric optics condition of Bardos, Lebeau, and Rauch. As far as we know, the result in Theorem 5.5 is completely new (in fact we are not aware of any previous result on identification of point sources for the Euler–Bernoulli equation in two space dimensions).

In the case of more regular sources, specifically $g(t)=\lambda(t)f$ with $f\in H_0^1(\Omega)$, we establish, in Theorem 5.6, a stable identifiability result by measuring the trace of one of the functions $\frac{\partial^2 w}{\partial t\partial\nu}$ or $\frac{\partial\Delta w}{\partial\nu}$.

Note that, since the only assumption on A in Theorem 4.3 is that it generates a C^0 semigroup, our results can be applied not only to the plate equations but to a variety of problems including, for instance, the Schrödinger equation, the wave equation, and the Maxwell system, by using appropriate exact observability results.

Our second main result is Theorem 6.2, which establishes identifiability for the inverse source problem

$$\left\{\begin{array}{ll}\dfrac{\partial^2 w}{\partial t^2}+\Delta^2 w=\displaystyle\sum_{j=1}^N\lambda_j(t)\delta_{\xi_j} & \text{in } (0,\tau)\times\Omega,\\ w=\Delta w=0 & \text{on } (0,\tau)\times\partial\Omega,\\ w(0,x)=w_0(x),\ \dfrac{\partial w}{\partial t}(0,x)=w_1(x), & x\in\Omega,\end{array}\right.\tag{1.9}$$

with Ω as above, $\xi_j\in\Omega$, and $\lambda_j\in H^2((0,\tau))$ satisfying $\lambda_j(0)=\lambda_j'(0)=0$ and $\lambda\equiv 0$ after some time τ_1. More precisely, we show that conditions

$$\frac{\partial w^{(1)}}{\partial\nu}=\frac{\partial w^{(2)}}{\partial\nu}\text{ on }(0,\tau)\times\Gamma,\qquad\frac{\partial\Delta w^{(1)}}{\partial\nu}=\frac{\partial\Delta w^{(2)}}{\partial\nu}\text{ on }(0,\tau_1)\times\Gamma_1,$$

with Γ satisfying either (H1) or (H2) and with Γ_1 a nonempty open subset of Γ, imply that the completely unknown corresponding sources $\sum_{j=1}^{N^{(1)}}\lambda_j^{(1)}(t)\delta_{\xi_j^{(1)}}$ and $\sum_{j=1}^{N^{(2)}}\lambda_j^{(2)}(t)\delta_{\xi_j^{(2)}}$ are equal. Note that, although it makes use of observability properties, Theorem 6.2 is not a consequence of Theorem 4.3.

Our results on the plate equation use in an essential manner the classical exact observability results due to Lebeau [16] and the recent ones in Tenenbaum and Tucsnak [22]. It should noted that in [22], the authors tackle arbitrarily small observation regions, provided that Ω is a rectangle, so that the geometric optics condition is not necessary in this case.

In the case where the intensity λ in $g(t) = \lambda(t)f$ is known, our method is inspired by the work of Puel and Yamamoto [19], where they consider the wave equation and determine a source which is in $L^2(\Omega)$, provided that the domain of Γ of observation satisfies the geometric optics condition. Other works treating sources of the form $g(t) = \lambda(t)f$ with f a sum of Dirac masses at points located inside the equation domain are due to Komornik and Yamamoto [13, 14] for the wave equation and the heat equation, respectively, and to Nicaise and Zaïr [18, 17] for the beam equation and the wave equation in heterogeneous trees, respectively.

As regards the problem of finding more regular sources, we refer the reader to Yamamoto [29, 30] for the wave equation and a source of the form $g(t) = \lambda(t)f$ with $f \in L^2(\Omega)$ and Wang [25] for the plate equation and a source of the form $g(t) = \lambda(t)f$ with $f \in H_0^1(\Omega)$. Let us mention that in [25], the condition $\lambda \in C^3([0, \tau])$ and of constant sign is needed and the stability result is based on five boundary observations on

$$\{x \in \partial\Omega \ ; \ (x - x_0) \cdot \nu(x) > 0\}, \quad x_0 \in \mathbb{R}^2.$$

Our contribution in Theorem 5.6 consists in showing that, in the absence of lower order terms, we have stable identifiability with only one boundary observation and under assumptions which are weaker than those in [25].

Concerning our identifiability result for the problem (1.9) with the number of sources, their location, and their intensity all unknown, the approach we use follows the method of El Badia and Ha-Duong in [11], which is based on Fourier analysis and on appropriate uniqueness results. The identifiability and reconstruction of linear combinations of point sources has also been studied for the heat equation in El Badia and Ha-Duong [10] and for the Stokes equations in Alves and Silvestre [2].

The paper is organized as follows. In section 2, we present some concepts and preliminary results from functional analysis and, in particular, on exact observability and exact controllability. In section 3, we recall exact observability results for the Euler–Bernoulli plate equation with boundary observation. Section 4 is devoted to the new results for the stability of sources in the general system (1.1)–(1.3). In section 5, we apply the results of section 4 to the Euler–Bernoulli plate equation with known intensity. Finally, in section 6, we consider the identifiability of an unknown linear combination of point sources in the plate equation.

2. Notation and preliminaries.

2.1. Some background in functional analysis. In this subsection we gather, for the convenience of the reader, some known results on functional analysis and, in particular, on self-adjoint operators and scales of Hilbert spaces.

Throughout this section H stands for an infinite dimensional Hilbert space with the inner product and the corresponding norm simply denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Moreover, $A_0 : \mathcal{D}(A_0) \to H$ is supposed to be a strictly positive operator with compact resolvents. When saying that A_0 is *strictly positive* we mean that A_0 is self-adjoint and that there exists a constant $\gamma > 0$ such that

$$\langle A_0 \varphi, \varphi \rangle \geqslant \gamma \|\varphi\|^2 \qquad (\varphi \in \mathcal{D}(A_0)).$$

PROPOSITION 2.1. *With the above assumptions, the operator A_0 is diagonalizable with an orthonormal basis $(\varphi_k)_{k \geqslant 1}$ of eigenvectors and the corresponding family of* *positive eigenvalues* $(\lambda_k)_{k\geqslant 1}$ *satisfies* $\lim_{k\to\infty}\lambda_k=\infty$. *Moreover, we have*

$$\mathcal{D}(A_0)=\left\{z\in H \;\middle|\; \sum_{k\geqslant 1}\lambda_k^2\left|\langle z,\varphi_k\rangle\right|^2<\infty\right\}$$

and

$$A_0 z=\sum_{k\geqslant 1}\lambda_k\left\langle z,\varphi_k\right\rangle\varphi_k \qquad (z\in\mathcal{D}(A_0)).$$

The proof of the above result is classical (see, for instance, [24, section 3.2]).

For $\alpha\geqslant 0$ the operator A_0^α is defined by

$$\mathcal{D}(A_0^\alpha)=\left\{z\in H \;\middle|\; \sum_{k\geqslant 1}\lambda_k^{2\alpha}\left|\langle z,\varphi_k\rangle\right|^2<\infty\right\} \tag{2.1}$$

and

$$A_0^\alpha z=\sum_{k\geqslant 1}\lambda_k^\alpha\left\langle z,\varphi_k\right\rangle\varphi_k \qquad (z\in\mathcal{D}(A_0)).$$

For every $\alpha\geqslant 0$ we denote by H_α the space $\mathcal{D}(A_0^\alpha)$ endowed with the inner product

$$\langle\varphi,\psi\rangle_\alpha=\langle A_0^\alpha\varphi,A_0^\alpha\psi\rangle \qquad (\varphi,\ \psi\in H_\alpha).$$

The induced norm is denoted by $\|\cdot\|_\alpha$. From the above facts it follows that for every $\alpha\geqslant 0$ the operator A_0 is a unitary operator from $H_{\alpha+1}$ onto H_α and A_0 is strictly positive on H_α. For each $\alpha>0$, we denote by $H_{-\alpha}$ the dual of H_α with respect to the pivot space H. We have the following result (see, for instance, [24, Corollary 3.4.6]).

PROPOSITION 2.2. A_0 *has a unique extension such that*

$$A_0\in\mathcal{L}(H_{\frac12},H_{-\frac12}),$$

and this is unitary. Moreover, this extension of A_0 *can be regarded as a strictly positive (densely defined) operator on* $H_{-\frac12}$.

We recall below the well-known example of the Dirichlet Laplacian since it will be needed in the remaining part of this work.

PROPOSITION 2.3. *Let* $\Omega\subset\mathbb{R}^2$ *be an open bounded set with boundary of class* C^5 *or let* Ω *be a rectangle. Let*

$$H=L^2(\Omega),\quad \mathcal{D}(A_0)=H^2(\Omega)\cap H_0^1(\Omega),$$

and let $A_0:\mathcal{D}(A_0)\to H$ *be defined by*

$$A_0\varphi=-\Delta\varphi \qquad (\varphi\in\mathcal{D}(A_0)).$$

Then A_0 *is strictly positive and has compact resolvents. Moreover, we have*

$$H_{\frac12}=H_0^1(\Omega), \tag{2.2}$$

$$H_{\frac32}=\left\{\varphi\in H^3(\Omega)\mid \varphi=\Delta\varphi=0 \text{ on } \partial\Omega\right\}, \tag{2.3}$$

$$H_{\frac{5}{2}} = \left\{\varphi \in H^5(\Omega) \mid \varphi = \Delta\varphi = \Delta^2\varphi = 0 \textit{ on } \partial\Omega\right\}. \tag{2.4}$$

Proof. The proof of the fact that A_0 is strictly positive is classical. We refer the reader, for instance, to [24, Theorems 3.6.1 and 3.6.2] for the case of $\partial\Omega$ of class C^2 and to [24, Example 3.6.5] for the case of a rectangle.

The fact that A_0 has compact resolvents follows from the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$.

For a proof of (2.2) we again refer the reader to Theorem 3.6.1 in [24].

Finally, in order to prove (2.3) and (2.4) in the case of a C^5 boundary $\partial\Omega$, it suffices to combine (2.2) with classical elliptic regularity theory (see, for instance, [28]).

Assertions (2.3) and (2.4) also hold if Ω is a rectangle (say $\Omega = (0,a) \times (0,b)$). Indeed, this can be easily proved by using (2.1) combined with the facts that, in this case, the eigenvalues of A_0 are

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad (m, n \geqslant 1) \tag{2.5}$$

and that a corresponding orthonormal basis formed of eigenvectors of A_0 is given by

$$\varphi_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad (m, n \geqslant 1).$$

We will prove (2.4) following the ideas of Example 3.6.5 in [24]. To simplify the notation, we set

$$\mathcal{H} := \left\{\varphi \in H^5(\Omega) \mid \varphi = \Delta\varphi = \Delta^2\varphi = 0 \text{ on } \partial\Omega\right\}.$$

It is obvious that $\mathcal{H} \subseteq H_{\frac{5}{2}}$. To show the opposite inclusion, let us assume that $f = \sum_{m,n\geqslant 1} f_{mn}\varphi_{mn} \in H_{\frac{5}{2}}$. Then

$$\sum_{m,n\geqslant 1} (m^2 + n^2)^5 |f_{mn}|^2 < \infty. \tag{2.6}$$

For $p \in \mathbb{N}$, set $f_p := \sum_{m,n\geqslant 1}^{p} f_{mn}\varphi_{mn}$. It is clear that $\lim_{p\to\infty} \|f - f_p\|_{\frac{5}{2}} = 0$. Moreover, since (2.6) implies $\sum_{m,n\geqslant 1} |f_{mn}|^2 < \infty$, we have

$$\lim_{p\to\infty} \|f - f_p\|_{L^2(\Omega)} = 0. \tag{2.7}$$

By direct calculations, we see that $f_p \in \mathcal{H}$ for all $p \in \mathbb{N}$ and that, for $p, q \in \mathbb{N}$ with $q > p$,

$$\|f_q - f_p\|_{H^5(\Omega)}^2 \leqslant C \sum_{\substack{\theta_1,\, \theta_2\, \in\, \{0,\ldots,5\} \\ \theta_1 + \theta_2 \leqslant 5}} \sum_{m,n=p}^{q} m^{2\theta_1} n^{2\theta_2} |f_{mn}|^2$$

for some constant $C > 0$. Therefore (f_p) is a Cauchy sequence in $H^5(\Omega)$. This fact combined with (2.7) implies that $f \in \mathcal{H}$. $\square$

In the remaining part of this subsection we recall some results of functional analysis which will be needed for duality arguments. We first give the following lemma, which is a consequence of the closed graph theorem (see, for instance, Douglas [9]).

LEMMA 2.4. *Suppose that V_1 and V_2 are Hilbert spaces, let V_1' and V_2' be the corresponding dual spaces, and let $F \in \mathcal{L}(V_2, V_1)$. Then the following statements are equivalent:*

(a) *F maps V_2 onto V_1.*

(b) *There exists a constant $c > 0$ such that*

$$\|F^* z\|_{V_2'} \geqslant c\|z\|_{V_1'} \qquad (z \in V_1').$$

In the next results, the notations H, H_1, and H_2 stand for Hilbert spaces which will be identified with their duals. We give below two technical results which are slight variations of those from [24, section 2.9].

LEMMA 2.5. *Let $V_j,\ H_j,\ j \in \{1, 2\}$, be Hilbert spaces such that $V_j \subset H_j$ with continuous and dense embeddings. Let $L \in \mathcal{L}(H_1, H_2)$ such that $L(V_1) \subset V_2$. Then the restriction of L to V_1 is in $\mathcal{L}(V_1, V_2)$.*

Proof. We notice that as an operator from V_1 to V_2, L is closed (we have used the continuous embedding of V_j into H_j for $j \in \{1, 2\}$). Therefore, by the closed graph theorem, L is bounded as an operator from V_1 to V_2. ∎

PROPOSITION 2.6. *Let $V_j,\ H_j,\ j \in \{1, 2\}$, be Hilbert spaces such that $V_j \subset H_j$ with continuous and dense embeddings. Let $L \in \mathcal{L}(H_1, H_2)$ be such that $L^*(V_2) \subset V_1$. Then L can be extended to an operator $\widetilde{L} \in \mathcal{L}(V_1', V_2')$, where V_1' (respectively, V_2') is the dual of V_1 (respectively, of V_2) with respect to the pivot space H_1 (respectively, H_2). Moreover, if $L^*(V_2) = V_1$, then there exists $m > 0$ such that*

$$\|\widetilde{L} f\|_{V_2'} \geqslant m\|f\|_{V_1'} \qquad (f \in V_1'). \tag{2.8}$$

Proof. To avoid confusion, we use a different notation, namely L^d, for the restriction of L^* to V_2. We use Lemma 2.5 to conclude that $L^d \in \mathcal{L}(V_2, V_1)$. Hence, $L^{d*} \in \mathcal{L}(V_1', V_2')$. We claim that L^{d*} is an extension of L, i.e., that $L^{d*} z = Lz$ holds for all $z \in H_1$. For this, it will be enough to show that

$$\langle L^{d*} z, \varphi\rangle_{V_2', V_2} = \langle Lz, \varphi\rangle_{V_2', V_2} \qquad (z \in H_1,\ \varphi \in V_2). \tag{2.9}$$

Since both the right-hand side and the left-hand side of (2.9) can be written as $\langle Lz, \varphi\rangle_{H_2}$, the above identity is obviously true. Thus, $\tilde{L} = L^{d*}$ is an extension of L.

The uniqueness of $\tilde{L}$ follows from the density of H_1 in V_1'.

Finally, if $L^*(V_2) = V_1$, then estimate (2.8) follows by applying Proposition 2.4, with $F = L^d$. ∎

We also need the following known result (see, for instance, [24, Proposition 2.10.3]).

PROPOSITION 2.7. *Let $A : \mathcal{D}(A) \to X$ be a densely defined operator with resolvent set $\rho(A) \neq \emptyset$, let $\beta \in \rho(A)$, let X_1 be $\mathcal{D}(A)$ with the graph norm, and let X_{-1} be the completion of X with respect to the norm*

$$\|z\|_{-1} = \left\|(\beta I - A)^{-1} z\right\| \qquad (z \in X). \tag{2.10}$$

Then $A \in \mathcal{L}(X_1, X)$ and A has a unique extension to an operator in $\mathcal{L}(X, X_{-1})$, also denoted by A. Moreover,

$$(\beta I - A)^{-1} \in \mathcal{L}(X, X_1), \qquad (\beta I - A)^{-1} \in \mathcal{L}(X_{-1}, X),$$

and these two operators are unitary.

Remark 2.8. In the construction of X_1 we may replace A with A^* and β with $\overline{\beta}$, obtaining a space denoted X_1^d. Note that X_{-1} is the dual of X_1^d with respect to the pivot space X.

2.2. Some background on exact observability and exact controllability. In this subsection we first recall some known facts on exact observability and on exact controllability of infinite dimensional systems. At the end of the section we prove a result, which seems new, concerning an observability inequality involving weakened norms. We continue to use the notation from the previous subsection for A, X_1, X_1^d, and X_{-1}. In what follows, we assume that A is the generator of a strongly continuous semigroup $\mathbb{T}$.

Let Y be a Hilbert space, which will be identified with its dual, and let $C \in \mathcal{L}(X_1, Y)$. For each $\tau > 0$, we define the operator $\Psi_\tau \in \mathcal{L}(X_1, L^2((0,\tau); Y))$ by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \quad \text{for} \quad t \in [0,\tau] \quad \text{and} \quad \text{for } z_0 \in X_1. \tag{2.11}$$

Note that, for every $z_0 \in \mathcal{D}(A)$, we have $\Psi_\tau z_0 = y$, where z_0 and y are related by (1.4)–(1.5). We next recall a definition which is by now classical in infinite dimensional systems theory (see, for instance, Salamon [20, 21] and Weiss [26, 27]).

Definition 2.1. *The operator $C \in \mathcal{L}(X_1, Y)$ is called an* admissible observation operator *for $\mathbb{T}$ if for some (and hence for all) $\tau > 0$, Ψ_τ has a continuous extension to X.*

Equivalently, the operator $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ if and only if there exists a positive constant K_τ such that the solution (z, y) of (1.4)–(1.5) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 \, \mathrm{d}t \leqslant K_\tau^2 \|z_0\|_X^2 \quad \forall z_0 \in \mathcal{D}(A).$$

Definition 2.2. *Let $\tau > 0$ and let C be an admissible observation operator for $\mathbb{T}$. The pair (A, C) is* exactly observable in time τ *if Ψ_τ is bounded from below.*

In other words, the pair (A, C) is exactly observable in time τ if and only if there exists a positive constant k_τ such that the solution (z, y) of (1.4)–(1.5) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 \, \mathrm{d}t \geqslant k_\tau^2 \|z_0\|_X^2 \quad \forall z_0 \in \mathcal{D}(A).$$

Let U be a Hilbert space, which will be identified with its dual. Let $B \in \mathcal{L}(U, X_{-1})$. For each $\tau > 0$ we define the operator $\Phi_\tau \in \mathcal{L}(L^2((0,\tau); U), X_{-1})$ by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) \mathrm{d}\sigma. \tag{2.12}$$

Note that $\Phi_\tau u = z(\tau)$, where z is the solution in X_{-1} of the differential equation

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = 0.$$

Definition 2.3. *The operator $B \in \mathcal{L}(U; X_{-1})$ is called an* admissible control operator *for $\mathbb{T}$ if for some (and hence for all) $\tau > 0$, $\operatorname{Ran} \Phi_\tau \subset X$.*

From Lemma 2.5, if $B \in \mathcal{L}(U, X_{-1})$ is admissible, then for every $\tau \geqslant 0$ we have

$$\Phi_\tau \in \mathcal{L}(L^2((0,\tau); U), X).$$

Definition 2.4. *Let $\tau > 0$ and let B be an admissible control operator for $\mathbb{T}$. The pair (A, B) is* exactly controllable in time τ *if $\operatorname{Ran} \Phi_\tau = X$.*

If $B \in \mathcal{L}(U, X_{-1})$, then, using the duality between X_1^d (which is $\mathcal{D}(A^*)$ with the graph norm) and X_{-1} and identifying U with its dual, we have $B^* \in \mathcal{L}(X_1^d, U)$. The adjoint of the operator Φ_τ defined in (2.12) is in $\mathcal{L}(X_1^d, L^2((0,\infty);U))$ and can be expressed using B^*, and the exact controllability of the pair (A,B) is equivalent to the exact observability of the pair (A^*, B^*). This type of result goes back to Dolecki and Russell [8], and we state them below in a form borrowed from [24, Proposition 4.4.1, Theorem 4.4.3, and Theorem 11.2.1].

PROPOSITION 2.9. *If $B \in \mathcal{L}(U, X_{-1})$, then, for every $\tau > 0$ and every $z_0 \in X_1^d$,*

$$(\Phi_\tau^* z_0)(t) = \begin{cases} B^* \mathbb{T}_{\tau-t}^* z_0 & \text{for } t \in [0,\tau], \\ 0 & \text{for } t > \tau. \end{cases} \tag{2.13}$$

If B is an admissible control operator for $\mathbb{T}$, so that Φ_τ can also be regarded as an operator in $\mathcal{L}(L^2((0,\infty);U), X)$, then its adjoint in $\mathcal{L}(X, L^2((0,\infty);U))$ is given, for $z_0 \in \mathcal{D}(A^*)$, by the same formula (2.13).

PROPOSITION 2.10. *Suppose that $B \in \mathcal{L}(U, X_{-1})$. Then B is an admissible control operator for $\mathbb{T}$ if and only if B^* is an admissible observation operator for $\mathbb{T}^*$.*

The pair (A,B) is exactly controllable in time τ if and only if (A^*, B^*) is exactly observable in time τ.

If a pair (A,B) is exactly controllable in a time τ_0, a natural question is the characterization of the states which can be reached by more regular inputs. Before stating a result in this direction, we introduce additional notation. For a Hilbert space V, and for $\tau > 0$, we set

$$H_L^1((0,\tau);V) = \left\{ u \in H^1((0,\tau);V) \mid u(0) = 0 \right\},$$
$$H_R^1((0,\tau);V) = \left\{ u \in H^1((0,\tau);V) \mid u(\tau) = 0 \right\}.$$

The following result has been proved in Tucsnak and Weiss [23] in the case of a finite dimensional input space U and in the general form below in [24, Theorem 11.3.6].

PROPOSITION 2.11. *Suppose that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. Then, for all $\tau > 0$,*

$$\Phi_\tau \in \mathcal{L}(H_L^1((0,\tau);U), Z),$$

where

$$Z = X_1 + (\beta I - A)^{-1} BU = (\beta I - A)^{-1}(X + BU) \tag{2.14}$$

for some $\beta \in \rho(A)$.

Suppose, moreover, that the pair (A,B) is exactly controllable in time τ_0. Then, for all $\tau > \tau_0$, Φ_τ is onto from $H_L^1((0,\tau);U)$ to Z and there exists $M_\tau > 0$ such that, for every $z_0 \in Z$, the minimal norm control u with $\Phi_\tau u = z_0$ satisfies

$$\|u\|_{H_L^1((0,\tau);U)} \leqslant M_\tau \|z_0\|_Z. \tag{2.15}$$

Remark 2.12. Note that the space Z defined by (2.14) does not depend on the choice of β. The norm of Z is defined by

$$\|z\|_Z^2 = \inf \left\{ \|x\|_X^2 + \|u\|_U^2 \ ; \ x \in X, \, u \in U, \, z = (\beta I - A)^{-1}(x + Bu) \right\}.$$

If E is a dense subspace of X, we denote by E' the dual of E with respect to the pivot space X. Moreover, we denote by $\left[H_L^1((0,\tau);U)\right]'$ (respectively, by $\left[H_R^1((0,\tau);U)\right]'$) the dual of $H_L^1((0,\tau);U)$ (respectively, $H_R^1((0,\tau);U)$) with respect to the pivot space $L^2((0,\tau);U)$.

Let (A, C) be exactly observable in time τ_0. Another natural question, which is dual to that in Proposition 2.11, is to find a lower bound of $\|\Psi_\tau z_0\|_w$, where $\|\cdot\|_w$ is a norm which is weaker than the norm in $L^2((0,\tau);Y)$. A partial answer is given by the result below.

PROPOSITION 2.13. *Let $A : \mathcal{D}(A) \to X$ be a densely defined operator and let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. For $\tau > 0$ let Ψ_τ be the output map corresponding to the pair (A, C), as defined in* (2.11). *Then, for each $\tau > 0$, Ψ_τ has a unique continuous extension $\Psi_\tau \in \mathcal{L}((Z^d)', \left[H^1_R((0,\tau);Y)\right]')$, where*

$$Z^d = (\beta I - A^*)^{-1}(X + C^*Y). \tag{2.16}$$

Moreover, assume that (A, C) is exactly observable in some time $\tau_0 > 0$. Then, for each $\tau > \tau_0$, there exists a constant $m_\tau > 0$ such that, for every $f \in (Z^d)'$, we have

$$\|\Psi_\tau f\|_{\left[H^1_R((0,\tau);Y)\right]'} \geqslant m_\tau \|f\|_{(Z^d)'}. \tag{2.17}$$

Proof. Since C is an admissible observation operator for $\mathbb{T}$, Proposition 2.10 yields that C^* is an admissible control operator for $\mathbb{T}^*$. By applying Proposition 2.11, it follows that Φ^d_τ maps $H^1_L((0,\tau);Y)$ to Z^d. As a consequence $\Phi^d_\tau \boldsymbol{Я}_\tau$ maps $H^1_R((0,\tau);Y)$ to Z^d, with $\boldsymbol{Я}_\tau$ defined by

$$(\boldsymbol{Я}_\tau f)(s) = f(\tau - s) \qquad (s \in [0, \tau]).$$

On the other hand, by using Proposition 2.9 we have

$$\Phi^d_\tau \boldsymbol{Я}_\tau = \Psi^*_\tau,$$

so that the conclusion follows from Proposition 2.6.

If (A, C) is exactly observable in time τ_0, we can apply Proposition 2.10 to obtain that (A^*, C^*) is exactly controllable in time τ_0. Let $\tau > \tau_0$. By again applying Proposition 2.11, it follows that Φ^d_τ maps $H^1_L((0,\tau);Y)$ onto Z^d and the conclusion again follows from Proposition 2.6. ◻

Remark 2.14. In (2.16), A^* is extended to X as an operator from X to X^d_{-1} (the completed space of X for the norm $z_0 \mapsto \left\|(\beta I - A^*)^{-1} z_0\right\|_X$; see Proposition 2.7).

3. Some background on the plate equation. In this section we recall some known results for the Euler–Bernoulli plate equation and obtain several consequences which will be used in the proofs of the main results.

We first introduce some notation which will be used in the remaining part of this section. Let Ω be a domain of $\mathbb{R}^2$ with a C^5 boundary $\partial\Omega$ or let Ω be a rectangular domain. Let A_0 be the Dirichlet Laplacian introduced in Proposition 2.3. As in subsection 2.1, we use the scale of Hilbert spaces $(H_\alpha)_{\alpha \geqslant 0}$ with the corresponding norms denoted by $\|\cdot\|_\alpha$. Recall from subsection 2.1 that A_0 is a strictly positive operator on $H_{\frac{1}{2}} = H^1_0(\Omega)$ of domain $H_{\frac{3}{2}}$, where the space $H_{\frac{3}{2}}$ is given by (2.3). Moreover, using [24, Remark 3.3.7] and Proposition 2.3, it follows that A_0^2 is a strictly positive operator in $H_{\frac{1}{2}}$ of domain $H_{\frac{5}{2}}$, with $H_{\frac{5}{2}}$ given by (2.4).

Let $X = H_{\frac{3}{2}} \times H_{\frac{1}{2}}$, and let $A : \mathcal{D}(A) \to X$ be defined by

$$\mathcal{D}(A) = H_{\frac{5}{2}} \times H_{\frac{3}{2}}, \quad A = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}.$$

Since A_0^2 is a strictly positive operator on $H_{\frac{1}{2}}$ of domain $H_{\frac{5}{2}}$, we can use a well-known result (see, for instance, [24, Proposition 3.7.6]) to obtain that the operator A is skew-adjoint. Consequently, by Stone's theorem, A is the generator of a strongly continuous unitary group $\mathbb{T}$ in X.

Let Γ be an open nonempty subset of $\partial\Omega$ and let $Y = L^2(\Gamma)$. Recall from subsection 2.1 that $H_1 = H^2(\Omega) \cap H_0^1(\Omega)$ and let $C_0 \in \mathcal{L}(H_1, Y)$ be defined by

$$C_0\varphi = \left.\frac{\partial\varphi}{\partial\nu}\right|_\Gamma \qquad (\varphi \in H_1).$$

Let X_1 be $\mathcal{D}(A)$ endowed with the graph norm and let $\widetilde{C} \in \mathcal{L}(X_1, Y)$ be the observation operator defined by

$$\widetilde{C} = \begin{bmatrix} 0 & C_0 \end{bmatrix}. \tag{3.1}$$

In some (but not all) of the results stated below we assume that Γ satisfies the following *geometric optics condition*: for all $x \in \Omega$, any ray coming from x at initial time, propagating at velocity one and following the geometric optics laws, meets $\overline{\Gamma}$ in finite time. This condition is sufficient and almost necessary to have the exact observability of the wave equation. Lebeau proved in [16] that this condition is also sufficient (without being necessary) for the plate equation with boundary observation

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + \Delta^2 w = 0 & \text{in } (0,\tau) \times \Omega, \\ w = \Delta w = 0 & \text{on } (0,\tau) \times \partial\Omega, \\ w(0,x) = w_0(x), \ \dfrac{\partial w}{\partial t}(0,x) = w_1(x), & x \in \Omega. \end{cases} \tag{3.2}$$

More precisely, we have the following slight variation of Lebeau's result (see Tucsnak and Weiss [24, Proposition 7.5.5 and Remark 7.5.6] for a detailed proof).

THEOREM 3.1. *Let $\tau > 0$, let Ω be a bounded domain of $\mathbb{R}^2$ with $\partial\Omega$ of class C^5, and assume that Γ satisfies the geometric optics condition. Then the pair $(A, \widetilde{C})$ is exactly observable in time τ. In other terms, for every $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in \mathcal{D}(A)$ the system* (3.2) *has a unique solution $w \in C^0([0,\tau]; H_{\frac{5}{2}}) \cap C^1([0,\tau]; H_{\frac{3}{2}})$ which satisfies*

$$\int_0^\tau \left\| \frac{\partial^2 w}{\partial t \partial \nu} \right\|_{L^2(\Gamma)}^2 \, \mathrm{d}t \geqslant k_{\tau,\Gamma}^2 \left(\|w_0\|_{\frac{3}{2}}^2 + \|w_1\|_{\frac{1}{2}}^2 \right) \tag{3.3}$$

for some positive constant $k_{\tau,\Gamma}$.

More recently, in the case of a rectangular domain Ω, Tenenbaum and Tucsnak [22] obtained an exact observability result for the Schrödinger equation under much weaker assumptions on Γ. This result can be easily transposed to the Euler–Bernoulli plate equation with boundary observation. More precisely, the following results holds.

THEOREM 3.2. *Let $\tau > 0$ and let $\Omega = (0,a) \times (0,b)$ $(a, b > 0)$. Assume that Γ is an open subset of $\partial\Omega$ containing both a horizontal and a vertical segment of nonzero length. Then the pair $(A, \widetilde{C})$ is exactly observable in time τ. In other terms, for every $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in \mathcal{D}(A)$ the system* (3.2) *has a unique solution $w \in C^0([0,\tau]; H_{\frac{5}{2}}) \cap C^1([0,\tau]; H_{\frac{3}{2}})$. Moreover, there exists a constant $k_{\tau,\Gamma} > 0$ such that*

$$\int_0^\tau \left\| \frac{\partial^2 w}{\partial t \partial \nu} \right\|_{L^2(\Gamma)}^2 \, \mathrm{d}t \geqslant k_{\tau,\Gamma}^2 \left(\|w_0\|_{\frac{3}{2}}^2 + \|w_1\|_{\frac{1}{2}}^2 \right) \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(A) \right). \tag{3.4}$$

Proof. The existence and uniqueness of solutions having the claimed regularity is a consequence of the fact that A generates a C^0 group on X.

In order to prove (3.4) we consider the Schrödinger equation

$$\begin{cases} \dot{z} + i\Delta z = 0 & (x \in \Omega,\ t \geqslant 0), \\ z = 0 & (x \in \partial\Omega,\ t \geqslant 0), \\ z(x,0) = \psi(x) & (x \in \Omega). \end{cases} \tag{3.5}$$

According to Theorem 1.4 in [22], for every $\tau > 0$ and every Γ satisfying the assumptions of the theorem there exists a positive constant $C_{\tau,\Gamma}$ such that

$$C_{\tau,\Gamma}^2 \int_0^\tau \left\| \frac{\partial z}{\partial \nu} \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \geqslant \|\nabla\psi\|_{L^2(\Omega)}^2 \qquad (\psi \in H^2(\Omega) \times H_0^1(\Omega)). \tag{3.6}$$

Using the terminology introduced in subsection 2.2, inequality (3.6) means that the pair (iA_0, C), with state space $H_{\frac{1}{2}}$ and output space $Y = L^2(\Gamma)$, is exactly observable in any time $\tau > 0$. Since the eigenvalues λ_{mn} of A_0 satisfy $\sum_{m,n\geqslant 1} \lambda_{mn}^{-2} < \infty$ (this follows easily from (2.5)), we can use [24, Proposition 6.8.2]. This yields that the pair $(A, \widetilde{C})$, with state space X and output space Y, is exactly observable in any time $\tau > 0$. Using the definition of exact observability we obtain the conclusion (3.4). ▯

From the two above theorems we deduce the following exact observability result.

COROLLARY 3.3. *Let $\tau > 0$, assume that Ω is a bounded domain of $\mathbb{R}^2$, and let Γ be an open subset of $\partial\Omega$ such that one of the following assertions holds:*

1. *$\partial\Omega$ is of class C^5 and Γ satisfies the geometric optics conditions;*
2. *Ω is a rectangle $(0,a) \times (0,b)$ and Γ contains both a horizontal and a vertical segment of nonzero length.*

Then, there exists a positive constant $k_{\tau,\Gamma}$ such that the solutions $w \in C^0([0,\tau]; H_{\frac{5}{2}}) \cap C^1([0,\tau]; H_{\frac{3}{2}})$ of the system (3.2) satisfy

$$\int_0^\tau \left\| \frac{\partial \Delta w}{\partial \nu} \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \geqslant k_{\tau,\Gamma}^2 \left(\|w_0\|_{\frac{3}{2}}^2 + \|w_1\|_{\frac{1}{2}}^2 \right). \tag{3.7}$$

Proof. Using the fact that A_0 is unitary from $H_{\alpha+1}$ onto H_α for every $\alpha \geqslant 0$, we see that the operator $V \in \mathcal{L}(X)$ defined by

$$V \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} A_0^{-1}\psi \\ -A_0\varphi \end{bmatrix} \qquad \left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X \right)$$

is unitary on X. Moreover, it is easy to check that V commutes with the semigroup $\mathbb{T}$ generated by A. Using this fact combined with Theorem 3.1 (if $\partial\Omega$ is of class C^5) or with Theorem 3.2 (if Ω is a rectangle), we obtain that

$$\int_0^\tau \left\| \frac{\partial (A_0 w)}{\partial \nu} \right\|_Y^2 \geqslant k_{\tau,\Gamma}^2 \left\| V \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X^2, \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(A) \right),$$

which clearly implies (3.7). ▯

Remark 3.4. Again using the fact that A_0 is unitary from $H_{\alpha+1}$ onto H_α for every $\alpha \geqslant 0$ and the above corollary it follows that, under the assumptions of Corollary 3.3, the solutions $w \in C^0([0,\tau]; H_{\frac{3}{2}}) \cap C^1([0,\tau]; H_{\frac{1}{2}})$ of the system (3.2) satisfy

$$\int_0^\tau \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \geqslant k_{\tau,\Gamma}^2 \left(\|w_0\|_{\frac{1}{2}}^2 + \|w_1\|_{-\frac{1}{2}}^2 \right).$$

4. Stability for an inverse source problem with known intensity. Throughout this section we continue to use notation introduced in the previous ones. In particular, X, Y are Hilbert spaces, $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X, and $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is an admissible observation operator for $\mathbb{T}$.

We consider the differential equation

$$\dot{z}(t) = Az(t) + \lambda(t)f, \quad z(0) = z_0, \tag{4.1}$$

where $z_0 \in X$ and $f \in Z'$, with $Z = (\beta I - A^*)^{-1}(X + C^*Y)$. Assume $\tau > 0$ and that we are measuring

$$y(t) = Cz(t) \qquad (t \in [0, \tau]). \tag{4.2}$$

Our aim is to study the mapping $f \mapsto y$, assuming that λ and z_0 are given. It is convenient to recall that, in the case where $f \in X$ and $z_0 \in \mathcal{D}(A)$, the solution of (4.1) satisfies $z \in C^0([0, \tau]; \mathcal{D}(A)) \cap C^1([0, \tau]; X)$ and, by Duhamel formula, y satisfies

$$y(t) = \int_0^t \lambda(t-s) C\mathbb{T}_s f \mathrm{d}s + C\mathbb{T}_t z_0 = \int_0^t \lambda(t-s)\Psi_\tau f(s)\mathrm{d}s + \Psi_\tau z_0(t).$$

PROPOSITION 4.1. *Let $\tau > 0$, let Y be a Hilbert space, and let $\lambda \in H^1((0, \tau))$ with $\lambda(0) \neq 0$. Let $S : L^2((0, \tau); Y) \to H^1_L((0, \tau); Y)$ be defined by*

$$(Sg)(t) = \int_0^t \lambda(t-s)g(s)\mathrm{d}s. \tag{4.3}$$

Then S is an isomorphism from $L^2((0, \tau); Y)$ onto $H^1_L((0, \tau); Y)$. Moreover, the operator S admits a unique extension to an isomorphism $\widetilde{S}$ from $\left[H^1_R((0, \tau); Y)\right]'$ onto $L^2((0, \tau); Y)$.

Proof. The fact that S is an isomorphism from $L^2((0, \tau); Y)$ onto $H^1_L((0, \tau); Y)$ is well known from the the theory of Volterra integral operators (see, for instance, Kress [15, pp. 33–34]). Denote $\mathcal{X} = L^2((0, \tau); Y)$ and $\mathcal{X}_1 = H^1_L((0, \tau); Y)$ and let $\mathcal{A} \in \mathcal{L}(\mathcal{X}_1, \mathcal{X})$ be the inverse of S. Then $\mathcal{A}$ can be seen as an unbounded densely defined operator in $\mathcal{X}$ and $\mathcal{A}^* = (S^*)^{-1}$. It is easy to check that S^* maps $L^2((0, \tau); Y)$ onto $H^1_R((0, \tau); Y)$ so that $\mathcal{D}(\mathcal{A}^*) = H^1_R((0, \tau); Y)$. By applying Proposition 2.7 and Remark 2.8 to $\mathcal{A}$, we obtain that $\mathcal{A}$ has a unique extension to an isomorphism $\widetilde{\mathcal{A}} \in \mathcal{L}(L^2((0, \tau); Y), \left[H^1_R((0, \tau); Y)\right]')$. Consequently, $\widetilde{S} := \widetilde{\mathcal{A}}^{-1}$ is an isomorphism from $\left[H^1_R((0, \tau); Y)\right]'$ onto $L^2((0, \tau); Y)$ and it is an extension of S. $\square$

Now we can show that for less regular data, the mapping $f \mapsto y$ associated with system (4.1)–(4.2) is still well defined.

PROPOSITION 4.2. *Assume that $\lambda \in H^1((0, \tau))$, $\lambda(0) \neq 0$. Assume that $f \in Z'$ and that $z_0 \in X$. Then (4.1) admits a unique solution $z \in C^0([0, \tau]; X)$ such that $y \in L^2((0, \tau); Y)$.*

Proof. The first conclusion follows from [24, Theorem 4.1.6], by using the fact that the right-hand side of (4.1) belongs to $H^1((0, \tau); X_{-1})$.

Moreover, $y = (\widetilde{S} \circ \Psi_\tau)f + \Psi_\tau z_0 \in L^2((0, \tau); Y)$, where $\widetilde{S} : \left[H^1_R((0, \tau); Y)\right]' \to L^2((0, \tau); Y)$ is the extension of S defined in Proposition 4.1 and $\Psi_\tau : Z' \to \left[H^1_R((0, \tau); Y)\right]'$ is defined in Proposition 2.13. $\square$

In order to study the stability for the inverse source problem, we have to consider two sources $f^{(1)}$ and $f^{(2)}$ and the corresponding solutions $z^{(1)}$ and $z^{(2)}$ and observations $y^{(1)}$ and $y^{(2)}$. Due to the linearity of the problem, it is enough to consider the system

$$\dot{z}(t) = Az(t) + \lambda(t)f, \quad z(0) = 0, \tag{4.4}$$

$$y(t) = Cz(t) \qquad (t \in [0,\tau]). \tag{4.5}$$

Assume that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ and that $\lambda \in H^1((0,\tau))$ with $\lambda(0) \neq 0$. For each $\tau > 0$, we introduce the operator $\mathbb{E}_\tau \in \mathcal{L}(X, H^1_L((0,\tau);Y))$ defined by

$$(\mathbb{E}_\tau f)(t) = [(S \circ \Psi_\tau)f](t) = \int_0^t \lambda(t-s)\Psi_\tau f(s)\mathrm{d}s \qquad (t \in [0,\tau]).$$

By using Propositions 4.1 and 2.13, we extend $\mathbb{E}_\tau$ to an operator $\mathbb{F}_\tau \in \mathcal{L}(Z', L^2((0,\tau);Y))$ defined by

$$(\mathbb{F}_\tau f)(t) = [(\widetilde{S} \circ \Psi_\tau)f](t) \qquad (t \in [0,\tau]),$$

where $\widetilde{S}$ is the operator constructed in Proposition 4.1. The first main result of this paper is the following.

THEOREM 4.3. *Let X, Y be Hilbert spaces and assume that the pair (A, C) is exactly observable in some time $\tau_0 > 0$ and that $\lambda \in H^1((0,\tau))$ with $\lambda(0) \neq 0$. Then, the following properties hold:*

1. *for every $\tau \geqslant \tau_0$, $\mathbb{E}_\tau$ is one-to-one from X to $H^1_L((0,\tau);Y)$ and there exists a positive constant κ_τ such that*

$$\|f\|_X \leqslant \kappa_\tau \|\mathbb{E}_\tau f\|_{H^1_L((0,\tau);Y)} \qquad \forall f \in X; \tag{4.6}$$

2. *for every $\tau > \tau_0$, $\mathbb{F}_\tau$ is one-to-one from Z' to $L^2((0,\tau);Y)$ and there exists a positive constant $\tilde{\kappa}_\tau$ such that*

$$\|f\|_{Z'} \leqslant \tilde{\kappa}_\tau \|\mathbb{F}_\tau f\|_{L^2((0,\tau);Y)} \qquad \forall f \in Z'. \tag{4.7}$$

Proof. In the first case, since $\lambda(0) \neq 0$, $S : L^2((0,\tau);Y) \to H^1_L((0,\tau);Y)$ is an isomorphism and we have

$$\|\mathbb{E}f\|_{H^1_L((0,\tau);Y)} = \|(S \circ \Psi_\tau)f\|_{H^1_L((0,\tau);Y)} \geqslant M_S \|\Psi_\tau f\|_{L^2((0,\tau);Y)}.$$

From the exact observability of (A, C) in time τ, we deduce that

$$\|\Psi_\tau f\|_{L^2((0,\tau);Y)} \geqslant k_\tau \|f\|_X.$$

Combining the two above inequalities yields

$$\|\mathbb{E}f\|_{H^1_L((0,\tau);Y)} \geqslant \kappa_\tau \|f\|_X \qquad (f \in X).$$

For the second case the proof is similar. By using Propositions 4.1 and 2.13, we have

$$\|\mathbb{F}f\|_{L^2((0,\tau);Y)} = \left\|(\widetilde{S} \circ \Psi_\tau)f\right\|_{L^2((0,\tau);Y)} \geqslant M_{\widetilde{S}} \|\Psi_\tau f\|_{\left[H^1_R((0,\tau);Y)\right]'}$$

and

$$\|\Psi_\tau f\|_{\left[H^1_R((0,\tau);Y)\right]'} \geqslant m_\tau \|f\|_{Z'}.$$

Thus,

$$\|\mathbb{F}f\|_{L^2((0,\tau);Y)} \geqslant M_{\widetilde{S}} m_\tau \|f\|_{Z'} \qquad (f \in Z'). \qquad \square$$

To illustrate Theorem 4.3 we give below two simple examples of applications to PDEs. In spite of their simplicity, it does not seem that these examples have been tackled in the literature.

Example 4.4. First, let us consider the wave equation in a bounded domain Ω of class C^∞ of $\mathbb{R}^N$, $N \geqslant 1$:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = \lambda(t) f & \text{in } (0,\tau)\times\Omega, \\ w = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ w(0,x) = w_0(x),\ \dfrac{\partial w}{\partial t}(0,x) = w_1(x), & x \in \Omega, \end{cases} \tag{4.8}$$

where $\tau > 0$, $\lambda \in H^1((0,\tau))$ with $\lambda(0) \neq 0$, $\xi \in \Omega$ and $f \in H^{-1}(\Omega)$, $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$. Assume $\mathcal{O}$ is a nonempty open subset of Ω. Using Proposition 4.2, we obtain that the mapping

$$H^{-1}(\Omega) \to L^2((0,\tau); L^2(\mathcal{O})), \quad f \mapsto y = w\big|_{(0,\tau)\times\mathcal{O}} \tag{4.9}$$

is well defined. Moreover, using the second assertion of Theorem 4.3, we deduce that if $\mathcal{O}$ satisfies the geometric optics condition of Bardos, Lebeau, and Rauch [3] and if τ large enough, then there exists $K > 0$ such that

$$\|f^{(1)} - f^{(2)}\|_{H^{-1}(\Omega)} \leqslant K \left\| w^{(1)} - w^{(2)} \right\|_{L^2((0,\tau);L^2(\mathcal{O}))}.$$

In this inequality $w^{(1)}$ and $w^{(2)}$ are the solutions of (4.8) corresponding to $f^{(1)}$ and $f^{(2)}$.

More precisely, to apply Proposition 4.2 and Theorem 4.3, we set

$$X = H_{\frac{1}{2}} \times H, \quad \mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}, \quad X_{-1} = H \times H_{-\frac{1}{2}},$$

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad C \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \varphi\big|_{\mathcal{O}} \quad \left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A) \right),$$

where A_0 is defined in Proposition 2.3. Since C is a bounded operator in X, $Z = \mathcal{D}(A)$, and thus for all $f \in H_{-\frac{1}{2}}$,

$$\begin{bmatrix} 0 \\ f \end{bmatrix} \in Z'.$$

To apply Theorem 4.3, we use the result in [3] to deduce that (A, C) is exactly observable in X.

Example 4.5. This example is devoted to a Schrödinger equation in a rectangle with a measure consisting of the Dirichlet trace on an *arbitrarily small* part of the boundary.

Let $\tau > 0$, let $\lambda \in H^1((0,\tau)), \lambda(0) \neq 0$, and let Ω be the rectangle $(0,a) \times (0,b)$, with a, $b > 0$. For every $f \in L^2(\Omega)$, we consider the initial and boundary value problem

$$\begin{cases} \dot{w} + i\Delta w = \lambda(t) f & (x \in \Omega,\ t \in (0,\tau)), \\ \dfrac{\partial w}{\partial \nu} = 0 & (x \in \partial\Omega,\ t \in (0,\tau)), \\ w(x,0) = 0 & (x \in \Omega). \end{cases}$$

Moreover, let Γ be an open nonempty subset of $\partial\Omega$ and define

$$y = w|_{(0,\tau)\times\Gamma}.$$

Then the map $f \mapsto y$ is bounded and one-to-one from $L^2(\Omega)$ to $H^1_L((0,\tau); L^2(\Gamma))$ and there exists a constant $\delta_{\tau,\Gamma} > 0$ such that

$$\|y\|_{H^1_L((0,\tau);L^2(\Gamma))} \geqslant \delta_{\tau,\Gamma} \|f\|_{L^2(\Omega)} \qquad (f \in L^2(\Omega)).$$

Indeed, it suffices to apply the first assertion in Theorem 4.3 with

$$X = L^2(\Omega), \quad Y = L^2(\Gamma),$$

$$\mathcal{D}(A) = \left\{ \varphi \in H^2(\Omega) \ \middle| \ \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

$$A\varphi = -i\Delta\varphi \qquad (\varphi \in \mathcal{D}(A)),$$

$$C\varphi = \varphi|_\Gamma.$$

The only assumption in Theorem 4.3 which is not checked in a obvious manner is the exact observability of (A, C). For this property (in any time $\tau > 0$ and for any nonempty open set Γ) we refer the reader to Theorem 1.1 in [22].

5. Inverse source problems for the plate equation with known intensity. In this section we apply the general results obtained in the previous sections to the inverse source problem for the plate equation, assuming that the intensity λ is known and that $\Omega \subset \mathbb{R}^2$ is a bounded domain with a C^5 boundary or a rectangle. More precisely, the first subsection is devoted to system (1.7), whereas the second one tackles more regular sources.

5.1. Recovery of point sources. We first introduce some notation and prove several results which are necessary for the proof of Theorem 5.5, which is the main result of this subsection.

Recall the spaces introduced in Proposition 2.3 and consider the Hilbert space

$$X = H_{\frac{1}{2}} \times H_{-\frac{1}{2}} \tag{5.1}$$

and the skew-adjoint operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) = H_{\frac{3}{2}} \times H_{\frac{1}{2}}, \quad A = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}. \tag{5.2}$$

Also recall from Proposition 2.2 that $A_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$. Then, A is the generator of a strongly continuous unitary group $\mathbb{T}$ in X.

The dual of X_1 (i.e., of $\mathcal{D}(A)$ endowed with the graph topology) with respect to the pivot space X is

$$X_{-1} = H_{-\frac{1}{2}} \times H_{-\frac{3}{2}}. \tag{5.3}$$

Set $Y = L^2(\Gamma)$ and let $C \in \mathcal{L}(X_1, Y)$ be defined by

$$C \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma} \qquad \left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A) \right). \tag{5.4}$$

It is well known that C is an admissible observation operator for the semigroup $\mathbb{T}$ generated by A (see, for instance, [24, Proposition 7.5.5]).

We first note a lemma which is very similar to results used and proved in [13, 14]. However, since we are not able to apply directly the results in [13, 14], we give a precise statement and a short proof (with no claim of originality).

LEMMA 5.1. *For every $\varepsilon > 0$ there exists a positive constant $\gamma = \gamma(\Omega, \varepsilon)$ such that, for all $a, b \in \Omega$, with* $\operatorname{dist}(a, \partial\Omega) > \varepsilon$ *and* $\operatorname{dist}(b, \partial\Omega) > \varepsilon$,

$$|a - b| \leqslant \gamma \, \|\delta_a - \delta_b\|_{H_{-\frac{3}{2}}} .$$

Proof. We denote by Ω_ε the open set defined by

$$\Omega_\varepsilon = \{x \in \Omega \ ; \ \operatorname{dist}(x, \partial\Omega) > \varepsilon\} .$$

For ε small enough, Ω_ε is not empty. There exists a function $\varphi_1 \in W$ such that

$$\varphi_1(x_1, x_2) = x_1 \qquad ((x_1, x_2) \in \Omega_\varepsilon).$$

Since $a, b \in \Omega_\varepsilon$, we have

$$\langle \delta_a - \delta_b, \varphi_1 \rangle = a_1 - b_1.$$

Thus,

$$|a_1 - b_1| \leqslant \gamma_1(\Omega, \varepsilon) \, \|\delta_a - \delta_b\|_{H_{-\frac{3}{2}}} .$$

We can use a similar argument for the second coordinate to conclude the proof of the lemma. $\square$

The space Z obtained inserting the operators A and C above in (2.14) and its dual Z' with respect to the pivot space X have the properties below.

PROPOSITION 5.2. *With the above notation for X, Y, A, and C, the space Z defined by* (2.14) *satisfies*

$$Z \subset H_{\frac{3}{2}} \times H,$$

with continuous and dense embedding.

Proof. We can take $\beta = 0$ in (2.14) so that

$$Z = A^{-1}(X + C^* Y),$$

with A and C given by (5.1)–(5.2) and (5.4). To obtain the adjoint of C, we consider the operator $D \in \mathcal{L}(L^2(\Gamma); L^2(\Omega))$ defined by

$$\begin{cases} -\Delta(Dg) = 0 & \text{in } \Omega, \\ (Dg) = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ (Dg) = g & \text{on } \Gamma \end{cases}$$

for all $g \in L^2(\Gamma)$. In other words, Dg is the unique element of $L^2(\Omega)$ such that

$$\int_\Omega (Dg)\Delta\varphi \, \mathrm{d}x = \int_\Gamma g \frac{\partial \varphi}{\partial \nu} \, \mathrm{d}\sigma \qquad (\varphi \in H^2(\Omega) \cap H_0^1(\Omega)).$$

Recall that the operator A_0 and the spaces H_α are defined in Proposition 2.3. Let $(\varphi, \psi) \in \mathcal{D}(A)$ and $g \in L^2(\Gamma) = Y$. By definition of C, we have

$$\left\langle C \begin{bmatrix} \varphi \\ \psi \end{bmatrix}, g \right\rangle_{L^2(\Gamma)} = \int_\Gamma g \frac{\partial \varphi}{\partial \nu} \, \mathrm{d}\sigma,$$

and therefore

$$\left\langle C \begin{bmatrix} \varphi \\ \psi \end{bmatrix}, g \right\rangle_{L^2(\Gamma)} = \int_\Omega (Dg)\Delta\varphi \, \mathrm{d}x = -\langle A_0\varphi, Dg\rangle_H .$$

Now, notice that for $f \in \mathcal{D}(A_0)$,

$$\langle A_0\varphi, f\rangle_H = \langle \varphi, f\rangle_{\frac{3}{2}} = \langle \varphi, f\rangle_{H_{\frac{3}{2}}, H_{-\frac{1}{2}}, H_{\frac{1}{2}}} ,$$

where $\langle \cdot, \cdot \rangle_{V,V',H}$ denotes the duality product of an element of V and of an element of V', with respect to the pivot space H. By density of $\mathcal{D}(A_0)$ in H, we get

$$\langle A_0\varphi, f\rangle_H = \langle \varphi, f\rangle_{H_{\frac{3}{2}}, H_{-\frac{1}{2}}, H_{\frac{1}{2}}} \ (f \in H),$$

and thus

$$\left\langle C \begin{bmatrix} \varphi \\ \psi \end{bmatrix}, g \right\rangle_{L^2(\Gamma)} = -\left\langle \begin{bmatrix} \varphi \\ \psi \end{bmatrix}, \begin{bmatrix} Dg \\ 0 \end{bmatrix} \right\rangle_{\mathcal{D}(A), \mathcal{D}(A)^*} .$$

We deduce from the above relation that for all $g \in L^2(\Gamma)$,

$$C^* g = -\begin{bmatrix} Dg \\ 0 \end{bmatrix}.$$

This implies that $Z = X_1 + \{0\} \times DY \subset H_{\frac{3}{2}} \times H$, which clearly yields the announced embedding. ∎

COROLLARY 5.3. *With the assumptions and notation in Proposition* 5.2*, let Z' be the dual of Z with respect to the pivot space X. Then we have*

$$\begin{bmatrix} 0 \\ \delta_a \end{bmatrix} \in Z' \qquad (a \in \Omega).$$

Moreover, for every $\varepsilon > 0$ there exists a positive constant $\gamma = \gamma(\Omega, \varepsilon)$ such that, for all $a, b \in \Omega$, with $\operatorname{dist}(a, \partial\Omega) > \varepsilon$ *and* $\operatorname{dist}(b, \partial\Omega) > \varepsilon$,

$$|a - b| \leqslant \gamma \left\| \begin{bmatrix} 0 \\ \delta_a \end{bmatrix} - \begin{bmatrix} 0 \\ \delta_b \end{bmatrix} \right\|_{Z'} .$$

Proof. With the assumptions and notation in Proposition 5.2, the dual Z' of Z with respect to the pivot space X contains $H_{-\frac{1}{2}} \times H_{-\frac{3}{2}}$, with continuous and dense embedding. This fact and Lemma 5.1 clearly imply the conclusion. ∎

We have the following regularity result for the plate equation with a point source.

PROPOSITION 5.4. *Let $w_0 \in H_0^1(\Omega)$ and $w_1 \in H^{-1}(\Omega)$. Let Γ be an open subset of $\partial\Omega$. Then, for any $\tau > 0$ and $\lambda \in H^1((0,\tau))$, $\lambda(0) \neq 0$, the system* (1.7) *admits a unique solution*

$$w \in C^0([0,\tau]; H_0^1(\Omega)) \cap C^1([0,\tau]; H^{-1}(\Omega))$$

such that y defined by (1.8) *is in $L^2((0,\tau); L^2(\Gamma))$.*

Proof. We write

$$z = \begin{bmatrix} w \\ \frac{\partial w}{\partial t} \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \delta_\xi \end{bmatrix}.$$

We know from Corollary 5.3 that $f \in Z'$. Applying Proposition 4.2 with the above choice of spaces and operators, we obtain the stated regularity result. ∎

The main result of the section is the following.

THEOREM 5.5. *Let $\tau > 0$, assume that Ω is a bounded domain of $\mathbb{R}^2$, and let Γ be an open subset of $\partial\Omega$ such that one of the following assertions holds:*

1. *$\partial\Omega$ is of class C^5 and Γ satisfies the geometric optics conditions;*
2. *Ω is a rectangle $(0,a) \times (0,b)$ and Γ contains both a horizontal and a vertical segment of nonzero length.*

Let $\varepsilon > 0$ and let $\xi^{(1)}, \xi^{(2)} \in \Omega$ be two points in Ω, each one at distance at least ε from $\partial\Omega$. Assume that $\lambda \in H^1((0,\tau))$ with $\lambda(0) \neq 0$, $w_0 \in H_0^1(\Omega)$, and $w_1 \in H^{-1}(\Omega)$ and denote

$$y^{(j)} = \left. \frac{\partial w^{(j)}}{\partial \nu} \right|_{\Gamma}, \qquad j \in \{1,2\},$$

where $w^{(j)}$ is the solution of (1.7) *with $\xi = \xi^{(j)}$, $j \in \{1,2\}$.*

Then there exists $K > 0$, depending only on Ω, Γ, ε, and τ, such that

$$\|y^{(1)} - y^{(2)}\|_{L^2((0,\tau);L^2(\Gamma))} \geqslant K|\xi^{(1)} - \xi^{(2)}|,$$

where $|\cdot|$ stands for the standard norm in $\mathbb{R}^2$.

Proof. We write

$$w = w^{(1)} - w^{(2)}, \quad z = \begin{bmatrix} w \\ \frac{\partial w}{\partial t} \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \delta_{\xi^{(1)}} - \delta_{\xi^{(2)}} \end{bmatrix}.$$

Then the problem (1.7)–(1.8) can be written under the form (4.4)–(4.5), with the spaces and operators given by (5.1)–(5.4).

On the other hand, we know from Remark 3.4 that the pair (A, C) is exactly observable in any time $\tau > 0$, provided that condition 1 (respectively, condition 2) in the statement of the theorem is satisfied. Consequently Theorem 4.3 yields

$$\|y^{(1)} - y^{(2)}\|_{L^2(\Gamma)} \geqslant \kappa_\tau \|f\|_{Z'}$$

for some $\kappa_\tau > 0$. This, together with Corollary 5.3, implies the conclusion. ∎

5.2. Recovery of sources in $H_0^1(\Omega)$. We consider the initial value problem

$$
\left\{\begin{array}{ll}
\dfrac{\partial^2 w}{\partial t^2} + \Delta^2 w = \lambda(t) f & \text{in } (0,\tau)\times\Omega,\\
w = \Delta w = 0 & \text{on } (0,\tau)\times\partial\Omega,\\
w(0,x) = w_0(x),\ \dfrac{\partial w}{\partial t}(0,x) = w_1(x), & x\in\Omega,
\end{array}\right. \tag{5.5}
$$

where λ is given and satisfies $\lambda \in H^1((0,\tau))$ and $\lambda(0) \neq 0$. We aim to find $f \in H_0^1(\Omega)$ by knowing either

$$\frac{\partial^2 w}{\partial t \partial \nu} \quad \text{or} \quad \frac{\partial^2 \Delta w}{\partial t \partial \nu}.$$

In this case, we obtain the following stability result.

THEOREM 5.6. *Let Ω, Γ, w_0, w_1, τ, and λ satisfy the conditions of Theorem* 5.5. *Suppose that $w^{(j)}$ is the solution of* (1.7) *with $f = f^{(j)} \in H_0^1(\Omega)$, $j \in \{1,2\}$. Then there exists $K > 0$, depending only on Ω, Γ, and τ, such that*

$$\left\| \frac{\partial^2 w^{(1)}}{\partial t\partial \nu} - \frac{\partial^2 w^{(2)}}{\partial t\partial \nu} \right\|_{H^1((0,\tau);L^2(\Gamma))} \geqslant K \left\| f^{(1)} - f^{(2)} \right\|_{H_0^1(\Omega)}$$

and

$$\left\| \frac{\partial \Delta w^{(1)}}{\partial \nu} - \frac{\partial \Delta w^{(2)}}{\partial \nu} \right\|_{H^1((0,\tau);L^2(\Gamma))} \geqslant K \left\| f^{(1)} - f^{(2)} \right\|_{H_0^1(\Omega)}.$$

Proof. With the notation of Proposition 2.3, consider $A_0 : H_{\frac{3}{2}} \to H_{\frac{1}{2}}$ and $A_0^2 : H_{\frac{5}{2}} \to H_{\frac{1}{2}}$, which is a strictly positive operator. Let $X = H_{\frac{3}{2}} \times H_{\frac{1}{2}}$, and let $A : \mathcal{D}(A) \to X$ be defined by

$$\mathcal{D}(A) = H_{\frac{5}{2}} \times H_{\frac{3}{2}}, \quad A = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}.$$

Recall from subsection 2.1 that A is the generator of a strongly continuous unitary group $\mathbb{T}$ in X. Let $Y = L^2(\Gamma)$ and consider the observation operator $C : H_{\frac{5}{2}} \times H_{\frac{3}{2}} \to Y$ given by

$$C \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \left.\frac{\partial \psi}{\partial \nu}\right|_\Gamma \qquad \left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in H_{\frac{5}{2}} \times H_{\frac{3}{2}} \right).$$

Writing

$$z = \begin{bmatrix} w \\ \frac{\partial w}{\partial t} \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

the system (5.5) can be written as (4.4)–(4.5) and, since $F \in X$, from Theorems 3.1 and 3.2, the couple (A, C) is exactly observable for all $\tau > 0$. Using Theorem 4.3, we deduce that

$$\|y^{(1)} - y^{(1)}\|_{H_L^1((0,\tau);Y)} \geqslant \kappa_\tau \|f^{(1)} - f^{(2)}\|_{H_0^1(\Omega)}.$$

To treat the other case, we consider the observation operator given by

$$C \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \left.\frac{\partial \Delta \varphi}{\partial \nu}\right|_\Gamma \quad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in H_{\frac{5}{2}} \times H_{\frac{3}{2}}$$

and apply Corollary 3.3 to deduce that the couple (A, C) is exactly observable for all $\tau > 0$. ∎

6. Inverse source problems for the plate equation with unknown intensities. In this section, we establish an identifiability result for the unknown source term $\sum_{j=1}^{N} \lambda_j(t)\delta_{\xi_j}$ in the plate equation

$$\left\{\begin{array}{ll} \dfrac{\partial^2 w}{\partial t^2} + \Delta^2 w = \displaystyle\sum_{j=1}^{N} \lambda_j(t)\delta_{\xi_j} & \text{in } (0,\tau)\times\Omega, \\ w = \Delta w = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ w(0,x) = w_0(x),\ \dfrac{\partial w}{\partial t}(0,x) = w_1(x) & \text{for } x\in\Omega, \end{array}\right. \tag{6.1}$$

where now the number N of point sources, their locations $\xi_j \in \Omega$, and the functions λ_j are all unknown. The problem now involves more unknowns, and therefore we consider further boundary measurements than those used to solve the problem of subsection 5.1. The method employed here is different from the method used in the previous section. This explains that the assumptions are quite different. In particular, we assume that $\lambda_j(0) = 0$ and that there exists a time $\tau_1 \in (0,\tau)$ such that

$$\lambda_j(t) = 0 \qquad (t \geqslant \tau_1). \tag{6.2}$$

To deal with the corresponding inverse problem, we follow a method inspired by [11], based on the Fourier transformation.

We first present a regularity result for (6.1). In what follows, for $\varepsilon > 0$, we set

$$\Omega^\varepsilon = \{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega) < \varepsilon\}$$

and use the notation from subsection 5.1 for the space X and the operator A, i.e.,

$$X = H_{\frac{1}{2}} \times H_{-\frac{1}{2}}, \tag{6.3}$$

$$\begin{aligned} &\mathcal{D}(A) = H_{\frac{3}{2}} \times H_{\frac{1}{2}}, \\ &A\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix}\psi\\ -A_0^2\varphi\end{bmatrix} \qquad \left(\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\in\mathcal{D}(A)\right). \end{aligned} \tag{6.4}$$

We recall that, if we set $f_j = \begin{bmatrix}0\\ \delta_{\xi_j}\end{bmatrix}$, $g(t) = \sum_{j=1}^{N}\lambda_j(t) f_j$, and $z_0 = \left[\begin{smallmatrix} w_0\\ w_1\end{smallmatrix}\right]$, the function w is the solution of (6.1) if and only if $z = \left[\begin{smallmatrix} w\\ \frac{\partial w}{\partial t}\end{smallmatrix}\right]$ is the solution of

$$\begin{aligned} &\dot z(t) = Az(t) + g(t) \quad (t\geqslant 0), \\ &z(0) = z_0. \end{aligned} \tag{6.5}$$

Henceforth, we will also use the notation $\mathcal{G}(t) := \sum_{j=1}^{N}\lambda_j(t)\delta_{\xi_j}$.

PROPOSITION 6.1. *Let*

$$w_0 \in H^5(\Omega),\ w_0 = \Delta w_0 = \Delta^2 w_0 = 0 \textit{ on } \partial\Omega \quad \textit{and}$$
$$w_1 \in H^3(\Omega),\ w_1 = \Delta w_1 = 0 \textit{ on } \partial\Omega.$$

Let $\{\xi_1,\ldots,\xi_N\} \subset \Omega$ *and* $\varepsilon = \min_{j\in\{1,\ldots,N\}}\{\operatorname{dist}(\xi_j,\partial\Omega)\}$. *Let* Γ *be a nonempty open subset of* $\partial\Omega$. *Then, for any* $\tau > 0$ *and* $\lambda_j \in C^2([0,\tau])$ $(j = 0,\ldots,N)$ *with* $\lambda_j(0) = \dot\lambda_j(0) = 0$, *the system* (6.1) *admits a unique solution*

$$w \in C^0([0,\tau]; H_0^1(\Omega)) \cap C^0([0,\tau]; H^5(\Omega^\varepsilon)) \cap C^1([0,\tau]; H^3(\Omega^\varepsilon)) \tag{6.6}$$

such that

$$\frac{\partial w}{\partial \nu} \in L^2(0,\tau;L^2(\Gamma)), \quad \frac{\partial \Delta w}{\partial \nu} \in L^2(0,\tau;L^2(\Gamma)).$$

Proof. We consider the spaces and the operator defined by (6.3)–(6.4). According to Proposition 2.3, we have

$$X_2 = \mathcal{D}(A^2) = H_{\frac{5}{2}} \times H_{\frac{3}{2}}$$

and $z_0 = \left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in X_2$. Moreover, $g \in H^2((0,\tau);X_{-1})$, with $X_{-1} = H_{-\frac{1}{2}} \times H_{-\frac{3}{2}}$. Then $z = \left[\begin{smallmatrix} w \\ \frac{\partial w}{\partial t} \end{smallmatrix}\right] \in C^0([0,\tau];X)$. The additional assumptions on the data will allow us to improve this result.

Since the problem is linear, we can analyze separately the problem (6.1) with $\mathcal{G} \equiv 0$ and the problem (6.1) with $w_0 \equiv w_1 \equiv 0$.

In the first case, we get from classical theory on semigroups that $z = \left[\begin{smallmatrix} w \\ \frac{\partial w}{\partial t} \end{smallmatrix}\right] \in C^0([0,\tau];X_2)$, and therefore $\frac{\partial w}{\partial \nu}$ and $\frac{\partial \Delta w}{\partial \nu}$ have the stated summability properties. In the second case, we note that each λ_j can be written in the form

$$\lambda_j(t) = \int_0^t \left(\int_0^s \ddot{\lambda}_j(\theta) \mathrm{d}\theta \right) \mathrm{d}s,$$

and that the unique solution of (6.1) is given by

$$w(t,x) = \int_0^t \left(\int_0^s u(\theta,x) \mathrm{d}\theta \right) \mathrm{d}s,$$

where $u \in C^0([0,\tau];H_{\frac{1}{2}}) \cap C^1([0,\tau];H_{-\frac{1}{2}})$ solves (6.1) when $w_0 \equiv w_1 \equiv 0$ and the source term is $\mathcal{G}(t) = \sum_{j=1}^N \ddot{\lambda}_j(t) f_j$. Therefore, $w \in C^2([0,\tau];H^1(\Omega))$ and we deduce that

$$\frac{\partial^2 w}{\partial t^2} \in C^0([0,\tau];H^1(\Omega)).$$

From (6.1) and classical regularity results for the elliptic problem associated with the bi-Laplacian operator, we obtain

$$w \in C^0([0,\tau];H^5(\Omega^\varepsilon)),$$

which implies the desired trace properties. $\square$

Based on Proposition 6.1 we will consider measurements of $\frac{\partial w}{\partial \nu}$ and $\frac{\partial \Delta w}{\partial \nu}$ on parts of $\partial\Omega$ until time τ_1 and continue the measurement of $\frac{\partial w}{\partial \nu}$ until time τ. In accordance with the operator formulation (6.5) with (6.3)–(6.4) and $z_0 \in \mathcal{D}(A^2)$, we introduce the following observation operators for the inverse source problem for (6.1). The output spaces are $Y_1 = L^2(\Gamma) \times L^2(\Gamma_1)$ and $Y_2 = L^2(\Gamma)$, and the operators $C_1 \in \mathcal{L}(V_1, Y_1)$, with

$$V_1 = \left\{ u \in H_0^1(\Omega) \mid u|_{\Omega^\varepsilon} \in H^5(\Omega^\varepsilon) \right\} \tag{6.7}$$

and $C_2 \in \mathcal{L}(X_1, Y_2)$ defined by

$$C_1 \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial \varphi}{\partial \nu}\right|_{\Gamma} \\ \left.\dfrac{\partial \Delta\varphi}{\partial \nu}\right|_{\Gamma_1} \end{bmatrix} \qquad \left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in V_1 \right), \tag{6.8}$$

$$C_2 \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \frac{\partial \varphi}{\partial \nu}\bigg|_{\Gamma} \qquad \left(\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A) \right). \tag{6.9}$$

Hence, the output function corresponding to the measured data in this case is given by

$$y(t) = \begin{cases} C_1 z(t) & (t \in [0, \tau_1]), \\ C_2 z(t) & (t \in [\tau_1, \tau]). \end{cases}$$

The main result of this section is the following.

THEOREM 6.2. *Consider the sources* $\mathcal{G}^{(l)} = \sum_{j=1}^{N^{(l)}} \lambda_j^{(l)} \delta_{\xi_j^{(l)}}$, $l \in \{1,2\}$, *in the plate equation* (6.1) *and assume that* $\lambda_j^{(l)} \in C^2([0,\tau])$, $j \in \{1,\ldots,N\}$ *and* $l \in \{1,2\}$, *satisfy* $\lambda_j^{(l)}(0) = \dot{\lambda}_j^{(l)}(0) = 0$ *and* $\lambda_j^{(l)}(t) = 0$ *for* $t \geqslant \tau_1$. *Let* $w^{(1)}$ *and* $w^{(2)}$ *be the corresponding solutions of* (6.1) *with initial condition*

$$w_0 \in H^5(\Omega),\ w_0 = \Delta w_0 = \Delta^2 w_0 = 0 \ \text{on}\ \partial\Omega \quad \text{and}$$
$$w_1 \in H^3(\Omega),\ w_1 = \Delta w_1 = 0 \ \text{on}\ \partial\Omega.$$

Let $\Gamma,\ \Gamma_1$ *be two nonempty open subsets of* $\partial\Omega$ *with* $\Gamma \subseteq \Gamma_1$ *and assume that one of the following assumptions holds:*

1. $\partial\Omega$ *is smooth and* Γ *satisfies the geometric optics conditions;*
2. Ω *is a rectangle and* Γ *contains both a horizontal and a vertical segment of nonzero length.*

If

$$\frac{\partial w^{(1)}}{\partial \nu} = \frac{\partial w^{(2)}}{\partial \nu} \ \text{on}\ (0,\tau) \times \Gamma, \qquad \frac{\partial \Delta w^{(1)}}{\partial \nu} = \frac{\partial \Delta w^{(2)}}{\partial \nu} \ \text{on}\ (0,\tau_1) \times \Gamma_1,$$

then $\mathcal{G}^{(1)} = \mathcal{G}^{(2)}$.

Proof. We denote by $z^{(1)}, z^{(2)}$ the solutions of (6.5) for $g^{(1)}(t) = \sum_{j=1}^{N^{(1)}} \lambda_j^{(1)}(t) f_j^{(1)}$ and $g^{(2)}(t) = \sum_{j=1}^{N^{(2)}} \lambda_j^{(2)}(t) f_j^{(2)}$, respectively, and by $y^{(1)}, y^{(2)}$ the corresponding observations given by (6.8)–(6.9). We assume that $y^{(1)}(t) = y^{(2)}(t)$ for $t \in (0,\tau)$. Let us write $z(t) := z^{(1)}(t) - z^{(2)}(t)$ and $y(t) := y^{(1)}(t) - y^{(2)}(t)$. Since $g^{(l)}(t) = 0$, for $t \geqslant \tau_1$ ($l \in \{1,2\}$), the functions z and y satisfy

$$\begin{aligned} &\dot{z}(t) = Az(t) \quad (t \in (\tau_1, \tau)), \\ &z(\tau_1) \in X, \\ &y(t) = 0 \qquad\quad (t \in (\tau_1, \tau)). \end{aligned}$$

We notice that $y(t) = [\Psi_{\tau-\tau_1} z(\tau_1)]\,(t - \tau_1)$. We set $\tau_0 = \tau - \tau_1 > 0$. Since the pair (A, C_2), with A defined by (6.4) and C_2 defined by (6.9), is exactly observable in time $\tau_0 > 0$, we have

$$\|y\|_{L^2(\tau_1,\tau;Y)} \geqslant \kappa_\tau \|z(\tau_1)\|_X,$$

and therefore $z(\tau_1) = 0$. Thus, z satisfies

$$\begin{aligned} &\dot{z}(t) = Az(t) + g^{(1)} - g^{(2)} \quad (t \in (0, \tau_1)), \\ &z(0) = 0, \quad z(\tau_1) = 0. \end{aligned}$$

Extending $\lambda_j^{(l)}$, z, and y by zero outside $(0,\tau_1)$ and then applying Fourier transformation in the variable t yields

$$(i\varpi I - A)\widehat{z}(\varpi) = \sum_{j=1}^{N^{(1)}} \widehat{\lambda_j^{(1)}}(\varpi) f_j^{(1)} - \sum_{j=1}^{N^{(2)}} \widehat{\lambda_j^{(2)}}(\varpi) f_j^{(2)} \quad \forall \varpi \in \mathbb{R},$$

where the notation $\widehat{\cdot}$ indicates the Fourier transform of the extended function. Setting $\widehat{z}(\varpi) = \left[\begin{smallmatrix} v \\ \vartheta \end{smallmatrix}\right]$, we conclude that v satisfies

$$\Delta^2 v - \varpi^2 v = \widehat{\mathcal{G}} \text{ in } \Omega, \tag{6.10}$$

where we have set $\mathcal{G} = \mathcal{G}^{(1)} - \mathcal{G}^{(2)}$.

Since $\widehat{\mathcal{G}} \in \mathcal{E}'(\mathbb{R}^2)$ (distribution with compact support) with support contained in Ω, we can extend the left-hand side of (6.10) by zero outside Ω and get

$$\widetilde{\Delta^2 v} - \varpi^2 \widetilde{v} = \widehat{\mathcal{G}} \text{ in } \mathbb{R}^2, \tag{6.11}$$

where the notation $\widetilde{\cdot}$ indicates the extension to $\mathbb{R}^2$ with respect to the variable x. The next lemma gives a relation between $\widetilde{\Delta^2 v}$ and $\Delta^2 \widetilde{v}$. This result is an easy extension of a well-known result (see, for instance, Theorem 5.4.13 in [5]), so we omit the proof.

LEMMA 6.3. *Let $v \in V_1$ be such that $\Delta v = 0$ on $\partial\Omega$. Let $\widetilde{v}$ and $\widetilde{\Delta^2 v}$ be the extensions by zero outside Ω of v and $\Delta^2 v$, respectively. Then*

$$\widetilde{\Delta^2 v} = \Delta^2 \widetilde{v} + \frac{\partial \Delta v}{\partial \nu} \delta_{\partial\Omega} + \Delta \left(\frac{\partial v}{\partial \nu} \delta_{\partial\Omega} \right) \ \textit{in}\ \mathcal{D}'(\mathbb{R}^2).$$

In the above lemma, we have used the notation

$$\langle \delta_{\partial\Omega}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2),\mathcal{D}(\mathbb{R}^2)} = \int_{\partial\Omega} \varphi \ \mathrm{d}\Gamma.$$

Now we go back to (6.11) and use Lemma 6.3 to conclude that $\widetilde{v}$ satisfies the following relation in the distributional sense:

$$\Delta^2 \widetilde{v} - \varpi^2 \widetilde{v} = \widehat{\mathcal{G}} + \frac{\partial \Delta v}{\partial \nu} \delta_{\partial\Omega} + \Delta \left(\frac{\partial v}{\partial \nu} \delta_{\partial\Omega} \right) \text{ in } \mathbb{R}^2 \ (\varpi \in \mathbb{R}). \tag{6.12}$$

Let

$$\Phi_0(x) = \frac{1}{8\pi} |x|^2 \ln(|x|) \qquad (x \in \mathbb{R}^2 \setminus \{0\}) \tag{6.13}$$

and, for each $\varpi \in \mathbb{R} \setminus \{0\}$, let

$$\Phi_\varpi(x) = \frac{i}{8|\varpi|} \left(H_0^{(1)}(\sqrt{|\varpi|}|x|) - H_0^{(1)}(i\sqrt{|\varpi|}|x|) \right) \qquad (x \in \mathbb{R}^2 \setminus \{0\}), \tag{6.14}$$

where $H_1^{(0)}$ denotes the first kind Hänkel function of order 0. Then, as shown in Kitahara [12, p. 211], we have

$$(\Delta^2 - \varpi^2)\Phi_\varpi = \delta_0 \text{ in } \mathbb{R}^2;$$

i.e., Φ_ϖ is a fundamental solution of $\Delta^2 - \varpi^2 I$ in $\mathbb{R}^2$. We recall (see, for instance, Abramowitz and Stegun [1, p. 358]) that

$$H_0^{(1)}(y) = J_0(y) + iY_0(y) \qquad (y \in \mathbb{R}),$$

where J_0 is the Bessel function of the first kind and of order zero and Y_0 is the Bessel function of the second kind and of order zero. The function J_0 is analytic in $\mathbb{R}$ with series expansion

$$J_0(y) = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{2^{2k}(k!)^2} \qquad (y \in \mathbb{R}), \tag{6.15}$$

whereas $Y_0(y)$ can be represented as
(6.16)

$$Y_0(y) = \frac{2}{\pi}\left(\ln\left(\frac{y}{2}\right) + \gamma\right) J_0(y) + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k!)^2}\left(\sum_{m=1}^{k}\frac{1}{m}\right)\left(\frac{y^2}{2}\right)^k \qquad (y \in \mathbb{R}\setminus\{0\}),$$

with γ the Euler–Mascheroni constant; see [1, p. 360]. Therefore each Φ_ϖ is analytic in $\mathbb{R}^2 \setminus \{0\}$.

We note that

$$\mathcal{G} := \sum_{j=1}^{N^{(1)}} \lambda_j^{(1)} \delta_{\xi_j^{(1)}} - \sum_{j=1}^{N^{(2)}} \lambda_j^{(2)} \delta_{\xi_j^{(2)}}$$

can be written

$$\mathcal{G} = \sum_{j=1}^{N} \mu_j \delta_{\xi_j}, \quad \text{so that} \quad \widehat{\mathcal{G}} = \sum_{j=1}^{N} \widehat{\mu_j}\delta_{\xi_j}, \tag{6.17}$$

by taking $N = \max\{N^{(1)}, N^{(2)}\}$.

Using the fundamental solutions (6.14)–(6.13) and the relation (6.12), we can write $\widetilde{v}$ as

$$\widetilde{v} = \Phi_\varpi * \left(\widehat{\mathcal{G}} + \frac{\partial \Delta v}{\partial \nu}\delta_{\partial\Omega} + \Delta\left(\frac{\partial v}{\partial \nu}\delta_{\partial\Omega}\right)\right),$$

from which we obtain the representation

$$\widetilde{v}(x) = \sum_{j=1}^{N} \widehat{\mu_j}\Phi_\varpi(x-\xi_j) + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(y)\Delta\Phi_\varpi(x-y)\,\mathrm{d}\sigma_y + \int_{\partial\Omega}\frac{\partial\Delta v}{\partial\nu}(y)\Phi_\varpi(x-y)\,\mathrm{d}\sigma_y.$$

Since

$$\frac{\partial w^{(1)}}{\partial \nu} = \frac{\partial w^{(2)}}{\partial \nu} \text{ on } (0,\tau)\times\Gamma, \quad \frac{\partial \Delta w^{(1)}}{\partial \nu} = \frac{\partial \Delta w^{(2)}}{\partial \nu} \text{ on } (0,\tau_1)\times\Gamma_1,$$

it follows that

$$\frac{\partial v}{\partial \nu} = \frac{\partial \Delta v}{\partial \nu} = 0 \text{ on } \Gamma_1,$$

which, in turn, implies that

$$\widetilde{v}(x) = \sum_{j=1}^{N} \widehat{\mu_j} \Phi_\varpi(x-\xi_j) + \int_{\partial\Omega\setminus\Gamma_1} \frac{\partial v}{\partial \nu}(y) \Delta\Phi_\varpi(x-y)\,\mathrm{d}\sigma_y + \int_{\partial\Omega\setminus\Gamma_1} \frac{\partial \Delta v}{\partial \nu}(y) \Phi_\varpi(x-y)\,\mathrm{d}\sigma_y. \tag{6.18}$$

Since Φ_ϖ is analytic in $\mathbb{R}^2 \setminus \{0\}$, (6.18) shows that $\widetilde{v}$ is analytic in the connected domain

$$\mathcal{W} := \left[\mathbb{R}^2 \setminus (\{\xi_1, \ldots, \xi_N\} \cup \partial\Omega)\right] \cup \Gamma_1.$$

From the fact that $\widetilde{v}$ vanishes outside Ω, it follows that $\widetilde{v} \equiv 0$ in $\mathcal{W}$.

Our aim now is to show that $\widehat{\mu_j} \equiv 0$. Based on the expansions (6.15) and (6.16), we can show that each function $\Phi_\varpi(x)$ ($\varpi \neq 0$) satisfies

$$\begin{aligned}
\Phi_\varpi(x) &= \frac{i}{8|\varpi|}\left(H_0^{(1)}(\sqrt{|\varpi|}|x|) - H_0^{(1)}(i\sqrt{|\varpi|}|x|)\right)\\
&= \frac{-1}{4\pi|\varpi|}\left(\ln(\sqrt{|\varpi|}|x|/2) - \ln(i\sqrt{|\varpi|}|x|/2)\right) + O(|x|) \quad (|x| \to 0)\\
&= \frac{i}{8|\varpi|} + O(|x|) \quad (|x| \to 0),\\
\nabla\Phi_\varpi(x) &= \frac{i}{8\sqrt{|\varpi|}}\frac{x}{|x|}\left(\dot{H}_0^{(1)}(\sqrt{|\varpi|}|x|) - i\dot{H}_0^{(1)}(i\sqrt{|\varpi|}|x|)\right)\\
&= x\left(\frac{2\gamma-3}{8\pi} - \frac{i}{16} + \frac{1}{4\pi}\ln(\sqrt{|\varpi|}|x|/2) + O(|x|)\right) \quad (|x| \to 0),\\
\Delta\Phi_\varpi(x) &= \frac{i}{8\sqrt{|\varpi|}|x|}\left(\dot{H}_0^{(1)}(\sqrt{|\varpi|}|x|) - i\dot{H}_0^{(1)}(i\sqrt{|\varpi|}|x|)\right)\\
&\quad + \frac{i}{8}\left(\ddot{H}_0^{(1)}(\sqrt{|\varpi|}|x|) + \ddot{H}_0^{(1)}(i\sqrt{|\varpi|}|x|)\right)\\
&= \frac{\gamma-1}{2\pi} - \frac{i}{8} + \frac{1}{2\pi}\ln(\sqrt{|\varpi|}|x|/2) + O(|x|) \quad (|x| \to 0),
\end{aligned}$$

and therefore

$$\begin{aligned}
&\lim_{x\to\xi_j} \Phi_\varpi(x-\xi_j) = \frac{i}{8|\varpi|},\\
&\lim_{x\to\xi_j} \nabla\Phi_\varpi(x-\xi_j) = 0,\\
&\lim_{x\to\xi_j} |\Delta\Phi_\varpi(x-\xi_j)| = \infty,
\end{aligned}$$

when $\varpi \neq 0$. Moreover,

$$\begin{aligned}
\lim_{x\to\xi_j} \Phi_0(x-\xi_j) &= 0,\\
\lim_{x\to\xi_j} \nabla\Phi_0(x-\xi_j) &= 0,\\
\lim_{x\to\xi_j} |\Delta\Phi_0(x-\xi_j)| &= \infty.
\end{aligned}$$

Now, we multiply $\Delta\widetilde{v}$ by $\frac{1}{\Delta\Phi_{\varpi}(x-\xi_j)}$ and let $x \to \xi_j$, which, from (6.18), yields

$$\widehat{\mu_j} = \lim_{x\to\xi_j} \frac{\Delta\widetilde{v}(x)}{\Delta\Phi_{\varpi}(x-\xi_j)} = 0$$

for all $j \in \{1, \dots, N\}$, since

$$\lim_{x\to\xi_j} \frac{1}{\Delta\Phi_{\varpi}(x-\xi_j)} = 0$$

and since

$$\lim_{x\to\xi_j} \Delta \left(\int_{\partial\Omega\setminus\Gamma_1} \frac{\partial v}{\partial\nu}(y)\Delta\Phi_{\varpi}(x-y)\ \mathrm{d}\sigma_y + \int_{\partial\Omega\setminus\Gamma_1} \frac{\partial\Delta v}{\partial\nu}(y)\Phi_{\varpi}(x-y)\ \mathrm{d}\sigma_y \right)$$

is finite.

Applying the inverse Fourier transformation, we deduce that $\mu_j = 0$ for all $j \in \{1, \dots, N\}$ and conclude that $\mathcal{G}^{(1)} = \mathcal{G}^{(2)}$. ∎

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