CONTROL AND STABILIZATION OF THE KORTEWEG-DE VRIES EQUATION: RECENT PROGRESSES*

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Abstract The study of the control and stabilization of the KdV equation began with the work of Russell and Zhang in late 1980s. Both exact control and stabilization problems have been intensively studied since then and significant progresses have been made due to many people's hard work and contributions. In this article, the authors intend to give an overall review of the results obtained so far in the study but with an emphasis on its recent progresses. A list of open problems is also provided for further investigation.

Key words Exact controllability, Korteweg-de Vries equation, smoothing property, stabilizability, unique continuation.

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1 Introduction

To tell any story about the Korteweg-de Vries equation one cannot help but begin with John Scott Russell's vivid description of his discovery of solitary waves in his Report on Waves:†

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†Report of the 14th meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311–390, Plates XLVII-LVII

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

Following this discovery, John Scott Russell built a 30 feet water tank in his back garden and designed experiments generating long waves to investigate the phenomenon he had observed. He studied the form of the waves, their speed of propagation and stability. Then he challenged the mathematical community to prove theoretically the existence of his solitary wave ("the Wave of Translation") and to give an a priori demonstration a posteriori, i.e., to show the possible existence of a stable solitary wave propagating without change of form.

However, while Russell believed strongly that his solitary wave was of fundamental importance, he encountered resistance from established convictions. In particular, several prominent scientists and mathematicians, including Airy and Stokes, were convinced that the existence of such solitary waves was impossible. The issue kept unresolved for several decades until the positive answer was finally given, first by J. Boussinesq in 1871 (see, for example, [1–4]), some time later in 1876, by Lord Rayleigh[5] and in order to remove still existing doubts over the existence of the solitary wave by G. de Vries[6] and by D. J. Korteweg and G. de Vries[7] in 1895 .‡

In their famous paper [7], D. J. Korteweg and his PhD student G. de Vries derived the following equation (now known as the KdV equation)

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

to model water waves propagating along a shallow canal. Its original form is

$$\partial_t \eta = \frac{3}{2}\sqrt{\frac{g}{l}}\partial_x\Big(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\partial_x^2\eta\Big),$$

where η is the surface elevation above the equilibrium level l, α is a small constant related to the uniform motion of the liquid, g is the gravitational constant, and

$$\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$$

with surface capillary tension T and density ρ.

Korteweg and de Vries[7] proved that Russell was correct by finding explicit, closed-form, traveling-wave solutions to their equation that moreover decay rapidly and so represent a highly localized moving hump and therefore completely settled the long-standing open problem. Nevertheless, the KdV equation did not receive significant further attention and its study kept dormant for more than half century until 1965, when N. Zabusky and M. Kruskal[9] published the results of their numerical experimentation with the equation. Their computer generated approximate solutions to the KdV equation indicated that any localized initial profile that was

‡The interested readers are referred to [8] for an excellent article regarding the origin of the KdV equation.

allowed to evolve according to the KdV equation eventually consisted of a finite set of localized traveling waves of the same shape and furthermore, when two of the localized disturbances collided, they would emerge from the collision again as another pair of traveling waves with a shift in phase as the only consequence of their interaction. Since the "solitary waves" which made up these solutions seemed to behave like particles, Zabusky and Kruskal[9] coined the term "soliton" to describe them and thus opened the era of soliton. Shortly after that, another remarkable discovery was made concerning the KdV equation. C. Gardner, J. Greene, M. Kruskal, and R. Miura[10] demonstrated that it was possible to write many exact solutions to the equation by using ideas from scattering theory. It turns out that their method, now called the inverse scattering transform or the nonlinear Fourier transform, not only can be used to solve the KdV equation, but also can be adapted to solve effectively many other important nonlinear partial differential equations.

Because of the discovery of solitons and the invention of the inverse scattering transform, the KdV equation returned to the center stage of nonlinear sciences and has been intensively studied in the past fifty years from various aspects of both mathematics and physics. The equation has become the source of important breakthroughs in mechanics and nonlinear analysis and of many developments in algebra, analysis, geometry, and physics. It has been found that the equation is not only a good model for some water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects[11]. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems.

In this paper, we are mainly concerned with the study of the KdV equation from control point of view. Our attention , in particular, is given to the following three distributed parameter control systems described by the KdV equation.

(a) The KdV equation posed on a periodic domain $\mathbb{T}$ (a unit circle in the plane)[§] with a forcing function $f = f(x,t)$ added to the equation:

$$\partial_t u + u\partial_x u + \partial_x^3 u = f, \quad u(x,0) = \phi(x), \qquad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.1}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$;

(b) The KdV equation posed on a finite interval $(0, L)$ with the Dirichlet boundary conditions:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad u(x,0) = \phi(x), \qquad x \in (0,L),\ t > 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad \partial_x u(L,t) = h_3(t). \end{cases} \tag{1.2}$$

(c) The KdV equation posed on the half line $\mathbb{R}^+$:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t), \quad x \in \mathbb{R}^+, \quad t > 0. \end{cases} \tag{1.3}$$

In these systems, the external forcing terms $f,\ h_j,\ j = 1, 2, 3$, and h, are considered as control inputs. The purpose is to see whether one can force the solutions of those systems to have certain desired properties by choosing appropriate control inputs. The following two problems are fundamental in control theory.

[§]This is equivalent to imposing the periodic boundary conditions:

$$u(0,t) = u(2\pi,t), \quad \partial_x u(0,t) = \partial_x u(2\pi,t), \quad \partial_x^2 u(0,t) = \partial_x^2 u(2\pi,t).$$

Exact control problem Given an initial state ϕ and a terminal state ψ in a certain space, can one find an appropriate control input f so that the equation (1.1) admits a solution u which equals ϕ at time $t=0$ and equals ψ at time $t=T$?

If one can always find a control input f to guide the system described by (1.1) from any given initial state ϕ to any given terminal state ψ, then the system (1.1) is said to be exactly controllable.

Stabilization problem Can one find a feedback control law: $f = Ku$ so that the resulting closed-loop system

$$\partial_t u + u\partial_x u + \partial_x^3 u = Ku, \quad u(x,0) = \phi(x) \quad x \in \mathbb{T},\ t \in \mathbb{R}^+$$

is asymptotically stable as $\to +\infty$?

If such a feedback control law exists, then the system (1.1) is said to be stabilizable.

The same exact control and stabilization problems hold also for the systems (1.2) and (1.3).

The study of control and stabilization of the KdV equation began with the work of Russell[12] and Zhang[13] in late 1980s. Both exact control problem and stabilization problem have been intensively studied since then. Significant progresses have been made due to many people's hard work and contributions in the past twenty years. In this article we intend to give an overall review of the results obtained so far in the study but with an emphasis on its recent progresses.

The paper is organized as follows.

In Section 2, we consider distributive control of the KdV equation posed on the periodic domain $\mathbb{T}$ and demonstrate how harmonic analysis and micro-local analysis play important roles in establishing exact controllability and stabilizability of the system (1.1).

In Section 3, consideration is given to the boundary control system (1.2), which has some amazing properties. It is only null controllable if control input is only allowed to apply to the left end of the spatial domain; The system therefore behaves like a parabolic system. But the system is exactly controllable if control inputs are allowed to act on the right end of the spatial domain; The system thus behaves like a hyperbolic system. Moreover, depending on the length of the spatial domain, the associated linear system may not be controllable, but the nonlinear system is still exact controllable.

In Section 4, the boundary control system (1.3) will be discussed. Some general concluding remarks together with a list of open problems for further investigation will be provided.

2 The KdV Equation on a Periodic Domain

Consideration in this section is given to the KdV equation posed on the periodic domain $\mathbb{T}$:

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{2.1}$$

The equation is known to possess an infinite set of conserved integral quantities of which the first three are

$$I_1(t) = \int_{\mathbb{T}} u(x,t)dx,$$

$$I_2(t) = \int_{\mathbb{T}} u^2(x,t)dx,$$

and

$$I_3(t) = \int_{\mathbb{T}} \Big(u_x^2(x,t) - \frac{1}{3}u^3(x,t)\Big)dx.$$

From the historical origins[4,7,14] of the KdV equation, involving the behavior of water waves in a shallow channel, it is natural to think of I_1 and I_2 as expressing conservation of volume (or mass) and energy, respectively. The Cauchy problem of the Equation (2.4) has been intensively studied for many years (see [15–18] and the references therein). The best known result[19] so far is that the Cauchy problem is well-posed in the space $H^s(\mathbb{T})$ for any $s \geq -1$:

Let $s \geq -1$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the equation (2.1) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x).$$

Moreover, the corresponding solution map ($u_0 \to u$) is continuous from the space $H^s(\mathbb{T})$ to the space $C([0,T]; H^s(\mathbb{T}))$.¶

In this section, the Equation (2.1) will be studied from control point of view with a forcing function $f = f(x,t)$ added to the equation as a control input by which solutions are to be influenced:

$$\partial_t u + u\partial_x u + \partial_x^3 u = f, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \tag{2.2}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. In order to keep the mass $I_1(t)$ conserved, the control input $f(x,t)$ is chosen to be of the form

$$f(x,t) = [Gh](x,t) := g(x)\Big(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)dy\Big), \tag{2.3}$$

where h is considered as a new control input and $g(x)$ is a given nonnegative smooth function such that $\omega = \{g > 0\}$ and

$$2\pi[g] = \int_{\mathbb{T}} g(x)dx = 1.$$

For the chosen g, it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)dx = \int_{\mathbb{T}} f(x,t)dx = 0, \qquad \text{for any } t \in \mathbb{R}$$

for any solution $u = u(x,t)$ of the system

$$\partial_t u + u\partial_x u + \partial_x^3 u = Gh, \tag{2.4}$$

the mass of the system is therefore conserved.

Exact control problem Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input h so that the Equation (2.4) admits a solution u which coincides with u_0 at time $t = 0$ and with u_1 at time $t = T$?

Stabilization problem Can one find a feedback control law: $h = Ku$ so that the resulting closed-loop system

$$\partial_t u + u\partial_x u + \partial_x^3 u = GKu, \quad x \in \mathbb{T}, \ t \in \mathbb{R}^+$$

is asymptotically stable as $\to +\infty$?

Depending on the support of the function $g(x)$ in the domain $\mathbb{T}$, there are two different control situations. If the support of the function g is the whole spatial domain $\mathbb{T}$, then our control acts on the whole domain and we refer to it as global control. If the support of the function g is a proper subset of the spatial domain $\mathbb{T}$, our control acts only on a sub-domain and we refer to it as local control. Obviously, we have more control power in the global control

¶If $s > -\frac{1}{2}$, this solution map is, in fact, analytic.

situation than that in the local control case. On the other hand, the local control situation includes more cases of practical interest and is therefore more relevant in general.

These problems for the KdV equation were first studied by Komornik, and Russell and Zhang in [20], Russell and Zhang in [21–22]. In the case of global control, they showed that the system is globally exactly controllable and globally exponentially stabilizable.

Theorem 2.1[22] *Let $T > 0$ and $s \geq 1$ be given. Assume that*

$$|g(x)| \geq \beta > 0 \ \forall \ x \in \mathbb{T}.$$

Then, for any $u_0,\ u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$, there exists a $h \in L^1(0,T;H^{s-1}(\mathbb{T}))$ such that the system (2.4) *admits a solution $u \in C([0,T];H^s(\mathbb{T}))$ satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x).$$

Theorem 2.2[20] *Let $k \geq 0$ be a given integer. There exists a $\gamma > 0$ such that for any $u_0 \in H^k(\mathbb{T})$, the closed-loop system*

$$\partial_t u + u\partial_x u \partial_x^3 v = -GG^* u, \quad u(x,0) = u_0(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}$$

admits a unique solution $u \in C(\mathbb{R}^+;H^k(\mathbb{T}))$ which, moreover, satisfies

$$\|u(\cdot,t) - [u_0]\|_{H^k(\mathbb{T})} \leq \alpha_k(\|u_0\|_{H^s(\mathbb{T})})\mathrm{e}^{-\gamma t}$$

for any $t \geq 0$, where $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

The case of local control is much more challenging. In this situation, Russell and Zhang[21] considered first the associated linear open loop control system:

$$\partial_t v + \partial_x^3 v = Gh, \quad v(x,0) = v_0(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{2.5}$$

Let A denote the operator

$$Aw = -w'''$$

with its domain $\mathcal{D}(A) = H^3(\mathbb{T})$. The operator A generates a strongly continuous group $W(t)$ on the space $L^2(\mathbb{T})$; The eigenfunctions are simply the orthonormal Fourier basis functions in $L^2(\mathbb{T})$,

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}kx}, \quad k = 0, \pm 1, \pm 2, \cdots.$$

The corresponding eigenvalue of ϕ_k is

$$\lambda_k = \mathrm{i}k^3, \quad k = 0, \pm 1, \pm 2, \cdots.$$

Note that $A^* = -A$, $G^* = G$, and that $W^*(-t) = W(t)$ for any $t \in \mathbb{R}$. The linear system (2.5) is shown by Russell and Zhang[22] to be exactly controllable in $H^s(\mathbb{T})$ for any $s \geq 0$.

Theorem 2.3[22] *Let $s \geq 0$ and $T > 0$ be given. There exists a bounded linear operator*

$$\varPhi : H^s(\mathbb{T}) \times H^s(\mathbb{T}) \mapsto L^2(0,T;H^s(\mathbb{T}))$$

such that for any $v_0,\ v_1 \in H^s(\mathbb{T})$, with $[v_0] = [v_1]$,

$$W(T)v_0 + \int_0^T W(T-t)G(\varPhi(v_0,v_1))(t)\,dt = v_1$$

and

$$\|\varPhi(v_0,v_1)\|_{L^2(0,T;H^s(\mathbb{T}))} \le C(\|v_0\|_s + \|v_1\|_s),$$

where $C>0$ depends only on T and $\|g\|_s$.

The following estimate is a direct consequence of Theorem 2.3.

Corollary 2.4 *Let $T>0$ be given. There exists $\delta>0$ such that*

$$\int_0^T \|GW(t)\phi\|_0^2 dt \ge \delta \|\phi\|_0^2$$

for any $\phi \in L^2(\mathbb{T})$ with $[\phi]=0$.

In addition, choosing the following simple feedback law

$$h(v) = -G^* v.$$

Russell and Zhang[22] showed that the resulting closed-loop system

$$\partial_t v + \partial_x^3 v = -GG^* v, \quad v(x,0) = v_0(x), \quad x \in \mathbb{T} \tag{2.6}$$

is exponentially stable.

Theorem 2.5[22] *Let $s \ge 0$ be given. There exists a number $\kappa > 0$ independent of s such that for any $v_0 \in H^s(\mathbb{T})$, the corresponding solution v of (2.6) satisfies*

$$\|v(.,t) - [v_0]\|_s \le C\mathrm{e}^{-\kappa t} \|v_0 - [v_0]\|_s$$

for any $t \ge 0$ where $C>0$ is a constant depending only on s.

Note that in the above theorem, the decay rate is

$$\kappa = \inf\{-\mathrm{Re}\ \lambda,\ \lambda \in \sigma(A - GG^*)\}. \tag{2.7}$$

However, it is possible to choose an appropriate linear feedback law such that the decay rate of the resulting closed-loop system is as large as one desires.

For given $\lambda > 0$, define

$$D_\lambda \phi = \int_0^1 \mathrm{e}^{-2\lambda\tau} W(-\tau) GG^* W^*(-\tau)\phi \, d\tau$$

for any $\phi \in H^s(\mathbb{T})$. Clearly, D_λ is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$. Moreover, D_λ is a self-adjoint positive operator on $L_0^2(\mathbb{T}) = \{u \in L^2(\mathbb{T});\ [u]=0\}$, and so is its inverse D_λ^{-1}.

Choose the feedback control

$$h = -G^* D_\lambda^{-1}(v - [v]).$$

The resulting closed-loop system reads:

$$\partial_t v + \partial_x^3 v = -K_\lambda (v - [v]), \quad v(x,0) = v_0(x),\ x \in \mathbb{T} \tag{2.8}$$

with

$$K_\lambda := GG^* D_\lambda^{-1}.$$

If $\lambda = 0$, we define $K_0 = GG^*$. As a direct application of Theorem 2.1 in [23] due to Slemrod, we obtain the following result.

Theorem 2.6 *Let $s \geq 0$ and $\lambda > 0$ be given. For any $v_0 \in H^s(\mathbb{T})$, the system* (2.8) *admits a unique solution $v \in C(\mathbb{R}^+; H^s(\mathbb{T}))$. Moreover, there exists $M = M_s$ depending on s such that*

$$\|v(.,t) - [v_0]\|_s \leq M_s \mathrm{e}^{-\lambda t} \|v_0 - [v_0]\|_s$$

for any $t \geq 0$.

To extend the above linear results to the nonlinear system (2.4), we rewrite (2.4) in its integral form

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)[Gh](\tau)d\tau - \int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau.$$

For given $T > 0$ and $u_0,\ u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$, define

$$\omega(T,u) = \int_0^T W(T-\tau)[u\partial_x u](\tau)d\tau.$$

Select

$$h = \varPhi(u_0, u_1 + \omega(T,u))$$

and consider the map

$$\varGamma(u)(t) = W(t)u_0 + \int_0^t W(t-\tau)[G\varPhi(u_0, u_1 + \omega(T,u))](\tau)d\tau - \int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau$$

for any $u \in C([0,T]; H^s(\mathbb{T}))$. Note that

$$\varGamma(u)|_{t=0} = u_0, \qquad \varGamma(u)|_{t=T} = u_1 + \omega(T,u) - \omega(T,u) = u_1.$$

Thus, if one wants to show that the nonlinear system (2.4) is exactly controllable from the given initial state u_0 to the given terminal state u_1, it suffices to prove that the map $\varGamma$ is a contraction in an appropriate Banach space $\mathbb{X}_s^T$ which meets the following three requirements:

(a) $\mathbb{X}_s^T \subset C([0,T]; H^s(\mathbb{T}))$;

(b) There exists a constant C such that

$$\|W(t)\phi\|_{\mathbb{X}_s^T} \leq C\|\phi\|_s \qquad \text{for any } \ \phi \in H^s(\mathbb{T})$$

and

$$\Big\| \int_0^t W(t-\tau)f(\tau)d\tau \Big\|_{\mathbb{X}_s^T} \leq C\|f\|_{L^2(0,T;H^s(\mathbb{T}))} \qquad \text{for any } f \in L^2(0,T;H^s(\mathbb{T}));$$

(c) There exists a constant C such that

$$\Big\| \int_0^t W(t-\tau)\partial_x(uv)(\tau)d\tau \Big\|_{\mathbb{X}_s^T} \leq C\|u\|_{\mathbb{X}_s^T}\|v\|_{\mathbb{X}_s^T} \qquad \text{for any } u,\ v \in \mathbb{X}_s^T.$$

Theorem A *Let $s \geq 0$ and $T > 0$ be given and assume that* (a), (b)*, and* (c) *hold. There exists a $\delta > 0$. If $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$ satisfying*

$$\|u_0\|_s \leq \delta, \qquad \|u_1\|_s \leq \delta,$$

then one can find a control input $h \in L^2(0,T;H^s(\mathbb{T}))$ such that the system (2.4) *admits a solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x).$$

Proof Consider the nonlinear map

$$\Gamma(u)(t) = W(t)u_0 + \int_0^t W(t-\tau)[G\Phi(u_0, u_1 + \omega(T,u))](\tau)d\tau - \int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau$$

for $u \in \mathbb{X}_s^T$. Note that, by its definition,

$$\begin{aligned}\|\omega(T,u)\|_s &\le \Big\| \int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau)\Big\|_{C([0,T];H^s(\mathbb{T}))} \\ &\le C\Big\| \int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau)\Big\|_{\mathbb{X}_s^T} \\ &\le C'\|u\|^2_{\mathbb{X}_s^T}\end{aligned}$$

for some constants $C, C' > 0$. Thus, there exist positive constants C_1 and C_2 such that

$$\|\Gamma(u)\|_{\mathbb{X}_s^T} \le C_1(\|u_0\|_s + \|u_1\|_s) + C_2\|u\|^2_{\mathbb{X}_s^T}$$

for any $u \in \mathbb{X}_s^T$. Let

$$R_1 = 2C_1(\|u_0\|_s + \|u_1\|_s).$$

Then

$$\|\Gamma(u)\|_{\mathbb{X}_s^T} \le \Big(\frac{1}{2} + C_2R_1\Big)R_1 \le \frac{3}{4}R_1$$

if $\|u\|_{\mathbb{X}_s^T} \le R_1$ and

$$\frac{1}{2} + C_2R_1 \le \frac{3}{4}.$$

Consequently, if we choose $\delta > 0$ such that

$$\frac{1}{2} + 2C_1C_2\delta \le \frac{3}{4}$$

and, in addition, require that

$$\|u_0\|_s + \|u_1\|_s \le \delta,$$

then

$$\|\Gamma(u)\|_{\mathbb{X}_s^T} \le R_1$$

and

$$\|\Gamma(u) - \Gamma(v)\|_{\mathbb{X}_s^T} \le \alpha\|u - v\|_{\mathbb{X}_s^T}$$

if $u,\ v \in X_s^T$ with $\|u\|_{\mathbb{X}_s^T} \le R_1$ and $\|v\|_{\mathbb{X}_s^T} \le R_1$, where $0 < \alpha < 1$ is a constant independent of u and v, that is to say, the map Γ is a contraction whose fixed point is the desired solution. ∎

Thus, attention should turn to find such an appropriate space $\mathbb{X}_s^T$. However, at the time when Russell and Zhang were attacking the problem around 1990, it was a common opinion then that the selection of such a space $\mathbb{X}_s^T$ was extremely difficulty if not impossible. Indeed, one cannot choose

$$\mathbb{X}_s^T = C([0,T]; H^s(\mathbb{T}))$$

because the requirement (c) would not be satisfied for any $s \ge 0$. This is due to the fact that $f \in L^2(0,T]; H^s(\mathbb{T}))$ only implies that $u = \int_0^t W(t-\tau)f(\tau)d\tau$ belongs to the space

$C([0,T];H^s(\mathbb{T}))$. So, when $u\in C([0,T],H^s(\mathbb{T}))$, the function $f=u\partial_x u$ belongs to the space $L^2([0,T];H^{s-1}(\mathbb{T}))$, one can only have

$$\int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau \in C([0,T];H^{s-1}(\mathbb{T}))$$

at most. Certain smoothing property of the KdV equation on a periodic domain $\mathbb{T}$ is needed in searching such a space $\mathbb{X}_s^T$. Many failed attempts to find smoothing properties of the KdV equation on $\mathbb{T}$ led people to believe that such a smoothing property may not exist. It thus came as a big surprise that, in establishing the well-posedness of the Cauchy problem,

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, \quad x\in\mathbb{T},\ t\in\mathbb{R}, \\ u(x,0)=u_0(x) \end{cases}$$

in the space $H^s(\mathbb{T})$ for any $s\geq 0$, Bourgain[17] discovered a very subtle smoothing property of the equation on $\mathbb{T}$. In particular, Bourgain constructed a Banach space $X^T_{b,s}$, now called Bourgain space associated to the KdV equation, that has the property

$$\Big\| \int_0^t W(t-\tau)[u\partial_x u](\tau)d\tau \Big\|_{X^T_{\frac12,s}} \leq C\|u\|^2_{X^T_{\frac12,s}} \quad \text{for any } s\geq 0!$$

Because of the usefulness of the Bourgain space, let us have a more detail discussion. For a function $w:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$, define the quantity

$$\Lambda_{b,s}(w) := \Big(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k\rangle^{2s} \left\langle \tau - k^3 + \mu k\right\rangle^{2b} \left|\widehat{\widehat{w}}(k,\tau)\right|^2 d\tau \Big)^{\frac12}$$

for given $b,\ s\in\mathbb{R}$, where $\widehat{\widehat{w}}(k,\tau)$ denotes the Fourier transform of w with respect to the space variable x and the time variable t (by contrast, $\widehat{w}(k,t)$ denotes the Fourier transform in space variable x) and $\langle\cdot\rangle=\sqrt{1+|.|^2}$. Moreover, denote by D^r the operator defined on $\mathcal{D}'(\mathbb{T})$ by

$$\widehat{D^r u}(k) = \begin{cases} |k|^r \widehat{u}(k), & \text{if} \quad k\neq 0, \\ \widehat{u}(0), & \text{if} \quad k=0. \end{cases} \tag{2.9}$$

The Bourgain space $X_{b,s}$ associated to the KdV equation posed on $\mathbb{T}$ is the completion of the space $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ under the norm

$$\|v\|_{X_{b,s}} := \Lambda_{b,s}(v)$$

for any $v\in\mathcal{S}(\mathbb{T}\times\mathbb{R})$. For a given interval I, let $X_{b,s}(I)$ be the associated restriction space of $X_{b,s}$ to the interval I with the norm

$$\|w\|_{X_{b,s}(I)} = \inf\left\{ \|\widetilde{w}\|_{X_{b,s}} \,|\widetilde{w}=w \text{ on } \mathbb{T}\times I \right\}.$$

Note that for any $u\in X_{b,s}$,

$$\|u\|_{X_{b,s}} = \|W(-t)u\|_{H^b(\mathbb{R},H^s(\mathbb{T}))}.$$

The following properties of the space $X_{b,s}$ are easily verified:

(i) $X_{b,s}(I)$ is a Hilbert space.

(ii) $D^r u\in X_{b,s-r}(I)$ for any $u\in X_{b,s}(I)$.

(iii)If $b_1\leq b_2$ and $s_1\leq s_2$, then X_{b_2,s_2} is continuously imbedded in the space X_{b_1,s_1}.

(iv) For a given finite interval I, if $b_1 < b_2$ and $s_1 < s_2$, then the space $X_{b_2,s_2}(I)$ is compactly imbedded in the space $X_{b_1,s_1}(I)$.

For simplicity, we denote $X_{b,s}(I)$ by $X^T_{b,s}$ if $I=(0,T)$. The space $X^T_{b,s}$ possesses the following important properties.

Lemma 2.7 *Let $b,s\in\mathbb{R}$ with $b>1/2$ and $T>0$ be given. There exists a constant $C>0$ depending only on b such that*

(i) for any $\phi\in H^s(\mathbb{T})$,

$$\|W(t)\phi\|_{X^T_{b,s}}\le C\|\phi\|_s;$$

(ii) For any $f\in X^T_{b-1,s}$,

$$\Big\|\int_0^t W(t-\tau)f(\tau)d\tau\Big\|_{X^T_{b,s}}\le C\|f\|_{X^T_{b-1,s}}.$$

Lemma 2.8 (Bilinear estimates) *Let $s\ge 0$, $T>0$, and $u,v\in X^T_{\frac12,s}\cap L^2(0,T;L^2_0(\mathbb{T}))$. Then, the function $f(t)=\int_0^t W(t-\tau)(uv)_x(\tau)\,d\tau$ belongs to the space $C([0,T];H^s_0(\mathbb{T}))\cap X^T_{\frac12,s}$, and there exist some constants $\theta>0$ and $C>0$ independent of T (for $T\to 0$) and u,v such that*

$$\|(uv)_x\|_{X^T_{-\frac12,s}}\le CT^\theta\|u\|_{X^T_{\frac12,s}}\|v\|_{X^T_{\frac12,s}}, \tag{2.10}$$

$$\Big\|\int_0^t W(t-\tau)(uv)_x(\tau)\,d\tau\Big\|_{C([0,T];H^s(\mathbb{T}))\cap X^T_{\frac12,s}}\le C\|u\|_{X^T_{\frac12,s}}\|v\|_{X^T_{\frac12,s}}. \tag{2.11}$$

In the above lemma, $H^s_0(\mathbb{T})=\{f\in H^s(\mathbb{T});\ [f]=0\}$ and $H^0_0(\mathbb{T})=L^2_0(\mathbb{T})$. Thus, for given $s\ge 0$ and $T>0$, the space

$$\mathbb{X}^T_s=X^T_{\frac12,s}\cap C([0,T];H^s_0(\mathbb{T}))$$

meets all the three requirements (a), (b) and (c) listed earlier. The following local controllability result due to Russell and Zhang[22] follows consequently.

Theorem 2.9[22] *Let $s\ge 0$ and $T>0$ be given. There exists a $\delta>0$. If $u_0,u_1\in H^s(\mathbb{T})$ with $[u_0]=[u_1]$ satisfying*

$$\|u_0\|_s\le\delta,\qquad \|u_1\|_s\le\delta,$$

then one can find a control input $h\in L^2(0,T;H^s(\mathbb{T}))$ such that the system (2.4) admits a solution $u\in C([0,T];H^s(\mathbb{T}))$ satisfying

$$u(x,0)=u_0(x),\qquad u(x,T)=u_1(x).$$

In order to stabilize the system (2.4), Russell and Zhang[22] selected

$$h(x,t)=-G^*u(x,t). \tag{2.12}$$

The resulting closed-loop system

$$\partial_t u+u\partial_x u+\partial_x^3 u=-GG^*u,\quad x\in\mathbb{T},\quad t\in\mathbb{R} \tag{2.13}$$

is locally exponentially stable.

Theorem 2.10[22] *Let $s \geq 0$ be given. There exist positive constants M, δ, and γ such that if $u_0 \in H^s(\mathbb{T})$ satisfies*

$$\|u_0 - [u_0]\|_s \leq \delta, \tag{2.14}$$

then the corresponding solution u of (2.13) *satisfies*

$$\|u(\cdot,t) - [u_0]\|_s \leq M\mathrm{e}^{-\gamma t}\|u_0 - [u_0]\|_s$$

for any $t \geq 0$.

As in the proof of Theorem 2.9, the Bourgain space $\mathbb{X}_s^T$ plays a key role in the proof of Theorem 2.10.

Thus, one can always find an appropriate control input h to guide the system (2.4) from a given initial state u_0 to a terminal state u_1 so long as their amplitude are small and $[u_0] = [u_1]$. A question arises naturally.

Question 2.1 *Can one still guide the system by choosing appropriate control input h from a given initial state u_0 to a given terminal state u_1 when u_0 or u_1 have large amplitude*?

As for the closed-loop system (2.13), its small amplitude solutions decay at a uniform exponential rate to the corresponding constant state $[u_0]$ with respect to the norm in the space $H^s(\mathbb{T})$ as $t \to \infty$. One may wonder:

Question 2.2 *Do the large amplitude solutions of the closed-loop system* (2.13) *decay exponentially as $t \to \infty$*?

A further question is

Question 2.3 *For any given number $\lambda > 0$, can we design a linear feedback control law such that the exponential decaying rate of the resulting closed-loop system is λ*?

All these three questions have been studied recently by Laurent, Rosier and Zhang and positive answers have been provided in [24].

Theorem 2.11[24] *Let $s \geq 0$ and $R > 0$ be given. There exists a $T > 0$. If $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$ satisfy*

$$\|u_0\|_s \leq R, \qquad \|u_1\|_s \leq R,$$

then one can find a control input $h \in L^2(0,T;H^s(\mathbb{T}))$ such that the system (2.4) *admits a solution $u \in C([0,T];H^s(\mathbb{T}))$ satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x).$$

Remark 2.12 In Theorem 2.9, the control time T is independent of the initial state u_0 and the terminal state u_1 and can be, in fact, chosen arbitrarily small. By contrast, in Theorem 2.11, the control time T depends on the size of the initial state u_0 and the terminal state u_1 in the space $L^2(\mathbb{T})$. Such a global controllability is called large time global controllability.

Theorem 2.13[24] *Let $s \geq 0$ be given. There exists a constant $\kappa > 0$ such that for any $u_0 \in H^s(\mathbb{T})$, the corresponding solution u of the system* (2.13) *satisfies*

$$\|u(\cdot,t) - [u_0]\|_s \leq \alpha_s(\|u_0\|_0)\mathrm{e}^{-\kappa t}\|u_0 - [u_0]\|_s \qquad \textit{for all } \ t \geq 0, \tag{2.15}$$

where $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s.

The decay rate κ in this theorem has an upper bound

$$\kappa \leq \inf\{-\mathrm{Re}\,\lambda : \ \lambda \in \sigma_p(A_G)\},$$

where A_G is the operator defined by

$$A_G v = -v''' - GG^* v$$

with its domain $\mathcal{D}(A_G) = H^3(\mathbb{T})$. In order to have the decay rate κ being arbitrarily large, a different feedback control law is needed.

Theorem 2.14[24] *Let $\lambda > 0$ and $s \geq 0$ be given. There exists a number $\delta > 0$ and a linear bounded operator Q_λ from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$ such that if one chooses the feedback control law*

$$h = -Q_\lambda u$$

in system (2.3)–(2.4)*, then the solution u of the resulting closed-loop system*

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GQ_\lambda u, \quad u(x,0) = u_0(x),\ x \in \mathbb{T} \tag{2.16}$$

satisfies

$$\|u(\cdot,t) - [u_0]\|_s \leq C\mathrm{e}^{-\lambda t}\|u_0 - [u_0]\|_s \qquad \text{for all } \ t \geq 0,$$

whenever $\|u_0\|_s \leq \delta$, $C > 0$ denoting a constant independent of u_0.

Note that this is still a local stabilization result. However, the feedback laws in Theorems 2.13 and 2.14 may be combined into a time-varying feedback law (as in [25]) ensuring a global stabilization with an arbitrary large decay rate.

Theorem 2.15[24] *Let $\lambda > 0$ and $s \geq 0$ be given. There exists a smooth map Q_λ from $H^s(\mathbb{T}) \times \mathbb{R}$ to $H^s(\mathbb{T})$ which is periodic with respect to the second variable, and such that the solution u of the closed-loop system*

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GQ_\lambda(u,t), \qquad u(\cdot,0) = u_0$$

satisfies

$$||u(\cdot,t) - [u_0]||_s \leq \alpha_{s,\lambda}(||u_0||_s)\mathrm{e}^{-\lambda t}||u_0 - [u_0]||_s \qquad \text{for all } \ t \geq 0,$$

where $\alpha_{s,\lambda} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s and λ.

Remark 2.16 While the decay rates κ in Theorem 2.13 and λ in Theorem 2.15 are independent of u_0, the constants $\alpha_s(\|u_0\|_0)$ or $\alpha_{s,\lambda}(\|u_0\|_s)$ are likely not uniformly bounded; i.e., it may happen that

$$\lim_{r\to\infty} \alpha_s(r) = \infty \text{ or } \lim_{r\to\infty} \alpha_{s,\lambda}(r) = \infty.$$

As explained earlier, for the control problems discussed in this article for KdV equation, how to extend the results for the linear systems to the corresponding nonlinear systems turned out to be a challenging task. Indeed, after having published their linear results[21], Russell and Zhang had to wait for several years to present their proofs of Theorems 2.9 and 2.10 for the nonlinear systems (2.4) and (2.13) until Bourgain[17] discovered a subtle smoothing property of solutions of the KdV equation posed on a periodic domain $\mathbb{T}$. This then newly discovered smoothing property of the KdV equation has played an indispensable role in the proofs of Theorems 2.9 and 2.10. By contrast, to establish the global exact controllability and stabilizability for the nonlinear system (2.13) is even more challenging. After all, the results presented in Theorems 2.9 and Theorem 2.10 are essentially linear in nature; they are more or less small perturbations of the linear results. The global results presented in Theorems 2.11, 2.13, and 2.15 are truly nonlinear and their proofs demand new tools. The needed help turns out to be certain propagation properties of compactness and regularity for the KdV equation in the framework of Bourgain space as given by the propositions below, which are inspired by those established by Laurent in [26] for the Schrödinger equation.

Proposition 2.17 (Propagation of compactness) *Let $T > 0$ and $0 \leq b' \leq b \leq 1$ be given* (*with $b > 0$*) *and suppose that $u_n \in X^T_{b,0}$ and $f_n \in X^T_{-b,-2+2b}$ satisfy*

$$\partial_t u_n + \partial_x^3 u_n = f_n$$

for $n = 1, 2, \cdots$. *Assume that there exists a constant* $C > 0$ *such that*

$$\|u_n\|_{X^T_{b,0}} \leq C \qquad \text{for all} \quad n \geq 1, \tag{2.17}$$

and that

$$\|u_n\|_{X^T_{-b,-2+2b}} + \|f_n\|_{X^T_{-b,-2+2b}} + \|u_n\|_{X^T_{-b',-1+2b'}} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.18}$$

In addition, assume that for some nonempty open set $\omega \subset \mathbb{T}$, *it holds*

$$u_n \to 0 \text{ strongly in } L^2(0,T; L^2(\omega)).$$

Then

$$u_n \to 0 \text{ strongly in } L^2_{\text{loc}}((0,T); L^2(\mathbb{T})).$$

Proposition 2.18 (Propagation of regularity) *Let* $T > 0$, $0 \leq b < 1$, $r \in \mathbb{R}$ *and* $f \in X^T_{-b,r}$ *be given. Let* $u \in X^T_{b,r}$ *be a solution of*

$$\partial_t u + \partial_x^3 u = f.$$

If there exists a nonempty open set ω *of* $\mathbb{T}$ *such that* $u \in L^2_{\text{loc}}((0,T), H^{r+\rho}(\omega))$ *for some* ρ *with*

$$0 < \rho \leq \min\left\{1 - b, \frac{1}{2}\right\},$$

then $u \in L^2_{\text{loc}}((0,T), H^{r+\rho}(\mathbb{T}))$.

The proofs of Propositions 2.17 and 2.18 can be found in [24].

Corollary 2.19 *Let* $u \in X^T_{\frac{1}{2},0}$ *be a solution of*

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0 \quad \text{on} \quad \mathbb{T} \times (0,T). \tag{2.19}$$

Assume that $u \in C^\infty(\omega \times (0,T))$, *where* ω *is a nonempty open set in* $\mathbb{T}$. *Then* $u \in C^\infty(\mathbb{T} \times (0,T))$.

Corollary 2.20 *Let* ω *be a nonempty open set in* $\mathbb{T}$ *and let* $u \in X^T_{\frac{1}{2},0}$ *be a solution of*

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, & \text{on} \quad \mathbb{T} \times (0,T), \\ u = c, & \text{on} \quad \omega \times (0,T), \end{cases}$$

where $c \in \mathbb{R}$ *denotes some constant. Then* $u(x,t) = c$ *on* $\mathbb{T} \times (0,T)$

In order to demonstrate how the properties of propagation of compactness and regularity together with the Bourgain space analysis play key roles in establishing the global results as described in Theorems 2.11, 2.13, and 2.15, we provide a sketch of the proof of Theorem 2.13 in the case of $s = 0$. The interested readers are referred to [24] for the proofs of the others.

Note first that Theorem 2.13 with $s = 0$ is a direct consequence of the following observability inequality.

Proposition 2.21 *Let* $T > 0$ *and* $R_0 > 0$ *be given. There exists a constant* $\beta > 1$ *such that for any* $u_0 \in L^2_0(\mathbb{T})$ *satisfying*

$$\|u_0\|_0 \leq R_0,$$

the corresponding solution u *of* (2.13) *satisfies*

$$\|u_0\|_0^2 \leq \beta \int_0^T ||Gu||_0^2(t)dt. \tag{2.20}$$

Indeed, if (2.20) holds, then it follows from the energy estimate

$$\|u(\cdot,t)\|_0^2 = \|u_0\|_0^2 - \int_0^t ||Gu||_0^2(\tau)d\tau, \quad \forall t \ge 0 \tag{2.21}$$

that

$$\|u(\cdot,T)\|_0^2 \le (1-\beta^{-1})\|u_0\|_0^2.$$

Thus

$$\|u(\cdot,mT)\|_0^2 \le (1-\beta^{-1})^m\|u_0\|_0^2,$$

which gives (2.15) by the semigroup property. ∎

Now, we present a sketch of the proof of Proposition 2.21.

The sketch of the proof of Proposition 2.21 We prove the estimate (2.20) by contradiction. If (2.20) is not true, then for any $n \ge 1$, (2.13) admits a solution $u_n \in X^T_{\frac{1}{2},0}$ satisfying

$$\|u_n(0)\|_0 \le R_0$$

and

$$\int_0^T ||Gu_n||_0^2 dt < \frac{1}{n}\,\|u_{0,n}\|_0^2\,, \tag{2.22}$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n := \|u_{0,n}\|_0 \le R_0$, one can choose a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, such that

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

There are two possible cases: (i) $\alpha > 0$ and (ii) $\alpha = 0$.

(i) $\alpha > 0$

Note that the sequence $\{u_n\}$ is bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))$ and $X^T_{\frac{1}{2},0}$. By Lemma 2.8, the sequence $\{\partial_x(u_n^2)\}$ is bounded in the space $X^T_{-\frac{1}{2},0}$. On the other hand, the space $X^T_{\frac{1}{2},0}$ is compactly imbedded in the space $X^T_{0,-1}$. Therefore, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{aligned} &u_n \to u \quad \text{ weakly in } X^T_{\frac{1}{2},0}, \text{ and strongly in } X^T_{0,-1},\\ &-\frac{1}{2}\partial_x(u_n^2) \to f \quad \text{ weakly in } X^T_{-\frac{1}{2},0}, \end{aligned}$$

where $u \in X^T_{\frac{1}{2},0}$ and $f \in X^T_{-\frac{1}{2},0}$. Furthermore, it can be shown that one can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $-\frac{1}{2}\partial_x(u_n^2)$ converges to f strongly in $X^T_{-\frac{1}{2},-1}$. It follows from (2.22) that

$$\int_0^T ||Gu_n||_0^2\, dt \longrightarrow \int_0^T ||Gu||_0^2\, dt = 0,$$

which implies that $u(x,t) = c(t)$ on $\omega \times (0,T)$ for some function $c(t)$. Thus, passing to the limit in (2.13), we obtain

$$\begin{cases} \partial_t u + \partial_x^3 u = f, & \text{on} \quad \mathbb{T}\times(0,T),\\ u = c(t), & \text{on} \quad \omega\times(0,T). \end{cases} \tag{2.23}$$

Let $w_n = u_n - u$ and $f_n = -\frac{1}{2}\partial_x(u_n^2) - f - K_0 u_n$. One can show that

$$\partial_t w_n + \partial_x^3 w_n = f_n$$

and

$$f_n \underset{X^T_{-\frac{1}{2},-1}}{\longrightarrow} 0, \quad w_n \underset{L^2(0,T;L^2(\widetilde{\omega}))}{\longrightarrow} 0,$$

where $\widetilde{\omega} := \{g > ||g||_{L^\infty(\mathbb{T})}/2\}$.

Applying Proposition 2.17 with $b = \frac{1}{2}$ and $b' = 0$ yields that

$$w_n \underset{L^2_{\mathrm{loc}}((0,T);L^2(\mathbb{T}))}{\longrightarrow} 0.$$

Consequently, u_n^2 tends to u^2 in $L^1_{\mathrm{loc}}((0,T);L^1(\mathbb{T}))$ and $\partial_x(u_n^2)$ tends to $\partial_x(u^2)$ in the distributional sense. Therefore, $f = -\frac{1}{2}\partial_x(u^2)$ and $u \in X^T_{\frac{1}{2},0}$ satisfies

$$\begin{cases} \partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x(u^2) = 0, & \text{on} \quad \mathbb{T} \times (0,T), \\ u = c(t), & \text{on} \quad \omega \times (0,T). \end{cases}$$

The first equation gives $c'(t) = 0$ which, combined to Corollary 2.20, yields that $u(x,t) \equiv c$ for some constant $c \in \mathbb{R}$. Since $[u] = 0$, $c = 0$, and u_n converges strongly to 0 in $L^2_{\mathrm{loc}}((0,T), L^2(\mathbb{T}))$. We can pick some time $t_0 \in [0,T]$ such that $u_n(t_0)$ tends to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|u_n(0)\|_0^2 = \|u_n(t_0)\|_0^2 + \int_0^{t_0} ||Gu_n||_0^2 dt,$$

it is inferred that $\alpha_n = \|u_n(0)\|_0 \to 0$, which is a contradiction to the assumption $\alpha > 0$.

(ii) $\alpha = 0$.

Note first that $\alpha_n > 0$ for all n. Set $v_n = u_n/\alpha_n$ for all $n \geq 1$. Then

$$\partial_t v_n + \partial_x^3 v_n + \mu \partial_x v_n + K_0 v_n + \frac{\alpha_n}{2}\partial_x(v_n^2) = 0$$

and

$$\int_0^T ||Gv_n||_0^2 dt < \frac{1}{n}. \tag{2.24}$$

Because of

$$\|v_n(0)\|_0 = 1, \tag{2.25}$$

the sequence $\{v_n\}$ is bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))$ and $X^T_{\frac{1}{2},0}$. We can extract a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that $v_n \to v$ weakly in the space $X^T_{\frac{1}{2},0}$ and strongly in the spaces $X^T_{-\frac{1}{2},-1}$ and $X^T_{0,-1}$. Moreover, the sequence $\{\partial_x(v_n^2)\}$ is bounded in the space $X^T_{-\frac{1}{2},0}$, and therefore $\alpha_n \partial_x(v_n^2)$ tends to 0 in the space $X^T_{-\frac{1}{2},0}$. Finally, $\int_0^T ||Gv||_0^2 dt = 0$. Thus, v solves

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, & \text{on} \quad \mathbb{T} \times (0,T) \\ v = c(t), & \text{on} \quad \omega \times (0,T). \end{cases} \tag{2.26}$$

Once again, we infer that $v(x,t) = c(t) = c$ by Holmgren Theorem, and that $c = 0$ because of $[v] = 0$.

According to (2.24),

$$\int_0^T ||Gv_n||_0^2 dt \longrightarrow 0$$

and so $K_0 v_n$ converges strongly to 0 in $X^T_{-\frac{1}{2},-1}$. Then, an application of Proposition 2.17 as in (i) shows that v_n converges to 0 in $L^2_{\text{loc}}((0,T),L^2(\mathbb{T}))$. Thus, we can pick a time $t_0 \in (0,T)$ such that $v_n(t_0)$ converges to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|v_n(0)\|_0^2 = \|v_n(t_0)\|_0^2 + \int_0^{t_0} ||Gv_n||_0^2 dt,$$

we infer from (2.24) that $\|v_n(0)\|_0 \to 0$, which is a contradiction to (2.25). ∎

3 The KdV Equation on a Finite Interval $(0, L)$

In this section our attention is focused on the KdV equation posed on a finite interval $(0,L)$ with the Dirichlet boundary conditions:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0,L),\ t>0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), & \partial_x u(L,t) = h_3(t). \end{cases} \tag{3.1}$$

Considering the boundary value functions $h_j,\ j=1,2,3$, as control inputs, we are mainly concerned with the controllability of the control system (3.1).

The boundary control problem of the KdV equation was first studied by Rosier[27] who considered the system (3.1) with only one boundary control input h_3 (with $h_1 = h_2 \equiv 0$) in action. He showed that the system (3.1) is locally exactly controllable in the space $L^2(0,L)$.

Theorem 3.1[27] *Let $T>0$ be given and assume*

$$L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2+l^2+jl}{3}};\quad j,l \in \mathbb{N}^* \right\}. \tag{3.2}$$

There exists a $\delta > 0$. If $\phi,\ \psi \in L^2(0,L)$ with

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \le \delta,$$

then one can find a control input $h_3 \in L^2(0,T)$ such that the system (3.1) (*with $h_1 = h_2 \equiv 0$*) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u(x,0) = \phi(x), \qquad u(x,T) = \psi(x).$$

The theorem was first proved for the associated linear system using the Hilbert Uniqueness Method without the smallness assumption on the initial state ϕ and the terminal state ψ. The linear result was then extended to the nonlinear system to obtain Theorem 3.1 by using the contraction mapping principle.

Remark 3.2 In the case of $L \in \mathcal{N}$, Rosier[27] demonstrated that the associated linear system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & x \in (0,L),\ t>0, \\ u(0,t) = 0, \quad u(L,t) = 0, & \partial_x u(L,t) = h(t) \end{cases} \tag{3.3}$$

is not exactly controllable; There exists a finite dimensional subspace M of $L^2(0,L)$ which is not reachable by the system (3.3) when starting from the origin. More precisely, for any $0 \neq \psi \in M$, the solution u of (3.3) satisfies

$$u(x,0)=0, \quad u(x,T) \neq \psi(x)$$

for any control input $h \in L^2(0,T)$.

From now on a spatial domain $(0,L)$ is called critical if its domain length $L \in \mathcal{N}$.

Remark 3.3 However, if one is allowed to use both h_2 and h_3 as control inputs, L. Rosier has pointed out in [27] that both the linear system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & x \in (0,L),\ t>0, \\ u(0,t)=0, \quad u(L,t)=h_2(t), & \partial_x u(L,t) = h_3(t) \end{cases}$$

and the corresponding nonlinear system

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0,L),\ t>0, \\ u(0,t)=0, \quad u(L,t)=h_2(t), & \partial_x u(L,t) = h_3(t) \end{cases}$$

are locally exactly controllable in the space $L^2(0,L)$ even if $L \in \mathcal{N}$.

When the spacial domain $(0,L)$ is critical, as the linear system (3.3) is not exactly controllable, one usually would not expect the corresponding nonlinear system

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0,L),\ t>0, \\ u(0,t)=0, \quad u(L,t)=0, & \partial_x u(L,t) = h(t) \end{cases} \tag{3.4}$$

to be exactly controllable. However, Coron and Crépeau[28] showed surprisingly that the system (3.4) with $L = 2k\pi \in \mathcal{N}$ $(k=1,2,\cdots)$ is locally exactly controllable in the space $L^2(0,L)$ though its associated linear system (3.3) is not exactly controllable!

Theorem 3.4[28] *Let $T>0$ and $L=2k\pi \in \mathcal{N}$ for some positive integer k. There exists a $\delta>0$. If $\phi,\ \psi \in L^2(0,L)$ with*

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \leq \delta,$$

then one can find a boundary control input $h \in L^2(0,T)$ such that the system (3.4) admits a solution

$$u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

satisfying

$$u(x,0)=\phi(x), \quad u(x,T)=\psi(x).$$

Note that when the spacial domain length $L = 2k\pi \in \mathcal{N}$ for some integer $k>0$, the unreachable subspace $\mathcal{M}$ of the linear system (3.3) is only one-dimension. Later, Cerpa[29] considered the case where the unreachable space $\mathcal{M}$ of the linear system (3.3) is two-dimensional, and showed that the nonlinear system is also locally large time exactly controllable in the space $L^2(0,L)$.

Let

$$\mathcal{N}' := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ (k,l) \in \mathbb{N}^* \times \mathbb{N}^* \text{ satisfying } k>l \text{ and } \right.$$
$$\left. \forall m,\ n \in \mathbb{N}^* \setminus \{k\},\ k^2+kl+l^2 \neq m^2+mn+n^2 \right\}.$$

Obviously, $\mathcal{N}' \subset \mathcal{N}$. Moreover, the corresponding unreachable subspace $\mathcal{M}$ is two-dimensional if $L \in \mathcal{N}'$.

Theorem 3.5[29] *Let $L \in \mathcal{N}'$ be given. There exists $T_L > 0$ such that for any $T > T_L$, there is a $\delta > 0$. If $\phi,\ \psi \in L^2(0,L)$ with*

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \le \delta,$$

then one can find a boundary control input $h \in L^2(0,T)$ such that the system (3.4) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u(x,0) = \phi(x), \quad u(x,T) = \psi(x).$$

Note that in Theorem 3.4, the control time T needed can be arbitrarily small while in Theorem 3.5, the needed control time T has to be large enough. That is why we call the exact controllability presented in Theorem 3.5 as large time controllability.

According to Theorems 3.4 and 3.5, the system (3.4) is still locally exactly controllable when the unreachable subspace $\mathcal{M}$ of the associated linear system is either one- or two-dimensional. However, the dimension of $\mathcal{M}$ though finite can be as large as one wants by choosing appropriate $L \in \mathcal{N}$. A question left open in [28–29] is if the system (3.4) is still locally exactly controllable for any other $L \in \mathcal{N}$. An affirmative answer is provided by Cerpa and Crépeau[30] recently.

Theorem 3.6[30] *Given $L \in \mathcal{N}$, there exists a $T_L > 0$ such that for any $T > T_L$, there is a $\delta > 0$ for which for any $\phi,\ \psi \in L^2(0,L)$ with*

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \le \delta,$$

one can find a boundary control input $h \in L^2(0,T)$ such that the system (3.4) *admits a solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u(x,0) = \phi(x), \quad u(x,T) = \psi(x).$$

Remark 3.7 Not like the local controllability result as stated in Theorem 3.1, which is a small perturbation of the linear controllability and therefore essentially a linear result, the controllability results presented in Theorems 3.4–3.6 are truly nonlinear.

Now, we return to consider the system (3.1) with more boundary control inputs in action. In [31], Zhang studied the nonlinear system (3.1) with all three boundary control inputs in action and showed the system is locally exactly controllable around any smooth solution v of the KdV equation posed on the spacial domain $(-\epsilon, L+\epsilon)$:

$$\partial_t v + \partial_x v + v\partial_x v + \partial_x^3 v = 0, \qquad x \in (-\epsilon, L+\epsilon), \quad t \ge 0.$$

Theorem 3.8[31] *Let $L > 0$, $T > 0$ and $s \ge 0$ be given. Suppose that*

$$v \in C^\infty((-\epsilon, L+\epsilon) \times (-\epsilon, T+\epsilon))$$

solves the KdV *equation on the region $(-\epsilon, L+\epsilon) \times (-\epsilon, T+\epsilon)$:*

$$\partial_t v + \partial_x v + v\partial_x v + \partial_x^3 v = 0, \qquad (x,t) \in (-\epsilon, L+\epsilon) \times (-\epsilon, T+\epsilon).$$

There exists a $\delta > 0$. *If* $\phi,\ \psi \in H^s(0,L)$ *satisfying*

$$\|\phi(\cdot) - v(\cdot,0)\|_{H^s(0,L)} \le \delta, \quad \|\psi(\cdot) - v(\cdot,T)\|_{H^s(0,L)} \le \delta,$$

then one can find control inputs $h_j \in C([0,T])$ *for* $j = 1,2$ *and* $h_3 \in L^2(0,T)$ ‖ *with*

$$h_1(t) = w(0,t), \quad h_2(t) = w(L,t), \quad h_3(t) = \partial_x w(L,t)$$

for some $w \in C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+1}_{\mathrm{loc}}(\mathbb{R}))$ *such that the system* (3.1) *admits a solution* $u \in C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$ *satisfying*

$$u(x,0) = \phi(x), \qquad u(x,T) = \psi(x), \quad \text{on the interval } (0,L).$$

To prove his theorem, Zhang considered an alternative initial value control problem of the KdV equation posed on the domain $\mathbb{R}\times\mathbb{R}$:

$$\begin{cases} \partial_t w + \partial_x w + w\partial_x w + \partial_x^3 w = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ w(x,0) = h(x), \quad x \in \mathbb{R}. \end{cases} \tag{3.5}$$

Let $T > 0$ and $s \ge 0$ be given. For any $\phi,\ \psi \in H^s(\mathbb{R})$, can we find a $h \in H^s(\mathbb{R})$ such that the corresponding solution u the IVP (3.5) satisfies

$$u(x,0) = \phi(x), \qquad u(x,T) = \psi(x)$$

on the interval $(0,L)$?

An affirmative answer to this initial value control problem leads to a positive answer to the boundary control problem asked earlier. Indeed, for any given $\phi,\ \psi \in H^s(0,L)$, we can extend ϕ and ψ to be functions in $H^s(\mathbb{R})$ (let us still write their extensions as ϕ and ψ). If we can find a $h \in H^s(\mathbb{R})$ such that the corresponding unique solution u of the IVP (3.5) satisfies

$$w(x,0) = \phi(x), \qquad w(x,T) = \psi(x)$$

on the interval $(0,L)$, then we can just choose

$$h_1(t) = w(0,t), \quad h_2(t) = w(L,t), \quad h_3(t) = \partial_x w(L,t)$$

to guide the boundary control system (3.1) from the initial state ϕ to the terminal state ψ.

The following theorem for the initial value control problem was established in [31].

Theorem 3.9[31] *Let* $T > 0$ *and* $s \ge 0$ *be given. Suppose that* $v \in C_0^\infty(\mathbb{R}\times\mathbb{R})$ *is a given function satisfying*

$$\partial_t v + \partial_x v + v\partial_x v + \partial_x^3 v = 0$$

for $(x,t) \in [0,L]\times[0,T]$. *There exists a* $\delta > 0$. *If* $\phi,\ \psi \in H^s(\mathbb{R})$ *with*

$$\|\phi(\cdot) - v(\cdot,0)\|_{H^s(0,L)} \le \delta, \quad \|\psi(\cdot) - v(\cdot,T)\|_{H^s(0,L)} \le \delta,$$

then one can find $h \in H^s(\mathbb{R})$ *such that the corresponding solution* w *of the IVP* (3.5) *satisfies*

$$w(x,0) = \phi(x), \qquad w(x,T) = \psi(x) \text{ on the interval } (0,L).$$

Instead of considering the initial control problem of the KdV equation posed on the whole line $\mathbb{R}$, one may consider the following distributive control of the KdV equation posed on the whole line $\mathbb{R}$:

‖In fact, $h_1,\ h_2 \in H^{\frac{s+1}{3}}(0,T)$, and $h_3 \in H^{\frac{s}{3}}(0,T)$.

$$\partial_t w + \partial_x w + w\partial_x w + \partial_x^3 w = f(x,t)g(x), \qquad x \in \mathbb{R},\ t \in (0,T), \tag{3.6}$$

where $g \in C^\infty(\mathbb{R})$ is a given function satisfying

$$g(x) = \begin{cases} 0, & \text{if} \quad x \in [0, L]; \\ 1, & \text{if} \quad x \notin [-1, L+1], \end{cases}$$

and $f = f(x,t)$ is considered as a control input:

Let $T > 0$ and $s \geq 0$ be given. For any $\phi,\ \psi \in H^s(\mathbb{R})$, can we find a function $f \in L^2(0,T; H^s(\mathbb{R}))$ such that the equation (3.6) admits a solution $w \in C([0,T]; H^s(\mathbb{R}))$ satisfying

$$w(x,0) = \phi(x), \qquad w(x,T) = \psi(x)$$

on the interval $(0, L)$?

An affirmative answer to this distributive control problem leads obviously to a positive answer to the boundary control problem asked earlier.**

There are two major advantages in these approaches dealing with boundary control problem of the KdV equation. First, by considering the KdV equation on the whole line $\mathbb{R}$, one may benefit a lot from various smoothing properties of the KdV equation discovered in the past twenty years, in particular, the Bourgain smoothing property[17] for the KdV equation discovered in early 1990s. Secondly, one can avoid some difficult trace regularity problems associated with the boundary value problem. On the other hand, there is a drawback to these approaches: one has to use all the three boundary control inputs.

In Theorem 3.8, all three boundary control inputs have been employed. It is interesting to study the cases that less control inputs are used.

In [33], Rosier considered the system (3.1) with only one boundary control input $h_1(t)$ acting on the left end of the spatial domain:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \qquad x \in (0, L),\ t > 0, \\ u(0,t) = h_1(t), \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0. \end{cases} \tag{3.7}$$

Using a new Carleman estimate for KdV, he showed that the system (3.7) is locally exactly trajectory controllable and, in particular, is locally exactly null controllable.

Theorem 3.10[33] *Let $L > 0$ and $T > 0$ be given. Let*

$$v \in C([0,T]; \quad H^2(0,L)) \cap C^1([0,T]; \quad L^2(0,L)) \cap H^1(0,T; H^1(0,L))$$

be a given function satisfying

$$\begin{cases} \partial_t v + \partial_x v + v\partial_x v + \partial_x^3 v = 0, \quad x \in (0,L),\ t \in (0,T), \\ v(L,t) = \partial_x v(L,t) = 0, \quad t \in (0,T), \\ v(x,0) = v_0(x), \quad x \in (0,L). \end{cases}$$

Then, there exists a $\delta > 0$ such that for any $\phi \in H^3(0,L)$ with $\phi(L) = \phi'(L) = 0$ and

$$\|\phi - v_0\|_{H^3(0,L)} \leq \delta,$$

there exists a function

$$u \in L^2(0,T; H^3(0,L)) \cap H^1(0,T; H^1(0,L))$$

**This is the approach used recently by Rosier and Zhang[32] to establish the exact boundary controllability of the nonlinear Schrödinger equation posed on a bounded domain Ω in $\mathbb{R}^n$.

which solves

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad x \in (0, L),\ t \in (0, T), \\ u(L, t) = \partial_x u(L, t) = 0, \quad t \in (0, T), \\ u(x, 0) = \phi(x), \qquad u(x, T) = v(x, T), \quad x \in (0, L). \end{cases} \tag{3.8}$$

Note that only two boundary conditions have been specified in system (3.8). To have a closed system, we have to add to these boundary conditions a third boundary condition involving the control, for instance

$$u(0, t) = h_1(t).$$

In that case, the control input h_1 may be taken in the class $H^1(0, T)$. On the other hand, if we are interested in the generation of waves by a wavemaker, a physically relevant boundary condition is

$$\left(u - \frac{1}{6}u^2 + u_{xx}\right)\Big|_{x=0} = h.$$

The control input h can then be chosen in $C([0, T])$ (see [33]). The theorem was improved later by Glass and Guerrero[34] as far as the regularity of the trajectory is concerned.

Theorem 3.11[34] *Let* $L > 0$ *and* $T > 0$ *be given. Let* $v \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ *be a given function satisfying*

$$\begin{cases} \partial_t v + \partial_x v + v\partial_x v + \partial_x^3 v = 0, \quad x \in (0, L),\ t \in (0, T), \\ v(0, t) = v(L, t) = \partial_x v(L, t) = 0, \quad t \in (0, T), \\ v(x, 0) = v_0(x), \quad x \in (0, L). \end{cases}$$

Then, there exists a $\delta > 0$ *such that for any* $\phi \in L^2(0, L)$ *with*

$$\|\phi - v_0\|_{L^2(0,L)} \leq \delta,$$

there exists $h_1 \in H^{\frac{1}{2}-\epsilon}(0, T)$ *such that the system* (3.7) *admits a solution*

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = \phi(x), \quad u(x, T) = v(x, T).$$

Glass and Guerrero also considered in [34] the system (3.1) using two control inputs $h_1(t)$ and $h_2(t)$:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \qquad x \in (0, L),\ t > 0, \\ u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad \partial_x u(L, t) = 0. \end{cases} \tag{3.9}$$

They showed that the system (3.9) is locally exactly controllable in the space $L^2(0, L)$ even if the spacial domain length L is critical.

Theorem 3.12[34] *Let* $L > 0$ *and* $T > 0$ *be given. There exists a* $\delta > 0$ *such that for any* $\phi,\ \psi \in L^2(0, L)$ *with*

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \leq \delta,$$

there exist $h_1,\ h_2 \in L^2(0, T)$ *such that the system* (3.9) *admits a solution*

$$u \in C([0, T]; H^{-1}(0, L)) \cap L^2(0, T; L^2(0, L))$$

satisfying

$$u(x, 0) = \phi(x), \quad u(x, T) = \psi(x).$$

More recently, Glass and Guerrero[35] considered the system (3.1) with only a single boundary control input acting on the right end of the spacial domain:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0, L),\ t > 0, \\ u(0,t) = 0, \quad u(L,t) = h_2(t), & \partial_x u(L,t) = 0 \end{cases} \tag{3.10}$$

and showed that if the spacial domain length

$$L \notin \mathcal{N}^* : = \{L \in \mathbb{R}^{+*};\ \exists (a,b) \in \mathbb{C}^2 \text{ such that } \\ a\mathrm{e}^a = b\mathrm{e}^b = -(a+b) \text{ and } L^2 = -(a^2 + ab + b^2)\},$$

then the system (3.10) is locally exactly controllable in the space $L^2(0, L)$.

Theorem 3.13 (Glass and Guerrero) *Let $T > 0$ and $0 < L \notin \mathcal{N}^*$ be given. There exists a $\delta > 0$ such that for any $\phi,\ \psi \in L^2(0, L)$ with*

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \le \delta,$$

there exists $h_2 \in H^{\frac{1}{6}}(0, T)$ such that the system (3.10) *admits a solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ satisfying*

$$u(x, 0) = \phi(x), \qquad u(x, T) = \psi(x).$$

If only boundary control input $h_1(t)$ on the left end of the spatial domain is allowed to use, then the system (3.1) is locally null controllable in the space $L^2(0, L)$ as demonstrated in Theorem 3.11, i.e., by choosing appropriate control input h_1, one can guide the system (3.1) from a given small initial state to zero in a given time T. One may wonder if one can guide the system from a given small initial state to another given small terminal state or more precisely, is the system (3.1) with $h_2 = h_3 \equiv 0$ locally exactly controllable? It was noticed by Rosier in [33] that the answer is negative for the linearized system (i.e., without the nonlinear term $u\partial_x u$ in KdV). The answer is still negative for the nonlinear system.

Theorem 3.14 *If only the boundary control input $h_1(t)$ is allowed to use, then the system* (3.1) (*with $h_2 = h_3 \equiv 0$*) *is not locally exactly controllable in the space $L^2(0, L)$.*

The theorem follows as a corollary from the result given below, which was obtained recently by Glass and Zhang[36], regarding interior regularity of solutions of the IBVP of the KdV equation on $(0, L)$:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad x \in (0, L),\ t \in (0, T), \\ u(x, 0) = \phi(x), \\ u(0, t) = h_1(t),\ u(L, t) = \partial_x u(L, t) = 0. \end{cases} \tag{3.11}$$

Theorem 3.15[36] *Let $L > 0$ and $T > 0$ be given. For any $\phi \in L^2(0, L)$ and $h_1 \in H^{\frac{1}{3}}(0, T)$, the unique solution u of IBVP* (3.11) *in the space $C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ also belongs to the space $C^\infty((\epsilon, L) \times (\epsilon, T))$.*

Remark 3.16 Thus, the boundary control system (3.1), if only the left control input h_1 is in action, behaves like a parabolic system and is only null controllable. However, if the system is allowed to control from the right end of the spacial domain, then the system behaves like a hyperbolic system and is exactly controllable.

Next, we turn to consider stabilization problem of the system (3.1). First, consider the case that no boundary control inputs are in action:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad x \in (0, L),\ t \in (0, T), \\ u(x, 0) = \phi(x), \\ u(0, t) = 0,\ u(L, t) = \partial_x u(L, t) = 0. \end{cases} \tag{3.12}$$

The system has been shown by Perla Menzala, Vasconcellos, and Zuazua[37], to be locally exponentially stable if the spacial domain length $L \notin \mathcal{N}$.

Theorem 3.17[37] *Assume that $L \notin \mathcal{N}$. There exist $\delta > 0$ and $\gamma > 0$ such that for $\phi \in L^2(0,L)$ with $\|\phi\|_{L^2(0,L)} \le \delta$, the system* (3.12) *admits a unique solution $u \in C(\mathbb{R}^+; L^2(0,L)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(0,L))$ satisfying*

$$\|u(\cdot,t)\|_{L^2(0,L)} \le \alpha_0(\|\phi\|_{L^2(0,L)})\mathrm{e}^{-\gamma t}, \quad \forall\, t > 0,$$

where $\alpha_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

In order to handle the case of $L \in \mathcal{N}$ and to have the solutions of (3.12) with large amplitude stabilized, Perla Menzala, Vasconcellos, and Zuazua[37] introduced an extra damping term $b(x)u$ to the equation in (3.12) to get the following system:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u + b(x)u = 0, \quad x \in (0,L),\ t \in (0,T), \\ u(x,0) = \phi(x), \\ u(0,t) = 0,\ u(L,t) = \partial_x u(L,t) = 0. \end{cases} \tag{3.13}$$

Here, $b \in L^\infty(0,L)$ is assumed to be a nonnegative function satisfying $b(x) \ge b_0 > 0$ a.e. in an open, nonempty subset ω of $(0,L)$.

Theorem 3.18[37] *For any given $\phi \in L^2(0,L)$, the system* (3.13) *admits a unique solution $u \in C(\mathbb{R}^+; L^2(0,L)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(0,L))$. Moreover, if*

$$\omega \text{ contains two sets of the form } (0,\delta) \text{ and } (L-\delta, L) \text{ for some } \delta > 0, \tag{3.14}$$

then, for any $L > 0$ and $R > 0$, there exist $C > 0$ and $\mu > 0$ such that for any $\phi \in L^2(0,L)$) with $\|\phi\|_{L^2(0,L)} \le R$, the corresponding solution u of (3.13) *satisfies*

$$\|u(\cdot,t)\|_{L^2(0,L)} \le C\|\phi\|_{L^2(0,L)}\mathrm{e}^{-\mu t} \qquad \forall\, t \ge 0. \tag{3.15}$$

Remark 3.19 (a) The result of Menzala, Vasconcellos, and Zuazua presented in Theorem 3.18 represents a significant advance in the subject of stabilization of the KdV equation. Indeed, all the previous results except [20], in which the feedback control acts on the whole domain, are local in the sense that only small amplitude solutions have been shown to decay exponentially; They are essentially linear stability results. In contrast, the stability result presented in Theorem 3.18 is global; All solutions of (3.13), large or small, decay exponentially in the space $L^2(0,L)$. It is a truly nonlinear stability result.

(b) Perla Menzala, Vasconcellos, and Zuazua have conjectured in [37] that Theorem 3.18 still holds without the assumption (3.14). A. F. Pazoto has proved in [**?**] that this is indeed true. The idea of the proof is as follows: A. F. Pazoto showed that the UCP (Unique Continuation Property) holds whatever be ω, in proving by multipliers and compactness arguments that any solution vanishing on a subinterval is necessarily smooth.

(c) In Theorem 3.18, the determination of the decay rate ν depends on the size of the initial value ϕ in the space $L^2(0,L)$. The system (3.13) is locally uniformly exponentially stable in $L^2(0,L)$.

Later, Rosier and Zhang[39] considered a system described by the generalized Korteweg-de Vries (GKdV) equation posed on the finite domain $(0,L)$ with a localized damping:

$$\begin{cases} \partial_t u + \partial_x u + a(u)\partial_x u + \partial_x^3 u + b(x)u = 0, \quad x \in (0,L),\ t \in (0,T), \\ u(x,0) = \phi(x), \\ u(0,t) = 0,\ u(L,t) = \partial_x u(L,t) = 0. \end{cases} \tag{3.16}$$

Here, $b \equiv b(x)$ is as given before and the function $a \equiv a(\mu)$ is a given smooth function satisfying the growth condition:

$$a(0) = 0, \qquad |a^{(j)}(\mu)| \le C(1 + |\mu|^{p-j}), \qquad \forall\ \mu \in \mathbb{R} \tag{3.17}$$

for $j = 0, 1$ if $1 \le p < 2$ and for $j = 0, 1, 2$ if $p \ge 2$. They showed that the system with $p < 4$ is globally exponentially stable.

Theorem 3.20[39] *Suppose that the growth condition* (3.17) *for* $a = a(\mu)$ *is satisfied with*

$$1 \le p < 4$$

and the support ω *of the function* b *contains a nonempty open subset of* $(0, L)$*. Then, there exists a number* $\nu > 0$ *depending only on* L *and a nondecreasing continuous function* $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for a given* $\phi \in L^2(0, L)$*, any solution of the system* (3.16) *in the class*

$$u \in C_w(\mathbb{R}^+; L^2(0, L)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1(0, L))$$

satisfies

$$\|u(\cdot, t)\|_{L^2(0,L)} \le \beta\big(\|\phi\|_{L^2(0,L)}\big)\mathrm{e}^{-\nu t}, \qquad \forall\ t \ge 0. \tag{3.18}$$

Remark 3.21 (i) Theorem 3.20 confirms positively the conjecture of Perla Menzala, Vasconcellos, and Zuazua[37] for the generalized KdV equation.

(ii) The decay rate ν in Theorem 3.20 depends only on L and does not, in particular, depend on the size of the initial value ϕ in the space $L^2(0, L)$. The system (3.16) is globally uniformly exponentially stable.

(iii) The uniqueness of the solution u of (3.16) in the class

$$u \in C(\mathbb{R}^+; L^2(0, L)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1(0, L))$$

can be established when $p < 2$. Whether the uniqueness still holds for $2 \le p < 4$ is an open question.

(iv) For the critical nonlinearity $a(u)\partial_x u = u^4 \partial_x u$, F. Linares and A. F. Pazoto have proved in [40] that the origin is locally exponentially stable in $L^2(0, L)$ for (3.16). The semiglobal exponential stability is open in that case. The asymptotic behavior of (3.16) for $a(u) = u^p$ with $p > 4$ is completely unknown.

Finally, we return to consider boundary controllability again of the boundary control system (3.1). All the controllability results presented so far are local. One may wonder

Does the system (3.1) possess any global controllability?

Combining global stability result of Theorem 3.20 with the local controllability results discussed earlier provides a positive answer to the question; The system (3.1) is large time globally controllable. To be more precise, let us consider first the system (3.1) with only control input $h_1(t)$ in action:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0, L),\ t > 0, \\ u(0, t) = h_1(t), \quad u(L, t) = 0, \quad \partial_x u(L, t) = 0. \end{cases} \tag{3.19}$$

Theorem 3.22 *Let* $L > 0$ *be given. For any* $R > 0$*, there exists a* $T = T_R > 0$ *such that if* $\phi \in L^2(0, L)$ *satisfies*

$$\|\phi\|_{L^2(0,L)} \le R,$$

then one can find $h_1 \in L^2(0, T_R)$ *such that the system* (3.19) *admits a solution*

$$u \in C([0, T_R]; L^2(0, L)) \cap L^2(0, T_R; H^1(0, L))$$

satisfying

$$u(x,0)=\phi(x), \qquad u(x,T_R)=0.$$

As we see, one can always guide the system (3.19) from a given initial state ϕ to zero, but the time T needed depends on the size of ϕ in the space $L^2(0,L)$.

The sketch of the proof For given $\phi \in L^2(0,L)$ with $\|\phi\|_{L^2(0,L)} \le R$, let ϕ' be its zero extension from $(0,L)$ to $(-1,L)$. Obviously, we have $\phi' \in L^2(-1,L)$ and $\|\phi'\|_{L^2(-1,L)} \le R$. Let us consider the following IBVP of the KdV equation posed the domain $(-1,L)$:

$$\begin{cases} \partial_t v + \partial_x v + v\partial_x v + \partial_x^3 v + b(x)v = 0, \qquad x \in (0,L),\ t>0, \\ v(x,0)=\phi'(x),\ x\in(-1,L), \\ v(-1,t)=0, \quad v(L,t)=0, \quad \partial_x v(L,t)=0, \end{cases} \tag{3.20}$$

where b is assumed to be supported in a nonempty open subset ω of $(-1,0)$. According to Theorem 3.20, for any $\delta>0$, there exists a $T*>0$ depending only on R and δ such that the solution v of the IBVP (3.20) belongs to the space $C(\mathbb{R}^+, L^2(-1,L)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(-1,L))$ and

$$\|v(\cdot,T^*)\|_{L^2(-1,L)} \le \delta.$$

Let us define

$$\phi^*(x) = v(x,T^*),\ x\in(0,L).$$

Then

$$\|\phi^*\|_{L^2(0,L)} \le \delta.$$

Consider the boundary control system:

$$\begin{cases} \partial_t w + \partial_x w + w\partial_x w + \partial_x^3 w = 0, \qquad x\in(0,L),\ t>T^*, \\ w(0,t)=q(t), \quad w(L,t)=0, \quad \partial_x w(L,t)=0. \end{cases} \tag{3.21}$$

By Theorem 3.11, if δ is small enough, for any $T'>0$, there exists a $q\in L^2(T^*,T^*+T')$ such that the system (3.21) admits a solution $w\in C([T^*,T^*+T'];L^2(0,L)) \cap L^2(T^*,T^*+T';H^1(0,L))$ satisfying

$$w(x,T^*)=\phi^*(x), \quad w(x,T^*+T')=0.$$

Consequently, if we select $T=T^*+T'$ and choose

$$h_1(t) = \begin{cases} v(0,t), & \text{if} \quad t\in(0,T^*); \\ q(t), & \text{if} \quad t\in[T^*,T^*+T') \end{cases}$$

as control input in the system (3.19), then the system admits a solution $u\in C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$ satisfying

$$u(x,0)=\phi(x), \quad u(x,T)=0. \qquad \blacksquare$$

Similarly, we can show that, if the two boundary control inputs on the right end of the spacial domain $(0,L)$ are allowed to use, then the system (3.1) is large time globally exactly controllable.

Theorem 3.23 *Let $L>0$ be given. For any $R>0$, there exists a $T=T_R>0$ such that if $\phi,\ \psi\in L^2(0,L)$ with*

$$\|\phi\|_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} \le R,$$

then one can find $h_2,\ h_3 \in L^2(0,T_R)$ such that the system

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0,L),\ t > 0, \\ u(0,t) = 0, \quad u(L,t) = h_2(t), \quad \partial_x u(L,t) = h_3(t) \end{cases}$$

admits a solution

$$u \in C([0,T_R]; L^2(0,L)) \cap L^2(0,T_R; H^1(0,L))$$

satisfying

$$u(x,0) = \phi(x), \qquad u(x,T_R) = \psi(x).$$

4 Concluding Remarks and Open Problems

In an elementary partial differential equation class, partial differential equations are usually classified as elliptic equations, parabolic equations and hyperbolic equations which are represented by the Laplace equation ($\Delta u = 0$), the heat equation ($\partial_t u - \Delta u = 0$), and the wave equation ($\partial_t^2 u - \Delta u = 0$), respectively. The classification is based on the shape of the equations themselves. However, there is another way to classify partial differential equations which is based on the solutions of the equations rather than the shape of the equations. For instance, consider the following general partial differential equations:

$$P(\partial_t, \partial_x)u = 0, \quad x,\ t \in \mathbb{R}, \tag{4.1}$$

where $P = P(T,X)$ is a two-variables complex-valued polynomial. It is easy to check that for given wavenumber $k \in \mathbb{R}$ and frequency $\omega \in \mathbb{C}$,

$$u(x,t) = \mathrm{e}^{\mathrm{i}(kx+\omega t)}$$

is a solution of the equation (4.1) if and only if

$$P(\mathrm{i}\omega, \mathrm{i}k) = 0$$

from which one can solve ω in terms of k:

$$\omega(k) = \alpha(k) + \mathrm{i}\beta(k)$$

with $\alpha(k)$ and $\beta(k)$ denoting the real and imaginary parts of $\omega(k)$, respectively.

Definition 4.1 The Equation (4.1) is called diffusive if $\alpha(k) \equiv 0$ and $\beta(k) > 0$; The Equation (4.1) is called dispersive if $\beta(k) \equiv 0$ and $\alpha''(k) \neq 0$.

By this definition, the heat equation is diffusive, the linear Schrödinger equation $i\partial_t u + \partial_x^2 u = 0$ is dispersive while the wave equation is neither diffusive nor dispersive. Many important partial differential equations in mathematical physics are dispersive. Among them the best known is the Korteweg-de Vries (KdV) equation:

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0.$$

Mathematically, as a classical representative of the dispersive wave equations, the KdV equation behaves like a hyperbolic equation in some aspects, but like a parabolic equation in some other aspects. For instance, the KdV equation possesses an infinite propagation speed and various smoothing properties (some of them are quite subtle) (see [17,41–43]). In those aspects, the equation behaves like a parabolic equation. On the other hand, the Cauchy problem of the KdV equation is usually well-posed both forward and backward in time. In this aspect, it behaves like a hyperbolic equation. The focus of this article is some properties of the KdV equation from a control point of view. In particular, we are mainly concerned with control and stabilization of the distributed parameter control systems described by the Kortweg-de Vries equation posed either on a periodic domain $\mathbb{T}$ with a mass conserved controller Gh:

$$\partial_t u + u\partial_x u + \partial_x^3 u = Gh, \quad x \in \mathbb{T};\ t \in \mathbb{R} \tag{4.2}$$

or posed on a finite domain $(0, L)$ with the Dirichlet boundary value functions as control inputs:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in (0, L),\ t > 0, \\ u(0, t)h_1(t), \quad u(L, t) = h_2(t), & \partial_x u(L, t) = h_3(t). \end{cases} \tag{4.3}$$

The following two fundamental control problems are the focus of our discussion.

(1) Are the systems controllable?

(2) Are the systems stabilizable?

Started by Russell[12] and Zhang[13] in late 1980s, both problems have been extensively studied in the past twenty years. Relative satisfactory answers have been provided for both problems through many people's intensive work and contributions. For the system (4.2):

(i) It is (short time) locally exactly controllable and large time globally exactly controllable;

(ii) It is globally exponentially stabilizable and, moreover, an appropriate linear feedback control law can be introduced so that the resulting closed-loop system is (locally) exponentially stable with an arbitrarily large decay rate.

As for the boundary control system (4.3), it possesses some amazing control properties; It behaves either like a parabolic system or a hyperbolic system depending on how control inputs are applied. More precisely,

(a) If a control is only allowed to act on the left end of the spatial domain $(0, L)$, then the system (4.3) (with $h_2 = h_3 \equiv 0$) is short time locally null controllable and large time globally null controllable, but not exactly controllable;

(b) If one of the controls from the right end of the spatial domain is in action, then the system is short time locally exactly controllable;

(c) If both controls on the right end of the spatial domain are employed, then the system is large time globally exactly controllable.

Being different from both parabolic system and hyperbolic system, the length L of the spatial domain may play a crucial role in determining the controllability of the system (4.3) when only one right control input is allowed to use. If the length L is critical, the associated linear system is not exactly controllable. Nevertheless, the corresponding nonlinear system (4.3) may still be locally exactly controllable!

In addition to consider the KdV equation posed on the periodic domain $\mathbb{T}$ and the finite domain $(0, L)$, one may also consider the KdV equation posed on the half line $\mathbb{R}^+$:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, & x \in \mathbb{R}^+,\ t > 0, \\ u(x, 0) = \phi(x), \quad u(0, t) = h(t), & x \in \mathbb{R}^+, \quad t > 0, \end{cases} \tag{4.4}$$

which serves as a model for long-crested small-amplitude long waves propagating on the surface of water along a channel with a wave maker mounted at one end of the channel. Naturally, one may ask:

What waves the wave maker can make?

In terms of control terminology, considering the boundary value function $h(t)$ as a control input, the question is:

What is the controllability of the system (4.4)?

Note that as initial-boundary-value problem, the well-posedness of the system (4.4) is well-established in the literature (see [44–47] and the references therein). The following result is due to Bona, Sun, and Zhang[47].

Theorem 4.1[47] *Let $\nu > 0$ and $s > -1$ be given with $s \neq 3m + \frac{1}{2}$, $m = 0, 1, 2, \cdots$. Let*

$$H^s_\nu(\mathbb{R}^+) := \{f \in H^s(\mathbb{R}^+);\ \mathrm{e}^{\nu x} f \in H^s(\mathbb{R}^+)\}.$$

For any given compatible pair $(\phi, h) \in H^s_\nu(\mathbb{R}^+) \times H_{loc}^{\frac{s+1}{3}}(\mathbb{R}^+)$, there exists a $T > 0$ such that the IBVP (4.4) admits a unique mild solution†† *$u \in C([0, T]; H^s_\nu(\mathbb{R}^+))$. Moreover, the corresponding solution map is (real) analytic.*

The following result follows immediately as a consequence of Theorem 4.1.

Theorem 4.2 *The system (4.4) is not exactly controllable in the space $H^s(\mathbb{R}^+)$ for any $s > -1$.*

Indeed, let us consider the initial state $\phi = 0$ and $\psi \in H^s(\mathbb{R}^+)$ for some $s > -1$, but $\psi \notin H^{s'}_\nu(\mathbb{R}^+)$ for any $\nu > 0$ and $s' > -1$. Then, no matter what control input h is chosen from the space $H_{\mathrm{loc}}^{\frac{s'+1}{3}}(\mathbb{R}^+)$, the corresponding solution u of (4.4) is not equal to ψ for any $t > 0$.

A further question is:

Is the system (4.4) null controllable?

The answer is most likely negative since the associated linear system is not null controllable. The following result is due to Rosier[49].

Theorem 4.3[49] *Let $T > 0$ be given. Then, there exists a $\phi \in L^2(\mathbb{R}^+)$ such that if $u \in L^\infty(0, T; L^2(\mathbb{R}^+))$ solves*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & in\ \ \mathcal{D}'((0, +\infty) \times (0, T)), \\ u|_{t=0} = \phi, \end{cases}$$

then

$$u_{|t=T} \neq 0.$$

On the other hand, when the bounded energy condition ($u \in L^\infty(0, T, L^2(\mathbb{R}^+))$) is dropped, the exact boundary controllability of the linear KdV holds true, as is shown in the following theorem, again due to Rosier[49].

Theorem 4.4[49] *Let T, ϵ, and b be positive numbers, with $\epsilon < \frac{T}{2}$. Let $\phi \in H^0(\mathbb{R}^+) = L^2(\mathbb{R}^+)$ and $\psi \in H^0_{-b}(\mathbb{R}^+)$. Then, there exists*

$$u \in L^2_{\mathrm{loc}}([0, \infty) \times [0, T]) \cap C([0, \epsilon]; L^2(\mathbb{R}^+)) \cap C([T - \epsilon, T], H^0_{-b}(\mathbb{R}^+))$$

which solves

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & in\ \ \mathcal{D}'((0, +\infty) \times (0, T)) \\ u|_{t=0} = \phi, \quad u|_{t=T} = \psi. \end{cases}$$

Thus, the price to be paid to obtain the exact controllability on the half-line is a (possible) bad behavior of the trajectories when $x \to \infty$. A fundamental solution with a compact support in time for the linear KdV equation is provided in [50]. This allows to construct an explicit trajectory (with a bad behavior at infinity) driving any function u_0 to 0.

††The reader is referred to [48] for the definition of mild solution.

In the case of $h \equiv 0$, any smooth solution of (4.4) satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^+} |u(x,t)|^2 dx = -|\partial_x u(0,t)|^2$$

and thus the L^2-energy of the solution decays so long as $\partial_x u(0,t) \neq 0$; a certain kind of dissipation mechanism is introduced to the system through the imposition of the boundary condition $u(0,t) = 0$. One may wonder whether that dissipation mechanism is strong enough to drive the solution of the system to 0 as $t \to \infty$.

In [51], Hayashi et al. considered the system (4.4) without the drifting term $\partial_x u$:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ u(x,0) = \phi(x), \qquad u(0,t) = 0, \quad x \in \mathbb{R}^+, \quad t > 0. \end{cases} \tag{4.5}$$

Its solution is shown to tend to zero pointwisely as $t \to \infty$.

Theorem 4.5[51] *If* $\phi \in H_1^{0,2} \cap H_2^{0,\frac{3}{2}}$ *with the norm*

$$\|\phi\|_{H_1^{0,2}} + \|\phi\|_{H_2^{0,\frac{3}{2}}}$$

sufficiently small, where

$$H_p^{s,k} = \{f \in L^2(\mathbb{R}^+);\ \|f\|_{H_p^{s,k}} := \|(1+|x|^2)^{\frac{k}{2}} D_x^s f\|_{L^p(\mathbb{R}^+)} < \infty,$$

then (4.5) *admits a unique solution*

$$u \in C^0(\mathbb{R}^+; H_1^{0,2}) \cap L^\infty(\mathbb{R}^+; H_1^{0,\frac{3}{2}}) \cap C^0(\mathbb{R}^+; H_2^{2,0}) \cap L^\infty(\mathbb{R}^+; H_2^{3,0}).$$

Moreover, the solution u has the following asymptotic form:

$$u(x,t) = \frac{1}{t} B \frac{x}{\sqrt[3]{t}} Ai\Big(\frac{x}{\sqrt[3]{t}}\Big) + O\Big(t^{-1-\frac{\delta}{3}} \max\Big(1, \frac{x}{\sqrt[3]{t}}\Big)\Big)$$

for $t \to \infty$ uniformly with respect to $x > 0$ where $B,\ \delta$ are constants with $0 < \delta < \frac{1}{2}$, $Ai(q)$ is the Airy function defined by

$$Ai(q) = \int_{\mathbb{R}} \mathrm{e}^{iz^3 + qiz} dz.$$

Recently, Linares and Pazoto[52] considered the following damped KdV equation on $\mathbb{R}^+$:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u + b(x)u = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t),\ x \in \mathbb{R}^+, \quad t > 0, \end{cases} \tag{4.6}$$

where $b = b(x)$ is a nonnegative function. Assuming that there exist $\delta > 0,\ \beta > 0,\ \alpha > \delta$ such that

$$b \in L^\infty(\mathbb{R}^+), \quad b(x) \geq \beta \ \text{ a.e. in } \ (0,\delta) \cup (\alpha, \infty), \tag{4.7}$$

they showed that the system (4.6) is exponentially stable.

Theorem 4.6[52] *Assume that the damping function $b = b(x)$ satisfies* (4.7). *For any $R > 0$ there exist positive constants $C = C(R)$ and $\mu = \mu(R)$ such that for any $\phi \in L^2(\mathbb{R}^+)$ with*

$$\|\phi\|_{L^2(\mathbb{R}^+)} \leq R,$$

the corresponding solution u of the system (4.6) *satisfies*

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^+)} \le C\|\phi\|_{L^2(\mathbb{R}^+)}\mathrm{e}^{-\mu t} \qquad \textit{for any} \ \ t \ge 0.$$

The above global exponential stability result has been extended to the weighted Sobolev spaces $H^s_\nu(\mathbb{R}^+)$ $(\nu > 0)$ and $H_2^{s,k}$ by A. F. Pazoto and L. Rosier in [53].

While significant progresses have been made in the past twenty years in studying control and stabilization problems of the Korteweg-de Vries equation, there are still many problems left open for further study. We list below a few of them with some remarks to end this paper.

Open Problems

1. All global controllability results established for the (nonlinear) KdV equation are large time controllability and, in contrast, most local controllability results obtained are short time controllability except Theorems 3.5 and 3.6.

(1-a) Do the distributive control system (4.2) and the boundary control system (4.3) possess any large time global controllability?

In addition, the system (4.3) with $h_1 = h_2 \equiv 0$ posed on a critical domain is known to be large time locally exactly controllable in general (Theorem 3.6).

(1-b) Is it short time (locally or globally) exactly controllable?

Since the KdV equation possesses an infinite propagation speed[43], one would usually expect that the needed control time could be as short as one wishes. This is in general true in the case of linear control systems. However, it may not be the case for nonlinear control systems. For instance, Beauchard and Coron have considered in [54] a bilinear control system described by the Schrödinger equation and shown that the system is large time locally exactly controllable along the ground state trajectory. But Coron proved in [55] and Theorem 9.8 in [56] that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation.

In [57], Chapouly considered recently the boundary control systems

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = g(t), \qquad x \in (0,L),\ t > 0; \\ u(0,t) = h_1(t), \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0 \end{cases} \tag{4.8}$$

and

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = g(t), \qquad x \in (0,L),\ t > 0; \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad \partial_x u(L,t) = 0, \end{cases} \tag{4.9}$$

where $g = g(t)$ is independent of the spatial variable x and is considered as a new control input. Recall that for $g \equiv 0$, the system (4.8) is short time locally exactly trajectory controllable (Theorem 3.10) and large time globally null controllable (Theorem 3.22). As for the system (4.9), it is short time locally exactly controllable (Theorem 3.12). However, with an extra control input $g(t)$ in action, Chapouly was able to show that system (4.8) is short time globally exactly trajectory controllable and that the system (4.9) is short time globally exactly controllable.

2. Consider again the system (4.3) but with all boundary control inputs being at rest:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \qquad x \in (0,L),\ t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad \partial_x u(L,t) = 0. \end{cases} \tag{4.10}$$

The system (4.10) is known to be locally exponentially stable if the spatial domain $(0,L)$ is not critical.

(2-a) Is the system (4.10) posed on a critical spatial domain $(0, L)$ asymptotically stable?

If the spatial domain is critical, the associated linear system (by dropping the nonlinear term $u\partial_x u$ from the equation) is not asymptotically stable; The linear system possesses a family of non-trivial solutions of the form $e^{\lambda t}v_\lambda(x)$ for some $\lambda \in \mathbb{C}$, where v_λ solves

$$\begin{cases} \lambda v_\lambda + v'_\lambda + v'''_\lambda = 0, \\ v_\lambda(0) = v_\lambda(L) = v'_\lambda(0) = v'_\lambda(L) = 0. \end{cases}$$

See e.g. [27]. It would be very interesting to see if the nonlinear system is asymptotically stable while its associated linear system is not.

(2-b) Is the system (4.10) globally asymptotically stable?

The initial value problem of the system (4.10) is known to be globally well-posed in the space $H^s(0, L)$ for any $s > -1$ (see [58]) but its asymptotical behavior is not clear at all even when the spatial domain is not critical, except the trivial fact that its solution is uniformly bounded (with respect to time t) in the space $L^2(0, L)$.

3. Consider boundary control of the KdV equation posed on the half line $\mathbb{R}^+$ with the Dirichlet boundary condition:

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}^+, \ t > 0, \\ u(x, 0) = \phi(x), \quad u(0, t) = h(t). \end{cases} \tag{4.11}$$

The associated linear system is not short time null controllable (see Theorem 4.4) and both linear and nonlinear systems are not exactly controllable (see Theorem 4.3).

(3-a) Is the associated linear system large time null controllable? More precisely, given $\phi \in L^2(\mathbb{R}^+)$, is there a $T > 0$ such that one can find an appropriate boundary control h such that the system (4.11) admits a solution $u \in C([0, T]; L^2(\mathbb{R}^+))$ satisfying

$$u(x, 0) = \phi(x), \qquad u(x, T) \equiv 0?$$

One may ask further:

(3-b) Is the nonlinear system (4.11) large time null controllable?

(3-c) How to characterize the reachable set of the system (4.11)?

4. Now, we consider the systems

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u - \alpha\partial_x^2 u + \gamma u = 0, \qquad x \in (0, L), \ t > 0; \\ u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad \partial_x u(L, t) = h_3(t) \end{cases} \tag{4.12}$$

and

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u - \alpha\partial_x^2 u + \gamma u = 0, \qquad x \in \mathbb{R}^+, \ t > 0; \\ u(0, t) = h(t) \end{cases} \tag{4.13}$$

from a different point of view.

Questions *Assume that the boundary forcing terms h_j, $j = 1, 2, 3$, and h are given periodic functions* (*of period τ*).

(i) *Do the systems* (4.12) *and* (4.13) *admit a time-periodic solution $u(x, t)$ of periodic τ*? *Such a time-periodic solution, if it exists, is called a forced oscillation.*

(ii) *If such a time-periodic solution exists, how many are there*?

(iii) *What are their stability if those forced oscillations exist*?

In [45], assuming the damping coefficient $\gamma > 0$ and $\alpha = 0$ [‡‡], Bona, Sun and Zhang showed that if the amplitude of the boundary forcing h is small, then the solution u of (4.13) is asymptotically time-periodic, satisfying

$$\|u(\cdot, t+\tau) - u(\cdot, t)\|_{L^2(\mathbb{R}^+)} \leq C\mathrm{e}^{-\beta t} \qquad \text{for any } t \geq 0,$$

where C and β are two positive constants. In addition, they demonstrated that the equation in (4.13) admits a unique time-periodic solution $u^*(x,t)$ of period τ satisfying the boundary condition, which is shown to be globally exponentially stable in the space $H^s(\mathbb{R}^+)$ with $s \geq 1$, i.e., for a given initial data $\phi \in H^s(\mathbb{R}^+)$, the unique solution $u(x,t)$ of (4.13) satisfying

$$u(x,0) = \phi(x)$$

has the property

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{H^s(\mathbb{R}^+)} \leq C\mathrm{e}^{-\beta t}$$

for any $t \geq 0$ where $C > 0$ is a constant depending only on $\|\phi\|_{H^s(\mathbb{R}^+)}$. The condition $\gamma > 0$ is crucial in the proof presented in [45].

(4-a) What answers can one have for the above questions for system (4.13) in the case of $\gamma = 0$?

A more challenging problem is

(4-b) What answers can one have for the above questions for system (4.13) in the case of $\gamma = 0$ and $\alpha = 0$?

As for the system (4.12), it has been shown in [59–62] that if the amplitudes of the boundary forcing terms h_j, $j = 1, 2, 3$, are small, then the system (4.12) admits a unique time-periodic solution u^* which is globally exponentially stable if $\alpha^2 + \gamma^2 \neq 0$ and locally exponentially stable if $\alpha = \gamma = 0$.

(4-c) Is the time-periodic solution u^* globally exponentially stable in the case $\alpha = \gamma = 0$?

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