A survey of controllability and stabilization results for partial differential equations

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ABSTRACT. *This paper surveys several issues related to the control of partial differential equations (PDE). The main focus is on the exact controllability property, which corresponds to the question of whether the solution of a PDE can be driven to a given state at a given final time by means of a control acting on a subregion of the domain or of the boundary. It is demonstrated that such a property is equivalent to an observability property for the adjoint system. The study of the exact controllability is detailed on several examples, including the wave equation, the heat equation, and the plate equation in dimension one. The controllability of the Korteweg-de Vries equation is also detailed in order to give an insight of the ideas involved in the control of a nonlinear PDE. The last part of the paper is devoted to the stabilization issue and to its connections with the controllability properties.*

RÉSUMÉ. *Cet article passe en revue diverses questions liées au contrôle des équations aux dérivées partielles (EDP). La question principale sur laquelle se focalise le papier est celle de la contrôlabilité exacte, qui correspond au fait que la solution d'une EDP peut être amenée à un état donné au bout d'un temps donné au moyen d'un contrôle agissant sur une sous-région du domaine ou de la frontière. On montre qu'une telle propriété est équivalente à une propriété d'observabilité pour le système adjoint. L'étude de la contrôlabilité exacte est détaillée sur plusieurs exemples, incluant l'équation des ondes, l'équation de la chaleur, et l'équation des plaques en une dimension d'espace. La contrôlabilité de l'équation de Korteweg-de Vries est également détaillée afin de fournir certaines idées mises en jeu dans le contrôle d'une EDP non linéaire. La dernière section du papier est dévolue à la question de la stabilisation et à ses liens avec les propriétés de contrôlabilité.*

KEYWORDS: *exact controllability, null controllability, frequency domain test, exponential stabilizability, multiplier method, Ingham lemma, Carleman estimate, compactness-uniqueness method.*

MOTS-CLÉS : *contrôlabilité exacte, contrôlabilité à zéro, tests fréquentiels, stabilisation exponentielle, méthode des multiplicateurs, lemme d'Ingham, inégalité de Carleman, méthode d'unicité-compacité.*

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1. Introduction

The aim of this paper is to survey several issues related to the control of Partial Differential Equations (PDE). Among the numerous applications of the control of PDE one may mention: the noise reduction (wave equation); the vibrations reduction (plate equations); the turbulence reduction (Navier-Stokes equation); the laser control of chemical reactions (Schrödinger equation).

Whereas the control of Ordinary Differential Equations (ODE), which goes back to the invention by J. Watt of his steam engine, is well understood in a linear framework, the control of PDE is a quite recent and very active field of investigation, even in a *linear* framework. One reason is that a linear PDE arising in an evolution problem may be of *hyperbolic* type (wave equation, Maxwell equations), of *dispersive* type (plate equation, Schrödinger equation, Korteweg-de Vries equation), or of *parabolic* type (heat equation, Stokes equation), and that the corresponding flow inherits very specific properties of the PDE: the Huygens' principle and the singularities propagation property hold for hyperbolic equations, whereas the infinite speed propagation property together with a weak (resp. strong) smoothing effect hold for dispersive (resp. parabolic) equations. As we shall see, these properties have a strong influence on the control properties of the different PDE. Even if the controllability of a PDE may always be reduced to an observability inequality, the proof of that inequality uses tools adapted to the PDE under investigation; e.g. the microlocal analysis giving sharp results for the wave equation, or the Carleman estimates for the null-controllability of the heat equation. Other methods have been developed for the control of wave equations, e.g. the nonharmonic Fourier series method (well adapted for one-dimensional PDE), the multiplier method, and the uniqueness-compactness method.

As the control theory for ODE is well developed, one may be tempted, when one has to control a physical phenomenon described by a PDE, to perform a reduction to a finite-dimensional model (e.g. by a Galerkin procedure, or by a discretization by finite differences, or merely by identification), and next to control the ODE by classical tools.

This approach presents several drawbacks:

1) the physics is not taken into account; actually, we shall see that the location and the duration of the control play a great role in the control of a PDE, and this fact may be completely hidden in a finite dimension analysis;

2) the control for the finite-dimensional system may not converge towards an exact control input for the PDE as the dimension of the finite-dimensional system tends to infinity (see (Zuazua, 2005)).

In this survey we shall see how to control a PDE in a direct way, *i.e.*, without the aid of any finite-dimensional control theory result. Due to an obvious lack of place, only the controllability and stabilization issues will be discussed here. The controllability is the property the most studied for a PDE, probably because a controllable PDE is also stabilizable, and that the converse holds true for a wide class of PDE including the wave equation and Schrödinger equation. Notice that for a PDE we have at our disposal three concepts of controllability; namely the *exact controllability* (any pair of state vectors may be connected by a trajectory), the *null controllability* (any state vector may be steered to 0) and the *approximate controllability* (any state vector me be steered arbitrarily close to another state vector). In the above definitions, the functional space in which the state lives (e.g. the space $H = L^2(\Omega)$ of square integrable functions for the distribution of temperature at time t for the heat equation), and the duration of the control have to be specified. For the stabilization issue, we shall mainly consider the *strong stability* (the state vector tends to 0) and the *exponential stability* (the convergence is uniform and exponential). Let us now describe more precisely the content of the survey.

Section 2 is devoted to the controllability of a PDE. The various concepts of controllability, which agree in finite dimension but not in general for a PDE, are introduced and next characterized thanks to the classical duality approach (see (Dolecky *et al.*, 1977), (Lions, 1988)). For instance, the exact controllability of a system is shown to be equivalent to the observability of the adjoint system. The proof given here is based on the Hilbert Uniqueness Method (HUM) due to J.-L. Lions. The controllability tests given here may be seen as natural extensions of Kalman rank criterion. When working in the frequency domain, the popular Hautus test of controllability may also be extended in infinite-dimensional spaces. Several frequency domain tests are provided here, and some applications to the control of PDE are pointed out. The section proceeds with an analysis of the approximate controllability of a PDE through a classical unique continuation theorem due to Holmgren. Finally, the main (expected) control properties of a PDE are summarized in a table.

Section 3 deals with the controllability of the most popular PDE in the families described above, namely the wave equation (hyperbolic), the plate equation (dispersive), and the heat equation (parabolic). For each of these equations, a controllability result is stated and (partially) proved in dimension one, for the sake of simplicity. Some important results when the spatial variable lives in an open set of $\mathbb{R}^N$ are given together with references.

The boundary controllability of the 1D wave equation is proved here by using Fourier series. The same is true for the 1D plate equation (the so-called *beam equation*). The controllability in any positive time, due to the infinite speed of propagation of a dispersive equation, is obtained thanks to a variant of Ingham's Lemma. The null controllability of the heat equation is derived from some Carleman estimate. The section ends with the exact controllability of a nonlinear PDE, namely the *Korteweg-de Vries equation*, for which new arguments are presented: the *multiplier method* to derive *a priori* estimates, the *compactness-uniqueness argument* (due to E. Zuazua (Lions, 1988)) to prove the observability inequality, and finally the *contraction principle* to get the exact controllability of the nonlinear equation.

The last section is devoted to the stabilization issue. The beginning of the section deals with the stability concepts in infinite dimension. Frequency domain tests are given for the strong stability, the polynomial stability, and the exponential stability.

The stabilizability of a PDE is next defined, and related to the previous controllability concepts. A special attention is paid to PDE with a skew-adjoint infinitesimal generator (the wave equation and the plate equations are concerned) for whose the controllability and stabilizability concepts considered here agree.

Let us end this introduction with a few notations. Let $P(D)$ denote a differential operator, with $P \in \mathbb{C}[\tau, \xi_1, ..., \xi_n]$, and $D = (-i\partial_t, -i\partial_{x_1}, ..., -i\partial_{x_N})$ (where $\partial_t = \partial/\partial_t$, etc.). E.g. $P = -\tau^2 + |\xi|^2$ gives the wave operator $P(D) = \partial_t^2 - \Delta$. Let $\Omega \subset \mathbb{R}^N$ be a bounded (sufficiently smooth) open set, whose boundary $\partial\Omega$ is denoted by Γ. The Sobolev space $H^m(\Omega)$, for $m \geq 0$, is defined as the set of the functions $f \in L^2(\Omega)$ whose partial derivatives up to the order m belong to $L^2(\Omega)$. $H_0^1(\Omega)$ is the set of the functions in $H^1(\Omega)$ which vanish on Γ, and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Two types of control problems are considered here.

Internal control problem:

Given some open set $\omega \subset \Omega$ with a smooth boundary $\Gamma = \partial\Omega$, and a set of boundary conditions, merely written $B(D)z = 0$, we consider the control problem

$$\begin{aligned} P(D)z &= \chi_\omega f && t > 0,\ x \in \Omega, \\ B(D)z &= 0 && t > 0,\ x \in \Gamma, \\ z(0,x) &= z_0(x) && x \in \Omega. \end{aligned}$$

Here, $f = f(t,x)$ is the internal control, $z = z(t,x)$ is the unknown function. For the controllability problem, given z_0 and z_1 in some functional space H, we seek for a control $f \in L^2(0,T;U)$ (U being another functional space) such that the solution z of the system satisfies $z(T,x) = z_1(x)$.

Boundary control problem:

Given some open set $\gamma \subset \Gamma = \partial\Omega$, and two sets of boundary conditions $B_1(D)z = \chi_\gamma f$, $B_2(D)z = 0$, we consider the control problem

$$\begin{aligned} P(D)z &= 0 && t > 0,\ x \in \Omega, \\ B_1(D)z &= \chi_\gamma f && t > 0,\ x \in \Gamma, \\ B_2(D)z &= 0 && t > 0,\ x \in \Gamma, \\ z(0,x) &= z_0(x) && x \in \Omega. \end{aligned}$$

Here $f = f(t,x)$ is the boundary control. In general ω (resp. γ) is a strict subset of Ω (resp. Γ)

Using a domain extension together with classical trace results, one may often derive boundary control results from internal control results.

Example 1.1. 1-D heat equation
Let us look at the classical control of the temperature of a rod. The temperature is kept constant at one extremity ($x = 0$), whereas the heat flux is controlled at the other extremity ($x = L$). The system reads

$$
\begin{aligned}
z_t - z_{xx} &= 0 \qquad t \in (0,T),\ x \in (0,L), && [1]\\
z_x(t,L) &= h(t) \qquad t \in (0,T), && [2]\\
z(t,0) &= 0 \qquad t \in (0,T), && [3]\\
z(0,x) &= z_0(x) \qquad x \in (0,L). && [4]
\end{aligned}
$$

This is a boundary control problem in the domain $\Omega = (0,L)$. We may also consider the following internal control problem on the domain $\Omega' = (0,L')$ with $L' > L$

$$
\begin{aligned}
y_t - y_{xx} &= \chi_{(L,L')} f \qquad t \in (0,T),\ x \in (0,L'), && [5]\\
y(t,0) = y(t,L') &= 0 \qquad t \in (0,T), && [6]\\
y(0,x) &= z_0(x) \qquad x \in (0,L'), && [7]
\end{aligned}
$$

z_0 being extended by 0 on (L,L'). Then, we shall see that we may design a control input $f \in L^2(0,T;L^2(L,L'))$ steering the solution y of the [5]-[7] to 0 in time T. Classical trace theorems (see e.g. (Evans, 1998, Chapter 7)) give that for $z_0 \in H_0^1(0,L)$ and f as above, $y \in L^2(0,T;H^2(0,L'))$, hence $y_x(.,L) \in L^2(0,T)$. Taking $h(t) = y_x(t,L)$ in [1]-[4] results again in $z(T,.) \equiv 0$.

Here we shall mainly be concerned with linear problems. (For the control of nonlinear systems in finite or infinite dimension, see (Coron, to appear) and the references therein.) We shall develop a general theory for control systems assuming the classical control form

$$\Sigma_{A,B} \qquad \dot{z} = Az + Bu. \qquad [8]$$

Here $A : D(A) \subset H \to H$ is an *unbounded* operator incorporating the differential operator together with the homogeneous boundary conditions, and $B : U \to H$ is a bounded operator (*i.e.* $B \in L(U,H)$) representing the effect of the control u, with H and U denoting appropriate (real or complex) Hilbert spaces of functions. We shall assume throughout that A generates a strongly continuous semigroup of operators on H, denoted by $(S(t))_{t\geq 0}$ (or $(e^{tA})_{t\geq 0}$ at some places). This setting is convenient to represent any (linear) internal control problem.

Example 1.2. The internal control of the heat equation may be written $\dot{z} = Az + Bu$, with $H = L^2(0,L')$, $U = L^2(L,L')$, $Az := z_{xx}$, $D(A) := H^2(0,L') \cap H_0^1(0,L')$, $Bu = \chi_{(L,L')} u$ (*i.e.* Bu is the extension of u by 0 outside (L,L')).

Remark 1.3. 1) Given some function $a \in L^\infty(\Omega)$ (often compactly supported in Ω), we may as well consider the bounded operator $Bu = au$, where e.g. $U = L^2(\Omega)$. If $a \in C_0^\infty(\omega)$, this "forces" the control Bu to vanish on $\partial\omega$.

2) Boundary control problems may also be put in the form [8], provided that B is viewed as an *bounded* operator from U to a larger space $\tilde{H}$ (*i.e.* $H \subset \tilde{H}$). (See (Weiss, 1989), (Coron, to appear).) Examples of boundary control problems will be given here without the aid of this abstract framework.

2. Controllability and observability

2.1. *Concepts of Controllability*

For given $z_0 \in H$, $u \in L^2(0,T;U)$, we consider the solution $z : [0,T] \to H$ of the Cauchy problem

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0. \end{cases} \qquad [9]$$

Recall that for any $z_0 \in D(A)$ and any $u \in W^{1,1}(0,T;U)$, the Cauchy problem [9] admits a unique classical solution $z \in C([0,T];D(A)) \cap C^1([0,T];H)$ given by Duhamel formula

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,ds \quad \forall t \in [0,T].$$

For $z_0 \in H$ and $u \in L^1(0,T;U)$, the above formula is still meaningful and defines the *mild solution* of [9].

Definition 2.1. – $\Sigma_{A,B}$ is *exactly controllable in time* T if for any $z_0, z_T \in H$, there exists $u \in L^2(0,T;U)$ such that the solution z of [9] fulfills $z(T) = z_T$;

– $\Sigma_{A,B}$ is *null controllable in time* T if for any $z_0 \in H$, there exists $u \in L^2(0,T;U)$ such that the solution z of [9] fulfills $z(T) = 0$;

– $\Sigma_{A,B}$ is *approximatively controllable in time* T if for any $z_0, z_T \in H$ and any $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the solution z of [9] fulfills $||z(T) - z_T||_H < \varepsilon$.

If the exact controllability holds for any time T, we say that $\Sigma_{A,B}$ is *exactly controllable*.

Let us introduce the operator $L_T : L^2(0,T;U) \to H$ defined by

$$L_T u = \int_0^T S(T-s)Bu(s)\,ds$$

Then

$$\text{exact controllability in time } T \iff \text{Im } L_T = H \qquad [10]$$

$$\text{zero controllability in time } T \iff S(T)H \subset \text{Im } L_T \qquad [11]$$

$$\text{approximate controllability in time } T \iff \overline{\text{Im } L_T} = H \qquad [12]$$

In finite dimension (*i.e.*, $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times M}$), the three concepts are equivalent, and equivalent to a purely algebraic condition, the famous Kalman rank condition: rank$(B, AB, ..., A^{N-1}B) = N$. As a consequence, the time T plays no role.

The situation is more tricky for PDE:

- there is no algebraic test for the controllability;
- the control time plays a role for hyperbolic PDE;
- the converses of

exact controllability $\Rightarrow$ zero controllability [13]

exact controllability $\Rightarrow$ approximate controllability [14]

are not true in general.

However, the following result holds true.

Proposition 2.2. *If A generates a continuous* group $(S(t))_{t\in\mathbb{R}}$, *then*

$$\text{exact controllability in time } T \iff \text{zero controllability in time } T$$

Proof of $\Leftarrow$: We may assume that $z_0 = 0$ (a control driving 0 to $z_T - S(T)z_0$ also drives z_0 to z_T). For any $z_T \in H$, there exists a control u steering $S(-T)z_T$ to 0, *i.e.*

$$0 = S(T)(S(-T)z_T) + \int_0^T S(T-s)Bu(s)\,ds$$

which yields, by the semigroup property,

$$z_T = \int_0^T S(T-s)B(-u(s))ds.$$

It follows that $\Sigma_{A,B}$ is exactly controllable in time T. ■

Proposition 2.2 applies to the wave equation and to the plate equation without damping, *not* to the heat equation.

2.2. *Adjoint operators*

The adjoint of the bounded operator $B \in L(U,H)$ is the operator $B^* \in L(H,U)$, defined by $(B^*z, u)_U = (z, Bu)_H$ for all $z \in H$, $u \in U$.

Example 2.3.

1) If $Bu = \chi_\omega u$, with $\omega \subset \Omega$ and $U = L^2(\omega)$, $H = L^2(\Omega)$, then $B^*z = z_{|_\omega}$.

2) If $Bu = au$, with $a \in L^\infty(\Omega)$ and $H = U = L^2(\Omega)$, then $B^* = B$.

The adjoint of the (unbounded) operator A is the unbounded operator A^* with domain

$$D(A^*) = \{z \in H |\, \exists C \in \mathbb{R}^+, \quad |(Ay, z)_H| \leq C||y||_H \quad \forall y \in D(A)\}$$

and defined by

$$(Ay, z)_H = (y, A^* z)_H \quad \forall y \in D(A),\ \forall z \in D(A^*).$$

A^* generates also a continuous semigroup $(e^{tA^*})_{t\geq 0}$ fulfilling $e^{tA^*} = S^*(t) \quad \forall t \geq 0$. If $A^* = A$ (resp. $A^* = -A$) the operator A is said to be self-adjoint (resp. skew-adjoint). Recall that a skew-adjoint operator generates a continuous group of isometries.

2.3. *Controllability tests*

Theorem 2.4. *The system* $\Sigma_{A,B}$ *is* exactly controllable in time $T > 0$ *if and only if there exists a constant* $c > 0$ *such that*

$$\int_0^T ||B^* S^*(t) y_0||_U^2 \, dt \geq c||y_0||_H^2 \quad \forall y_0 \in H. \qquad [15]$$

[15] is called an *observability inequality*. Such inequality means that the map $y_0 \mapsto B^* S^*(\cdot) y_0$ is boundedly invertible; *i.e.*, it is possible to recover a complete information about the initial state y_0 from a measure on $[0, T]$ of the output $B^*[S^*(t)y_0]$ (observability property).

Theorem 2.5. *The system* $\Sigma_{A,B}$ *is* null controllable in time $T > 0$ *if and only if there exists a constant* $c > 0$ *such that*

$$\int_0^T ||B^* S^*(t) y_0||_U^2 \, dt \geq c||S^*(T) y_0||_H^2 \quad \forall y_0 \in H. \qquad [16]$$

[16] is a weak observability inequality: only $S^*(T)y_0$ may be recovered, not y_0.

Theorem 2.6. *The system* $\Sigma_{A,B}$ *is* approximately controllable in time $T > 0$ *if and only if the only* $y_0 \in H$ *for which*

$$B^* S^*(t) y_0 = 0 \quad \forall t \in [0, T] \qquad [17]$$

is $y_0 = 0$.

[17] is called a *Unique Continuation Property (UCP)*.

We readily infer from Theorem 2.5 and Theorem 2.6 that a system $\Sigma_{A,B}$ which is null-controllable for any $T > 0$ is also approximately controllable for any $T > 0$.

The proofs of Theorems 2.4, 2.5 and 2.6 are classical, and they may be found e.g. in (Zabczyk, 1992) and (Coron, to appear). We give however in the next section a proof of Theorem 2.4 which is based on the Hilbert Uniqueness Method (HUM) due to J.-L. Lions (see (Lions, 1988)). The proof is quite short and it provides an explicit control input u.

2.4. *Proof of the Exact Controllability test by HUM*

We associate to the boundary-initial value problem

$$\Sigma \qquad \begin{cases} \dot{z} & = Az + Bu, \\ z(0) & = 0. \end{cases}$$

its adjoint problem, obtained by taking the distributional adjoint of the operator $\partial_t - A$, namely $-\partial_t - A^*$:

$$\Sigma^* \qquad \begin{cases} \dot{y} & = -A^* y \\ y(T) & = y_T. \end{cases}$$

Note that Σ^* is without control and backwards in time. For any $y_T \in H$, the solution y of Σ^* reads $y(t) = S^*(T-t)y_T$.

KEY IDENTITY. $(z(T), y_T)_H = \int_0^T (u, B^* y)_U \, dt$.

Proof. Assume first that $u \in C^1([0,T], U)$ and $y_T \in D(A^*)$, so that both z and y are in $C^1([0,T], H)$. Therefore, integrating by parts, we have

$$\begin{aligned} 0 &= \int_0^T (\dot{z} - Az - Bu, y)_H \, dt \\ &= -\int_0^T (z, \dot{y} + A^* y)_H \, dt + [(z,y)_H]_0^T - \int_0^T (u, B^* y)_U \, dt \\ &= (z(T), y_T)_H - \int_0^T (u, B^* y)_U \, dt. \end{aligned}$$

The result extends to arbitrary $u \in L^2(0,T;U)$ and $y_T \in H$ by a density argument.

$\Rightarrow$ Assume that $\Sigma_{A,B}$ is EC, that is, the map $L_T : u \mapsto z(T)$ is onto. Let $\Lambda : H \to L^2(0,T;U)$ denote the inverse of the restriction of L_T to (Ker $L_T)^\perp$, so that

$u = \Lambda(z_T)$ steers 0 to z_T for Σ. Pick any $y_T \in H$ and take $z_T = y_T$, $u = \Lambda(z_T)$. Then

$$\begin{aligned} ||y_T||^2 &= (z(T), y_T)_H \\ &= \int_0^T (\Lambda(z_T), B^* y)_U \, dt \\ &\leq ||\Lambda|| \cdot ||y_T||_H \left(\int_0^T ||B^* y||_U^2 \, dt \right)^{\frac{1}{2}} . \end{aligned}$$

$\Leftarrow$ Conversely, assume that $\int_0^T ||B^*y||_U^2 \, dt \geq c||y_T||_H^2$. For any $y_T \in H$, we set $u(t) := B^*y(t)$ (where y solves Σ^*), and consider the solution z of Σ corresponding to that control u. This defines a bounded operator $\Gamma : y_T \in H \mapsto z(T) = L_T(B^*y(\cdot)) \in H$. Since

$$(\Gamma y_T, y_T)_H = \int_0^T ||B^*y(t)||_U^2 \, dt \geq c||y_T||_H^2,$$

it follows from Lax-Milgram theorem that Γ is invertible. Thus, for any $z_T \in H$, the state $y_T = \Gamma^{-1}(z_T)$ is such that the control $u(t) = B^*y(t)$ steers 0 to z_T. ■

Remark 2.7. 1) Notice that the evolution equation in the *adjoint problem* $\dot{y} = -A^*y$ differs from the one for the *adjoint operator* $\dot{y} = A^*y$ by a sign minus. Solutions of the second one give solutions of the first one just by changing t into $T - t$ inside.

2) HUM provides a bounded operator $\Lambda : z_T \mapsto u$ giving the control.

3) In general we don't need to explicit B and B^*. The important ingredients in HUM are the key identity and the observability inequality.

2.5. *Frequency domain tests*

A popular frequency domain test in finite dimension is given by *Hautus lemma*: a control system $\dot{z} = Az + Bu$, with $A \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{N\times M}$, is controllable if and only if

$$\text{rank}[A - \lambda I|B] = n \qquad \forall \lambda \in \mathbb{C}. \qquad [18]$$

The algebraic condition [18] may also be written in the following analytic form: there exists a constant $\delta > 0$ such that

$$||(A^* - \lambda I)z||^2 + ||B^*z||^2 \geq \delta ||z||^2 \qquad \forall z \in \mathbb{C}^N, \ \forall \lambda \in \mathbb{C}, \qquad [19]$$

where A^* (resp. B^*) denotes the conjugate of the transpose of the matrix A (resp. B). Indeed,

$$\text{rank}[A - \lambda I|B] = \text{rank} \begin{bmatrix} A^* - \overline{\lambda} I \\ B^* \end{bmatrix}$$

and the linear map $z \in \mathbb{C}^N \mapsto \begin{bmatrix} A^* - \overline{\lambda} I \\ B^* \end{bmatrix} z \in \mathbb{C}^{N+M}$ has full rank if and only if $||(A^* - \overline{\lambda} I)z||^2 + ||B^* z||^2 \geq \delta ||z||^2$ for some constant $\delta > 0$.

It turns out that Hautus criterion in the form [19] is still valid in infinite dimension, as it has been observed in (Liu, 1997) and in (Burq *et al.*, 2004).

Theorem 2.8. *Let $\Sigma_{A,B}$ be as in* [8]. *Then the system $\Sigma_{A,B}$ is exactly controllable in some time $T > 0$ if and only if there exists some constant $\delta > 0$ such that*

$$||(A^* - \lambda I)z||^2 + ||B^* z||^2 \geq \delta ||z||^2 \qquad \forall z \in D(A),\ \forall \lambda \in \mathbb{C}. \qquad [20]$$

Remark 2.9. 1) An obvious advantage of [20] is that we don't have to compute the solutions associated with the adjoint operator A^*. In a certain sense, the time t has been replaced by the frequency λ.

2) Theorem 2.8 is actually valid for boundary controls (B not bounded), see (Burq *et al.*, 2004).

3) Since the operator $A^* - \lambda I$ is boundedly invertible when λ is not in the spectrum $\sigma(A^*)$ of A^*, we may assume in [20] that $\lambda \in \sigma(A^*)$.

From now on we assume that A is skew-adjoint operator ($A^* = -A$) with a compact resolvent. Then the spectrum of A is given by $\sigma(A) = \{i\mu_n |\ n \in \mathbb{N}\}$ for a sequence (μ_n) of real numbers fulfilling $|\mu_n| \nearrow +\infty$, and there exists an orthonormal basis $(e_n)_{n\geq 0}$ of H constituted of eigenfunctions of A: $Ae_n = i\mu_n e_n\ \forall n \geq 0$. Notice that [20] may be written

$$||(A - i\mu)z||^2 + ||B^* z||^2 \geq \delta ||z||^2 \qquad \forall z \in D(A),\ \forall \mu \in \mathbb{R}, \qquad [21]$$

or

$$||(A - i\mu_n)z||^2 + ||B^* z||^2 \geq \delta ||z||^2 \qquad \forall z \in D(A),\ \forall n \geq 0. \qquad [22]$$

Since $(A - i\mu_n)e_n = 0$, we obtain at once the following necessary condition for the exact controllability of $\Sigma_{A,B}$: there exists some number $\delta > 0$ such that

$$||B^* e_n||^2 \geq \delta ||e_n||^2 \qquad \forall n \geq 0. \qquad [23]$$

It could be tempting to conjecture that the condition [23] is also sufficient for the exact controllability of $\Sigma_{A,B}$, for $||(A - i\mu_n)z|| > 0$ when z is not an eigenvector of A associated with the eigenvalue $i\mu_n$. The following example shows that it is not true.

Example 2.10. Let H be any complex separable Hilbert space endowed with an orthonormal basis $(e_n)_{n\geq 0}$, and let $A : D(A) \subset H \to H$ be the operator defined by $A(\sum_{n\geq 0} c_n e_n) = \sum_{n\geq 0} i\mu_n c_n e_n$, with $D(A) = \{z = \sum_{n\geq 0} c_n e_n |\ \sum_{n\geq 0} |\mu_n c_n|^2 < \infty\}$ and $(\mu_n)_{n\geq 0}$ being (for the moment) any sequence of real numbers such that $\mu_n \nearrow \infty$. Then A is skew-adjoint, and $(i\mu_n)_{n\geq 0}$ is the sequence of its eigenvalues. Consider now the operator $B \in L(H, H)$ whose adjoint B^* is defined by $B^*(\sum_{n\geq 0} c_n e_n) = \sum_{p\geq 0}(c_{2p} - c_{2p+1})e_{2p}$. Notice that [23] is satisfied, since $||B^* e_n|| = ||e_n|| = 1$ for each n. Let $z_p := e_{2p} + e_{2p+1}$. Then $B^*(z_p) = 0$, $||z_p||^2 = 2$ for any p, and

$$\begin{aligned} ||(A - i\mu)z_p||^2 &= ||(i\mu_{2p} - i\mu)e_{2p} + (i\mu_{2p+1} - i\mu)e_{2p+1}||^2 \\ &= (\mu_{2p} - \mu)^2 + (\mu_{2p+1} - \mu)^2. \end{aligned}$$

Assume now that for a sequence $p_k \nearrow \infty$, $\mu_{2p_k} - \mu_{2p_k+1} \to 0$. (This is e.g. the case for the wave equation with Dirichlet boundary condition on a rectangle $\Omega = (0,a) \times (0,b)$ with $b/a \notin \mathbb{Q}$, see (Chen *et al.*, 1991).) Then

$$||(A - i\mu_{2p_k})z_{p_k}||^2 = (\mu_{2p_k+1} - \mu_{2p_k})^2 \to 0 \quad as \quad k \to \infty,$$

hence [22] does not hold.

Thus [23] is a necessary, but not sufficient, condition for the exact controllability in some time T of $\Sigma_{A,B}$. It is however possible to obtain a necessary and sufficient condition of exact controllability by considering, instead of a single eigenfunction as in [23], a sum of eigenfunctions associated with eigenvalues living in intervals of fixed length. To state the criterion let us first introduce some notation. For any $\mu \in \mathbb{R}$ and any $\varepsilon > 0$ let $J_\varepsilon(\mu) = \{n \in \mathbb{N} | \ |\mu_n - \mu| < \varepsilon\}$. Then the following frequency domain test holds.

Theorem 2.11. *(Chen* et al.*, 1991), (Ramdani* et al.*, 2005)*
Assume that A is a skew-adjoint operator with compact resolvent and spectrum $\sigma(A) = \{i\mu_n | \ n \in \mathbb{N}\}$. Then the system $\Sigma_{A,B}$ is exactly controllable in some time $T > 0$ if and only if there exist two numbers $\varepsilon > 0$, $\delta > 0$ such that for any $\mu \in \mathbb{R}$ and any $z = \sum_{n \in J_\varepsilon(\mu)} c_n e_n$ we have

$$||B^* z|| \geq \delta ||z||.$$

As above, we obtain an equivalent condition when μ is assumed to range over the set $\{\mu_n\}$. On the other hand, if there is a *spectral gap* (*i.e.* $\mu_{n+1} - \mu_n \geq \varepsilon_0 \ \forall n$), then $J_\varepsilon(\mu_n) = \{\mu_n\}$ for $\varepsilon < \varepsilon_0$. In that case, [23] turns out to be a necessary and sufficient condition for the exact controllability in some time T of $\Sigma_{A,B}$.

To conclude this section, we point out that the above frequency domain tests are useful when the eigenfunctions are known and sufficiently simple. This is the case when Ω is e.g. a rectangle or a disk (see (Chen *et al.*, 1991), (Ramdani *et al.*, 2005)).

2.6. *Approximate controllability*

In this section we investigate the approximate controllability of systems with an internal control, namely

$$\dot{z} = Az + \chi_\omega u \tag{[24]}$$

$$z(0) = z_0. \tag{[25]}$$

ω denotes an open subset of Ω, and both ω and Ω are assumed to be convex. According to Theorem 2.6, the approximate controllability of [24] is equivalent to the unique continuation property for the adjoint equation; *i.e.,* for any solution y of the adjoint equation $\dot{y} = -A^*y$

$$y \equiv 0 \quad \text{in } (0,T)\times\omega \;\Rightarrow\; y \equiv 0 \quad \text{in } (0,T)\times\Omega.$$

Such property is often obtained as a byproduct of some Carleman estimate when the operator A incorporates variable coefficients. The situation is simpler when the differential operator has only constant coefficients.

Let us introduce a few notations and definitions. If $x_1, ..., x_N$ denote independent variables in $\mathbb{R}^N$, let $\partial_j = \partial/\partial x_j$ and $D_j = -i\partial_j$ (hence $\partial_j = iD_j$) for $j = 1, ..., N$. For any polynomial function $P(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ (the coefficients a_α being analytic in x), we set $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$, where $D^\alpha = D_1^{\alpha_1}\cdots D_N^{\alpha_N}$. Notice that $\xi_1, ..., \xi_N$ are the Fourier variables associated with $x_1, ..., x_N$, respectively.

P is termed the *symbol* of the operator $P(x,D)$. The principal symbol of $P(x,D)$ is defined as $P_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.

A C^1 surface $S \subset \mathbb{R}^N$ with normal ξ at x is said to be *non-characteristic* at x for $P(x,D)$ if $P_m(x,\xi) \neq 0$.

A classical result giving the Unique Continuation Property (UCP) for a PDE with analytic coefficients is the Holmgren Uniqueness Theorem (see (Hörmander, 2003, Theorem 8.6.5)).

Theorem 2.12. *(Holmgren Uniqueness Theorem) If $u \in \mathcal{D}'(\Omega)$ is a solution of a differential equation $P(x,D)u = 0$ with analytic coefficients, then $u = 0$ in a neighborhood of a non-characteristic C^1 surface if this true on one side.*

The following corollary of Holmgren Uniqueness theorem (see (Hörmander, 2003, Theorem 8.6.8)) provides a useful characterization of the UCP for a differential operator with *constant* coefficients.

Theorem 2.13. *Let $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be a differential operator with constant coefficients, and let Ω_1 and Ω_2 be two convex open sets in $\mathbb{R}^N$ with $\Omega_1 \subset \Omega_2$. The following properties are equivalent.*

(1) (UCP) Every $u \in \mathcal{D}'(\Omega_2)$ *satisfying* $P(D)u = 0$ *in* Ω_2 *and vanishing in* Ω_1 *must also vanish on* Ω_2.
(2) Every characteristic hyperplan for $P(D)$ *which intersects* Ω_2 *has to intersect* Ω_1.

Example 2.14. 1) The adjoint equation of the heat equation $z_t - \Delta z = 0$ reads $-y_t - \Delta y = 0$, that is $P(D)y = 0$ with $P(\tau, \xi) = -i\tau + |\xi|^2$ ($\xi = (\xi_1, ..., \xi_N)$), and $D = (-i\partial_t, -i\partial_{x_1}, ..., -i\partial_{x_N})$. Therefore the principal symbol reads $P_2(\tau, \xi) = |\xi|^2$ and the characteristic hyperplans are those defined by the equations $t = C$. Clearly, a characteristic hyperplan intersects $\Omega_1 := (0, T) \times \omega$ if and only if it intersects $\Omega_2 := (0, T) \times \Omega$ (actually this occurs if and only if $0 < C < T$). It follows that the UCP holds, hence the heat equation is approximately controllable in arbitrary time $T > 0$.

2) The adjoint equation of the Schrödinger equation $iz_t + \Delta z = 0$ reads $-iy_t + \Delta y = 0$, that is $P(D)y = 0$ with $P(\tau, \xi) = \tau - |\xi|^2$. The principal symbol reads $P_2(\tau, \xi) = -|\xi|^2$ and the characteristic hyperplans are again those defined by the equations $t = C$. It follows that the Schrödinger equation is also approximately controllable in arbitrary time $T > 0$.

2.7. *A view of the results*

We report in Table 1 the main features of the controllability results for hyperbolic, dispersive, or parabolic equations with internal control.

Type of the PDE	Controllability	Control Region	Time Control
Hyperbolic (e.g. the wave eq.)	Exact	Depends on Ω	$T \geq T_0$
Dispersive (e.g. the Schrödinger eq.)	Exact	Depends on Ω	Any $T > 0$
Parabolic (e.g. the heat eq.)	Null	Any ω	Any $T > 0$

Table 1. *Controllability results for linear PDE*

3. Controllability of some PDE

3.1. *Wave equation*

Let us investigate the boundary controllability of the 1-D wave equation, namely

$$\begin{aligned} z_{tt} - z_{xx} &= 0 \qquad 0 < t < T,\ 0 < x < \pi && [26] \\ z(t,0) &= 0 \qquad 0 < t < T, && [27] \\ z(t,\pi) &= h(t) \quad 0 < t < T, && [28] \\ (z(0,.), z_t(0,.)) &= (z_0, z_1) && [29] \end{aligned}$$

where $h \in L^2(0,T)$ is the control input and (z_0, z_1) stand for the initial data. (The length of the spatial domain is π to simplify the computations with Fourier series.) We shall prove the

Theorem 3.1. *The system* [26]-[29] *is exactly controllable in* $L^2(0,\pi) \times H^{-1}(0,\pi)$ *in time* T *if and only if* $T \geq 2\pi$.

Sketch of the proof. The uncontrolled problem

$$\begin{aligned} z_{tt} - z_{xx} &= 0 \quad 0 < t < T,\ 0 < x < \pi, && [30] \\ z(t,0) = z(t,\pi) &= 0 \quad 0 < t < T, && [31] \\ (z(0,.), z_t(0,.)) &= (z_0, z_1) && [32] \end{aligned}$$

may be written $(z, z_t)_t = A(z, z_t)$ with $A(z_0, z_1) = (z_1, (z_0)_{xx})$ on the domain $D(A) = (H^2(0,\pi) \cap H_0^1(0,\pi)) \times H_0^1(0,\pi) \subset H := H_0^1(0,\pi) \times L^2(0,\pi)$. It may be seen that A is skew-adjoint, so that it generates a group of isometries on H. Actually, if $z_0 = \sum_{k\geq 1} a_k \sin(kx)$ and $z_1 = \sum_{k\geq 1} b_k \sin(kx)$, then the solution z of [30]-[32] reads

$$z(t,x) = \sum_{k\geq 1} [a_k \cos(kt) + \frac{b_k}{k} \sin(kt)] \sin(kx). \qquad [33]$$

More generally, one may prove that A generates a group of isometries in each space $H^\alpha \times H^{\alpha-1}$ ($\alpha \in \mathbb{R}$) where

$$H^\alpha := \{z(x) = \sum_{k\geq 1} a_k \sin(kx) |\ \sum_{k\geq 1} |k^\alpha a_k|^2 < \infty\}.$$

is endowed with its natural norm.

To identify the adjoint problem and B^*, we multiply [26] by y, integrate over $(0,T)\times(0,\pi)$ and perform integrations by parts, assuming that the functions y and z are sufficiently regular. We obtain

$$\begin{aligned} 0 &= \int_0^T\int_0^\pi (z_{tt}-z_{xx})y\,dxdt \\ &= \int_0^T\int_0^\pi (y_{tt}-y_{xx})z\,dxdt + [\int_0^\pi (z_t y - z y_t)\,dx]_0^T \\ &\qquad + \int_0^T [-z_x y + z y_x]_0^\pi dt. \end{aligned}$$

If y is a solution of [30]-[31], then we obtain

$$-\int_0^T h(t)y_x(t,\pi)\,dt = [\int_0^\pi (z_t y - z y_t)\,dx]_0^T. \qquad [34]$$

Assume in addition that $(y(T,.),y_t(T,.)) = (y_{0,T},y_{1,T}) \in H$, with $y_{0,T} = \sum_{k\ge1} c_k\sin(kx)$, $y_{1,T} = \sum_{k\ge1} d_k\sin(kx)$, then

$$y(t,x) = \sum_{k\ge1}\big[c_k\cos(k(t-T)) + \frac{d_k}{k}\sin(k(t-T))\big]\sin(kx),$$

hence

$$y_x(t,\pi) = \sum_{k\ge1}\big[kc_k\cos(k(t-T)) + d_k\sin(k(t-T))\big](-1)^k. \qquad [35]$$

If $T=2\pi$, then by the orthogonality properties of the functions $\sin(kt)$ and $\cos(kt)$ in $L^2(0,2\pi)$, we see that

$$\int_0^{2\pi} |y_x(.,\pi)|^2\,dt = \pi\sum_{k\ge1}(|kc_k|^2+|d_k|^2) < \infty. \qquad [36]$$

It follows that $y_x(.,\pi)\in L^2_{loc}(\mathbb{R})$, hence the integral term in [34] is meaningful.

To define a solution of [26]-[29], we use the *transposition method*. First, note that the dual of the space H is $H' = H^{-1}(0,\pi)\times L^2(0,\pi)$, and that the duality pairing $\langle .,.\rangle_{H',H}$ is defined by

$$\langle (z_1,z_0),(y_0,y_1)\rangle_{H',H} = \langle z_1,y_0\rangle_{H^{-1}(0,\pi),H_0^1(0,\pi)} + \langle z_0,y_1\rangle_{L^2(0,\pi),L^2(0,\pi)}.$$

Replacing T by any t, [34] may be rewritten as

$$\begin{aligned} &\langle(-z_t(t),z(t)),(y(t),y_t(t))\rangle_{H',H} \\ &\qquad = \int_0^t h(s)y_x(s,\pi)\,ds + \langle(-z_1,z_0),(y(0),y_t(0))\rangle_{H',H}. \end{aligned} \qquad [37]$$

As the map $(y_{0,T}, y_{1,T}) \in H \mapsto (y(t), y_t(t)) \in H$ is an isomorphism of Hilbert spaces, [37] defines $(z_t(t), z(t))$ in H' in a unique way, and $(z_t, z) \in C([0,T]; H')$. z is termed the solution by transposition of [26]-[29]. Assume now that $(z_0, z_1) = (0,0)$. Then

$$\langle(-z_t(T), z(T)), (y_{0,T}, y_{1,T})\rangle_{H',H} = \int_0^T h(t) y_x(t,\pi)\, dt.$$

Then it is easily seen that [26]-[29] is exactly controllable in $H^{-1}(0,\pi) \times L^2(0,\pi)$ if and only if the following observability inequality

$$\int_0^T |y_x(t,\pi)|^2 dt \geq C||(y_{0,T}, y_{1,T})||_H^2. \qquad [38]$$

holds. This is the case for $T = 2\pi$ by [36], hence also for any $T \geq 2\pi$. To see that [38] does not hold for $T = 2\pi - 2\delta < 2\pi$, it is sufficient to consider any solution y of [30]-[31] such that the functions $y(\frac{T}{2}, .)$ and $y_t(\frac{T}{2}, .)$ are supported in $(0, \delta)$ (not both null). (See Figure 1.) Then $y_x(., \pi) \equiv 0$ on $(0, T)$, whereas $y \not\equiv 0$. This completes the proof of Theorem 3.1. ■

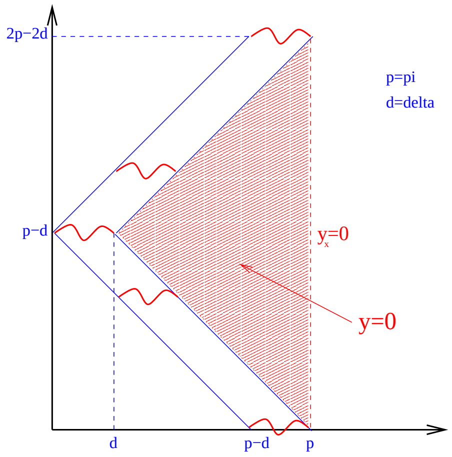

Figure 1. *Unobservable solution of the 1D wave equation*

Let us now turn to the control of the wave equation in several space dimensions. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary Γ of class C^∞. Let γ denote an open set in Γ, and consider the boundary control problem:

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } (0,T) \times \Omega \\ z = h\chi_\gamma & \text{on } (0,T) \times \Gamma \\ z(0,.) = z_0, \ z_t(0,.) = z_1. \end{cases} \qquad [39]$$

In [39], $z = z(t,x)$ is the state function, $h = h(t,x)$ is the control function, and χ_γ stands for the characteristic function of γ.

When $h \in L^2(0,T;L^2(\gamma))$ and $(z_0,z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, then a solution by transposition z of [39] may be defined in the class $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$.

By time-reversibility of the wave equation, the system [39] is exactly controllable in $H = L^2(\Omega) \times H^{-1}(\Omega)$ in time T if and only if any pair $(z_0,z_1) \in H$ may be steered to $(0,0)$ by a control $h \in L^2(0,T;L^2(\gamma))$.

An application of HUM shows that the exact controllability is equivalent to the following observability inequality

$$||(y_0,y_1)||^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq C \int_0^T \int_\gamma \left|\frac{\partial y}{\partial n}\right|^2 d\sigma dt \qquad [40]$$

for any solution y of the uncontrolled wave equation with $(y_0,y_1) \in H^1_0(\Omega) \times L^2(\Omega)$ as initial data. There is an important literature about that problem. We quote only a few references (see e.g. (Komornik, 1994), (Coron, to appear), or (Zuazua, 2005) for more references).

– (Ho, 1986) proved by the multiplier method that the observability inequality [40] holds true if γ takes the form

$$\gamma = \Gamma(x^0) := \{x \in \Gamma |\ (x - x^0) \cdot n(x) > 0\} \qquad [41]$$

for some $x^0 \in \mathbb{R}^N$, and if $T > 0$ is large enough. In [41], $n(x)$ denotes the outward normal vector to Ω at $x \in \Gamma$, and the point stands for the usual scalar product in $\mathbb{R}^N$. If e.g. Ω is the unit disk in $\mathbb{R}^2$, then the control region γ may be any arc of length larger that π (see Figure 2).

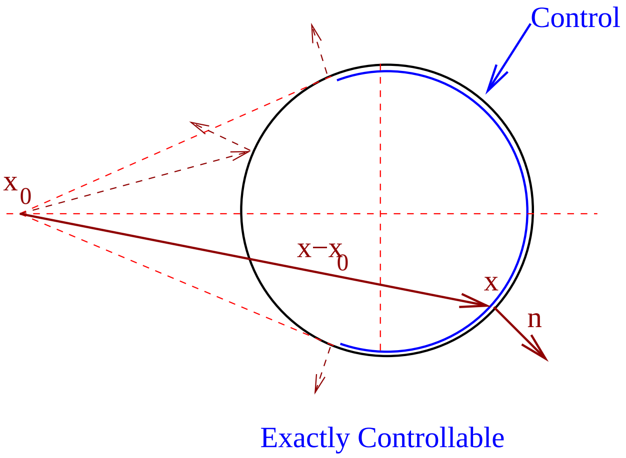

Figure 2. *The set $\Gamma(x^0)$ for a disk*

– With the same $\gamma = \Gamma(x^0)$, (Lions, 1988) proved that [40] holds provided that

$$T > T(x^0) := 2||x - x^0||_{L^\infty(\Omega)}. \qquad [42]$$

This estimation of the control time is sharp (only) in dimension 1. (In the domain $\Omega = (0, \pi)$ with $\gamma = \{\pi\}$, take $x^0 = -\varepsilon < 0$ with $\varepsilon > 0$ small. Then $T(x^0) = 2(\pi + \varepsilon)$.)

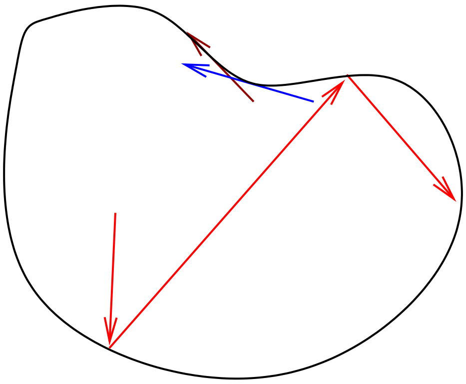

Figure 3. *Laws of geometric optics*

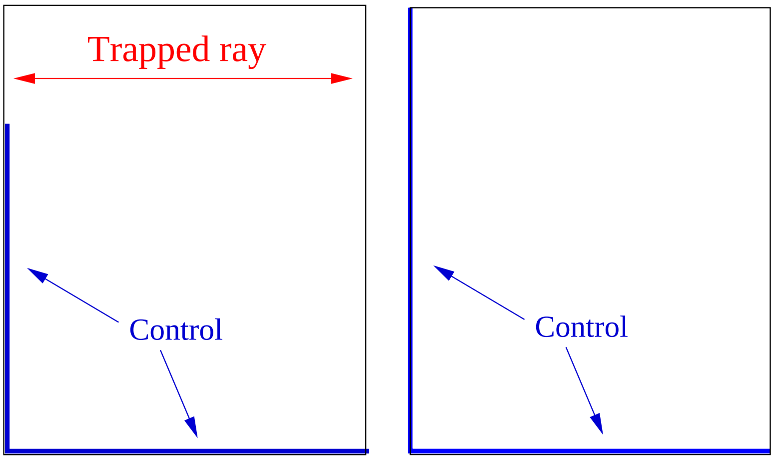

Figure 4. *Boundary controllability of the wave equation in a rectangle: No Exact Controllability at the left, Exact Controllability at the right*

$-$ (Bardos *et al.*, 1992) proved that the observability inequality [40] holds if and only if the pair (γ, T) satisfies the so-called *Geometric Control Condition* (GCC): each ray which propagates in Ω and is reflected on Γ according to the laws of geometric optics has to meet γ in time less than T (see Figure 3).

Example 3.2. 1) In 1D, for $\Omega = (0, L)$ the control time has to be at least $2L$ with one control ($\gamma = \{\pi\}$), and at least L with two controls ($\gamma = \{0, \pi\}$).

2) In 2D, when Ω is a rectangle, the control region needs to contain a point of each line parallel to one side. Otherwise, there may exist trapped rays that support solutions that are not observable (see Figure 4).

3) In 2D, when Ω is a ball, the control region γ needs to contain a point of each diameter, without being necessarily connected. The time control T has to be larger than $2d$, where d denotes the diameter of the ball (see Figure 5).

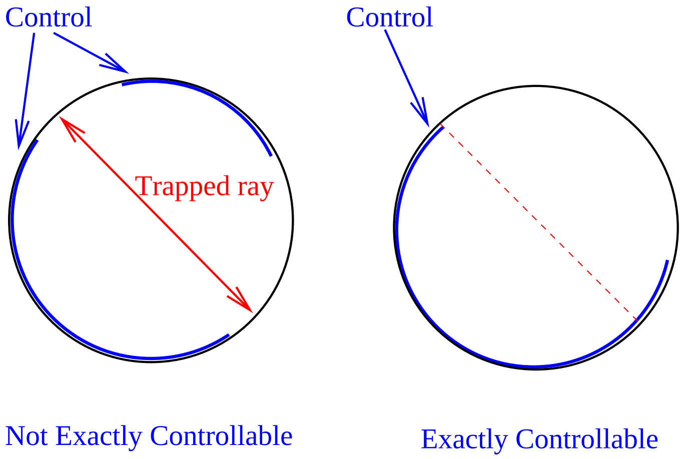

Figure 5. *Boundary controllability of the wave equation in a disk*

– The result in (Bardos *et al.*, 1992) was proved by means of microlocal analysis and only C^∞ domains were concerned. The proof has been simplified in (Burq *et al.*, 1997) by using the microlocal defect measures introduced in (Gérard, 1991). On the other hand, it has been proved by Burq that the GCC implies the EC when the domain Ω is merely of class C^3 (see (Burq, 1997)), or piecewise analytic with exterior corners (see (Burq, 1998)).

3.2. *Plate equation*

We first investigate the boundary controllability of the beam equation

$$\begin{array}{rcll} z_{tt}+z_{xxxx} & = & 0 & 0<t<T,\ 0<x<\pi \qquad [43]\\ z(t,0)=z(t,\pi)=z_{xx}(t,0) & = & 0 & 0<t<T, \qquad [44]\\ z_{xx}(t,\pi) & = & h(t) & 0<t<T, \qquad [45]\\ (z(0,.),z_t(0,.)) & = & (z_0,z_1) & \qquad [46]\end{array}$$

where $h\in L^2(0,T)$ stands for the control input. We shall prove the

Theorem 3.3. *The system* [43]-[46] *is exactly controllable in* $H^1_0(0,\pi)\times H^{-1}(0,\pi)$ *in any time* $T>0$.

Sketch of the proof. The uncontrolled problem

$$
\begin{aligned}
z_{tt} + z_{xxxx} &= 0 \quad 0 < t < T,\ 0 < x < \pi && [47]\\
z(t,0) = z(t,\pi) = z_{xx}(t,0) = z_{xx}(t,\pi) &= 0, \quad 0 < t < T && [48]\\
(z(0,.), z_t(0,.)) &= (z_0, z_1) && [49]
\end{aligned}
$$

may be written $(z, z_t)_t = A(z, z_t)$ with $A(z_0, z_1) = (z_1, -(z_0)_{xxxx})$ on the domain $D(A) = \{z \in H^3(0,1) \cap H_0^1(0,\pi) | z_{xx} \in H_0^1(0,\pi)\} \times H_0^1(0,\pi) \subset H := H_0^1(0,\pi) \times H^{-1}(0,\pi)$. It may be seen that A is skew-adjoint, so that it generates a group of isometries on H. Actually, if $z_0 = \sum_{k\geq 1} a_k \sin(kx)$ and $z_1 = \sum_{k \geq 1} b_k \sin(kx)$, then the solution z of [47]-[49] reads

$$
z(t,x) = \sum_{k\geq 1} [a_k \cos(k^2 t) + \frac{b_k}{k^2} \sin(k^2 t)] \sin(kx). \qquad [50]
$$

Once again, one may prove that A generates a group of isometries in each space $H^\alpha \times H^{\alpha-2}$ ($\alpha \in \mathbb{R}$).

To identify the adjoint problem and B^*, we multiply [43] by y, integrate over $(0,T) \times (0,\pi)$ and perform integrations by parts, assuming that the functions y and z are sufficiently regular. We obtain

$$
\begin{aligned}
0 &= \int_0^T \int_0^\pi (z_{tt} + z_{xxxx}) y \, dxdt \\
&= \int_0^T \int_0^\pi (y_{tt} + y_{xxxx}) z \, dxdt + [\int_0^\pi (z_t y - z y_t) \, dx]_0^T \\
&\qquad + \int_0^T [z_{xxx} y - z_{xx} y_x + z_x y_{xx} - z y_{xxx}]_0^\pi dt.
\end{aligned}
$$

If y is a solution of [47]-[48], then we obtain

$$
\int_0^T h(t) y_x(t,\pi) \, dt = [\int_0^\pi (z_t y - z y_t) \, dx]_0^T. \qquad [51]
$$

Assume in addition that $(y(T,.), y_t(T,.)) = (y_{0,T}, y_{1,T}) \in H$, with $y_{0,T} = \sum_{k\geq 1} c_k \sin(kx)$, $y_{1,T} = \sum_{k\geq 1} d_k \sin(kx)$, then

$$
y(t,x) = \sum_{k\geq 1} \left[c_k \cos(k^2(t-T)) + \frac{d_k}{k^2} \sin(k^2(t-T))\right] \sin(kx)
$$

hence

$$
y_x(t,\pi) = \sum_{k\geq 1} \left[k c_k \cos(k^2(t-T)) + \frac{d_k}{k} \sin(k^2(t-T))\right] (-1)^k. \qquad [52]
$$

If $T = 2\pi$, then by the orthogonality properties of the functions $\sin(k^2t)$ and $\cos(k^2t)$ in $L^2(0, 2\pi)$, we see that

$$\int_0^{2\pi} |y_x(t, \pi)|^2 \, dt = \sum_{k\geq 1}(|kc_k|^2 + |\frac{d_k}{k}|^2) < \infty. \qquad [53]$$

It follows that $y_x(., \pi) \in L^2_{loc}(\mathbb{R})$, hence the integral term in [51] is meaningful.

A solution of [43]-[46] is defined by the transposition method. First, note that the dual of the space H is $H' = H^{-1}(0, \pi) \times H^1_0(0, \pi)$ (as the Hilbert space $H^1_0(0, \pi)$ is equal to its bidual), and that the duality pairing $\langle ., .\rangle_{H',H}$ is defined by

$$\langle (z_1, z_0), (y_0, y_1)\rangle_{H',H} = \langle z_1, y_0\rangle_{H^{-1}(0,\pi),H^1_0(0,\pi)} + \langle y_1, z_0\rangle_{H^{-1}(0,\pi),H^1_0(0,\pi)}.$$

Replacing T by any t, [51] may be rewritten as

$$\langle (z_t(t), -z(t)), (y(t), y_t(t))\rangle_{H',H} = \int_0^t h(s)y_x(s, \pi) \, ds + \langle (z_1, -z_0), (y(0), y_t(0))\rangle_{H',H}. \qquad [54]$$

As the map $(y_{0,T}, y_{1,T}) \in H \mapsto (y(t), y_t(t)) \in H$ is an isomorphism of Hilbert spaces for any $t \in \mathbb{R}$, [54] defines $(z_t(t), z(t))$ in H' in a unique way, and $(z_t, z) \in C([0, T]; H')$. z is called the *solution by transposition* of [43]-[46].

Assume now that $(z_0, z_1) = (0, 0)$. Then

$$\langle (z_t(T), -z(T)), (y_{0,T}, y_{1,T})\rangle_{H',H} = \int_0^T h(t)y_x(t, \pi) \, dt.$$

Then it is easily seen that [43]-[46] is exactly controllable in $H^1_0(0, \pi) \times H^{-1}(0, \pi)$ in time T if and only if the following observability inequality holds

$$\int_0^T |y_x(t, \pi)|^2 dt \geq C||(y_{0,T}, y_{1,T})||^2_H. \qquad [55]$$

This is the case for $T = 2\pi$ by [53], hence also for any $T \geq 2\pi$. To see that [55] holds for *any* $T > 0$, we use a result due to Ball-Slemrod (Ball *et al.*, 1979), which extends slightly a famous result in nonharmonic analysis due to Ingham (Ingham, 1936).

Theorem 3.4. *Let* a_k, $-\infty < k < \infty$, *be a sequence of complex numbers such that* $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$, *and let* μ_k, $-\infty < k < \infty$ *be a sequence a real numbers fulfilling the following properties*

$$\mu_{k+1} - \mu_k \geq \gamma > 0 \qquad \text{for } |k| \geq K, \qquad [56]$$

$$\mu_{k+1} - \mu_k \geq \rho > 0 \qquad \forall k \in \mathbb{Z} \qquad [57]$$

where K is some integer. Then if $T > (2\pi)/\gamma$, one may find two positive constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \le \int_0^T \big|\sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\big|^2 dt \le C_2 \sum_{k\in\mathbb{Z}} |a_k|^2.$$

Some comments are in order.

1) For $\mu_k = k$ and $T = 2\pi$ (harmonic setting) we have by Parseval theorem

$$\int_0^{2\pi} \big|\sum_{k\in\mathbb{Z}} a_k e^{ikt}\big|^2 dt = 2\pi \sum_{k\in\mathbb{Z}} |a_k|^2.$$

Theorem 3.4 holds in a *nonharmonic setting*, that is, when the functions $e^{i\mu_k t}$ are not pairwise orthogonal in $L^2(0,T)$.

2) When $\rho = \gamma$, Theorem 3.4 is just Ingham's lemma. The infimum of the $\mu_{k+1} - \mu_k$ is called the *spectral gap*. The larger spectral gap, the smaller observability time T. With Ball-Slemrod's improvement of Ingham's lemma, only the *asymptotic spectral gap* γ has to be considered. When $\lim_{|k|\to\infty} \mu_{k+1} - \mu_k = \infty$, then γ may be taken as large as wanted, and so T as small as wanted.

To apply Theorem 3.4, we set $\mu_k = \text{sgn }(k)|k|^2$ for any $k \in \mathbb{Z}$, and

$$a_k = \begin{cases} \frac{1}{2}(kc_k - \frac{d_k}{ik})(-1)^k & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ \frac{1}{2}(|k|c_{|k|} + \frac{d_{|k|}}{i|k|})(-1)^k & \text{if } k < 0. \end{cases}$$

A straightforward computation gives that

$$\int_0^T |y_x(t,\pi)|^2 dt = \int_0^T \left|\sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\right|^2 dt.$$

For any given $T > 0$, we pick γ and K so that $\mu_{k+1} - \mu_k \ge \gamma > \frac{2\pi}{T}$ whenever $|k| \ge K$. (This is possible, as for $k > 0$ $\mu_{k+1} - \mu_k = 2k+1$, and for $k < 0$ $\mu_{k+1} - \mu_k = -(2k+1)$, so $\lim_{|k|\to\infty} \mu_{k+1} - \mu_k = \infty$.) Thus $T > 2\pi/\gamma$ and $\rho = 1$. An application of Theorem 3.4 gives then

$$\begin{aligned} \int_0^T |y_x(t,\pi)|^2 dt &\ge C_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \\ &\ge \frac{C_1}{2} \sum_{k\ge 1} \left(|kc_k|^2 + |\frac{d_k}{k}|^2 \right) \\ &\ge \frac{C_1}{2} ||(y_{0,T}, y_{1,T})||_H^2. \end{aligned}$$

This completes the proof of Theorem 3.3. ■

Let us now pass to the control of the plate equation ($z_{tt} + \Delta^2 z = 0$) in several space dimensions. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary Γ of class C^∞. Let γ denote an open set in Γ, and let consider the boundary control problem:

$$\left\{\begin{array}{ll} z_{tt} + \Delta^2 z = 0 & \text{in } (0,T)\times\Omega \\ z = 0 & \text{on } (0,T)\times\Gamma \\ \Delta z = \chi_\gamma h & \text{on } (0,T)\times\Gamma \\ z(0,.) = z_0,\ z_t(0,.) = z_1. & \end{array}\right. \qquad [58]$$

In [58], $z = z(t,x)$ is the state function, and $h = h(t,x)$ is the control input.

When $h \in L^2(0,T;L^2(\gamma))$ and $(z_0,z_1) \in H := H_0^1(\Omega)\times H^{-1}(\Omega)$, then a solution by transposition z of [58] may be defined in the class $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$.

An application of HUM shows that the exact controllability in H in time T is equivalent to the following observability inequality

$$||(y_0,y_1)||^2_{H_0^1(\Omega)\times H^{-1}(\Omega)} \le C\int_0^T\int_\gamma \left|\frac{\partial y}{\partial n}\right|^2 d\sigma dt \qquad [59]$$

for any solution y of the uncontrolled plate equation with $(y_0,y_1)\in H$ as initial data.

Besides this boundary control problem, the internal controllability of the plate equation has attracted much attention in the 90's. The problem in question was the exact controllability in $\tilde{H} = (H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$ of the system

$$\left\{\begin{array}{ll} z_{tt} + \Delta^2 z = \chi_\omega f & \text{in } (0,T)\times\Omega \\ z = 0 & \text{on } (0,T)\times\Gamma \\ \Delta z = 0 & \text{on } (0,T)\times\Gamma \\ z(0,.) = z_0,\ z_t(0,.) = z_1. & \end{array}\right. \qquad [60]$$

Here, $\omega \subset \Omega$ is an open set and $f \in L^2(0,T;L^2(\omega))$ is the control function.

$-$ As the plate operator is merely the composition of the Schrödinger operator with its conjugate: $z_{tt} + \Delta^2 z = (-i\partial_t + \Delta)(i\partial_t + \Delta)z$, the control properties of the plate equation and of Schrödinger equation are almost the same. The exact controllability in $H^{-1}(\Omega)$ of the system

$$\left\{\begin{array}{ll} iz_t + \Delta z = 0 & \text{in } (0,T)\times\Omega \\ z = \chi_\gamma h & \text{on } (0,T)\times\Gamma \\ z(0,.) = z_0, & \end{array}\right. \qquad [61]$$

was established in (Machtyngier, 1994) when $\gamma = \Gamma(x^0)$ for some $x^0 \in \mathbb{R}^N$, in (Lebeau, 1992) when the GCC is fulfilled, and in (Burq *et al.*, 2004) for domains with holes under appropriate assumptions on the control region.

– (Haraux, 1989) proved that when Ω is a rectangle in $\mathbb{R}^2$ (say $\Omega = (a,b) \times (c,d)$) and ω is a strip of the form $(a',b') \times (c,d)$ with $a \leq a' < b' \leq b$, then the exact controllability of [60] holds in $\tilde{H}$ for any $T > 0$ (see Figure 6). The proof was based upon nonharmonic Fourier series. (The proof of Theorem 3.3 is inspired from it.)

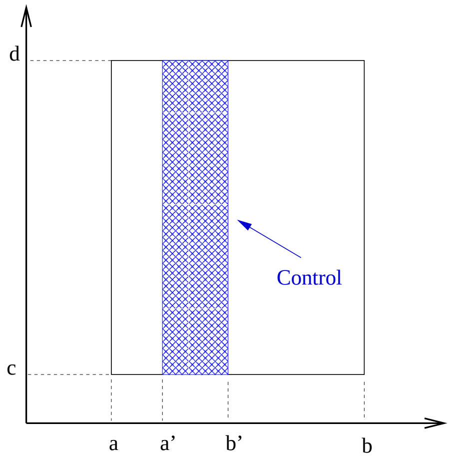

Figure 6. *Control region in Haraux' theorem*

– Later, using a theorem by Kahane on nonharmonic Fourier series, S. Jaffard improved Haraux's result in (Jaffard, 1990) by showing that the exact controllability of the plate equation in a rectangle Ω still holds (in any time T) when the control region ω is an arbitrary open subset of Ω (see Figure 7).

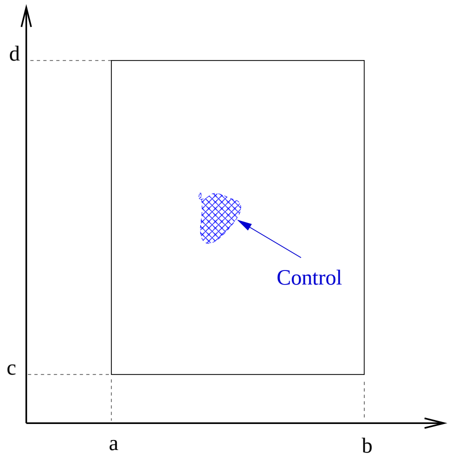

Figure 7. *Control region in Jaffard's theorem*

– Jaffard's result is not valid for *any* domain Ω. Indeed, it follows from results in (Chen *et al.*, 1991) that if Ω is the unit disk in $\mathbb{R}^2$ and $\omega = \{x| \, |x| < r\}$ for some $r < 1$, then [61] is not exactly controllable in $H^{-1}(\Omega)$, whereas [61] is exactly controllable in $H^{-1}(\Omega)$ if $\omega = \{x| \, r < |x| < 1\}$ for some $r < 1$ (see Figure 8).

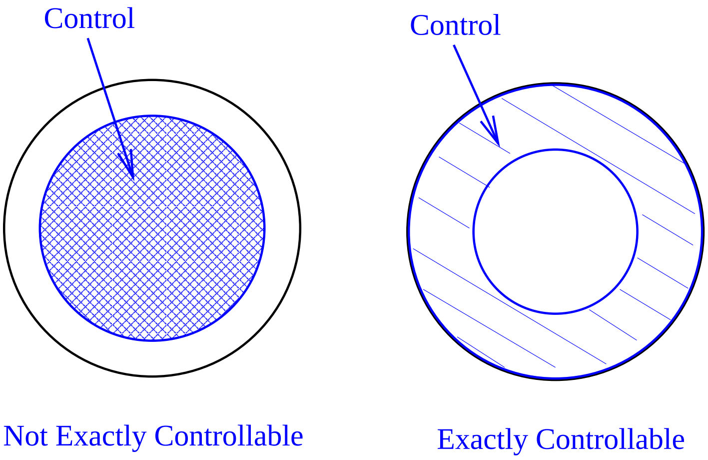

Figure 8. *Internal controllability of Schrödinger equation in a disk*

– For the boundary control problem [58], it has been proved in (Ramdani *et al.*, 2005) that for the domain $\Omega = (0, \pi)^2$, the exact controllability holds in H (in any time $T > 0$) if and only if the control region γ contains both a horizontal segment and a vertical segment of nonzero lengths (see Figure 9).

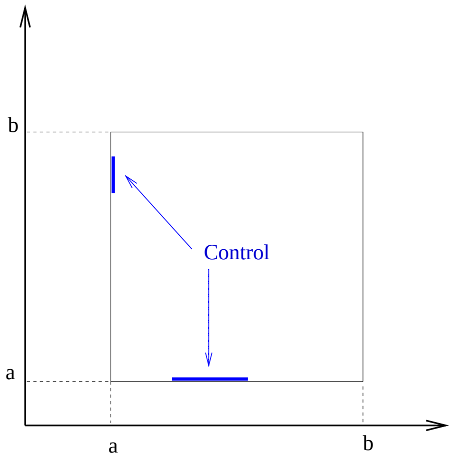

Figure 9. *Boundary control region for the plate equation in a square*

3.3. *Heat equation*

In this part we are concerned with the controllability properties of the heat equation. We consider the following internal control problem

$$\begin{aligned} z_t - z_{xx} + d(x)z &= \chi_{(a,b)} f, & [62] \\ z(t,0) = z(t,L) &= 0, & [63] \\ z(0,x) &= z_0 & [64] \end{aligned}$$

in which $z = z(t,x)$ is the state, $f = f(t,x)$ is the control input, and $d = d(x)$ is some potential. We assume that $d \in L^\infty(0,L)$, that $f \in L^2(0,T;L^2(a,b))$, and that $z_0 \in L^2(0,L)$. Here, a and b are given numbers with $0 \le a < b \le L$.

By standard regularity results for parabolic equations (see e.g. (Evans, 1998, Chapter 7)), the weak solution z of [62]-[64] fulfills

$$z \in L^2(0,T;H_0^1(0,L)) \cap H^1(0,T;H^{-1}(0,L)),$$

and for any $t_0 \in (0,T)$

$$z \in L^2(t_0,T;H^2(0,L)) \cap H^1(t_0,T;L^2(0,L)).$$

In particular, $z \in C([0,T];L^2(0,L)) \cap C((0,T],H_0^1(0,L))$. It follows that an exact controllability result cannot hold in $L^2(0,L)$, for $z(T) \in H_0^1(0,L)$. Actually, if $d \equiv 0$, then $z \in C^\infty((0,T) \times (0,a))$ by hypoellipticity of the heat operator, hence $z(T)$ has to be smooth on $(0,a)$ and on (b,L).

The following result may however be proved.

Theorem 3.5. *The system* [62]-[64] *is null controllable in* $L^2(0,L)$ *for any* $T > 0$.

Proof. The system [62]-[64] may be put in the form $\dot z = Az + Bf$, with $Az := d^2z/dx^2 - d(x)z$ on the domain $D(A) = H^2(0,L) \cap H_0^1(0,L)$ and $B : f \in L^2(a,b) \mapsto \chi_{(a,b)} f \in L^2(0,L)$ (*i.e.*, Bf is the extension of f by 0). Clearly $A^* = A$, so $S^*(t) = S(t)$. According to Theorem 2.5 we have to check that [16] is valid, that is, there exists some constant $C > 0$ such that for any $z_0 \in L^2(0,L)$

$$\int_0^T \int_a^b |z(t,x)|^2 \, dxdt \ge C \int_0^L |z(T,x)|^2 \, dx \qquad [65]$$

where $z(t,x)$ denotes the solution of

$$\begin{aligned} z_t - z_{xx} + d(x)z &= 0, & [66] \\ z(t,0) = z(t,L) &= 0, & [67] \\ z(0,x) &= z_0. & [68] \end{aligned}$$

By a density argument, we may as well assume that $z_0 \in D(A) = H^2(0,L) \cap H_0^1(0,L)$, so that $z \in C([0,T]; D(A)) \cap C^1([0,T]; L^2(0,L))$. The key ingredient is the following Carleman estimate.

Lemma 3.6. *There exist two positive numbers s_0, C_0, and a function $\psi \in C^2([0,L])$ with $\psi(x) > 0 \quad \forall x \in [0,L]$ such that for any $z \in L^2(0,T; H^2(0,L) \cap H_0^1(0,L)) \cap H^1(0,T; L^2(0,L))$ and for any $s \geq s_0$ we have*

$$\begin{aligned}
&\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} \left(\frac{t(T-t)}{s}(|z_t|^2 + |z_{xx}|^2) \right.\\
&\qquad \left. + \frac{s}{t(T-t)}|z_x|^2 + \frac{s^3}{t^3(T-t)^3}|z|^2 \right) dxdt \\
&\leq C_0 \left(\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} |z_t - z_{xx}|^2 \, dxdt \right.\\
&\qquad\qquad \left. + \int_0^T \int_a^b e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3}|z|^2 \, dxdt \right). \qquad [69]
\end{aligned}$$

See (Fernández-Cara *et al.*, 2006) for a proof.

Let us now see how [65] may be deduced from [69]. Pick any $z_0 \in D(A)$, so that the solution $z(t,x)$ of [66]-[68] fulfills all the requirements in Lemma 3.6. Pick also any $s \geq s_0$ such that

$$\frac{s^3}{t^3(T-t)^3} \geq C_0 ||d||^2_{L^\infty(0,L)} + 1 \quad \forall t \in (0,T). \qquad [70]$$

Since $z_t - z_{xx} = -d(x)z$, it follows from [69] that

$$\begin{aligned}
&\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3}|z|^2 \, dxdt \\
&\leq C_0 \left(\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} |d(x)z|^2 \, dxdt \right.\\
&\qquad \left. + \int_0^T \int_a^b e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3}|z|^2 \, dxdt \right) \\
&\leq C_0 \left(||d||^2_{L^\infty(0,L)} \int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} |z|^2 \, dxdt \right.\\
&\qquad \left. + \int_0^T \int_a^b e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3}|z|^2 \, dxdt \right).
\end{aligned}$$

Using [70], we obtain

$$\int_0^T \int_0^L e^{-2s\frac{\psi(x)}{t(T-t)}} |z|^2 \, dxdt \le C_0 \int_0^T \int_a^b e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3} |z|^2 \, dxdt.$$

As $\psi(x) > 0$ on $[0, L]$, we may find some positive constants C_1, C_2 such that

$$e^{-2s\frac{\psi(x)}{t(T-t)}} \ge C_1 \quad \forall (t,x) \in (\frac{T}{3}, \frac{2T}{3}) \times (0, L)$$

and

$$e^{-2s\frac{\psi(x)}{t(T-t)}} \frac{s^3}{t^3(T-t)^3} \le C_2 \quad \forall (t,x) \in (0,T) \times (0,L).$$

We arrive to

$$\int_{\frac{T}{3}}^{\frac{2T}{3}} \int_0^L |z|^2 \, dxdt \le \frac{C_0 C_2}{C_1} \int_0^T \int_a^b |z|^2 \, dxdt. \qquad [71]$$

Since A generates a strongly continuous semigroup $(S(t))_{t\ge 0}$ on $L^2(0,L)$, we may write for some positive constants M, μ

$$||S(t)|| \le M e^{\mu t} \quad \forall t \ge 0,$$

hence, for any $t \in (\frac{T}{3}, \frac{2T}{3})$

$$||z(T)||^2_{L^2(0,L)} = ||S(T-t)z(t)||^2_{L^2(0,L)} \le M^2 e^{2\mu(T-t)} ||z(t)||^2_{L^2(0,L)}.$$

Integrating over $(\frac{T}{3}, \frac{2T}{3})$ yields

$$\frac{T}{3} ||z(T)||^2_{L^2(0,L)} \le M^2 e^{4\mu T/3} \int_{\frac{T}{3}}^{\frac{2T}{3}} ||z(t)||^2_{L^2(0,L)} dt. \qquad [72]$$

Gathering [71] and [72], we obtain [65]. ■

The null-controllability of the heat equation was derived in (Russell, 1973) from an exact controllability result for the wave equation. In particular, the null controllability of the heat equation was established in dimension 1 for any time T by means of a boundary control applied to both extremities. Later, (Èmanuilov, 1995) and (Lebeau *et al.*, 1995) proved independently the zero controllability of the heat equation in any dimension by using Carleman estimates. Since then, Carleman estimates have been used to prove the zero controllability of certain *semilinear* heat equations, and of the Navier-Stokes equations. (See (Fernández-Cara *et al.*, 2006) and the references therein.)

3.4. *Korteweg-de Vries equation*

We are interested in the boundary control properties of the Korteweg-de Vries (KdV) equation. (Most of the material of this section comes from (Rosier, 1997).) More precisely, we investigate the system

$$\begin{aligned} y_t + y_x + y_{xxx} + yy_x &= 0, && [73] \\ y(t,0) = y(t,L) &= 0, && [74] \\ y_x(t,L) &= h(t), && [75] \\ y(0,x) &= y_0(x), && [76] \end{aligned}$$

where $h \in L^2(0,T)$ stands for the control input. In a first step we investigate the boundary controllability of the linearized system (the term yy_x being removed in [73])

$$\begin{aligned} y_t + y_x + y_{xxx} &= 0, && [77] \\ y(t,0) = y(t,L) &= 0, && [78] \\ y_x(t,L) &= h(t), && [79] \\ y(0,x) &= y_0(x). && [80] \end{aligned}$$

The outline of the study is as follows:

1) study of the Cauchy problem for the linear KdV equation;

2) *a priori* estimates obtained by the *multiplier method*;

3) application of HUM to the linear KdV equation;

4) observability inequality proved by the *compactness-uniqueness method*;

5) exact controllability of the (nonlinear) KdV equation proved by a *fixed-point argument*.

(1) Cauchy problem for the linear KdV equation

We first have a look at the Cauchy problem for the linearized system without control ($h \equiv 0$):

$$\begin{aligned} y_t + y_x + y_{xxx} &= 0, && [81] \\ y(t,0) = y(t,L) = y_x(t,L) &= 0, && [82] \\ y(0,x) &= y_0(x). && [83] \end{aligned}$$

Let A denote the operator $Aw = -w''' - w'$ on the (dense) domain $D(A) \subset L^2(0,L)$ defined by

$$D(A) = \{w \in H^3(0,L) |\ w(0) = w(L) = w'(L) = 0\}.$$

The following result holds:

Proposition 3.7. *A generates a strongly continuous semigroup of contractions in $L^2(0,L)$.*

Proof. It is easy to see that A is closed. Let $w \in D(A)$. Then

$$\begin{aligned}
(w, Aw)_{L^2(0,L)} &= \int_0^L w(x)\big(-w'''(x) - w'(x)\big)dx \\
&= \int_0^L w'(x)w''(x)dx - [ww'']_0^L - [\frac{w^2}{2}]_0^L \\
&= [\frac{w'^2}{2}]_0^L \\
&= -\frac{w'(0)^2}{2} \leq 0.
\end{aligned}$$

Hence A is dissipative. It may be seen that $A^*(w) = w' + w'''$ with domain $D(A^*) = \{w \in H^3(0,L)|\ w(0) = w(L) = w'(0) = 0\}$, so that $(w, A^*(w))_{L^2(0,L)} = -\frac{w'(L)^2}{2} \leq 0$ and A^* is dissipative too. Now the result follows from (Pazy, 1983, cor. 4.4 chapter 1). ■

(2) *A priori* estimates by the multiplier method

From now on we let $(S(t))_{t\geq 0}$ denote the semigroup of contractions associated with A, and we let $\mathcal{B}$ denote the (Banach) space $C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$ endowed with the norm

$$\|y\|_{\mathcal{B}} := \sup_{t\in[0,T]} \|y(t,.)\|_{L^2(0,L)} + \left(\int_0^T \|y(t,.)\|^2_{H^1(0,L)}dt\right)^{\frac{1}{2}}.$$

To obtain useful estimates for the mild solutions of [81]-[83] we use the *multiplier method*, which goes back to (Morawetz, 1962). The multiplier method has also proved to be useful to establish observability inequalities for the wave equation and the plate equation. (See e.g. (Komornik, 1994).)

Proposition 3.8. *1) For any $y_0 \in L^2(0,L)$, $S(\cdot)y_0 \in \mathcal{B}$, and the map $y_0 \in L^2(0,L) \mapsto S(\cdot)y_0 \in \mathcal{B}$ is continuous.*
2) For $y_0 \in L^2(0,L)$, $y_x(.,0)$ makes sense in $L^2(0,T)$, and

$$\|y_x(.,0)\|_{L^2(0,T)} \leq \|y_0\|_{L^2(0,L)}, \tag{84}$$

$$\|y_0\|^2_{L^2(0,L)} \leq \frac{1}{T}\|S(\cdot)y_0\|^2_{L^2((0,T)\times(0,L))} + \|y_x(\cdot,0)\|^2_{L^2(0,T)}. \tag{85}$$

Proof. 1) For $y_0 \in L^2(0,L)$ we denote by y the mild solution $S(\cdot)y_0$ of [81]-[83]. By Proposition 3.7, $y \in C([0,T];L^2(0,L))$ and

$$\|y\|_{C([0,T];L^2(0,L))} \leq \|y_0\|_{L^2(0,L)}. \qquad [86]$$

To see that $y \in L^2(0,T;H^1(0,L))$ we first assume that $y \in D(A)$. Let $q \in C^\infty([0,T]\times[0,L])$. We multiply [81] by qy (q being for the moment an arbitrary multiplier), integrate over $(0,T)\times(0,L)$ and integrate by parts to obtain

$$\begin{aligned} - & \int_0^T\int_0^L (q_t+q_x+q_{xxx})\frac{y^2}{2}dxdt + \int_0^L (q\frac{y^2}{2})(T,x)dx \\ - & \int_0^L q(0,x)\frac{y_0^2\,(x)}{2}dx + \frac{3}{2}\int_0^T\int_0^L q_x y_x^2 dxdt \\ + & \int_0^T (q\frac{y_x^2}{2})(t,0)dt = 0. \end{aligned} \qquad [87]$$

Choosing $q(t,x)=x$ leads to

$$-\int_0^T\int_0^L y^2dxdt + \int_0^L xy^2(T,x)dx - \int_0^L xy_0^2(x)dx + 3\int_0^T\int_0^L y_x^2dxdt \;=\; 0.$$

Hence

$$\int_0^T\int_0^L y_x^2\,dxdt \leq \frac{1}{3}\left(\int_0^T\int_0^L y^2\,dxdt + L\int_0^L y_0^2(x)\,dx\right)$$

and then, using [86]

$$\|y\|_{L^2(0,T,H^1(0,L))} \leq \left(\frac{4T+L}{3}\right)^{\frac{1}{2}} \|y_0\|_{L^2(0,L)}. \qquad [88]$$

By the density of $D(A)$ in $L^2(0,L)$ the result extends to an arbitrary $y_0 \in L^2(0,L)$.

2) We again assume that $y_0 \in D(A)$ and take $q=1$ in [87]. We get

$$\int_0^T y_x^2(t,0)\,dt = \int_0^L y_0^2(x)\,dx - \int_0^L y^2(T,x)\,dx \leq \int_0^L y_0^2(x)\,dx. \qquad [89]$$

On the other hand the choice $q(t,x)=T-t$ yields

$$\int_0^T\int_0^L y^2\,dxdt - \int_0^L Ty_0^2(x)\,dx + \int_0^T (T-t)y_x^2(t,0)\,dt = 0,$$

hence

$$\int_0^L y_0^2(x)\,dx \leq \frac{1}{T}\int_0^T\int_0^L y^2\,dxdt + \int_0^T y_x^2(t,0)\,dt. \qquad [90]$$

Thanks to [89] there exists a unique continuous (linear) extension of the map $y_0 \in D(A) \mapsto y_x(.,0) \in L^2(0,T)$ to the whole space $L^2(0,L)$. In what follows, we will again denote by $y_x(.,0)$ the value of this map at any $y_0 \in L^2(0,L)$. Obviously [89] and [90] are valid for any $y_0 \in L^2(0,L)$. ■

(3) HUM applied to the linear KdV equation

We first need to define precisely what is a solution of [77]-[80]. The following result is proved by means of estimates similar to those given in Proposition 3.8.

Proposition 3.9. *There exists a unique linear continuous map* $\Psi : L^2(0,L) \times L^2(0,T) \to \mathcal{B}$ *such that, for* $y_0 \in D(A)$ *and* $h \in C^2([0,T])$ *with* $h(0) = 0$, $\Psi(y_0, h)$ *is the unique (classical) solution of* [77]-[80].

Proof. See (Rosier, 1997).

To apply HUM, we have to write the key identity and the observability inequality to be proved. Multiplying [77] by u, integrating over $(0,T) \times (0,L)$ and performing integrations by parts, we arrive to

$$0 = -\int_0^T \int_0^L y(u_t + u_x + u_{xxx})\, dxdt + \int_0^L [yu]_0^T\, dx + \int_0^T [-y_x u_x + y_{xx} u]_0^L\, dt.$$

Assuming in addition that u is the solution of the backwards system

$$u_t + u_x + u_{xxx} = 0 \quad 0 < t < T,\ 0 < x < L \qquad [91]$$

$$u(t,0) = u(t,L) = u_x(t,0) = 0 \quad 0 < t < T \qquad [92]$$

$$u(T,.) = u_T \qquad [93]$$

and that $y_0 = 0$, we conclude that

$$\int_0^L u_T(x) y(T,x) dx = \int_0^T u_x(t,L) h(t) dt.$$

Therefore, the exact controllability of [77]-[80] in $L^2(0,L)$ holds if and only if the following observability inequality is satisfied:

$$||u_T||_{L^2(0,L)} \le C ||u_x(.,L)||_{L^2(0,T)}.$$

Performing the change of variables $\tau = T - t$, $\xi = L - x$, which transforms [91]-[93] into [81]-[83], we see that the above observability inequality is equivalent to the following one:

$$||y_0||_{L^2(0,L)} \le C ||y_x(.,0)||_{L^2(0,T)} \qquad [94]$$

for any solution $y(\cdot)$ of the forward problem [81]-[83].

(4) Observability inequality

To establish the observability inequality [94], we use the compactness-uniqueness method introduced in (Lions, 1988, Appendix 1). Roughly, the idea is to show, by using some compactness, that if the observability inequality is not valid, then one may find a nontrivial solution of the system with a vanishing observer (here, $y_x(.,0) \equiv 0$). Thus, the problem is reduced to a *unique continuation property*. In a second step, following a method introduced in (Bardos *et al.*, 1992), we show how to eliminate the time t and reduce the issue to a spectral problem.

The following result shows that the length of the domain plays an important role in the controllability results for KdV.

Proposition 3.10. *Let* $\mathcal{N} = \{2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l \in \mathbb{N}^*\}$. *Then for any* $L \in (0,+\infty)\setminus \mathcal{N}$ *and any* $T > 0$, *there exists a constant* $C = C(L,T) > 0$ *such that*

$$\forall y_0 \in L^2(0,L)\ \ \|y_0\|_{L^2(0,L)} \leq C\|y_x(.,0)\|_{L^2(0,T)}.$$

Proof. If the statement is false, then there exists a sequence $(y_0^n)_{n\geq 0}$ in $L^2(0,L)$ such that $\|y_0^n\|_{L^2(0,L)} = 1$ for any n but $\|y_x^n(.,0)\|_{L^2(0,T)} \to 0$ as $n \to \infty$, where $y^n := S(\cdot)y_0^n$. Clearly, y^n is bounded in $L^2(0,T;H^1(0,L))$ (see [88]). On the other hand $y_t^n = -(y_x^n + y_{xxx}^n)$ is bounded in $L^2(0,T;H^{-2}(0,L))$. Since the first embedding in

$$H^1(0,L) \subset L^2(0,L) \subset H^{-2}(0,L)$$

is compact, it follows from a common version of Aubin's lemma (see below Lemma 3.11) that the set $\{y_n\}$ is relatively compact in $L^2(0,T;L^2(0,L))$.

Lemma 3.11. *Let* $H_1 \subset H_2 \subset H_3$ *denote three Hilbert spaces with continuous embeddings, and such that the first embedding* $H_1 \subset H_2$ *is compact. Let* (f_n) *be a sequence of functions* $(0,T) \to H_1$. *If* (f_n) *is bounded in* $L^2(0,T;H_1)$ *and* (df_n/dt) *is bounded in* $L^2(0,T;H_3)$, *then* (f_n) *possesses a subsequence which converges in* $L^2(0,T;H_2)$ *for the strong topology.*

For more general statements and proofs of Aubin's lemma, see e.g. (Simon, 1987).

Without loss of generality we may assume that the whole sequence (y^n) is convergent in $L^2(0,T;L^2(0,L))$. We infer from [85] that (y_0^n) is a Cauchy sequence in $L^2(0,L)$. Let $y_0 = \lim_{n\to\infty} y_0^n$ and $y = S(\cdot)y_0$. By [84] $y_x^n(.,0) \to y_x(.,0)$ in $L^2(0,T)$. Thus $\|y_0\|_{L^2(0,L)} = 1$ and $y_x(.,0) = 0$, but such a function does not exist because of the following:

Lemma 3.12. *For* $T > 0$ *let* N_T *denote the space of the (initial) states* $y_0 \in L^2(0,L)$ *such that the mild solution* $y = S(\cdot)y_0$ *of* [81]-[83] *satisfies* $y_x(\cdot,0) = 0$ *in* $L^2(0,T)$. *Then for* $L \in (0,+\infty)\setminus\mathcal{N}$, $N_T = (0)\ \ \forall T > 0$.

Proof of Lemma 3.12. Obviously $T < T' \Rightarrow N_{T'} \subset N_T$. On the other hand for any $T > 0$ N_T is a finite-dimensional vector space: Indeed if (y_0^n) is a sequence in the unit ball $\{y \in N_T | \; \|y\|_{L^2(0,L)} \leq 1\}$, then the same argument as above shows that there exists a convergent subsequence. Since the unit ball is compact, the space N_T is finite dimensional according to Riesz theorem. Let $T' > 0$ be given. To prove $N_{T'} = (0)$, it is sufficient to find $0 < T < T'$ such that $N_T = (0)$. Since the map $T \mapsto \text{dim}(N_T) \in \mathbb{N}$ is nonincreasing, there exist $T, \epsilon > 0$ such that $T < T + \epsilon < T'$ and $\text{dim}N_T = \text{dim}N_{T+\epsilon}$. (Hence $N_t = N_T$ for $T \leq t \leq T + \epsilon$.) Let $y_0 \in N_T$, $y = S(\cdot)y_0$ and $0 < t < \epsilon$. Since $S(\tau)\big(S(t)y_0\big) = S(\tau + t)y_0$ for $0 \leq \tau \leq T$ and $y_0 \in N_{T+\epsilon}$, we see that

$$\frac{S(t)y_0 - y_0}{t} \in N_T. \tag{[95]}$$

Let

$$M_T := \{\tilde{y} = S(\tau)\tilde{y}_0;\ 0 \leq \tau \leq T,\ \tilde{y}_0 \in N_T\} \subset C([0,T]; L^2(0,L)).$$

Since $y \in H^1(0, T + \epsilon; H^{-2}(0,L))$,

$$\lim_{t \to 0^+} \frac{y(t + \cdot) - y}{t} = y' \ \text{ in } L^2(0,T; H^{-2}(0,L)).$$

On the other hand (thanks to [95]) $\frac{y(t+\cdot)-y}{t} \in M_T$ for $0 < t < \epsilon$ and M_T is closed in $L^2(0,T; H^{-2}(0,L))$ (since $dim(M_T) < \infty$). It follows that $y' \in M_T \subset C([0,T]; L^2(0,L))$ and $y \in C^1([0,T]; L^2(0,L))$, hence we may write

$$y'(0) = \lim_{t \to 0^+} \frac{S(t)y_0 - y_0}{t} \ \text{ in } L^2(0,L).$$

This in turn implies that

$$y_0 \in D(A),\ \ A(y_0) = y'(0) \in N_T \ \text{ and } \ y_x(.,0) \in C([0,T]).$$

Thus

$$\left(\frac{dy_0}{dx}\right)_{\big| x=0} = y_x(0,0) = 0.$$

If $N_T \neq (0)$, the map $y_0 \in \mathbb{C}N_T \mapsto A(y_0) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denotes the complexification of N_T) has (at least) one eigenvalue, hence there exist $\lambda \in \mathbb{C}$, $y_0 \in H^3(0,L) \setminus \{0\}$ such that (prime denoting here spatial derivative)

$$\begin{aligned} &\lambda\, y_0 = -{y_0}' - {y_0}''', \\ &y_0(0) = y_0(L) = y_0'(0) = y_0'(L) = 0. \end{aligned}$$

We prove in the following lemma that this does not hold if $L \notin \mathcal{N}$.

Lemma 3.13. *Let* $L \in (0,+\infty)$. *Consider the following assertion:*

$$(\mathcal{A}) \quad \exists \lambda \in \mathbb{C},\ \exists y_0 \in H^3(0,L)\setminus\{0\} \quad \textit{s.t.} \quad \left\{ \begin{array}{l} \lambda\, y_0 + y_0{}' + y_0{}''' = 0, \\ y_0(0) = y_0(L) = y_0'(0) = y_0'(L) = 0. \end{array} \right.$$

Then $(\mathcal{A}) \iff L \in \mathcal{N}$.

Proof of Lemma 3.13. Let y_0 be as in $(\mathcal{A})$, and let us denote by $u \in H^2(\mathbb{R})$ its prolongation by 0. Then

$$\lambda u + u' + u''' = y_0{}''(0)\delta_0 - y_0{}''(L)\delta_L \ \text{ in } \mathcal{D}'(\mathbb{R}),$$

where δ_{x_0} denotes the Dirac measure at x_0. It is easy to see that $(\mathcal{A})$ is equivalent to the existence of complex numbers α, β, λ (with $(\alpha,\beta) \neq (0,0)$) and of a function $u \in H^2(\mathbb{R})$ with compact support in $[-L, L]$ such that

$$\lambda u + u' + u''' = \alpha\delta_0 - \beta\delta_L \text{ in } \mathcal{D}'(\mathbb{R}).$$

Taking Fourier transform we get

$$\big(\lambda + (i\xi) + (i\xi)^3\big)\hat{u}(\xi) = \alpha - \beta e^{-iL\xi},$$

hence (setting $\lambda = -ip$)

$$\hat{u}(\xi) = i\frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}.$$

Using Paley-Wiener theorem (see (Yosida, 1995)) and the usual characterization of $H^2(\mathbb{R})$ functions by means of their Fourier transforms, we see that $(\mathcal{A})$ is equivalent to the existence of $p \in \mathbb{C}$ and of $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that the map

$$f(\xi) := \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}$$

satisfies :

(i) f is an entire function in $\mathbb{C}$;

(ii) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2 \big(1+|\xi|^2\big)^2 \, d\xi < \infty$;

(iii) $\forall \xi \in \mathbb{C} \quad |f(\xi)| \le C(1+|\xi|)^N\, e^{L|\mathrm{Im}\,\xi|}$ for some positive constants C, N.

Since the roots of $\alpha - \beta e^{-iL\xi}$ are simple (unless $\alpha = \beta = 0$), (i) holds provided that the roots of $\xi^3 - \xi + p$ are simple and also roots of $\alpha - \beta e^{-iL\xi}$. Notice that if (i) holds true, then (ii) and (iii) are satisfied. It follows that $(\mathcal{A})$ is equivalent to the existence of complex numbers p, μ_0 and of positive integers k, l such that, setting

$$\mu_1 := \mu_0 + k\frac{2\pi}{L} \text{ and } \mu_2 := \mu_1 + l\frac{2\pi}{L},$$

we have

$$\xi^3 - \xi + p \equiv (\xi - \mu_0)(\xi - \mu_1)(\xi - \mu_2),$$

that is

$$\begin{aligned} \mu_0 + \mu_1 + \mu_2 &= 0, \\ \mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2 &= -1, \\ \mu_0\mu_1\mu_2 &= -p. \end{aligned}$$

Easy calculations lead to

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}, \qquad [96]$$

$$\mu_0 = -\frac{1}{3}(2k + l)\frac{2\pi}{L}, \qquad [97]$$

$$p = -\mu_0(\mu_0 + k\frac{2\pi}{L})(\mu_0 + (k + l)\frac{2\pi}{L}). \qquad [98]$$

Hence $(\mathcal{A}) \iff L \in \mathcal{N}$. This completes the proof of Lemmas 3.13, 3.12 and of Proposition 3.10. ■

Remark 3.14. 1) For $L \in \mathcal{N}$, if p is given by [98] and y_0 (with $\mathrm{Re}\, y_0 \neq 0$) is as in $(\mathcal{A})$ (with $\lambda = -ip$), then $y(t, x) := \mathrm{Re}\left(e^{-ipt} y_0(x)\right)$ is a nontrivial (smooth) solution of [81]-[83] such that $y_x(., 0) \equiv 0$. Thus the result in Proposition 3.10 holds true if and *only if* $L \notin \mathcal{N}$.

2) In the particular situation where $k = l$, the system [96]-[98] reduces to $L = 2k\pi,\ \mu_0 = -1,\ p = 0$ (hence $\lambda = 0$). This yields the (unobservable) steady solutions of [81]-[83]: $y(t, x) = a(\cos x - 1)$, $a \in \mathbb{R}$.

As a consequence of Proposition 3.10, we obtain the following result:

Theorem 3.15. *The linear KdV system* [77]-[80] *is exactly controllable in* $L^2(0, L)$ *if and only if* $L \notin \mathcal{N}$.

(5) Exact controllability of the KdV equation

To prove the exact controllability of the (nonlinear) KdV system [73]-[76], we apply the Banach fixed-point theorem in some closed ball of the Hilbert space

$L^2(0,T;H^1(0,L))$. We assume that $L \notin \mathcal{N}$, so that by Theorem 3.15 the linear KdV system [77]-[80] is exactly controllable in $L^2(0,L)$. Furthermore, as a byproduct of HUM, we may construct a bounded operator $\Lambda : y_T \in L^2(0,L) \mapsto h \in L^2(0,T)$ such that the solution of [77]-[80] with $y_0 = 0$ and $h = \Lambda(y_T)$ satisfies $y(T) = y_T$.

We first write a solution of [73]-[76] in the form $y = S(t)y_0 + y_1 + y_2$ where $(S(t))_{t\geq 0}$ denotes the semigroup associated with the operator A, and y_1 and y_2 are the respective solutions of the following nonhomogeneous problems

$$\begin{aligned} y_{1t} + y_{1x} + y_{1xxx} &= 0 && [99]\\ y_1(t,0) = y_1(t,L) &= 0 && [100]\\ y_{1x}(t,L) &= h(t) && [101]\\ y_1(0,x) &= 0 && [102] \end{aligned}$$

and

$$\begin{aligned} y_{2t} + y_{2x} + y_{2xxx} &= f && [103]\\ y_2(t,0) = y_2(t,L) &= 0 && [104]\\ y_{2x}(t,L) &= 0 && [105]\\ y_2(0,x) &= 0 && [106] \end{aligned}$$

with $f = -yy_x$ in [103]. We now define the (nonlinear) map F from $L^2(0,T;H^1(0,L))$ into itself in the following way: given any $y \in L^2(0,T;H^1(0,L))$, we set $f := -yy_x$, and denote by y_2 the solution of [103]-[106] associated with that f. Next, we set $h := \Lambda(y_T - S(T)y_0 - y_2(T))$ and denote by y_1 the solution of [99]-[102] associated with that control input h. Finally, we set $F(y) = S(\cdot)y_0 + y_1 + y_2$. We first notice that any fixed-point of F is a solution of [73]-[76] connecting y_0 to y_T in time T. It may be seen (see (Rosier, 1997)) that for $||y_0||_{L^2(0,L)}$ and $||y_T||_{L^2(0,L)}$ small enough and for a convenient choice of the radius R, F maps the closed ball $\overline{B}(0,R)$ (in $L^2(0,T;H^1(0,L))$) into itself and is a contraction on $\overline{B}(0,R)$. Therefore, F possesses a fixed-point in $\overline{B}(0,R)$. The following result has been obtained.

Theorem 3.16. *For any* $L \notin \mathcal{N}$ *and any* $T > 0$, *there exists a number* $r_0 > 0$ *such that for any* $y_0, y_T \in L^2(0,L)$ *with* $||y_0||_{L^2(0,L)} \leq r_0$ *and* $||y_T||_{L^2(0,L)} \leq r_0$, *one may find a control input* $h \in L^2(0,T)$ *such that the solution* $y(\cdot)$ *of* [73]-[76] *fulfills* $y(T) = y_T$.

Let us end this section with a few references about the control of KdV. (Russell *et al.*, 1993) is the first paper devoted to the control of the KdV equation. In that paper, the domain was periodic (for the boundary control, an expression involving $y_x(t,0)$ and $y_x(t,2\pi)$ was assumed to be controlled), and exact controllability results as well as exponential stability results were obtained for the linear KdV equation. The results were partially extended to the (nonlinear) KdV equation in (Russell *et al.*, 1995). Local exact controllability results and local exponential stability results were given in (Russell *et al.*, 1996) for the (nonlinear) KdV equation with an internal control in a periodic domain. After (Rosier, 1997), (Zhang, 1999) showed that the KdV equation was EC in any Sobolev space $H^s(0,L)$, $s \geq 0$, whatever be the length L of the domain, provided that the three traces $y(t,0)$, $y(t,L)$ and $y_x(t,L)$ are controlled. Recall that when the length is critical (*i.e.*, $L \in \mathcal{N}$), then the linear KdV system [77]-[80] fails to be controllable. However, the space of reachable states (from 0) is of finite codimension in $L^2(0,L)$, that is, the space of missing directions is of finite dimension. Brocket's system shows that a system whose linearization fails to be controllable, may be controllable. One may wonder whether the system [73]-[76] may be controllable for a critical length thanks to the nonlinear term yy_x. (Crépeau, 2001) showed that for a critical length $L = 2k\pi$, $k \in \mathbb{N}^*$, the system [73]-[76] is locally controllable around a nonnull stationary solution of [73]. Later, E. Crépeau and J.-M. Coron proved in (Coron *et al.*, 2004) that for a critical length $L = 2k\pi$ with $k \in \mathbb{N}^*$ fulfilling

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^* \text{ with } m^2 + mn + n^2 = 3k^2 \text{ and } m \neq n,$$

then the system [73]-[76] is exactly controllable around 0. For such lengths the space of missing directions is only one-dimensional. A similar result has been obtained by E. Cerpa in (Cerpa, preprint) when

$$L \neq 2\pi n \quad \forall n \in \mathbb{N}^* \text{ and } L = 2\pi\sqrt{(k^2 + kl + l^2)/3}$$

for a unique couple $(k,l) \in \mathbb{N}^* \times \mathbb{N}^*$ with $k > l$. In that case, the space of missing directions is two-dimensional.

The boundary controllability of the linear KdV equation on the half-line ($\Omega = (0,+\infty)$) was investigated in (Rosier, 2000). It was proved in that paper that an exact controllability result cannot hold in the *energy space* $L^2(0,+\infty)$, and however that two states in $L^2(0,+\infty)$ may be connected by a solution z of the linear KdV equation living in the space $L^2_{loc}((0,T) \times (0,+\infty))$. The proof was based upon a new global Carleman estimate for KdV. An estimation of the growth at infinity of such trajectory was given in (Rosier, 2002). The previous Carleman estimate was improved in (Rosier, 2004) and applied to the wavemaker problem: roughly, it was proved in (Rosier, 2004) that solitons on the surface of water in a channel can been annihilated by certain waves generated by a moving wall (the so-called "wavemaker").

4. Stabilizability

In this last part we investigate the stabilizability of a control system

$$\Sigma_{A,B} \qquad \dot{z} = Az + Bu, \qquad [107]$$

where A generates a continuous semigroup of operators $(S(t))_{t\geq 0}$ on some Hilbert space H, and B is a linear, bounded operator acting from a Hilbert space U to H.

4.1. *Finite-dimensional systems*

We recall some well-known stability properties when $A \in \mathbb{C}^{N\times N}$ and $B \in \mathbb{C}^{N\times M}$. We denote by $\sigma(A)$ the *spectrum* of A, that is, the set of its eigenvalues.

– The origin is *asymptotically stable* ($\iff$ *exponentially stable*) for the system $\dot{x} = Ax$ if and only if $\sigma(A) \subset \mathbb{C}_- := \{z \in \mathbb{C} |\ \text{Re}\ z < 0\}$. Furthermore

$$\sup\{\text{Re}\ \lambda |\ \lambda \in \sigma(A)\} = \inf\{\mu \in \mathbb{R} |\ \exists C > 0,\ ||e^{tA}|| \leq Ce^{\mu t}\ \forall t \geq 0\}.\ [108]$$

– (Wonham's Theorem) The control system $\Sigma_{A,B}$ is controllable if and only if it is exponentially stabilizable with an arbitrary exponential decay rate, *i.e.*,

$$\forall \mu \in (-\infty, 0),\ \exists K \in \mathbb{C}^{M,N},\ \exists C > 0 \quad ||e^{t(A+BK)}|| \leq Ce^{\mu t} \quad \forall t \geq 0.$$

4.2. *Stability properties for infinite-dimensional systems*

Assume now that A is the generator of a continuous semigroup $(S(t))_{t\geq 0}$ on H.

The *resolvent set* of A, denoted by $\rho(A)$, is the set of complex numbers λ for which the operator $\lambda I - A$ is boundedly invertible (*i.e.*, its inverse $(\lambda I - A)^{-1} : H \to D(A)$ exists and is bounded from H to H). As A is closed, an application of the closed graph theorem shows at once that $\lambda \in \rho(A)$ if and only if $\lambda I - A : D(A) \to H$ is onto and one-to-one. The map $\lambda \in \rho(A) \mapsto (\lambda I - A)^{-1} \in L(H, H)$ is called the *resolvent* of A. The *spectrum* of A, denoted by $\sigma(A)$, is the complement of the resolvent set: $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Remark 4.1. 1) If H is real, it is complexified and A is extended in an obvious way in order to define the resolvent of A.

2) λ is in the spectrum of A if and only if $\lambda I - A$ is not one-to-one (*i.e.*, λ is an eigenvalue of A) or $\lambda I - A$ is not onto. Thus, the spectrum is not reduced to the set of the eigenvalues of A in general. (See below Example 4.6.)

Let us consider the following properties:
(i) for some constants $C > 0$, $\mu > 0$, and all $t \geq 0$ $||S(t)|| \leq Ce^{-\mu t}$;
(ii) for any $z_0 \in H$ $S(t)z_0 \to 0$ exponentially as $t \to +\infty$;
(iii) for any $z_0 \in H$, $\int_0^{+\infty} ||S(t)z_0||_H^2\, dt < \infty$;
(iv) for any $z_0 \in H$ $S(t)z_0 \to 0$ as $t \to +\infty$;
(v) $\sup\{\text{Re}\ \lambda |\ \lambda \in \sigma(A)\} < 0$.

In finite dimension, these properties are all equivalent. In infinite dimension, the situation is more tricky. The following result, due to R. Datko (Datko, 1972), shows the links between these properties. (See also (Zabczyk, 1992) for a proof).

Theorem 4.2. *1) We have* $(i) \iff (ii) \iff (iii)$.
2) We only have $(i) \Rightarrow (iv)$ *and* $(i) \Rightarrow (v)$.

In an infinite dimensional setting, [108] has to be replaced by

$$\omega \leq \omega_0 \qquad [109]$$

where

$$\begin{aligned} \omega &:= \sup\{\text{Re } \lambda |\ \lambda \in \sigma(A)\} \quad \text{is the } \textit{spectral abscissa}, \text{ and} \\ \omega_0 &:= \inf\{\mu \in \mathbb{R}|\ \exists C > 0,\ ||S(t)|| \leq Ce^{\mu t}\ \forall t \geq 0\} \text{ is the } \textit{best decay rate}. \end{aligned}$$

To prove [109], we pick any $\mu > \omega_0$, so that $||S(t)|| \leq Ce^{\mu t}$ for some constant $C > 0$ and all $t \geq 0$. Then, by a well-known result in semigroup theory, for any $\lambda \in \mathbb{C}$ with Re $\lambda > \mu$, we have that $\lambda \in \rho(A)$ and $(\lambda I - A)^{-1} = \int_0^{+\infty} e^{-\lambda t} S(t)\, dt$. Therefore $\sigma(A) \subset \{z \in \mathbb{C}|$ Re $z \leq \mu\}$, and $\omega \leq \mu$.

The existence of a possible gap between ω and ω_0 is attested by the following lemma (see (Zabczyk, 1992, Lemma 3.1 p. 224)).

Lemma 4.3. *Let* $(\lambda_k)_{k\geq 1}$ *be an arbitrary sequence of real numbers such that* $|\lambda_k| \to \infty$. *Then there exists a semigroup* $(S(t))_{t\geq 0}$ *on* $H = l^2_{\mathbb{C}}(\mathbb{N})$ *such that*

$$||S(t)|| = e^t \quad \forall t \geq 0 \quad \textit{and} \quad \sigma(A) = \{i\lambda_k;\ k = 1, 2, ...\}$$

where A *denotes the generator of* $(S(t))_{t\geq 0}$. *In particular,* $\omega = 0 < \omega_0 = 1$.

Definition 4.4. If (i) (or equivalently (ii) or (iii)) holds, then we say that the semigroup $(S(t))_{t\geq 0}$ is *exponentially stable*. When (iv) holds, we say that the semigroup $(S(t))_{t\geq 0}$ is *strongly stable*.

We now review classical stability results based on frequency-domain considerations.

For the strong stability, we have at our disposal a result which is very useful in applications (see (Arendt *et al.*, 1988)).

Theorem 4.5. *(Arendt-Batty) If* $(S(t))_{t\geq 0}$ *is bounded and* $i\mathbb{R} \subset \rho(A)$, *then* $(S(t))_{t\geq 0}$ *is strongly stable.*

The condition $i\mathbb{R} \subset \rho(A)$ is not necessary for the strong stability to hold, as it is shown in next example.

Example 4.6. Let $Au = \frac{du}{dx}$ with domain $D(A) = H^1(0, +\infty)$ in $H = L^2(0, +\infty)$. Then it is easy to see that A generates the (shift) semigroup $(S(t))_{t\geq 0}$ defined by $[S(t)u](x) = u(x+t)$, that $(S(t))_{t\geq 0}$ is strongly stable (as $u(\cdot + t) \to 0$ in $L^2(0, +\infty)$ as $t \to \infty$), and that $i\mathbb{R} \subset \sigma(A)$. To check the last claim, it is sufficient to observe that every $\lambda \in \mathbb{C}_-$ is an eigenvalue of A, with corresponding eigenfunction $u(x) = \exp(\lambda x)$ ($u \in H^1(0, +\infty)$), and to use the fact that $\sigma(A)$ is closed.

Notice that only the location of the spectrum plays a role in Theorem 4.5. When we look at the decay rate, then the behavior of the resolvent $(\lambda I - A)^{-1}$ as $\lambda \to \infty$ has also to be considered, as is demonstrated in next result (see (Huang, 1985), (Prüss, 1984)).

Theorem 4.7. *(Huang-Prüss) Assume that* $(S(t))_{t\geq 0}$ *is bounded. Then* $(S(t))_{t\geq 0}$ *is exponentially stable if and only if* $i\mathbb{R} \subset \rho(A)$ *and* $\sup_{\beta\in\mathbb{R}} ||(i\beta I - A)^{-1}|| < \infty$.

Finally, a polynomial decay rate may be asserted when the growth of the resolvent on the imaginary axis is polynomial (see (Liu *et al.*, 2005)):

Theorem 4.8. *(Liu-Rao) Assume that* $(S(t))_{t\geq 0}$ *is bounded, that* $i\mathbb{R} \subset \rho(A)$*, and that for some number* $s > 0$

$$\sup_{|\beta|\geq 1} \frac{1}{|\beta|^s} ||(i\beta I - A)^{-1}|| < \infty.$$

Then, for any $k \in \mathbb{N}^*$ *there exists a constant* $C_k > 0$ *such that*

$$||S(t)z||_H \leq C_k \left(\frac{\ln t}{t}\right)^{\frac{k}{s}} (\ln t)||z||_{D(A^k)} \quad \forall z \in D(A^k).$$

Remark 4.9. 1. The smoother z, the faster decay.
2. For $k = 1$, $||S(t)z|| \leq C(\ln t)^{1+1/s}t^{-1/s}||z||_{D(A)}$. Notice that $||z||_{D(A)}$ cannot be replaced by $||z||_H$! In fact we have the (quite surprising but easy) result, whose proof is left to the reader:

Proposition 4.10. *If* $||S(t)x||_H \leq f(t)||x||_H$ *with* $\lim_{t\to+\infty} f(t) = 0$*, then* $(S(t))_{t\geq 0}$ *is exponentially stable.*

4.3. *Stabilizability of infinite-dimensional systems*

For any bounded operator $K \in L(H,U)$, we let A_K denote the operator $A_K z = Az + BKz$ with domain $D(A_K) = D(A)$, and by $(S_K(t))_{t\geq 0}$ the semigroup generated by A_K.

Definition 4.11. The control system $\Sigma_{A,B}$ is said to be

– *exponentially stabilizable* if there exists a feedback $K \in L(H,U)$ such that the operator A_K is exponentially stable; *i.e.*, for some constants $C > 0,\ \mu > 0$,

$$||S_K(t)|| \leq Ce^{-\mu t} \quad \forall t \geq 0.$$

– *completely stabilizable* if it is exponentially stabilizable with an arbitrary exponential decay rate; *i.e.*, for arbitrary $\mu \in \mathbb{R}$, there exists a feedback $K \in L(H,U)$ and a constant $C > 0$ such that

$$||S_K(t)|| \leq Ce^{-\mu t} \quad \forall t \geq 0.$$

The first result is due to Datko (1972). (See also (Zabczyk, 1992, Theorem 3.3 p. 227).)

Theorem 4.12. *If the system* $(\Sigma_{A,B})$ *is null controllable, then it is exponentially stabilizable.*

The next result gives an infinite dimensional version of Wonham's theorem.

Theorem 4.13. *Assume that* A *generates a group* $(S(t))_{t\in\mathbb{R}}$ *of operators. Then the following properties are equivalent.*

1) $(\Sigma_{A,B})$ *is exactly controllable in some time* $T>0$*;*

2) $(\Sigma_{A,B})$ *is null controllable in some time* $T>0$*;*

3) $(\Sigma_{A,B})$ *is completely stabilizable.*

The implication $(2)\Rightarrow(3)$ is due to Slemrod (1974), whereas the implication $(3)\Rightarrow(1)$ is due to Megan (1975) (see also (Zabczyk, 1992, Theorem 3.4 p. 229)). $(1)\Rightarrow(2)$ is obvious.

Theorem 4.13 applies in particular to a skew-adjoint operator A, which generates a group of isometries on H. In fact more can be said. First, we have the "controllability via stabilizability" principle due to Russell (1978). On the other hand, explicit exponentially stabilizing feedback laws may be given.

Corollary 4.14. *(Liu, 1997, Theorem 2.3) Assume that* A *is skew-adjoint. Then the following properties are equivalent.*

1) $(\Sigma_{A,B})$ *is exactly controllable in some time* $T>0$*;*

2) $(\Sigma_{A,B})$ *is null controllable in some time* $T>0$*;*

3) $(\Sigma_{A,B})$ *is completely stabilizable;*

4) $(\Sigma_{A,B})$ *is exponentially stabilizable;*

5) For every positive definite self-adjoint operator $S\in L(U)$*, the operator* $A-BSB^*$ *generates an exponentially stable semigroup on* H*.*

As an application, following Liu (1997), we consider an abstract second-order control system

$$\ddot{w}+Lw = a(x)u, \qquad [110]$$

$$(w(0),\dot{w}(0)) = (w_0,w_1), \qquad [111]$$

where L is a positive definite self-adjoint operator with domain $D(L)\subset H$, $u\in U=H$, and $a\in L^\infty(\Omega,\mathbb{R})$ is such that $aH\subset H$. We introduce the space $\hat{H}=D(L^{\frac{1}{2}})\times H$ endowed with the scalar product

$$\langle (f_1,g_1),(f_2,g_2)\rangle_{\hat{H}} = \langle L^{\frac{1}{2}}f_1, L^{\frac{1}{2}}f_2\rangle_H + \langle g_1,g_2\rangle_H.$$

The second-oder control system [110]-[111] may be written in the form [107], with

$$A = \left[\begin{array}{cc} 0 & I \\ -L & 0 \end{array}\right] \quad \text{defined on the domain} \quad D(A) = D(L) \times D(L^{\frac{1}{2}}),$$

and $B \in L(U, \hat{H})$ is defined by $B(u) = (0, au)$.

Example 4.15. 1) The wave equation with internal control

$$\begin{array}{rcll} \ddot{w} - \Delta w & = & a(x)u & \text{in } (0,T) \times \Omega \\ w & = & 0 & \text{on } (0,T) \times \partial\Omega \\ (w(0), w_t(0)) & = & (w_0, w_1) & \end{array}$$

is concerned, with $H = U = L^2(\Omega)$, $L = -\Delta$, $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$, $D(L^{\frac{1}{2}}) = H_0^1(\Omega)$, a being any function in $L^\infty(\Omega)$.

2) The plate equation with internal control

$$\begin{array}{rcll} \ddot{w} + \Delta^2 w & = & a(x)u & \text{in } (0,T) \times \Omega \\ w = \Delta w & = & 0 & \text{on } (0,T) \times \partial\Omega \\ (w(0), w_t(0)) & = & (w_0, w_1) & \end{array}$$

is also concerned, with $H = U = L^2(\Omega)$, $L = \Delta^2$, $D(L) = \{z \in H^4(\Omega) |\ z, \Delta z \in H_0^1(\Omega)\}$, $D(L^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$, a being any function in $L^\infty(\Omega)$.

Notice that $B^*(w, v) = av$. Letting $S = I$ in the property (5) of Corollary 4.14, we see that the exact controllability of [107] in $\hat{H}$ in some time T is equivalent to the exponential stability of the system

$$\left(\begin{array}{c} w \\ v \end{array}\right)_t = (A - BB^*) \left(\begin{array}{c} w \\ v \end{array}\right) = \left[\begin{array}{cc} 0 & I \\ -L & -a(x)^2 \end{array}\right] \left(\begin{array}{c} w \\ v \end{array}\right).$$

which is the first-order reduction of the damped second-order system

$$\ddot{w} + Lw + a^2 \dot{w} = 0.$$

Setting $d(x) = a(x)^2$, we have that $|a(x)| > a_0 > 0$ in some region $\omega \subset \Omega$ if and only if $d(x) > d_0 := a_0^2 > 0$ in the same region. In particular, if $a = \chi_\omega$, then also $d = \chi_\omega$.

Example 4.16. Consider the wave equation in the domain $\Omega = (0, \pi) \times (0, \pi)$ with internal control $\chi_\omega u$, with $\omega = (a, b) \times (0, \pi)$, $0 < a < b < \pi$. The GCC being violated, the system is not EC (for any time T) in the space $\hat{H} = H_0^1(\Omega) \times L^2(\Omega)$. According to Corollary 4.14, the closed-loop system

$$\left\{ \begin{array}{l} \ddot{w} - \Delta w + \chi_\omega \dot{w} = 0 \\ w_{|\partial\Omega} = 0 \end{array} \right.$$

is not exponentially stable. Applying Theorem 4.8, Liu and Rao have however proved in (Liu *et al.*, 2005) that a polynomial decay rate holds in that case, namely

$$||(w, w_t)||_{\hat{H}} \leq C_k \left(\frac{\ln t}{t}\right)^{\frac{k}{2}} (\ln t)||(w_0, w_1)||_{D(A^k)} \quad \forall (w_0, w_1) \in D(A^k).$$

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