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GLOBAL STABILIZATION OF THE GENERALIZED KORTEWEG–DE VRIES EQUATION POSED ON A FINITE DOMAIN*

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Abstract. This paper is concerned with the internal stabilization of the generalized Korteweg–de Vries equation on a bounded domain. The global well-posedness and the exponential stability are investigated when the exponent in the nonlinear term ranges over the interval $[1, 4)$. The global exponential stability is obtained whatever the location where the damping is active, confirming positively a conjecture of Perla Menzala, Vasconcellos, and Zuazua [*Quart. Appl. Math.*, 60 (2002), pp. 111–129].

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1. Introduction.

In this paper we study the generalized Korteweg–de Vries (GKdV) equation posed on a finite interval $I = (0, L)$ with a localized damping

$$\partial_t u + \partial_x u + a(u)\partial_x u + \partial_x^3 u + b(x)u = 0, \quad 0 \le x \le L, \quad t \ge 0, \tag{1.1}$$

satisfying the homogeneous boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0, \quad t \ge 0, \tag{1.2}$$

and the initial condition

$$u(x,0) = \phi(x), \qquad 0 \le x \le L. \tag{1.3}$$

Here $b \equiv b(x) \in L^2(I)$ is a given nonnegative function with its support ω being a subset of I, and the function $a \equiv a(\mu)$ is a given smooth function satisfying the growth condition

$$a(0) = 0, \qquad |a^{(j)}(\mu)| \le C(1 + |\mu|^{p-j}) \qquad \forall\, \mu \in \mathbb{R} \tag{1.4}$$

for $j = 0, 1$ if $1 \le p < 2$ and for $j = 0, 1, 2$ if $p \ge 2$.

We are mainly concerned with two issues regarding the initial boundary value problem (IBVP) (1.1)–(1.3). One is the well-posedness of the IBVP (1.1)–(1.3) in the classical Sobolev space $H^s(I)$. That is, we aim to establish existence, uniqueness, and persistence properties of solutions corresponding to the given initial data ϕ, together with continuous dependence of solution upon the initial data ϕ. The other one is the long time behavior of solutions of the IBVP (1.1)–(1.3). More precisely, we will

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investigate if the solution of (1.1)–(1.3) tends to zero as $t \to \infty$ and under what rate it decays.

When $a(\mu) = \mu$ and $b \equiv 0$, (1.1) is the celebrated Korteweg–de Vries (KdV) equation

$$\partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \tag{1.5}$$

which was derived by Korteweg and de Vries [18] in 1895 as a model for propagation of surface water waves along a channel. The equation has been studied extremely intensively from various aspects of both mathematics and physics since the early 1960s when *soliton* was discovered through the KdV equation and the *inverse scattering transform*, a so-called nonlinear Fourier transform, was invented to solve the initial value problem of the KdV equation (cf. [19, 21]). In particular, the pure initial value problem for the KdV equation posed on the whole line $\mathbb{R}$ or on a periodic domain $\mathbb{S}$ has received a lot of attention in the past three decades for its well-posedness problem in the classical Sobolev space $H^s(\mathbb{R})$ or $H^s(\mathbb{S})$ (see [1, 2, 5, 6, 10, 13, 14, 15, 16] and the references therein). So far, the best known results are that the pure initial value problem of the KdV equation, when posed on the real line $\mathbb{R}$, is well-posed in the space $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$ [16, 10] and is, when posed on the periodic domain $\mathbb{S}$, well-posed in the space $H^s(\mathbb{S})$ for $s \geq -1$ [12].

For the KdV equation posed on a finite interval, the study of its IBVP began with Bubnov [7], who investigated the general two-point boundary value problem

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = f(x,t), \qquad u(x,0) = 0, \\ \alpha_1 \partial_x^2 u(0,t) + \alpha_2 \partial_x u(0,t) + \alpha_3 u(0,t) = 0, \\ \beta_1 \partial_x^2 u(1,t) + \beta_2 \partial_x u(1,t) + \beta_3 u(1,t) = 0, \\ \xi_1 \partial_x u(1,t) + \xi_2 u(1,t) = 0 \end{cases} \tag{1.6}$$

posed on the interval $(0,1)$. Here $\alpha_j,\ \beta_j,\ \xi_i \in \mathbb{R},\ j = 1,2,3,\ i = 1,2$, are constants, and assumptions are imposed so that the L^2-norm of the solutions of the linear version of (1.6) (obtained by dropping the nonlinear term $u\partial_x u$) is decreasing. It was shown in [7] that for given $T > 0$ and $f \in H^1([0,T]; L^2(0,1))$, there exists a $T^* > 0$ depending on $\|f\|_{H^1([0,T];L^2(0,1))}$ such that (1.6) admits a unique solution

$$u \in L^2([0,T^*]; H^3(0,1)), \qquad u_t \in L^\infty([0,T^*]; L^2(0,1)) \cap L^2([0,T^*]; H^1(0,1)).$$

In [4], Bona, Sun, and Zhang studied the following nonhomogeneous boundary value problem for the KdV equation posed on the interval $(0,1)$:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \qquad u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad \partial_x u(1,t) = h_3(t). \end{cases} \tag{1.7}$$

The IBVP (1.7) was shown in [4] to be well-posed in the space $H^s(0,1)$ for $s \geq 0$ with $h_j \in H_{loc}^{(s+1)/3}(\mathbb{R}^+)$, $j = 1,2$, and $h_3 \in H_{loc}^{s/3}(\mathbb{R}^+)$. Earlier, in the case of $h_j = 0$, $j = 1,2,3$, Zhang [38] showed that the IBVP (1.7) is well-posed in the space $H^{3k+1}(0,1)$ for $k = 0,1,2,\ldots$, and Perla Menzala, Vasconcellos, and Zuazua [24] showed that the IBVP (1.7) is well-posed in the space $L^2(0,1)$. The reader is referred to [3, 9, 11] for more references on the IVBP for the KdV equation posed either on a finite interval or on the half-line $\mathbb{R}^+$.

The study of long time behavior of solutions of the KdV equation posed on a finite interval was also started by Bubnov [8]. In [17], Komornik, Russell, and Zhang considered the stabilization problem for the KdV equation posed on finite interval $(0, \pi)$ with the periodic boundary conditions

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = f, \qquad u(x,0) = \phi(x), \\ \partial_x^j u(0,t) = \partial_x^j u(\pi,t), \qquad j = 0,1,2, \end{cases} \tag{1.8}$$

where $f \equiv f(x,t)$ is considered as a control input acting on the whole domain $(0, \pi)$. A special feedback control law $f = \mathcal{B}u$ is designed to ensure the conservation of mass, and the resulting closed loop system

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = \mathcal{B}u, \qquad u(x,0) = \phi(x), \\ \partial_x^j u(0,t) = \partial_x^j u(\pi,t), \qquad j = 0,1,2, \end{cases} \tag{1.9}$$

is demonstrated to be exponentially stable; its solution converges exponentially to the average value of its initial datum ϕ over the domain $(0, \pi)$ as $t \to \infty$. The same problem was studied by Russell and Zhang [29, 31], assuming that the control input f acted only on an open subdomain of $(0, \pi)$. The resulting closed loop system is shown to be locally exponentially stable in the sense that the initial data ϕ is required to be small in a certain sense. In [30], Russell and Zhang studied boundary stabilization of the KdV equation posed on a finite interval $(0, \pi)$ with the periodic boundary conditions. The resulting closed loop system appears in the following form:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \qquad u(x,0) = \phi(x), \\ u(0,t) = u(\pi,t), \quad \partial_x^2 u(0,t) = \partial_x^2 u(\pi,t), \quad \partial_x u(\pi,t) = \alpha\partial_x u(0,t), \end{cases} \tag{1.10}$$

where $-1 < \alpha < 1$ is a constant. The system was shown by Russell and Zhang [30] to be locally exponentially stable when $\alpha \neq -\frac{1}{2}$. For the exceptional case $\alpha = -\frac{1}{2}$, the system (1.10) was also shown to be locally exponentially stable later by Sun [34] via a different approach. A similar boundary stabilization problem was studied by Zhang [38] for the KdV equation posed on a finite interval $(0, 1)$ with the Dirichlet boundary condition. The resulting closed loop system is of the form

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \qquad u(x,0) = \phi(x), \\ u(0,t) = 0, \quad u(1,t) = 0, \quad \partial_x u(1,t) = \gamma\partial_x u(0,t) \end{cases} \tag{1.11}$$

with $0 \leq |\gamma| < 1$. When $0 < |\gamma| < 1$, the system (1.11) was shown to be locally exponentially stable.

For the KdV equation posed on a finite interval $(0, L)$ with the homogeneous boundary conditions,

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad 0 \leq x \leq L, \quad t \geq 0, \\ u(x,0) = \phi(x), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0; \end{cases} \tag{1.12}$$

it is easily seen that any smooth solution u of (1.12) satisfies

$$\frac{d}{dt}\int_0^L |u(x,t)|^2 dx = -|\partial_x u(0,t)|^2.$$

This leads one to guess that any solution u of (1.12) may decay to zero as $t \to \infty$. However, this may not always be the case. In [25], Rosier discovered that if the length L of the domain $(0, L)$ belongs to the set

$$\mathcal{E} = \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2 + kl + l^2},\ k \text{ and } l \text{ are positive integers} \right\},$$

then the linear system associated with the IBVP (1.12) possesses a solution with a constant L^2-norm. It is thus reasonable to say that not all solutions of (1.12) decay to 0 as $t \to \infty$.

If the length L of the interval I does not belong to the set $\mathcal{E}$, it has been demonstrated in [24] by Perla Menzala, Vasconcellos, and Zuazua that there exist $\delta > 0$ and $\gamma > 0$ such that if $\phi \in L^2(I)$ with

$$\|\phi\|_{L^2(I)} \le \delta,$$

then

$$\|u(\cdot,t)\| \le C\|\phi\|_{L^2(I)} e^{-\gamma t} \qquad \forall\, t \ge 0,$$

where C depends only on $\|\phi\|_{L^2(I)}$.

In order to handle the case of $L \in \mathcal{E}$ and to have the solutions of (1.12) with large amplitude stabilized, Perla Menzala, Vasconcellos, and Zuazua [24] introduced an extra damping term $b(x)u$ to the equation in (1.12) to get the following system:

$$\left\{ \begin{array}{l} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u + b(x)u = 0, \quad 0 \le x \le L, \quad t \ge 0, \\ u(x,0) = \phi(x), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0. \end{array} \right. \tag{1.13}$$

Here $b \in L^\infty(I)$ is assumed to be a nonnegative function satisfying $b(x) \ge b_0 > 0$ a.e. in an open, nonempty subset ω of I. Perla Menzala, Vasconcellos, and Zuazua [24] showed the following theorem.

THEOREM 1.1 (Perla Menzala, Vasconcellos, and Zuazua). *For any given $\phi \in L^2(I)$, the IBVP* (1.13) *admits a unique solution $u \in C(\mathbb{R}^+; L^2(I)) \cap L^2_{loc}(\mathbb{R}^+; H^1(I))$. Moreover, if*

$$\omega \text{ contains two sets of the form } (0,\delta) \text{ and } (L-\delta, L) \quad \text{for some } \delta > 0, \tag{1.14}$$

then, for any $L > 0$ and $N > 0$, there exist $C > 0$ and $\mu > 0$ such that for any $\phi \in L^2(I)$ with $\|\phi\|_{L^2(I)} \le N$, the corresponding solution u of (1.13) *satisfies*

$$\|u(\cdot,t)\|_{L^2(I)} \le C\|\phi\|_{L^2(I)} e^{-\mu t} \qquad \forall\, t \ge 0. \tag{1.15}$$

Remarks.

(a) The result of Perla Menzala, Vasconcellos, and Zuazua presented in Theorem 1.1 represents a significant advance in the subject of stabilization of the KdV equation. Indeed, all the previous results except [17], in which the feedback control acts on the whole domain, are local in the sense that only *small amplitude* solutions have been shown to decay exponentially; they are essentially linear stability results. In contrast, the stability result presented in Theorem 1.1 is global; all solutions of (1.13), large or small, decay exponentially in the space $L^2(I)$. It is truly a nonlinear stability result.

(b) Perla Menzala, Vasconcellos, and Zuazua have conjectured in [24] that Theorem 1.1 still holds without the assumption (1.14). Pazoto [22] has proved that this is indeed true. The idea of the proof is as follows: Pazoto shows that the unique continuation property holds, whatever ω is, in proving by multipliers and compactness arguments that any solution vanishing on a subinterval is necessarily smooth.

(c) In Theorem 1.1, the determination of the decay rate ν depends on the size of the initial value ϕ in the space $L^2(I)$. The system (1.12) is locally uniformly exponentially stable in $L^2(I)$.

In this paper, motivated by Perla Menzala, Vasconcellos, and Zuazua's work, we will consider the IBVP of the GKdV equation as described by (1.1)–(1.3) for its well-posedness and long time behavior of its solutions. We will first extend the well-posedness result established in [4] for the IBVP (1.7) for the KdV equation to the IBVP (1.1)–(1.3) for the GKdV equation.

THEOREM 1.2. *Suppose the growth condition* (1.4) *for $a = a(\mu)$ is satisfied with*

$$1 \leq p < 2.$$

Let $0 \leq s \leq 3$ and $T > 0$ be given. Then for any given $\phi \in H^s(I)$ satisfying the s-compatibility conditions

$$\begin{cases} \phi(0) = \phi(L) = 0 \quad \text{when } 1/2 < s \leq 3/2, \\ \\ \phi(0) = \phi(L) = \phi'(L) = 0 \quad \text{when } 3/2 < s \leq 3, \end{cases} \tag{1.16}$$

the IBVP (1.1)–(1.3) *admits a unique solution*

$$u \in C([0,T]; H^s(I)) \cap L^2(0,T; H^{s+1}(I)).$$

Moreover, the solution u depends on its initial value continuously in the corresponding spaces.[1]

Remarks.

(i) We may extend the theorem to include the nonhomogeneous boundary conditions as in [4]. But we choose not to do so here since our main concern is the long time behavior of solutions of (1.1)–(1.3).

(ii) The property that $\phi \in H^s(I)$ implies that the corresponding solution $u \in L^2(0,T; H^{s+1}(I))$ reveals a type of global Kato smoothing effect of the GKdV equation posed on a finite interval. This global Kato smoothing effect together with the property of persistence of regularity presented in Theorem 1.2 yields the following strong smoothing properties for solutions of (1.1)–(1.3):

$$\phi \in L^2(I) \Longrightarrow u \in C([\epsilon,T]; H^3(I)) \cap L^2(\epsilon,T; H^4(I)) \quad \text{for any } \epsilon > 0.$$

---

[1] If a is a real analytic function, then the solution u depends on its initial value analytically in the corresponding spaces (cf. [4, 39]).

In the case of $p \geq 2$, we have the following local well-posedness result.

THEOREM 1.3. *Suppose the growth condition* (1.4) *for* $a = a(\mu)$ *is satisfied with*

$$p \geq 2.$$

Then for any given $\phi \in H^s(I)$ *with* $s = 3$ *or with* 3 *satisfying the* s*-compatibility condition* (1.16) *there exists a* $T^* > 0$ *depending on* $\|\phi\|_{H^s(I)}$ *such that the IBVP* (1.1)–(1.3) *admits a unique solution* $u \in C([0, T^*]; H^s(I)) \cap L^2(0, T^*; H^{s+1}(I))$ *which depends on its initial value continuously.*

In order to get the global well-posedness results in the space $H^1(I)$ or $H^3(I)$, one needs some a priori H^1- or H^3-global estimate. However, those a priori estimates are not available. In fact, the only available a priori estimate for the IBVP (1.1)–(1.3) is an L^2-a priori estimate which is not sufficient to yield a global well-posedness result because we do not know if the IBVP (1.1)–(1.3) is locally well-posed in the space $L^2(I)$. Whether Theorem 1.3 can be extended to a global well-posedness result at least in the case of $2 \leq p < 4$ remains an interesting open question. On the other hand, when $2 \leq p < 4$, the following a priori estimates are known to hold for solutions of (1.1)–(1.3) (cf. Lemma 2.5 in section 2):

$$\|u(\cdot, T)\|^2_{L^2(I)} + \int_0^T (\partial_x u)^2(0, \tau) d\tau + \int_0^T \int_0^L b(x) u^2(x, \tau) dx d\tau = \|\phi\|^2_{L^2(I)}$$

and

$$\int_0^T \int_0^L |\partial_x u|^2 \, dx dt \leq \frac{L + (C+1)T}{2} ||\phi||^2_{L^2(I)} + C_p T ||\phi||^{\frac{8+2p}{4-p}}_{L^2(I)}.$$

Taking advantage of these estimates, we are able to establish the global existence of solutions of (1.1)–(1.3) in the space $L^2(I)$.

THEOREM 1.4. *Suppose the growth condition* (1.4) *for* $a = a(\mu)$ *is satisfied with*

$$1 \leq p < 4.$$

Then for any given $\phi \in L^2(I)$, *the IBVP* (1.1)–(1.3) *admits a solution*

$$u \in C_w(\mathbb{R}^+; L^2(I)) \cap L^2_{loc}(\mathbb{R}^+; H^1(I)).$$

Note that when $1 \leq p < 2$, a solution given by Theorem 1.4 is identical to the unique solution provided by Theorem 1.2.

With the well-posedness results in hand, we may then investigate the long time behavior of solutions. The following theorem is the main result of this paper.

THEOREM 1.5. *Under the assumptions of Theorem* 1.4, *assume additionally that the support* ω *of the function* b *in* (1.1) *contains a nonempty open subset of* $(0, L)$. *Then there exists a number* $\nu > 0$ *depending only on* L *and a nondecreasing continuous function* $\beta:\ \mathbb{R}^+ \to \mathbb{R}^+$ *such that for a given* $\phi \in L^2(I)$, *any solution* u *of* (1.1)–(1.3) *provided by Theorem* 1.4 *satisfies*

$$\|u(\cdot, t)\|_{L^2(I)} \leq \beta\big(\|\phi\|_{L^2(I)}\big) e^{-\nu t} \qquad \forall\ t \geq 0. \tag{1.17}$$

Remarks.

(1) Theorem 1.5 confirms positively the conjecture of Perla Menzala, Vasconcellos, and Zuazua [24] for the GKdV equation.

(2) The decay rate ν in Theorem 1.5 depends only on L and does not, in particular, depend on the size of the initial value ϕ in the space $L^2(I)$. The system (1.1)–(1.3) is globally uniformly exponentially stable.

Theorem 1.5 will be proved by the same compactness-uniqueness argument used by Perla Menzala, Vasconcellos, and Zuazua [24]. The key is to establish the following unique continuation property for the solution of the GKdV equation.

Unique continuation property. Let ω be the support of b in the interval I. If $v \in L^\infty(0,T;L^2(I)) \cap L^2(0,T;H^1(I))$ solves

$$\begin{cases} \partial_t v + \partial_x v + a(v)\partial_x v + \partial_x^3 v = 0, \quad 0 \le x \le L, \quad t \ge 0, \\ \\ v(0,t) = 0, \quad v(L,t) = 0, \quad \partial_x v(L,t) = 0 \end{cases}$$

and, in addition, satisfies

$$\partial_x v(0,t) = 0 \qquad \forall\ t \in (0,T)$$

and

$$v(x,t) = 0 \qquad \forall\ (x,t) \in \omega \times (0,T),$$

then v vanishes on $I \times (0,T)$.

When $1 \le p < 2$, the solution v, in fact, belongs to the space $C([\epsilon,T];H^3(I))$ for any $0 < \epsilon < T$ thanks to the strong smoothing property of the GKdV equation (cf. Remark (ii) of Theorem 1.2). Thus v vanishes on the whole domain $I \times (0,T)$ by the standard unique continuation property of the KdV-type equation (cf. [32, 37]). When $2 \le p < 4$, we cannot use the standard unique continuation property because we do not know if $v \in C([\epsilon,T];H^3(I))$. Instead, we will use a new unique continuation property for the GKdV equation as described below.

Theorem 1.6. *Let $0 < l < L$ and $T > 0$. If $w \in L^\infty(0,T;H^1(0,l))$ solves*

$$\begin{cases} \partial_t w + \partial_x w + a(w)\partial_x w + \partial_x^3 w = 0 & \text{in } (0,l) \times (0,T), \\ w(0,t) = 0 & \text{for a.e. } t \in (0,T), \\ w \equiv 0 & \text{in } (l',l) \times (0,T) \end{cases}$$

with $a \in C^0(\mathbb{R};\mathbb{R})$ and $0 < l' < l$, then $w \equiv 0$ in $(0,l) \times (0,T)$.

The proof of this unique continuation property is mainly based on a key Carleman estimate for the KdV equation established earlier by Rosier [27] (see also Lemma 3.4 in section 3).

When $1 \le p < 2$, for a given $\phi \in L^2(I)$, (1.1)–(1.3) admits a unique solution u which not only decays exponentially in the space $L^2(I)$ but also in the space $H^3(I)$ as described by the following theorem.

Theorem 1.7. *Under the assumption of Theorem 1.2, assume additionally that the support ω of the function b in (1.1) contains a nonempty open subset of $(0,L)$. Then there exists a number $\nu > 0$ depending only on L and a continuous function $\alpha : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\varepsilon > 0$ and any $\phi \in L^2(I)$, the corresponding solution u of (1.1)–(1.3) satisfies*

$$\|u(\cdot,t)\|_{H^3(I)} \le \alpha\big(\|\phi\|_{L^2(I)}, \epsilon\big) e^{-\nu t} \qquad \forall\ t \ge \epsilon. \tag{1.18}$$

The paper is organized as follows. In section 2, we study the well-posedness problem of the IBVP (1.1)–(1.3) and provide the proofs of Theorems 1.2, 1.3, and 1.4. In section 3, the long time behavior of solutions of (1.1)–(1.3) is investigated.

2. Well-posedness.

In this section, attention will be given to the nonlinear IBVP

$$
\left\{ \begin{array}{l} \partial_t u + \partial_x u + a(u)\partial_x u + \partial_x^3 u + b(x)u = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \\ \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0 \end{array} \right. \tag{2.1}
$$

for its well-posedness in the classical Sobolev space $H^s(I)$.

Considered first is the linear problem

$$
\left\{ \begin{array}{l} \partial_t u + \partial_x u + \partial_x^3 u = 0, \quad u(x,0) = \phi(x)\,, \quad x \in (0,L), \\ \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0 \end{array} \right. \tag{2.2}
$$

with homogeneous boundary conditions and no forcing. Let A be the linear operator defined by

$$
Af = -f''' - f'
$$

with the domain

$$
\mathcal{D}(A) = \{ f \in H^3(I),\ f(0) = f(L) = f'(L) = 0 \}.
$$

The IBVP (2.2) can be written as the initial value problem of an abstract evolution equation in the space $L^2(I)$, namely

$$
\frac{du}{dt} = Au, \qquad u(0) = \phi, \tag{2.3}
$$

where the spatial variable is suppressed. It is easily verified that both A and its adjoint A^* are dissipative, i.e.,

$$
\Big\langle Af, f \Big\rangle_{L^2(I)} \le 0, \qquad \Big\langle A^* g, g \Big\rangle_{L^2(I)} \le 0
$$

for any $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(A^*)$, where $A^* g = g''' + g'$ and

$$
\mathcal{D}(A^*) = \{ f \in H^3(0,L);\ f(0) = f'(0) = f(L) = 0 \}.
$$

Thus the operator A is the infinitesimal generator of a C_0-semigroup $W(t)$ in the space $L^2(I)$ (see [23]). By semigroup theory (see [23]), in this situation in the overlying space $L^2(I)$, for any $\phi \in L^2(I)$,

$$
u(t) = W(t)\phi
$$

belongs to the space $C_b(\mathbb{R}^+; L^2(I))$. The function u thus defined is called a *mild solution* of (2.2). Such solutions certainly solve (2.2) in the sense of distributions. If $\phi \in \mathcal{D}(A)$, then $u(t) = W(t)\phi$ belongs to the space $C(\mathbb{R}^+; H^3(I)) \cap C^1(\mathbb{R}^+; L^2(I))$ and $u(t) \in \mathcal{D}(A)$ for all $t \ge 0$. Moreover, the equation is satisfied in the sense of $C(\mathbb{R}^+; L^2(I))$ and, in particular, pointwise a.e. Such solutions are called *strong solutions.*

LEMMA 2.1. *For any $\phi \in L^2(I)$, $u(t) = W(t)\phi$ satisfies*

$$
\|u(\cdot,t)\|^2_{L^2(I)} + \int_0^t (\partial_x u)^2(0,\tau)d\tau = \|\phi\|^2_{L^2(I)}, \tag{2.4}
$$

$$\int_0^L xu^2(x,t)dx + 3\int_0^t \int_0^L (\partial_x u)^2(x,\tau)dxd\tau \le (L+t)\int_0^L \phi^2(x)dx \tag{2.5}$$

for any $t \ge 0$. *Moreover,* u *has the property*

$$\partial_x u \in C_b([0,L]; L_t^2(\mathbb{R}^+)),$$

and there exists a constant C *such that*

$$\sup_{x\in I} \|\partial_x u(x,\cdot)\|_{L^2(\mathbb{R}^+)} \le C\|\phi\|_{L^2(I)}. \tag{2.6}$$

Proof. See [4]. □

Note that estimate (2.6) for $W(t)$ reveals a sharp Kato smoothing effect of the system described by the IBVP (2.2). Next, attention is turned to the inhomogeneous linear problem

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = f(x,t), \qquad u(x,0) = 0\,, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0\,. \end{cases} \tag{2.7}$$

In terms of the operator A defined above, one may write (2.7) as an initial value problem for an abstract nonhomogeneous evolution equation, namely

$$\frac{du}{dt} = Au + f, \qquad u(0) = 0. \tag{2.8}$$

By standard semigroup theory (see again [23]), for any $f \in L^1_{loc}(\mathbb{R}^+; L^2(I))$,

$$u(t) = \int_0^t W(t-\tau)f(\tau)d\tau \tag{2.9}$$

belongs to the space $C(\mathbb{R}^+; L^2(I))$ and is called a mild solution of (2.8). It is a weak solution of (2.7) in the sense of distributions. In addition, if $f(t) \in \mathcal{D}(A)$ for $t > 0$ and $Af \in L^1_{loc}(\mathbb{R}^+; L^2(I))$, then $u(t)$ given by (2.9) solves (2.8) a.e. on $[0,T)$ and is called a strong solution of (2.8).

LEMMA 2.2. *There exists a constant* C *such that for any* $f \in L^1_{loc}(\mathbb{R}^+; L^2(I))$, *the solution* u *of* (2.7) *satisfies*

$$\|u(\cdot,t)\|_{L^2(I)} + \|\partial_x u(0,\cdot)\|_{L^2(0,t)} \le C\|f\|_{L^1(0,t;L^2(I))} \tag{2.10}$$

and

$$\int_0^L xu^2(x,t)dx + \int_0^t \int_0^L (\partial_x u)^2(x,\tau)dxd\tau \le C(L+t)\|f\|^2_{L^1(0,t;L^2(I))} \tag{2.11}$$

for any $t \ge 0$. *Moreover, the solution* u *has the property*

$$\partial_x u \in C_b([0,L]; L_t^2(\mathbb{R}^+)),$$

and there exists a constant C *such that*

$$\sup_{x\in I} \|\partial_x u(x,\cdot)\|_{L^2(\mathbb{R}^+)} \le C\|f\|_{L^1(0,t;L^2(I))}. \tag{2.12}$$

Proof. See [4]. □

For any $T > 0$ and $0 \leq s \leq 3$, let X_s be the collection of all functions ϕ in the space $H^s(I)$ satisfying the compatibility conditions (1.16) with its usual topology, and let $Y_{s,T}$ be the collection of

$$v \in C([0,T]; X_s) \cap L^2([0,T]; H^{s+1}(I))$$

with $\partial_x v \in C([0,L]; L^2(0,T))$. A norm $\| \cdot \|_{Y_{s,T}}$ on the space $Y_{s,T}$ is defined by

$$\|v\|_{Y_{s,T}} := \left(\|v\|^2_{C([0,T];H^s(I))} + \|v\|^2_{L^2([0,T];H^{s+1}(I))} + \|\partial_x v\|^2_{C([0,L];L^2(0,T))} \right)^{1/2}$$

for $v \in Y_{s,T}$.

Lemma 2.3. *Let a be a C^0 function satisfying*

$$|a(\mu)| \leq C(1 + |\mu|^p) \quad \textit{for any } \mu \in \mathbb{R}$$

with $0 \leq p \leq 2$. There exists a constant C such that for any $T > 0$ and u, $v \in Y_{0,T}$,

$$\int_0^T \|a(u(\cdot,t))\partial_x v(\cdot,t)\|_{L^2(I)} dt$$
$$\leq CT^{(2-p)/4} \|u\|^p_{Y_{0,T}} \|v\|_{Y_{0,T}} + CT^{1/2} \left(1 + \|u\|^p_{Y_{0,T}}\right) \|v\|_{Y_{0,T}}.$$

Proof. Using the assumption on the function a and the inequality

$$\|u(\cdot,t)\|_{L^\infty(I)} \leq C \left(\|u(\cdot,t)\|_{L^2(I)} + \|u(\cdot,t)\|^{\frac{1}{2}}_{L^2(I)} \|\partial_x u(\cdot,t)\|^{\frac{1}{2}}_{L^2(I)} \right),$$

one obtains

$$\begin{aligned} \|a(u(\cdot,t))\partial_x v(\cdot,t)\|_{L^2(I)} &\leq C\|u(\cdot,t)\|^p_{L^\infty(I)} \|\partial_x v(\cdot,t)\|_{L^2(I)} + C\|\partial_x v(\cdot,t)\|_{L^2(I)} \\ &\leq C\|u(\cdot,t)\|^{p/2}_{L^2(I)} \|\partial_x u(\cdot,t)\|^{p/2}_{L^2(I)} \|\partial_x v(\cdot,t)\|_{L^2(I)} \\ &\quad + C\left(1 + \|u(\cdot,t)\|^p_{L^2(I)}\right) \|\partial_x v(\cdot,t)\|_{L^2(I)}. \end{aligned}$$

The first term, when integrated with respect to t, is bounded as follows:

$$\begin{aligned} &\int_0^T \|u(\cdot,t)\|^{p/2}_{L^2(I)} \|\partial_x u(\cdot,t)\|^{p/2}_{L^2(I)} \|\partial_x v(\cdot,t)\|_{L^2(I)} dt \\ &\leq \sup_{0\leq t\leq T} \|u(\cdot,t)\|^{p/2}_{L^2(I)} \left(\int_0^T \|\partial_x u(\cdot,t)\|^2_{L^2(I)} dt \right)^{p/4} \left(\int_0^T \|\partial_x v(\cdot,t)\|^{4/(4-p)}_{L^2(I)} dt \right)^{(4-p)/4} \\ &\leq T^{(2-p)/4} \|u\|^p_{Y_{0,T}} \|v\|_{Y_{0,T}} \end{aligned}$$

if $0 \leq p \leq 2$. Thus

$$\int_0^T \|a(u(\cdot,t))\partial_x v(\cdot,t)\|_{L^2(I)} dt$$
$$\leq C2^{p/2} T^{(2-p)/4} \|u\|^p_{Y_{0,T}} \|v\|_{Y_{0,T}} + CT^{1/2} \left(1 + \|u\|^p_{Y_{0,T}}\right) \|v\|_{Y_{0,T}}.$$

The proof is complete. □

Lemma 2.4. *There exists a constant C depending only on L such that for any $T > 0$, $1 \le p \le 2$, $b \in L^2(I)$, and u, v, $w \in Y_{0,T}$,*

$$\int_0^T \|bu\|_{L^2_x(I)} dt \le CT^{1/2} \|b\|_{L^2(I)} \|u\|_{Y_{0,T}}, \tag{2.13}$$

$$\int_0^T \|u\partial_x w\|_{L^2_x(I)} dt \le CT^{1/4} \|u\|_{Y_{0,T}} \|w\|_{Y_{0,T}}, \tag{2.14}$$

$$\int_0^T \|u|w|^{p-1}\partial_x w\|_{L^2_x(I)} dt \le CT^{(2-p)/4} \|u\|_{Y_{0,T}} \|w\|^p_{Y_{0,T}}, \tag{2.15}$$

and

$$\int_0^T \|u|v|^{p-1}\partial_x w\|_{L^2_x(I)} dt \le CT^{(2-p)/4} \|u\|_{Y_{0,T}} \|w\|_{Y_{0,T}} \|v\|^{p-1}_{Y_{0,T}}. \tag{2.16}$$

Proof. The estimate (2.13) follows from the direct calculation

$$\begin{aligned}\int_0^T \left(\int_0^L |b(x)|^2 |u(x,t)|^2 dx\right)^{\frac{1}{2}} dt &\le \int_0^T \sup_{x\in(0,L)} |u(x,t)| \left(\int_0^L |b(x)|^2 dx\right)^{\frac{1}{2}} dt \\ &\le CT^{\frac{1}{2}} \|b\|_{L^2(I)} \|u\|_{Y_{0,T}}.\end{aligned}$$

The other estimates can be proved by a similar argument as that in the proof of Lemma 2.3 and are therefore omitted. The proof is complete. ☐

Now we turn to the nonlinear problem (2.1). We first present the following global a priori estimates for solutions of the IBVP (2.1).

Lemma 2.5. *Let a be a C^0 function satisfying*

$$|a(\mu)| \le C(1 + |\mu|^p) \quad \textit{for any } \mu \in \mathbb{R}$$

with $0 \le p < 4$. There exists a constant C_p such that for any smooth solution u of (2.1), the following estimates hold:

$$\|u(\cdot,T)\|^2_{L^2(I)} + \int_0^T (\partial_x u)^2(0,\tau) d\tau + 2\int_0^T \int_0^L b(x)u^2(x,\tau) dx d\tau = \|\phi\|^2_{L^2(I)} \tag{2.17}$$

and

$$\int_0^T \int_0^L |\partial_x u|^2 \, dxdt \le \frac{L + (C+1)T}{2} ||\phi||^2_{L^2(I)} + C_p T ||\phi||^{\frac{8+2p}{4-p}}_{L^2(I)} \tag{2.18}$$

for any $T > 0$.

Proof. Multiply both sides of the equation in (2.1) by u and integrate with respect to x over the interval I and about t over the interval $(0, T)$. An integration by parts leads directly to the equality (2.17). To prove the inequality (2.18), let us first introduce the functions

$$A(u) := \int_0^u a(v)\, dv, \qquad \tilde{A}(u) := \int_0^u va(v)\, dv.$$

Multiplying both sides of the equation in (2.1) by xu and integrating over $(0,L)\times(0,T)$, we obtain after some integrations by parts

$$\frac{1}{2}\left(\int_0^L x|u(x,T)|^2dx-\int_0^L x|\phi(x)|^2dx\right)+\frac{3}{2}\int_0^T\int_0^L|\partial_x u|^2dxdt-\frac{1}{2}\int_0^T\int_0^L|u|^2dxdt$$
$$-\int_0^T\int_0^L\tilde{A}(u)\,dxdt+\int_0^T\int_0^L xb(x)|u|^2dxdt=0.$$

Combining this equality with (2.17), we obtain

$$\frac{3}{2}\int_0^T\int_0^L|\partial_x u|^2dxdt\le\frac{L+T}{2}||\phi||^2_{L^2(I)}+\int_0^T\int_0^L\tilde{A}(u)\,dxdt. \tag{2.19}$$

We infer from the assumption on a that

$$|\tilde{A}(u)|\le C\left(\frac{u^2}{2}+\frac{|u|^{p+2}}{p+2}\right).$$

Hence

$$\begin{aligned}\left|\int_0^T\int_0^L\tilde{A}(u)\,dxdt\right|&\le\frac{C}{2}\int_0^T\int_0^L|u|^2dxdt+\frac{C}{p+2}\int_0^T\int_0^L|u|^{p+2}dxdt\\&\le\frac{CT}{2}||\phi||^2_{L^2(I)}+\frac{C}{p+2}\int_0^T||u||^2_{L^2(I)}\cdot||u||^p_{L^\infty(I)}dt\\&\le\frac{CT}{2}||\phi||^2_{L^2(I)}+\frac{2^{\frac{p}{2}}C}{p+2}\int_0^T||u||^{2+\frac{p}{2}}_{L^2(I)}\cdot||\partial_x u||^{\frac{p}{2}}_{L^2(I)}\,dt\\&\le\frac{CT}{2}||\phi||^2_{L^2(I)}+\frac{2^{\frac{p}{2}}CT^{1-\frac{p}{4}}}{p+2}||\phi||^{2+\frac{p}{2}}_{L^2(I)}\left(\int_0^T||\partial_x u||^2_{L^2(I)}dt\right)^{\frac{p}{4}}\\&\le\frac{CT}{2}||\phi||^2_{L^2(I)}+C_pT||\phi||^{\frac{8+2p}{4-p}}_{L^2(I)}+\frac{1}{2}\int_0^T\int_0^L|\partial_x u|^2dxdt\end{aligned} \tag{2.20}$$

for some positive constant C_p which depends only on C and p. The inequality (2.18) then follows from (2.19) and (2.20). The proof is complete. □

We now present our first well-posedness result for the IBVP (2.1).

PROPOSITION 2.6. *Let a be a C^1 function satisfying*

$$|a(\mu)|\le C(1+|\mu|^p),\quad |a'(\mu)|\le C(1+|\mu|^{p-1})\quad \textit{for any } \mu\in\mathbb{R}$$

with $1\le p<2$, and let $T>0$ be given. For any $\phi\in X_0$, the IBVP (2.1) admits a unique solution $u\in Y_{0,T}$, which also satisfies

$$\|u\|_{Y_{0,T}}\le\beta_0(\|\phi\|_{X_0})\|\phi\|_{X_0}, \tag{2.21}$$

where $\beta_0:\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing continuous function. Moreover, the corresponding solution map is locally Lipschitz continuous; for any ϕ, $\psi\in X_0$, the corresponding solutions u and v of (2.1) satisfy

$$\|u-v\|_{Y_{0,T}}\le\beta_0(\|\phi\|_{X_0}+\|\psi\|_{X_0})\|\phi-\psi\|_{X_0}. \tag{2.22}$$

Proof. Write the IBVP (2.1) in its integral equation form

$$u(t) = W(t)\phi - \int_0^t W(t-\tau)\Big[(a(u)\partial_x u)(\tau) + (bu)(\tau)\Big]d\tau, \tag{2.23}$$

where the spatial variable is suppressed throughout. For given $\phi \in X_0$, let $r > 0$ and $\theta > 0$ be constants to be determined. Let

$$S_{\theta,r} = \{v \in Y_{0,\theta},\ \|v\|_{Y_{0,\theta}} \leq r\}.$$

The set $S_{\theta,r}$ is a closed, convex, and bounded subset of the space $Y_{0,\theta}$ and therefore is a complete metric space in the topology induced from $Y_{0,\theta}$. Define a map Γ on $S_{\theta,r}$ by

$$\Gamma(v) = W(t)\phi - \int_0^t W(t-\tau)\Big[(a(v)\partial_x v)(\tau) + (bv)(\tau)\Big]d\tau$$

for $v \in S_{\theta,r}$. Applying Lemmas 2.2, 2.3, and 2.4, it is adduced that there are constants C_0, C_1, and C_2 for which

$$\begin{aligned}\|\Gamma(v)\|_{Y_{0,\theta}} &\leq C_0\|\phi\|_{X_0} + C_1\int_0^\theta \Big[\|a(v)\partial_x v(\cdot,\tau)\|_{L^2(I)} + \|bv(\cdot,\tau)\|_{L^2(I)}\Big]d\tau \\ &\leq C_0\|\phi\|_{X_0} + C_1\theta^{(2-p)/4}\|v\|^{p+1}_{Y_{0,\theta}} + C_2\theta^{\frac{1}{2}}(\|b\|_{L^2(I)}+1)\|v\|_{Y_{0,\theta}} \\ &\leq C_0\|\phi\|_{X_0} + C_1\theta^{(2-p)/4}r^p\|v\|_{Y_{0,\theta}} + C_2\theta^{\frac{1}{2}}(\|b\|_{L^2(I)}+1)\|v\|_{Y_{0,\theta}}\end{aligned}$$

for any $v \in S_{r,\theta}$. In addition, for any $v,\ w \in S_{r,\theta}$,

$$\begin{aligned}\Gamma(v) - \Gamma(w) &= \int_0^t W(t-\tau)\big[a(v)\partial_x v - a(w)\partial_x w + b(v-w)\big]d\tau \\ &= \int_0^t W(t-\tau)\big[a(v)\partial_x(v-w) + (a(v)-a(w))\partial_x w + b(v-w)\big]d\tau.\end{aligned}$$

Thus

$$\begin{aligned}\|\Gamma(v)-\Gamma(w)\|_{Y_{0,\theta}} &\leq C_1\int_0^\theta \big[\|a(v)\partial_x(v-w)\|_{L^2_x(I)} \\ &\qquad + \|(a(v)-a(w))\partial_x w\|_{L^2_x(I)} + \|b(v-w)\|_{L^2_x(I)}\big]\,d\tau \\ &\leq C_1\int_0^\theta \big[\|a(v)\partial_x(v-w)\|_{L^2_x(I)} + \|\left(1+|v|^{p-1}+|w|^{p-1}\right)(v-w)\partial_x w\|_{L^2_x(I)} \\ &\qquad + \|b(v-w)\|_{L^2_x(I)}\big]\,d\tau \\ &\leq C_1\theta^{\frac{2-p}{4}}\Big(\|v\|^p_{Y_{0,\theta}} + \|w\|^p_{Y_{0,\theta}} + \|v\|^{p-1}_{Y_{0,\theta}}\|w\|_{Y_{0,\theta}}\Big)\|v-w\|_{Y_{0,\theta}} \\ &\qquad + C_1\theta^{1/4}\|w\|_{Y_{0,\theta}}\|v-w\|_{Y_{0,\theta}} + C_2\theta^{\frac{1}{2}}(\|b\|_{L^2(I)}+1)\|v-w\|_{Y_{0,\theta}} \\ &\leq 3C_1\theta^{\frac{2-p}{4}}r^p\|v-w\|_{Y_{0,\theta}} + C_1\theta^{1/4}r\|v-w\|_{Y_{0,\theta}} + C_2\theta^{\frac{1}{2}}(\|b\|_{L^2(I)}+1)\|v-w\|_{Y_{0,\theta}}.\end{aligned}$$

Choosing $r > 0$ and $\theta > 0$ so that

$$\left\{ \begin{array}{l} r = 2C_0\|\phi\|_{X_0}, \\ \\ 3C_1\theta^{(2-p)/4}r^p + C_1\theta^{1/4}r + C_2\theta^{1/2}(\|b\|_{L^2(I)} + 1) \leq \frac{1}{2}, \end{array} \right. \tag{2.24}$$

then

$$\|\Gamma(v)\|_{Y_{0,\theta}} \leq r, \qquad \|\Gamma(v) - \Gamma(w)\|_{Y_{0,\theta}} \leq \frac{1}{2}\|v - w\|_{Y_{0,\theta}}$$

for any v, $w \in S_{\theta,r}$. Thus, with such a choice of r and θ, Γ is a contraction mapping of $S_{r,\theta}$. Its fixed point $u = \Gamma(u)$ is the unique solution of the IBVP (2.1) in $S_{\theta,r}$. Note that θ depends only on $\|\phi\|_{X_0}$ and

$$\sup_{0\leq t\leq\theta} \|u(\cdot,t)\|_{L^2(I)} \leq \|\phi\|_{X_0}.$$

By the standard extension argument, one may extend θ to T. The proof is complete. ☐

Next, we show that the IBVP (2.1) is well-posed in the space X_3.

PROPOSITION 2.7. *Let $b \in H^1(I)$, let a be a C^1 function satisfying*

$$|a(\mu)| \leq C(1 + |\mu|^p), \quad |a'(\mu)| \leq C(1 + |\mu|^{p-1}) \quad \text{for any } \mu \in \mathbb{R}$$

with $1 \leq p < 2$, and let $T > 0$ be given. For any $\phi \in X_3$, the IBVP (2.1) *admits a unique solution $u \in Y_{3,T}$. Moreover, there exists a nondecreasing continuous function $\beta_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u\|_{Y_{3,T}} \leq \beta_3(\|\phi\|_{X_0})\|\phi\|_{X_3}. \tag{2.25}$$

Proof. By Proposition 2.6, (2.1) admits a unique solution $u \in Y_{0,T}$. We just need to show further that $u \in Y_{3,T}$. For this purpose, let $v = u_t$. Then the function v is a solution of

$$\left\{ \begin{array}{l} \partial_t v + \partial_x v + \big(a'(u)\partial_x u\big)v + a(u)\partial_x v + \partial_x^3 v + b(x)v = 0, \\ \\ v(x,0) = \phi^*(x), \\ \\ v(0,t) = 0, \quad v(L,t) = 0, \quad \partial_x v(L,t) = 0 \end{array} \right. \tag{2.26}$$

with

$$\phi^*(x) = -\phi'''(x) - \phi'(x) - a(\phi(x))\phi'(x) - b(x)\phi(x).$$

Observe that $\phi^* \in X_0$ and that there exists a constant $C = C(\|\phi\|_{X_0})$ such that

$$\|\phi^*\|_{X_0} \leq C\|\phi\|_{X_3}.$$

We may write (2.26) in its integral form

$$v(t) = W(t)\phi^* - \int_0^t W(t-\tau)\Big(a'(u(\cdot,\tau))\partial_x u(\cdot,\tau)v(\cdot,\tau) + a(u(\cdot,\tau))\partial_x v(\cdot,\tau) + b(\cdot)v(\cdot,\tau)\Big)d\tau.$$

Using Lemma 2.4 and proceeding as in the proof of Proposition 2.6, we see that (2.26) admits a unique solution $v \in Y_{0,T}$. Notice that $u_t = v$, $b \in H^1(I)$, and $u \in$

$Y_{0,T}$. Since $u \in H^1(0,T;H^1(I)) \subset C([0,T];C(\overline{I}))$, we obviously have that $a(u)\partial_x u \in L^2(0,T;L^2(I))$. It then follows from

$$\partial_x^3 u = -\partial_t u - \partial_x u - b(x)u - a(u)\partial_x u = -v - \partial_x u - b(x)u - a(u)\partial_x u \tag{2.27}$$

that $u \in L^2(0,T;H^3(I))$; hence $u \in C([0,T];H^2(I))$. This implies further that $\partial_x u$, bu, and $a(u)\partial_x u$ all belong to the space $C([0,T];L^2(I)) \cap L^2(0,T;H^1(I))$. Using (2.27) again yields that $u \in Y_{3,T}$. The proof is complete. $\square$

According to Propositions 2.6 and 2.7, the IBVP (2.1) defines a continuous nonlinear map K from the space X_j to $Y_{j,T}$ for $j = 0, 3$. Next, we show that K is a continuous map from X_s to $Y_{s,T}$ for $0 \le s \le 3$ by using the following nonlinear interpolation theory (cf. [35, 1]), which implies the well-posedness of the IBVP (2.1) in the space X_s for $0 \le s \le 3$.

Let B_0 and B_1 be two Banach spaces such that $B_1 \subset B_0$ with the inclusion map continuous. Let $f \in B_0$ and, for $t \ge 0$, define

$$K(f,t) = \inf_{g \in B_1} \{\|f-g\|_{B_0} + t\|g\|_{B_1}\}.$$

For $0 < \theta < 1$ and $1 \le p \le +\infty$, define

$$[B_0, B_1]_{\theta,p} = B_{\theta,p} = \left\{ f \in B_0 : \ \|f\|_{\theta,p} = \left(\int_0^{+\infty} K(f,t)^p t^{-\theta p - 1} dt \right)^{1/p} < +\infty \right\}$$

with the usual modification for the case $p = +\infty$. Then $B_{\theta,p}$ is a Banach space with norm $\| \ \|_{\theta,p}$. Given two pairs of indices (θ_1, p_1) and (θ_2, p_2) as above, then $(\theta_1, p_1) \prec (\theta_2, p_2)$ means

$$\begin{cases} \theta_1 < \theta_2 & \text{or} \\ \theta_1 = \theta_2 & \text{and } p_1 > p_2. \end{cases}$$

If $(\theta_1, p_1) \prec (\theta_2, p_2)$, then $B_{\theta_2,p_2} \subset B_{\theta_1,p_1}$ with the inclusion map continuous.

THEOREM 2.8. *Let* B_0^j *and* B_1^j *be Banach spaces such that* $B_1^j \subset B_0^j$ *with continuous inclusion mappings,* $j = 1, 2$. *Let* λ *and* q *lie in the ranges* $0 < \lambda < 1$ *and* $1 \le q \le +\infty$. *Suppose* A *is a mapping such that*

(i) $A: \ B_{\lambda,q}^1 \to B_0^2$ *and for* $f, \ g \in B_{\lambda,q}^1$,

$$\|Af - Ag\|_{B_0^2} \le C_0(\|f\|_{B_{\lambda,q}^1} + \|g\|_{B_{\lambda,q}^1})\|f-g\|_{B_0^1}$$

and

(ii) $A: \ B_1^1 \to B_1^2$ *and for* $h \in B_1^1$,

$$\|Ah\|_{B_1^2} \le C_1(\|h\|_{B_{\lambda,q}^1})\|h\|_{B_1^1},$$

where $C_j: \ \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous nondecreasing functions,* $j = 0, 1$.
Then if $(\theta, p) \ge (\lambda, q)$, A *maps* $B_{\theta,p}^1$ *into* $B_{\theta,p}^2$ *and for* $f \in B_{\theta,p}^1$,

$$\|Af\|_{B_{\theta,p}^2} \le C(\|f\|_{B_{\lambda,q}^1})\|f\|_{B_{\theta,p}^1},$$

where for $r > 0$, $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^{\theta}$.

Remark. This theorem is identical to Theorem 2 of Tartar [35] except that Tartar makes the more restrictive assumption that the nondecreasing functions C_0 and C_1

depend only on the B_0^1 norms of the functions in question. Theorem 2.8 was used by Bona and Scott [1] to prove the global well-posedness of the pure initial value problem for the KdV equation on the whole line in fractional order Sobolev spaces $H^s(\mathbb{R})$.

Here is the promised well-posedness result for the IBVP (2.1) in X_s.

Theorem 2.9. *Let a be a C^1 function satisfying*

$$|a(\mu)| \leq C(1+|\mu|^p), \quad |a'(\mu)| \leq C(1+|\mu|^{p-1}) \quad \text{for any } \mu \in \mathbb{R} \tag{2.28}$$

with $1 \leq p < 2$, and let $T > 0$ and $0 \leq s \leq 3$ be given. In addition, assume $b \in L^2(I)$ when $s = 0$ and $b \in H^1(I)$ when $s > 0$. Then, for any $\phi \in X_s$, the IBVP (2.1) *admits a unique solution $u \in Y_{s,T}$. Moreover, there exists a nondecreasing continuous function $\beta_s: \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u\|_{Y_{s,T}} \leq \beta_s(\|\phi\|_{X_0})\|\phi\|_{X_s}.$$

Proof. Choose

$$B_0^1 = X_0, \quad B_1^1 = X_3, \quad B_0^2 = Y_{0,T}, \quad B_1^2 = Y_{3,T}.$$

Let A be the solution map of the IBVP (2.1): $u = A(\phi)$. For given s with $0 < s < 3$, choose $p = 2$ and $\theta = s/3$. Then

$$B_{\theta,p}^2 = Y_{s,T}, \qquad B_{\theta,p}^1 = X_s.$$

In this case, assumptions (i) and (ii) of Theorem 2.8 are (2.22) and (2.25), respectively, which we have already proved. The proof is then completed by invoking Theorem 2.8. □

By Theorem 2.9, the condition $\phi \in X_s$ implies that the corresponding solution $u \in L^2(0,T;H^{s+1}(I))$. Thus, for any $0 < \epsilon < T$, one can always find a time $t_1 \in (0,\epsilon)$ such that $u(\cdot,t_1) \in H^{s+1}(I)$. The following corollary, which reveals a strong smoothing property of the system (2.1), follows directly from Theorem 2.9.

Corollary 2.10. *Under the assumptions of Theorem* 2.9, *for any $\phi \in L^2(I)$, the corresponding solution u of* (2.1) *belongs to the space $C([\epsilon,T];H^3(I)) \cap L^2(\epsilon,T;H^4(I))$ for any $\epsilon > 0$.*

In Theorem 2.9, the nonlinear term $a(u)$ of the equation in (2.1) is required to satisfy (2.28) with $1 \leq p < 2$. We consider next the case of $p \geq 2$.

Let $\mathcal{B}$ be the linear operator defined by $\mathcal{B}f = b(x)f$. Consider $A_b = A + \mathcal{B}$ as an unbounded operator on $L^2(I)$ with the domain

$$\mathcal{D}(A_b) = \{f \in H^3(I),\ f(0) = f(L) = f'(L) = 0\}.$$

The operator A_b is the infinitesimal generator of a C_0-semigroup $W_b(t)$ in the space $L^2(I)$. For any $\phi \in L^2(I)$,

$$u(t) = W_b(t)\phi$$

belongs to the space $C_b(\mathbb{R}^+;L^2(I))$ and solves

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + b(x)u = 0, \qquad u(x,0) = \phi(x)\,, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0. \end{cases} \tag{2.29}$$

LEMMA 2.11. *Let $T > 0$ and $0 \leq s \leq 3$ be given. Assume that $b \in L^2(I)$ if $s = 0$ and $b \in H^1(I)$ if $s > 0$. Then there exists a constant C depending only on T and s such that*

$$\|W_b(t)\phi\|_{Y_{s,T}} \leq C\|\phi\|_{X_s} \quad \text{for any } \phi \in X_s \tag{2.30}$$

and

$$\left\| \int_0^t W_b(t-\tau) f(\cdot,\tau) d\tau \right\|_{Y_{s,T}} \leq C\|f\|_{L^1(0,T;H^s(I))} \quad \text{for any } f \in L^1(0,T;X_s). \tag{2.31}$$

Proof. The estimate (2.30) is established by using a similar (but simpler, because there is no nonlinear term in the equation) argument as that used in the proof of Theorem 2.9. To see that (2.31) is true, note that

$$\begin{aligned}\left\| \int_0^t W_b(t-\tau) f(\cdot,\tau) d\tau \right\|_{C([0,T];H^s(I))} &\leq \int_0^T \|W_b(t-\tau) f(\cdot,\tau)\|_{C([\tau,T]_t;H^s(I))}\, d\tau \\ &\leq C \int_0^T \|f(\cdot,\tau)\|_{H^s(I)} d\tau\end{aligned}$$

and

$$\begin{aligned}\left\| \int_0^t W_b(t-\tau) f(\cdot,\tau) d\tau \right\|_{L^2(0,T;H^{s+1}(I))} &\leq \int_0^T \|W_b(t-\tau) f(\cdot,\tau)\|_{L^2([\tau,T]_t;H^{s+1}(I))}\, d\tau \\ &\leq C \int_0^T \|f(\cdot,\tau)\|_{H^s(I)} d\tau.\end{aligned}$$

Moreover, using the same argument as that used in the proof of Proposition 2.17 in [4] yields

$$\sup_{x\in I} \left\| \partial_x \int_0^t W_b(t-\tau) f(\cdot,\tau) d\tau \right\|_{L^2(0,T)} \leq C \int_0^T \|f(\cdot,\tau)\|_{H^s(I)} d\tau.$$

Inequality (2.31) then follows by combining the above three estimates. The proof is complete. ◻

We have the following estimate by direct calculation.

LEMMA 2.12. *Let a be a C^1 function satisfying*

$$|a(\mu)| \leq C(1+|\mu|^p), \quad |a'(\mu)| \leq C(1+|\mu|^{p-1}) \quad \text{for any } \mu \in \mathbb{R}$$

with $p \geq 1$. There exists a constant C such that

$$\|a(u)\partial_x v\|_{L^1(0,T;H^1(I))} \leq CT^{1/2}\left(1 + \|u\|_{Y_{1,T}} + \|u\|_{Y_{1,T}}^p\right)\|v\|_{Y_{1,T}} \quad \text{for any } u,\ v \in Y_{1,T}.$$

Using the notation of the semigroup $W_b(t)$, the nonlinear IBVP (2.1) may be written in the following integral form:

$$u(t) = W_b(t)\phi - \int_0^t W_b(t-\tau) a(u(\cdot,\tau)) \partial_x u(\cdot,\tau) d\tau.$$

The same arguments as those in the proof of Propositions 2.6 and 2.7 lead to the following (local) well-posedness result for the IBVP (2.1).

THEOREM 2.13. *Let $b \in H^1(I)$, let a be a C^2 function satisfying*

$$|a(\mu)| \le C(1+|\mu|^p), \quad |a'(\mu)| \le C(1+|\mu|^{p-1}), \quad |a''(\mu)| \le C(1+|\mu|^{p-2}) \quad \text{for any } \mu \in \mathbb{R}$$

with $p \ge 2$, and let $j = 1$ or 3 be given. Then for any $\phi \in X_j$ there exists a $T^* > 0$ depending only on $\|\phi\|_{X_1}$ such that the IBVP (2.1) *admits a unique solution $u \in Y_{j,T^*}$. Moreover, there exists a nondecreasing continuous function $\alpha_j : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u\|_{Y_{j,T^*}} \le \alpha_j(\|\phi\|_{X_1})\|\phi\|_{X_j}.$$

Remark. The well-posedness result presented in Theorem 2.13 is local in the sense that the length of the time interval $(0, T^*)$ in which the solution exists depends on the norm of the initial value ϕ in the space X_1. In order to obtain the global well-posedness result, one needs to establish the corresponding global a priori estimate for the IBVP (2.1) in the space X_1, which is not available. In fact, the only available global a priori estimate for the IBVP (2.1) is the L^2-estimate presented in Lemma 2.5. On the other hand, whether the IBVP (2.1) is well-posed in the space $L^2(I)$ in the case of $p \ge 2$ is still an open question.

The next theorem shows that if $\phi \in X_0$, then the IBVP (2.1) admits one solution $u \in C_w(R^+; L^2(I)) \cap L^2_{loc}(R^+; H^1(I))$.

THEOREM 2.14. *Let a be a C^2 function satisfying*

$$|a(\mu)| \le C(1+|\mu|^p), \quad |a'(\mu)| \le C(1+|\mu|^{p-1}), \quad |a''(\mu)| \le C(1+|\mu|^{p-2}) \quad \text{for any } \mu \in \mathbb{R}$$

with $2 \le p < 4$. Then, for any given $\phi \in X_0$, the IBVP (2.1) *admits at least one solution $u \in C_w(\mathbb{R}^+; L^2(I)) \cap L^2_{loc}(\mathbb{R}^+; H^1(I))$.*

Proof. Let $\{a_n\}$ denote a sequence of functions in $C_0^\infty(\mathbb{R};\mathbb{R})$ such that

$$|a_n^{(j)}(\mu)| \le C(2 + |\mu|^{p-j}) \qquad \forall n \ge 0,\ \forall \mu \in \mathbb{R},\ j = 0, 1, 2; \tag{2.32}$$

$$a_n(\mu) \to a(\mu) \quad \text{uniformly on each compact set in } R. \tag{2.33}$$

Observe that $|a_n(\mu)| \le C(n)(1 + |\mu|)$ and $|a_n'(\mu)| \le C(n)(1 + |\mu|)$. According to Proposition 2.6, there exists a (unique) solution $u_n \in C(\mathbb{R}^+; L^2(I)) \cap L^2(\mathbb{R}^+; H_0^1(I))$ of the modified boundary initial value problem

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n + \partial_x u_n + a_n(u_n)\partial_x u_n + b u_n = 0, \\ u_n(0,t) = u_n(L,t) = \partial_x u_n(L,t) = 0, \\ u_n(x,0) = \phi(x). \end{cases} \tag{2.34}$$

Let us introduce the functions

$$A_n(u) := \int_0^u a_n(v)\,dv, \qquad \tilde{A}_n(u) := \int_0^u v a_n(v)\,dv.$$

According to Lemma 2.5,

$$||u_n(T)||^2_{L^2(I)} = ||\phi||^2_{L^2(I)} - \int_0^T |\partial_x u_n(0,t)|^2 dt - 2\int_0^T \int_0^L b(x)|u_n|^2 dxdt$$

and

$$\int_0^T \int_0^L |\partial_x u_n|^2\,dxdt \le \frac{L + (C+1)T}{2}||\phi||^2_{L^2(I)} + C_p T||\phi||^{\frac{8+2p}{4-p}}_{L^2(0,L)}$$

for any $T \geq 0$ and $n \geq 0$. Thus $\{u_n\}$ is bounded in $L^\infty(\mathbb{R}^+; L^2(I)) \cap L^2_{loc}(\mathbb{R}^+; H^1_0(I))$, and there exist a function $u \in L^\infty(\mathbb{R}^+; L^2(I)) \cap L^2_{loc}(\mathbb{R}^+; H^1(I))$ and a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{in } L^\infty(\mathbb{R}^+; L^2(I)) \ \text{ weak}\,*, \tag{2.35}$$

$$u_n \rightharpoonup u \quad \text{in } L^2_{loc}(\mathbb{R}^+; H^1(I)) \text{ weak}. \tag{2.36}$$

To pass to the limit in (2.34) we have to pay some attention to the nonlinear term $a_n(u_n)\partial_x u_n = \partial_x[A_n(u_n)]$. Let us introduce the function $A(u) := \int_0^u a(v)\,dv$. We aim to prove that

$$A_n(u_n) \to A(u) \quad \text{in } \mathcal{D}'(I_x \times (0, +\infty)_t).$$

We first prove the following claim.

Claim 1. Let $T > 0$ and $\alpha \in (1, \frac{6}{p+1}]$. Then $A_n(u_n)$ is bounded in $L^\alpha(I \times (0,T))$.

Proof of Claim 1. It follows from (2.32) that

$$|A_n(u)| \leq C\left(2|u| + \frac{|u|^{p+1}}{p+1}\right) \leq C'(1 + |u|^{p+1})$$

and

$$|A_n(u)|^\alpha \leq C''(1 + |u|^{\alpha(p+1)}),$$

where C' and C'' denote some positive constants which depend only on C, p, and α. Therefore

$$\begin{aligned}\int_0^T \int_0^L |A_n(u_n)|^\alpha dxdt &\leq C''\left(TL + \int_0^T \int_0^L |u_n|^{\alpha(p+1)} dxdt\right)\\ &\leq C''\left(TL + \int_0^T ||u_n||^2_{L^2(I)} \cdot ||u_n||^{\alpha(p+1)-2}_{L^\infty(I)} dt\right)\\ &\leq C''\left(TL + ||\phi||^{1+\frac{\alpha(p+1)}{2}}_{L^2(I)} \int_0^T ||\partial_x u_n||^{\frac{\alpha(p+1)-2}{2}}_{L^2(I)} dt\right).\end{aligned}$$

As $\frac{\alpha(p+1)-2}{2} \leq 2$, the result in Claim 1 follows from Lemma 2.5. □

Next we prove the following claim.

Claim 2. Let T and α be as in Claim 1. Then the sequences $\{a_n(u_n)\partial_x u_n\}$ and $\{\partial_t u_n\}$ are bounded in $L^\alpha(0,T; H^{-2}(I))$.

Proof of Claim 2. Notice first that $L^\alpha(I) \subset H^{-1}(I)$ (as $H^1_0(I) \subset L^\infty(I)$); hence $a_n(u_n)\partial_x u_n = \partial_x[A_n(u_n)]$ is bounded in $L^\alpha(0,T; H^{-2}(I))$. Clearly, $\partial_x^3 u_n$ is bounded in $L^2(0,T; H^{-2}(I))$ and $\partial_x u_n + bu_n$ is bounded in $L^2(0,T; L^2(I))$, so we conclude that $\partial_t u_n = -(\partial_x^3 u_n + \partial_x u_n + a_n(u_n)\partial_x u_n + bu_n)$ is bounded in $L^\alpha(0,T; H^{-2}(I))$. □

As the first embedding in $H^1_0(I) \subset L^2(I) \subset H^{-2}(I)$ is compact, we infer from [33, Cor. 4] that the sequence $\{u_n\}$ is relatively compact in $L^2(0,T; L^2(I))$ for each $T > 0$. Therefore, extracting a subsequence of $\{u_n\}$ again denoted by $\{u_n\}$, we have that

$$u_n \to u \text{ (strongly)} \quad \text{in } L^2_{loc}(\mathbb{R}^+; L^2(I)) \text{ and a.e.} \tag{2.37}$$

Using (2.33) and (2.37), it is easy to see that

$$A_n(u_n(x,t)) \to A(u(x,t)) \quad \text{for a.e. } (x,t) \in (0,L) \times \mathbb{R}^+.$$

On the other hand, picking any $\alpha \in (1, \frac{6}{p+1}]$, we deduce from Claim 1 that there exists a function $g \in L^\alpha_{loc}(\mathbb{R}^+; L^\alpha(I))$ such that (for a subsequence)

$$A_n(u_n) \rightharpoonup g \quad \text{in } L^\alpha((0,L)\times(0,T))$$

for any $T > 0$. The following lemma is an easy consequence of the Egoroff theorem.

Lemma 2.15. *Let Ω be an open set in R^N, and let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ (with $1 < p < \infty$) such that $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $f_n(x) \to g(x)$ a.e. Then $f(x) = g(x)$ a.e.*

We infer from Lemma 2.15 that $g(x,t) = A(u(x,t))$ a.e. Therefore,

$$A_n(u_n) \to A(u) \quad \text{in } \mathcal{D}'((0,L)\times R^+),$$

and taking the spatial derivative, we obtain

$$a_n(u_n)\partial_x u_n \to a(u)\partial_x u \quad \text{in } \mathcal{D}'((0,L)\times R^+). \tag{2.38}$$

Gathering (2.37) and (2.38) together, and taking the limit in (2.34), we conclude that u solves the equation in (2.1) in the sense of distributions.

Since u_n is bounded in $L^\infty(0,T;L^2(I))$ (see (2.17)) and $\partial_t u_n$ is bounded in $L^\alpha(0,T;\, H^{-2}(I))$ (with $\alpha > 1$), we infer from [33, Cor. 4] that for a subsequence, again denoted by $\{u_n\}$,

$$u_n \to u \quad \text{in } C([0,T],H^{-1}(I)) \quad \text{for any } T > 0. \tag{2.39}$$

In particular,

$$u(0) = \lim_{n\to+\infty} u_n(0) = \phi.$$

As $u \in L^\infty(0,T;L^2(I)) \cap C_w([0,T],H^{-1}(I))$, we deduce from [36, Lem. 1.4, p. 263] that $u \in C_w([0,T],L^2(I))$. ☐

Definition 2.16. *A function $u \in C_w(\mathbb{R}^+;L^2(I)) \cap L^2_{loc}(\mathbb{R}^+;H^1_0(I))$ is said to be a* weak solution *of* (2.1) *if there exist a sequence $\{a_n\}$ of functions in $C^\infty_0(\mathbb{R};\mathbb{R})$ satisfying* (2.32) *and* (2.33) *and a sequence of strong solutions u_n to* (2.34) *such that* (2.35), (2.36), (2.37), *and* (2.39) *hold true.*

3. Exponential stability.

In this section we study the long time behavior of weak solutions of (2.1). The goal is to show that any weak solution of (2.1) decays exponentially in the space $L^2(I)$. As a weak solution of (2.1) may fail to be unique, the concept of exponential stability has to be generalized in the following way.

Definition 3.1. *System* (2.1) *is said to be* locally uniformly exponentially stable *in $L^2(I)$ if for any $r > 0$ there exist two constants $C > 0$ and $\nu > 0$ such that for any $\phi \in L^2(I)$ with $||\phi||_{L^2(I)} < r$ and for any weak solution $u = u(x,t)$ of* (2.1), *it holds that*

$$||u(\cdot,t)||^2_{L^2(I)} \le C\,||\phi||^2_{L^2(I)} e^{-\nu t} \qquad \forall t \ge 0. \tag{3.1}$$

If the constant ν in (3.1) *is independent of r, the system* (2.1) *is said to be* globally uniformly exponentially stable *in $L^2(I)$.*

We first show that the system (2.1) is locally uniformly exponentially stable in $L^2(I)$.

PROPOSITION 3.2. *Assume that* $a = a(\mu)$ *is a* C^2 *function which satisfies*

$$|a(\mu)| \leq C(1+|\mu|^p), \quad |a'(\mu)| \leq C(1+|\mu|^{p-1}), \quad |a''(\mu)| \leq C(1+|\mu|^{p-2}) \quad \text{for any } \mu \in \mathbb{R}$$

with $1 \leq p < 4$ *and* $b \in L^2(I)$. *Then the system* (2.1) *is locally uniformly exponentially stable in* $L^2(I)$.

The following Carleman estimate [27, Prop. 2.3] will play a great role in establishing the unique continuation property of the GKdV equation as described by Lemma 3.4.

LEMMA 3.3. *Let* T *and* l *be positive numbers. Then there exist a smooth positive function* ψ *on* $[0, l]$ *and two constants* $C_0 > 0$, $s_0 > 0$ *such that for any*

$$q \in L^2(0,T;H^3(0,l)) \cap H^1(0,T;L^2(0,l)) \tag{3.2}$$

fulfilling

$$q(0,t) = q(l,t) = \partial_x q(l,t) = \partial_x^2 q(l,t) = 0$$

and for any $s \geq s_0$ *we have*

$$\begin{aligned}\int_0^T\int_0^l \left\{ \frac{s^5}{t^5(T-t)^5}|q|^2 + \frac{s^3}{t^3(T-t)^3}|\partial_x q|^2 + \frac{s}{t(T-t)}|\partial_x^2 q|^2 \right\} \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right) dxdt \\ \leq C_0 \int_0^T\int_0^l |\partial_t q + \partial_x^3 q + \partial_x q|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right) dxdt. \end{aligned} \tag{3.3}$$

(Notice that Lemma 3.3 is stated in [27] under the extra assumption that $q \in C^3([0,l] \times [0,T])$, but (3.2) is what is really needed to perform the computations.)

LEMMA 3.4. *Let* $l > 0$ *and* $T > 0$. *If* $v \in L^\infty(0,T;H^1(0,l))$ *solves*

$$\begin{cases} \partial_t v + \partial_x v + a(v)\partial_x v + \partial_x^3 v = 0 & \text{in } (0,l) \times (0,T), \\ v(0,t) = 0 & \text{for a.e. } t \in (0,T), \\ v \equiv 0 & \text{in } (l',l) \times (0,T) \end{cases}$$

with $a \in C^0(\mathbb{R};\mathbb{R})$ *and* $0 < l' < l$, *then* $v \equiv 0$ *in* $(0,l) \times (0,T)$.

Proof. As the function v is not expected to fulfill (3.2), we have to smooth it. For any function $u = u(x,t)$ and any $h > 0$, we set

$$u^{[h]}(x,t) = \frac{1}{h}\int_t^{t+h} u(x,s)\, ds.$$

Recall that if $u \in L^p(0,T;V)$, where $1 \leq p \leq +\infty$ and V denotes any Banach space, then $u^{[h]} \in W^{1,p}(0,T-h;V)$, $||u^{[h]}||_{L^p(0,T-h;V)} \leq ||u||_{L^p(0,T;V)}$, and for $p < \infty$ and $T' < T$

$$u^{[h]} \to u \quad \text{in } L^p(0,T';V) \text{ as } h \to 0.$$

Pick any $T' < T$. Then for any small enough number h, $v^{[h]} \in W^{1,\infty}(0,T';H_0^1(0,l))$, and $v^{[h]}$ solves

$$\partial_t v^{[h]} + \partial_x v^{[h]} + \partial_x^3 v^{[h]} + (a(v)\partial_x v)^{[h]} = 0 \quad \text{in } (0,l) \times (0,T'), \tag{3.4}$$

$$v^{[h]}(0,t) = 0 \quad \text{in } (0,T'), \tag{3.5}$$

$$v^{[h]} \equiv 0 \quad \text{in } (l',l) \times (0,T'). \tag{3.6}$$

As $v \in L^\infty(0,T;H^1(0,l))$, $a(v)\partial_x v \in L^\infty(0,T;L^2(0,l))$; hence it follows from (3.4) that $v^{[h]}_{xxx} \in L^\infty(0,T';L^2(0,l))$ and so $v^{[h]} \in L^\infty(0,T';H^3(0,l))$. It follows then from Lemma 3.3 that for any $s \geq s_0$ we have

$$\begin{aligned}(3.7)\quad &\int_0^{T'}\int_0^l \left\{ \frac{s^5}{t^5(T'-t)^5}|v^{[h]}|^2 + \frac{s^3}{t^3(T'-t)^3}|\partial_x v^{[h]}|^2 \right.\\ &\qquad \left. + \frac{s}{t(T'-t)}|\partial_x^2 v^{[h]}|^2 \right\} \exp\left(-\frac{2s\psi(x)}{t(T'-t)}\right) dxdt\\ &\leq C_0 \int_0^{T'}\int_0^l |(a(v)\partial_x v)^{[h]}|^2 \exp\left(-\frac{2s\psi(x)}{t(T'-t)}\right) dxdt\\ &\leq 2C_0 \int_0^{T'}\int_0^l |a(v)\partial_x v^{[h]}|^2 \exp\left(-\frac{2s\psi(x)}{t(T'-t)}\right) dxdt\\ &\quad + 2C_0 \int_0^{T'}\int_0^l |(a(v)\partial_x v)^{[h]} - a(v)\partial_x v^{[h]}|^2 \exp\left(-\frac{2s\psi(x)}{t(T'-t)}\right) dxdt\\ &\equiv I_1 + I_2.\end{aligned}$$

We first estimate I_1. Since $a(v) \in L^\infty(0,T;L^\infty(0,l))$, we have

$$(3.8)\qquad I_1 \leq C \int_0^{T'}\int_0^l |\partial_x v^{[h]}|^2 \exp\left(-\frac{2s\psi(x)}{t(T'-t)}\right) dxdt$$

for some constant C which does not depend on h. Comparing the powers of s in the right-hand side of (3.8) with those in the left-hand side of (3.7), we deduce that the term I_1 in (3.7) may be dropped by increasing the constants C_0 and s_0 in a convenient way. From now on, we fix s, say, to the value s_0. We claim that $I_2 \to 0$ as $h \to 0$. As $\exp(-\frac{2s_0\psi(x)}{t(T'-t)}) \leq 1$, it is sufficient to prove that

$$(3.9)\qquad (a(v)\partial_x v)^{[h]} \to a(v)\partial_x v \quad \text{in } L^2(0,T';L^2(0,l)),$$

$$(3.10)\qquad a(v)\partial_x v^{[h]} \to a(v)\partial_x v \quad \text{in } L^2(0,T';L^2(0,l)).$$

Property (3.9) is obvious for $a(v)\partial_x v \in L^2(0,T;L^2(0,l))$, and (3.10) follows from the fact that $a(v) \in L^\infty(0,T;L^\infty(0,l))$ and that $v \in L^2(0,T;H^1(0,l))$. We conclude that as $h \to 0$ the integral term

$$\begin{aligned}&\int_0^{T'}\int_0^l \left\{ \frac{s_0^5}{t^5(T'-t)^5}|v^{[h]}|^2 + \frac{s_0^3}{t^3(T'-t)^3}|\partial_x v^{[h]}|^2 \right.\\ &\qquad \left. + \frac{s_0}{t(T'-t)}|\partial_x^2 v^{[h]}|^2 \right\} \exp\left(-\frac{2s\psi(x)}{t(T'-t)}\right) dxdt\end{aligned}$$

tends to 0. On the other hand, $v^{[h]} \to v$ in $L^2(0,T';L^2(0,l))$. It follows that $v \equiv 0$ in $(0,l) \times (0,T')$. As T' may be taken arbitrarily close to T, we infer that $v \equiv 0$ in $(0,l) \times (0,T)$. This completes the proof of Lemma 3.4. □

The following unique continuation property is a direct consequence of Lemma 3.4.

Lemma 3.5. *Let $L > 0$ and $T > 0$ be two real numbers, and let $\omega \subset (0,L)$ be a nonempty open set. If $v \in L^\infty(0,T;H^1(I))$ solves*

$$\begin{cases} \partial_t v + \partial_x v + a(v)\partial_x v + \partial_x^3 v = 0 & \text{in } (0,L)\times(0,T),\\ v(0,t) = v(L,t) = 0 & \text{for a.e. } t \in (0,T),\\ v \equiv 0 & \text{in } \omega \times (0,T)\end{cases}$$

with $a \in C^0(\mathbb{R};\mathbb{R})$, then $v \equiv 0$ in $(0,L)\times(0,T)$.

Proof. Without loss of generality we may assume that $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Pick $l = (l_1 + l_2)/2$. Applying Lemma 3.4 to the function $v(x,t)$ on $(0,l) \times (0,T)$ and then to the function $v(L-x, T-t)$ on $(0, L-l) \times (0,T)$, we conclude that $v \equiv 0$ on $(0,L) \times (0,T)$. $\square$

Now we are in a position to present the proof of Proposition 3.2.

Proof of Proposition 3.2. First, notice that if u is a weak solution of (2.1) emanating from a given initial state $\phi \in L^2(I)$, then by Lemma 2.5 there exists a constant C_p depending only on p such that for any $T > 0$,

$$||u(\cdot,T)||^2_{L^2(I)} = ||\phi||^2_{L^2(I)} - \int_0^T |\partial_x u(0,t)|^2\,dt - 2\int_0^T \int_0^L b|u|^2\,dxdt \tag{3.11}$$

and

$$||\partial_x u||^2_{L^2(0,T;L^2(I))} \leq \frac{L + (C+1)T}{2} ||\phi||^2_{L^2(I)} + C_p T ||\phi||^{\frac{8+2p}{4-p}}_{L^2(I)}. \tag{3.12}$$

On the other hand, multiplying (2.1) by $(T-t)u$ yields

$$\begin{aligned} T||\phi||^2_{L^2(I)} = &\int_0^T \int_0^L |u(x,t)|^2\,dxdt + \int_0^T (T-t)|\partial_x u(0,t)|^2 dt \\ &+ 2\int_0^T \int_0^L (T-t) b|u(x,t)|^2 dxdt. \end{aligned} \tag{3.13}$$

We now proceed as in the proof of [24, Thm. 3.1]. Let $r > 0$ be given. The proof would be complete if the following claim is true.

Claim 3. For any $T > 0$ and any $r > 0$ there exists a positive constant $C = C(r,T)$ such that for any weak solution u issuing from a state $\phi \in L^2(I)$ with $||\phi||_{L^2(I)} < r$, it holds that

$$||\phi||^2_{L^2(I)} \leq C \left(\int_0^T |\partial_x u(0,t)|^2 dt + 2\int_0^T \int_0^L b(x)|u(x,t)|^2 dxdt \right). \tag{3.14}$$

Indeed, if Claim 3 is proved, then it follows from (3.11) and (3.14) that there exists a constant $\gamma \in (0,1)$ such that

$$||u(\cdot,kT)||^2_{L^2(I)} \leq \gamma^k ||\phi||^2_{L^2(I)} \qquad \forall k \geq 0.$$

Since $||u(\cdot,t)||_{L^2(I)} \leq ||u(\cdot,kT)||_{L^2(I)}$ for $kT \leq t < (k+1)T$, we readily obtain

$$||u(\cdot,t)||^2_{L^2(I)} \leq \frac{1}{\gamma} ||\phi||^2_{L^2(I)} e^{\frac{\log\gamma}{T} t}$$

from which the result stated in Proposition 3.2 follows.

To prove Claim 3, because of the identity (3.13), it is sufficient to show that there exists some constant $C_1 > 0$ such that

$$\int_0^T \int_0^L |u|^2 dxdt \leq C_1 \left(\int_0^T |\partial_x u(0,t)|^2 dt + 2\int_0^T \int_0^L b(x)|u|^2 dxdt \right), \tag{3.15}$$

provided that $||u(.,0)||_{L^2(I)} < r$. To this end, we argue by contradiction.

Suppose that (3.15) fails to be true. Then there exists a sequence of weak solutions $u_n \in C_w([0,T];L^2(I)) \cap L^2(0,T;H_0^1(I))$ of (2.1) with

$$||u_n(\cdot,0)||_{L^2(I)} < r \tag{3.16}$$

and such that

$$\lim_{n\to+\infty} \frac{||u_n||^2_{L^2(0,T;L^2(I))}}{\int_0^T |\partial_x u_n(0,t)|^2 dt + 2\int_0^T \int_0^L b(x)|u_n|^2 dxdt} = +\infty.$$

Let $\lambda_n := ||u_n||_{L^2(0,T;L^2(I))}$ and $v_n(x,t) := u_n(x,t)/\lambda_n$. Notice that λ_n is bounded from above, according to (3.11) and (3.16). Hence, extracting a subsequence if needed, we may assume that

$$\lambda_n \to \lambda \geq 0.$$

Then v_n fulfills

$$\begin{aligned}
&\partial_t v_n + \partial_x^3 v_n + \partial_x v_n + a(\lambda_n v_n)\partial_x v_n + bv_n = 0,\\
&v_n(0,t) = v_n(L,t) = \partial_x v_n(L,t) = 0,\\
&||v_n||_{L^2(0,T;L^2(I))} = 1,
\end{aligned}$$

and

$$\int_0^T |\partial_x v_n(0,t)|^2 dt + 2\int_0^T \int_0^L b(x)|v_n|^2 dxdt \to 0$$

as $n \to +\infty$. It follows from (3.13) that $||v_n(.,0)||_{L^2(I)} = ||u_n(.,0)/\lambda_n||_{L^2(I)}$ is bounded. Noticing that

$$|a(\lambda_n \mu)| \leq C(1 + |\lambda_n|^p |\mu|^p) \leq C'(1 + |\mu|^p),$$

where C' denotes a positive constant which does not depend on μ and n, we obtain from (3.12) applied to v_n that v_n is bounded in $L^2(0,T;H_0^1(I))$. Proceeding as in the proof of Claim 2, it is easily seen that $a(\lambda_n v_n)\partial_x v_n$ and $\partial_t v_n$ are bounded in $L^\alpha(0,T;H^{-2}(I))$ with $\alpha > 1$. Extracting a subsequence, we may assume that

$$v_n \rightharpoonup v \quad \text{in } L^\infty(0,T;L^2(I)) \text{ weak}\,*; \tag{3.17}$$

$$v_n \rightharpoonup v \quad \text{in } L^2(0,T;H_0^1(I)) \text{ weak}; \tag{3.18}$$

$$v_n \to v \quad \text{in } L^2(0,T;L^2(I)) \text{ and a.e.}; \tag{3.19}$$

$$v_n \to v \quad \text{in } C([0,T],H^{-1}(I)); \tag{3.20}$$

$$a(\lambda_n v_n)\partial_x v_n \to a(\lambda v)\partial_x v \text{ in } \mathcal{D}'(I \times (0,T)). \tag{3.21}$$

It follows that v solves

$$\partial_t v + \partial_x^3 v + \partial_x v + a(\lambda v)\partial_x v + bv = 0 \quad \text{in } \mathcal{D}'(I \times (0,T)), \tag{3.22}$$

$$v(0,t) = v(L,t) = \partial_x v(L,t) = 0. \tag{3.23}$$

Moreover, $||v||_{L^2(0,T;L^2(I))} = 1$ and $0 = \int_0^T \int_0^L b(x)|v|^2\, dxdt$; hence

$$v \equiv 0 \quad \text{on } \omega \times (0,T).$$

To conclude, the following result is needed.

LEMMA 3.6. *Let* $0 < t_1 < t_2 < T$. *Then there exists a subinterval* $(t_1', t_2') \subset (t_1, t_2)$ *such that* $v \in L^\infty(t_1', t_2'; H^1(I))$.

Proof of Lemma 3.6. C will denote here a constant which may vary from line to line. Using (2.39) for each weak solution w_n, we may pick a sequence $\{a_n\}$ in $C_0^\infty(\mathbb{R};\mathbb{R})$ fulfilling (2.32) and such that, if w_n denotes the solution of

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n + \partial_x w_n + a_n(\lambda_n w_n)\partial_x w_n + b w_n = 0, \quad x \in (0,L), \quad t \geq 0, \\ w_n(x,0) = v_n(x,0), \\ w_n(0,t) = w_n(L,t) = \partial_x w_n(L,t) = 0, \end{cases}$$

we have as $n \to \infty$

$$v_n - w_n \to 0 \quad \text{in } C([0,T], H^{-1}(I)). \tag{3.24}$$

As

$$||w_n||_{L^2(0,T;H^1(I))} \leq C,$$

we may pick a sequence $\{\tau_n\}$ in $(t_1, (t_1+t_2)/2)$ such that $\tau_n \to \tau$ and $||w_n(\tau_n)||_{H^1(I)} \leq C$. Using (3.20) and (3.24) we obtain that

$$w_n(\tau_n + \cdot) \to v(\tau + \cdot) \quad \text{in } C([0,\varepsilon], H^{-1}(I))$$

for any $\varepsilon < (t_2 - t_1)/2$. According to Theorem 2.13, for ε sufficiently small,

$$||w_n(\tau_n + \cdot)||_{L^\infty(0,\varepsilon;H^1(I))} \leq C;$$

hence $v \in L^\infty(\tau, \tau + \varepsilon; H^1(I))$. ◻

Let $t_1 \in (0,T)$, and let $t_2 \in (t_1, T)$. According to Lemma 3.6, $v \in L^\infty(t_1', t_2'; H^1(I))$ for some interval $(t_1', t_2') \subset (t_1, t_2)$. It follows then from Lemma 3.5 that $v \equiv 0$ on $(0,L) \times (t_1', t_2')$. As t_2 is arbitrarily close to t_1, we obtain by continuity of v in $H^{-1}(I)$ that $v(.,t) = 0$. Thus $v \equiv 0$, which contradicts the fact that $||v||_{L^2(0,T;L^2(I))} = 1$. The proof is complete. ◻

Now we present the main result of this paper, asserting that the system (2.1) is globally uniformly exponentially stable in the space $L^2(I)$.

THEOREM 3.7. *Assume that* $a = a(\mu)$ *is a* C^2 *function which satisfies*

$$|a(\mu)| \leq C(1+|\mu|^p), \quad |a'(\mu)| \leq C(1+|\mu|^{p-1}), \quad |a''(\mu)| \leq C(1+|\mu|^{p-2}) \quad \textit{for any } \mu \in \mathbb{R}$$

with $1 \leq p < 4$ *and* $b \in L^2(I)$. *Then the system* (2.1) *is globally uniformly exponentially stable in the space* $L^2(I)$; *i.e., there exist a* $\nu^* > 0$ *and a continuous nonnegative function* $\alpha_0 :\ \mathbb{R}^+ \to \mathbb{R}^+$ *such that for a given* $\phi \in L^2(I)$, *any weak solution* u *of* (2.1) *satisfies*

$$\|u(\cdot,t)\|_{L^2(I)} \leq \alpha_0(\|\phi\|_{L^2(I)})e^{-\nu^* t} \qquad \forall\ t \geq 0.$$

Proof. By Proposition 3.2, there exists a $\nu^* > 0$ such that if

$$\|\phi\|_{L^2(I)} \leq 1,$$

then the corresponding solution u of (2.1) satisfies

$$\|u(\cdot,t)\|_{L^2(I)} \leq C\|\phi\|_{L^2(I)}e^{-\nu^* t} \qquad \forall\ t \geq 0 \tag{3.25}$$

for some constant $C \geq 1$ which depends only on $\|\phi\|_{L^2(I)}$. In addition, for a given $r > 0$, there exist two constants $C_r > 0$ and $\nu_r > 0$ such that if $\|\phi\|_{L^2(I)} \leq r$, then any weak solution u of (2.1) satisfies

$$\|u(\cdot,t)\|_{L^2(I)} \leq C_r\|\phi\|_{L^2(I)}e^{-\nu_r t} \qquad \forall\ t \geq 0.$$

Consequently, setting $T_r := \nu_r^{-1}\ln(rC_r)$, we have that

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(I)} &\leq C\|u(\cdot,T_r)\|_{L^2(I)}e^{-\nu^*(t-T_r)} \qquad \forall\ t \geq T_r\\ &\leq CC_r\|\phi\|_{L^2(I)}e^{\nu^* T_r}e^{-\nu^* t} \qquad \forall\ t \geq 0.\end{aligned}$$

The proof is complete. $\square$

For a given function $u = u(x,t)$, a real number $s \geq 0$, and an interval (a,b), define

$$\|v\|^s_{Y_{(a,b)}} := \left(\|v\|^2_{C([a,b];H^s(I))} + \|v\|^2_{L^2([a,b];H^{s+1}(I))} + \|\partial_x v\|^2_{C([0,L];L^2(a,b))}\right)^{1/2}.$$

As a direct corollary of Theorem 3.7 and Lemma 2.5, we have the following result.

Corollary 3.8. *Under the assumptions of Theorem* 3.7, *there exists a* $\nu^* > 0$ *such that for any* $T > 0$ *and* $\phi \in L^2(I)$, *there exists a nonnegative continuous function* $\alpha_0 :\ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that any weak solution* u *of* (2.1) *satisfies*

$$\|u\|_{Y^0_{(t,t+T)}} \leq \alpha_0(\|\phi\|_{L^2(I)}, T)e^{-\nu^* t}$$

for any $t \geq 0$.

Under the assumptions of Theorem 2.9, if $\phi \in H^s(I)$ with $0 \leq s \leq 3$ satisfies the compatibility conditions (1.16), then the corresponding solution u of (2.1) belongs to the space $C(\mathbb{R}^+; H^s(I))$. Our next theorem shows that the system (2.1) is globally uniformly exponentially stable in the space $H^3(I)$. First, we show that the system (2.1) is globally uniformly exponentially stable in the space $H^3(I)$ if $\phi \in H^3(I)$ as described below.

Proposition 3.9. *Under the assumptions of Theorem* 2.9 *with* $a(0) = 0$ *additionally, there exist a* $\nu^* > 0$ *and a continuous nonnegative function* $\alpha_3 :\ \mathbb{R}^+ \to \mathbb{R}^+$ *such that for any* $\phi \in H^3(I)$ *satisfying the compatibility conditions, the corresponding solution* u *satisfies*

$$\|u(\cdot,t)\|_{H^3(I)} \leq \alpha_3(\|\phi\|_{L^2(I)})\|\phi\|_{H^3(I)}e^{-\nu^* t} \qquad \forall\ t \geq 0.$$

Proof. Let $v = u_t$. Then v solves the following IBVP linearized KdV equation:

$$\begin{cases} \partial_t v + \partial_x v + \partial_x(a(u)v) + bv + \partial_x^3 v = 0, \quad x \in (0,L), \quad t \geq 0,\\ v(x,0) = \phi^*(x),\\ v(0,t) = 0, \quad v(L,t) = 0, \quad v_x(L,t) = 0, \end{cases} \tag{3.26}$$

where

$$\phi^* = -\phi''' - a(\phi)\phi' - \phi' - b\phi.$$

Note that $\phi \in H^3(I)$ implies that $\phi^* \in L^2(I)$ and that there exists a constant $C = C(\|\phi\|_{L^2(I)})$ such that $\|\phi^*\|_{L^2(I)} \leq C(\|\phi\|_{H^3(I)})$ for any $\phi \in H^3(I)$. The same argument as that in the proof of Proposition 2.6 shows that

$$\|v\|_{Y_{0,T}} \leq \sigma(\|u\|_{Y_{0,T}})\|\phi^*\|_{L^2(I)}, \tag{3.27}$$

where $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function. By the semigroup property of the system (3.26), we can rewrite (3.27) as

$$\|v\|_{Y_{0,[t,t+T]}} \leq \sigma(\|u\|_{Y_{0,[t,t+T]}})\|v(\cdot,t)\|_{L^2(I)} \tag{3.28}$$

for any $t \geq 0$, where $Y_{0,[t,t+T]}$ is defined as $Y_{0,T} =: Y_{0,[0,T]}$. Using the notation of the semigroup $W_b(t)$, the solution v of (3.26) may be written as

$$v(t) = W_b(t)\phi^* - \int_0^t W_b(t-\tau)(a(u)v)_x(\tau)d\tau.$$

According to Proposition 3.2 (with $a \equiv 0$), $u_1(x,t) = W_b(t)\phi^*(x)$ satisfies

$$\|u_1(\cdot,t)\|_{L^2(I)} \leq Ce^{-\alpha t}\|\phi^*\|_{L^2(I)} \qquad \forall t \geq 0,$$

where C and α are some positive constants independent of ϕ^*. In addition, by Lemma 2.3, $u_2(x,t) = \int_0^t W_b(t-\tau)(\partial_x(a(u)v))(\tau)d\tau$ satisfies

$$\|u_2(\cdot,T)\| \leq C_T\|u\|^p_{Y_{0,T}}\|v\|_{Y_{0,T}}.$$

Thus,

$$\|v(\cdot,T)\|_{L^2(I)} \leq C_1e^{-\alpha T}\|\phi^*\|_{L^2(I)} + C_2\|u\|^p_{Y_{0,T}}\sigma(\|u\|_{Y_{0,T}})\|\phi^*\|_{L^2(I)}.$$

Let $y_n = v(\cdot,nT)$ for $n = 0,1,2,\ldots$, and let w be the solution of the IBVP

$$\begin{cases} \partial_t w + \partial_x w + \partial_x(qw) + bw + \partial_x^3 w = 0, \quad x \in (0,L), \quad 0 \leq t, \\ w(x,0) = y_n, \\ w(0,t) = 0, \quad w(L,t) = 0, \quad w_x(L,t) = 0 \end{cases} \tag{3.29}$$

with $q(x,t) = a(u(x,t+nT))$. Then $y_{n+1}(x) = w(x,T)$ by the semigroup properties of the system (3.26). Consequently, we have the following estimate for y_{n+1}:

$$\|y_{n+1}\|_{L^2(I)} \leq C_1e^{-\alpha T}\|y_n\|_{L^2(I)} + C_2\|u\|^p_{Y_{0,[nT,(n+1)T]}}\sigma(\|u\|_{Y_{0,[nT,(n+1)T]}})\|y_n\|_{L^2(I)}$$

for any $n \geq 0$. Choose T and β such that

$$C_1e^{-\alpha T} = \gamma < 1$$

and

$$\gamma + C_2\beta^p\sigma(\beta) = r < 1.$$

In addition, according to Corollary 3.8, one can choose $N > 0$ such that

$$\|u\|_{Y_{0,[nT,(n+1)T]}} < \beta \qquad \forall\ n \geq N.$$

Consequently, for such chosen T and N,

$$\|y_{n+1}\|_{L^2(I)} \leq r\|y_n\|_{L^2(I)} \qquad \forall\ n \geq N.$$

This inequality implies that there exists a $\nu^* > 0$ such that

$$\|v(\cdot,t)\|_{L^2(I)} = \|\partial_t u(\cdot,t)\|_{L^2(I)} \leq C\alpha_3\left(\|\phi\|_{L^2(I)}\right)\|\phi\|_{H^3(I)}e^{-\nu^* t} \qquad \forall\ t \geq 0. \tag{3.30}$$

Then, using (3.28) and Corollary 3.8 yields that

$$\|v\|_{Y_{0,[t,t+T]}} \le \alpha_4\left(\|\phi\|_{L^2(I)}, T\right)\|\phi\|_{H^3(I)} e^{-\nu^* t} \tag{3.31}$$

for any $t \ge 0$. In particular,

$$\|u_t\|_{L^2(t,t+T;H^1(I))} \le \alpha_4\left(\|\phi\|_{L^2(I)}, T\right)\|\phi\|_{H^3(I)} e^{-\nu^* t}$$

and

$$\|u\|_{L^2(t,t+T;H^1(I))} \le \alpha_0\left(\|\phi\|_{L^2(I)}, T\right) e^{-\nu^* t}$$

for any $t \ge 0$. Consequently,

$$\|a(u(\cdot,t))\partial_x u(\cdot,t)\|_{L^2(I)} + \|\partial_x u(\cdot,t)\|_{L^2(I)} \le \alpha_5\left(\|\phi\|_{L^2(I)}, T\right)\|\phi\|_{H^3(I)} e^{-\nu^* t}$$

for any $t \ge 0$.

The conclusion of Proposition 3.9 follows from the above estimates, since

$$\partial_x^3 u = -\partial_t u - a(u)\partial_x u - \partial_x u - bu.$$

The proof is complete. $\square$

THEOREM 3.10. *Let a be a C^1 function satisfying $a(0) = 0$ and*

$$|a(\mu)| \le C(1+|\mu|^p), \quad |a'(\mu)| \le C(1+|\mu|^{p-1}) \quad \text{for any } \mu \in \mathbb{R}$$

with $1 \le p < 2$. Then there exists a $\nu > 0$ depending only on L such that for any $\phi \in L^2(I)$ and any $\varepsilon > 0$, the corresponding solution u of (2.1) belongs to the space $C(\mathbb{R}^{+*}; H^3(I))$ and satisfies

$$\|u(\cdot,t)\|_{H^3(I)} \le C_\varepsilon e^{-\nu t} \qquad \forall\ t \ge \varepsilon$$

for some constant $C_\varepsilon > 0$.

Proof. It follows directly from Corollary 2.10 and Proposition 3.9. $\square$

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