

LOCAL EXACT CONTROLLABILITY AND STABILIZABILITY OF THE NONLINEAR SCHRÖDINGER EQUATION ON A BOUNDED INTERVAL*

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Abstract. This paper studies the exact controllability and the stabilization of the cubic Schrödinger equation posed on a bounded interval. Both internal and boundary controls are considered, and the results are given first in a periodic setting, and next with Dirichlet (resp., Neumann) boundary conditions. It is shown that the systems with either an internal control or a boundary control are locally exactly controllable in the classical Sobolev space H^s for any $s \geq 0$. It is also shown that the systems with an internal stabilization are locally exponentially stabilizable in H^s for any $s \geq 0$.

Key words. Schrödinger equation, Bourgain spaces, exact boundary controllability, exact internal controllability, exponential stabilization

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1. Introduction. In this paper we study the nonlinear Schrödinger equation

$$(1.1) \quad iu_t + \lambda|u|^2u + u_{xx} = 0,$$

where $u = u(x, t)$ is a complex-valued function of two real variables x and t , the subscripts denote the corresponding partial derivatives, and the parameter λ is a nonzero real constant. The equation arises in various physical contexts as a model for propagation of nonlinear waves. In optics, it may serve as a model of wave propagation in fiber optics, the function u represents a wave, and the equation describes the propagation of the wave through a nonlinear medium. The equation is also used as a model for some water waves to describe the evolution of the envelope of modulated wave groups. The value of the nonlinearity parameter λ depends on the relative water depth. For deep water, with the water depth large compared to the wave length of the water waves, λ is positive and envelope solitons may occur [27].

Our main concern in this paper is control and stabilization of the system described by (1.1). Consideration will be first given to internal control of the Schrödinger equation

$$(1.2) \quad iu_t + \lambda|u|^2u + u_{xx} = f$$

posed on the finite interval $(-\pi, \pi)$ with periodic boundary conditions

$$(1.3) \quad u(-\pi, t) = u(\pi, t), \quad u_x(-\pi, t) = u_x(\pi, t),$$

or posed on the finite interval $(0, \pi)$ with either the Dirichlet boundary conditions

$$(1.4) \quad u(0, t) = 0, \quad u(\pi, t) = 0,$$

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or the Neumann boundary conditions

$$(1.5) \quad u_x(0, t) = 0, \quad u_x(\pi, t) = 0.$$

Here $f = f(x, t)$ is a given function considered as a control input. Without loss of generality, we assume that λ takes the value of 1 or -1 and restrict our attention to controls of the form

$$(1.6) \quad f(x, t) = iGh := ig(x)h(x, t),$$

where g , called a controller, is a given nonzero real-valued smooth function with its support contained in the domain $(-\pi, \pi)$ in the case of periodic boundary conditions, or in the domain $(0, \pi)$ in the case of Dirichlet boundary conditions or Neumann boundary conditions, and $h(x, t)$ is a new control input.

Then we will turn to boundary control of the nonlinear Schrödinger equation

$$(1.7) \quad iu_t + u_{xx} + \lambda|u|^2u = 0$$

posed on the finite interval $(0, \pi)$ with either the Dirichlet boundary conditions

$$(1.8) \quad u(0, t) = h(t), \quad u(\pi, t) = 0$$

or the Neumann boundary conditions

$$(1.9) \quad u_x(0, t) = h(t), \quad u_x(\pi, t) = 0,$$

where the boundary value function h will be considered as a control input.

In this paper, the focus of our study is the following two control problems.

Exact controllability problem: Let $T > 0$ be given. Given the initial state u_0 and the terminal state u_1 in an appropriate space, can one find a control h such that system (1.2)–(1.3) (resp., system (1.2)–(1.4) or system (1.2)–(1.5)) admits a solution $u(x, t)$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x)?$$

Stabilizability problem: Can one find a linear feedback control law $h = Ku$ such that the resulting closed-loop system is exponentially stable?

Control and stabilization problems of the Schrödinger equation have received a lot of attention in the past decade.¹ While significant progresses have been made for the linear Schrödinger equation on its controllability and stabilizability properties (cf., e.g., [4, 10, 8, 11, 12, 13, 14, 15, 17, 16, 20, 21]), there are only a few results for the nonlinear Schrödinger equation. Illner, Lange, and Teismann [6, 7] considered internal controllability of the nonlinear Schrödinger equation posed on a finite interval with periodic boundary conditions. They showed that the system (1.2)–(1.3) is locally exactly controllable in the space $H_p^1(-\pi, \pi) := \{v \in H^1(-\pi, \pi) : v(-\pi) = v(\pi)\}$. Their approach is based on the well-known Hilbert uniqueness method (HUM) and Schauder's fixed point theorem. Later, Lange and Teismann [9] considered internal control of the nonlinear Schrödinger equation (1.2) posed on a finite interval with the homogeneous Dirichlet boundary conditions (1.4) and established local exact controllability of the system (1.2)–(1.4) in the space $H_0^1(0, \pi)$ around a special ground state of the system.

¹The readers are referred to Zuazua [29] for an excellent review on recent progresses of this subject up to 2003.

Recently Dehman, Gérard, and Lebeau [5] studied internal control and stabilization of a class of defocusing nonlinear Schrödinger equations posed on a two-dimensional compact Riemannian manifold M without boundary. They demonstrated, in particular, that the system is semiglobally exact controllable and semiglobally exponentially stabilizable in the space $H^1(M)$ assuming both *the geometric control condition* and *the unique continuation property*² are satisfied.

There are two natural energy spaces associated with the nonlinear Schrödinger equation (1.2), namely $L^2(I)$ or $H^1(I)$. Here I stands for either the interval $(-\pi, \pi)$ or the interval $(0, \pi)$. Indeed, let u be a smooth solution of (1.2) with the control input $h \equiv 0$ satisfying one of the boundary conditions (1.3), (1.4), and (1.5). Then it satisfies the following two conservation laws:

$$E_0(t) := \int_I |u(x, t)|^2 dx = E_0(0)$$

and

$$E_1(t) := \int_I |u_x(x, t)|^2 dx - \frac{\lambda}{2} \int_I |u(x, t)|^4 dx = E_1(0)$$

for any $t \in \mathbb{R}$. While the local exact controllability of (1.2) has been established in the space $H^1(I)$ in [7] and [9], it would be interesting to know whether the nonlinear Schrödinger equation (1.2) is exactly controllable in the space $L^2(I)$ or in other Sobolev spaces $H^s(I)$ with $s \geq 0$.

One of our main objectives is to establish local exact controllability of (1.2) in the space $H^s(I)$ for any $s \geq 0$. In order to describe precisely our results, we introduce the following notations.

Let

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \quad k = 0, \pm 1, \pm 2, \dots$$

Then $\{\phi_k\}_{k=-\infty}^{+\infty}$ forms an orthonormal basis in the space $L^2(-\pi, \pi)$. We may define the Sobolev space $H_p^s := H_p^s(-\pi, \pi)$ of order s ($s \geq 0$) as the space of all 2π -periodic functions

$$v(x) = \sum_{k=-\infty}^{\infty} v_k \phi_k(x)$$

such that

$$(1.10) \quad \left\{ \sum_{k=-\infty}^{\infty} |v_k|^2 (1 + |k|)^{2s} \right\}^{\frac{1}{2}} < \infty.$$

The left-hand side of (1.10) is a Hilbert norm for H_p^s ; we denote it by $\|v\|_s$. In addition, let

$$C_{odd}^\infty(0, \pi) = \left\{ v \in C^\infty[0, \pi]; v^{(2k)}(0) = v^{(2k)}(\pi) = 0, \quad k = 0, 1, 2, \dots \right\}$$

and

$$C_{even}^\infty(0, \pi) = \left\{ v \in C^\infty[0, \pi]; v^{(2k+1)}(0) = v^{(2k+1)}(\pi) = 0, \quad k = 0, 1, 2, \dots \right\}.$$

²See [5] for exact descriptions of these two conditions.

Obviously, both $C_{odd}^\infty(0, \pi)$ and $C_{even}^\infty(0, \pi)$ are subspaces of $H^s(0, \pi)$ for any $s \geq 0$. Let $H_{odd}^s(0, \pi)$ and $H_{even}^s(0, \pi)$ be the closure of $C_{odd}^\infty(0, \pi)$ and $C_{even}^\infty(0, \pi)$ in the space $H^s(0, \pi)$, respectively. Note that $H_{odd}^0(0, \pi) = H_{even}^0(0, \pi) = L^2(0, \pi)$ and $H_{odd}^1(0, \pi) = H_0^1(0, \pi)$.

We have the following local controllability result for system (1.2)–(1.3).

THEOREM 1.1. *Let $T > 0$ and $s \geq 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H_p^s(-\pi, \pi)$ satisfying*

$$\|u_0\|_s \leq \delta, \quad \|u_1\|_s \leq \delta,$$

there exists a control $h \in L^2([0, T]; H_p^s(-\pi, \pi))$ such that the system (1.2)–(1.3) admits a solution $u \in C([0, T]; H_p^s(-\pi, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x).$$

For the Schrödinger equation (1.2) posed on the finite interval $(0, \pi)$ with the Neumann boundary conditions (1.5), we have the following local controllability result.

THEOREM 1.2. *Let $T > 0$ and $s \geq 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H_{even}^s(0, \pi)$ satisfying*

$$\|u_0\|_{H^s(0, \pi)} \leq \delta, \quad \|u_1\|_{H^s(0, \pi)} \leq \delta,$$

there exists a control $h \in L^2([0, T]; H^s(0, \pi))$ such that the system (1.2)–(1.5) admits a solution $u \in C([0, T]; H_{even}^s(0, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x).$$

A similar result holds for the Schrödinger equation (1.2) posed on the finite interval $(0, \pi)$ with the Dirichlet boundary conditions (1.4).

THEOREM 1.3. *Let $T > 0$ and $s \geq 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H_{odd}^s(0, \pi)$ satisfying*

$$\|u_0\|_{H^s(0, \pi)} \leq \delta, \quad \|u_1\|_{H^s(0, \pi)} \leq \delta,$$

there exists a control $h \in L^2([0, T]; H^s(0, \pi))$ such that the system (1.2)–(1.4) admits a solution $u \in C([0, T]; H_{odd}^s(0, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x).$$

For $s \geq 0$, let $H_{e, \pi}^s$ be the closure of the set $\{v \in C^\infty[0, \pi]; v^{(2k)}(\pi) = 0, k = 0, 1, 2, \dots\}$ in the space $H^s(0, \pi)$ and $H_{o, \pi}^s$ be the closure of the set $\{v \in C^\infty[0, \pi]; v^{(2k+1)}(\pi) = 0, k = 0, 1, 2, \dots\}$ in the space $H^s(0, \pi)$ for any $s \geq 0$. Then we have the following boundary controllability results for the systems (1.7)–(1.8) and (1.7)–(1.9).

THEOREM 1.4. *Let $s \geq 0$ and $T > 0$ be given. There exists a $\delta > 0$ such that*

(a) *for any $u_0, u_1 \in H_{e, \pi}^s(0, \pi)$ satisfying and*

$$\|u_0\|_{H^s(0, \pi)} \leq \delta, \quad \|u_1\|_{H^s(0, \pi)} \leq \delta,$$

there exists a boundary control h such that the system (1.7)–(1.8) admits a solution $u \in C([0, T]; H_{e, \pi}^s(0, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x);$$

(b) for any $u_0, u_1 \in H^s_{0,\pi}(0, \pi)$ satisfying

$$\|u_0\|_{H^s(0,\pi)} \leq \delta, \quad \|u_1\|_{H^s(0,\pi)} \leq \delta,$$

there exists a boundary control h such that the system (1.7)–(1.9) admits a solution $u \in C([0, T]; H^s_{0,\pi}(0, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x).$$

Remarks.

- (i) The same results hold if we apply a boundary control h on the right end of the domain $x = \pi$.
- (ii) As it will be demonstrated in the proof in section 3, the boundary control h is taken as the trace of a function $w \in C([0, T]; H^s(-\epsilon, \pi))$ at $x = 0$, which is a solution of the system

$$\begin{cases} iw_t + w_{xx} + \lambda|w|^2w = ig(x)h(x, t), & x \in (-\epsilon, \pi), \\ w(-\epsilon, t) = w(\pi, t), & w_x(-\epsilon, t) = w_x(\pi, t). \end{cases}$$

Our other main objective in this paper is to study stabilizability of system (1.2)–(1.3). We will show that it is locally stabilizable in $H^s_p(-\pi, \pi)$ ($s \geq 0$) by the feedback law

$$h(x, t) = -g(x)u(x, t).$$

THEOREM 1.5. *Let $s \geq 0$ be given. There exist some positive constants δ, C , and ν such that every solution of the system*

$$(1.11) \quad iu_t + u_{xx} + \lambda|u|^2u = -ig^2(x)u, \quad u(x, 0) = u_0(x)$$

$$(1.12) \quad u(-\pi, t) = u(\pi, t), \quad u_x(-\pi, t) = u_x(\pi, t)$$

issued from an initial state $u_0 \in H^s_p(-\pi, \pi)$ with $\|u_0\|_s \leq \delta$ satisfies

$$(1.13) \quad \|u(t)\|_s \leq Ce^{-\nu t} \|u_0\|_s \quad \forall t \geq 0.$$

Remarks.

- (i) In Theorem 1.5, the regularity of g may be weakened to $g \in L^\infty(-\pi, \pi)$ for $s = 0$, and to $g \in H^s_p(-\pi, \pi)$ for $s > 1/2$. The solution of (1.11)–(1.12) is proved to exist and to be unique in some restricted Bourgain space.
- (ii) The L^2 norm of any solution of (1.11)–(1.12) is nondecreasing whatever be λ . Indeed, a simple computation gives

$$\int_{-\pi}^{\pi} |u(T, x)|^2 dx - \int_{-\pi}^{\pi} |u_0(x)|^2 dx = -2 \int_0^T \int_{-\pi}^{\pi} |u(x, t)|^2 g(x) dx.$$

We conjecture that when $s = 0$ Theorem 1.5 is valid for any $\delta > 0$, i.e., that a semiglobal stabilization occurs in $L^2(-\pi, \pi)$.

- (iii) Similar results may be derived for the systems with either Dirichlet or Neumann boundary conditions.

Our next theorem presents a local stabilization result for a general nonlinearity in the Sobolev space $H^s_p(-\pi, \pi)$ with $s > 1/2$.

THEOREM 1.6. Let $s > 1/2$, and let $F : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function such that for some positive constants r , C , and N it holds

$$\|F(u) - F(v)\|_s \leq C (\|u\|_s^N + \|v\|_s^N) \|u - v\|_s$$

for all $u, v \in H_p^s(-\pi, \pi)$ with $\|u\|_s < r$, $\|v\|_s < r$. For any $\mu > 0$, let B_μ denote the space

$$B_\mu = \left\{ u \in C(\mathbb{R}^+; H_p^s(-\pi, \pi)); \|e^{\mu t} u(t)\|_{L^\infty(\mathbb{R}^+, H_p^s(-\pi, \pi))} < \infty \right\}$$

endowed with its natural norm. Let the function g be as in Theorem 1.5. Then there exist positive constants δ , μ , and K such that for any $u_0 \in H_p^s(-\pi, \pi)$ with $\|u_0\|_s < \delta$, the system

$$(1.14) \quad iu_t + u_{xx} + F(u) = -ig^2(x)u, \quad u(x, 0) = u_0(x)$$

$$(1.15) \quad u(-\pi, t) = u(\pi, t), \quad u_x(-\pi, t) = u_x(\pi, t)$$

admits a unique solution $u \in B_\mu$, and it holds

$$\|u(t)\|_s \leq Ke^{-\mu t} \|u_0\|_s \quad \forall t \geq 0.$$

To prove the above theorems we use the approach developed earlier by Russell and Zhang [25] in dealing with control and stabilization problem of the Korteweg-de Vries equation posed on a periodic domain. The associated linear systems are studied first using the classical moment method which enables us to establish the exact controllability of the associated linear systems in the space $H^s(I)$ for any $s \geq 0$. The linear results are then extended to the nonlinear systems. During this process, the Bourgain smoothing property [2, 3] for solutions of the Schrödinger equation posed on a periodic domain plays a key role. In particular, this Bourgain smoothing property seems indispensable in establishing the exact controllability of the system (1.2)–(1.3) in the space $L^2(-\pi, \pi)$. The proof of exact controllability for systems (1.2)–(1.4) and (1.2)–(1.5) is based on the following observation (see [1] for more detail and its application in establishing well-posedness of nonhomogeneous boundary value problems of the nonlinear Schrödinger equation posed on a bounded domain):

If $u \in C([0, T]; H_p^s(-\pi, \pi))$ is an odd (even) function with respect to x -variable and solves (1.1), then its restriction $w = w(x, t)$ on the interval $(0, \pi)$ belongs to the space $C([0, T]; H^s(0, \pi))$ and is a solution of the system (1.1)–(1.4) (system (1.1)–(1.5)). On the other hand, if $w \in C([0, T]; H^s(0, \pi))$ solves system (1.1)–(1.4) (or system (1.1)–(1.5)) and u is its odd (or even) extension to the interval $(-\pi, \pi)$, then $u \in C([0, T]; H_p^s(-\pi, \pi))$ and solves system (1.1)–(1.3).

Thus, one can reduce exact control problem of systems (1.2)–(1.4) and (1.2)–(1.5) to that of system (1.2)–(1.3). Theorem 1.2 and Theorem 1.3 can be considered as corollaries of Theorem 1.1. As for boundary control systems (1.7)–(1.8) and (1.7)–(1.9), their exact controllability follows from internal controllability of systems (1.2)–(1.4) and (1.2)–(1.5) by a standard procedure.

The paper is outlined as follows. In section 2, we establish the exact internal controllability of the linear Schrödinger equation with periodic boundary conditions by using the moment approach. In section 3, we derive the (internal or boundary) exact controllability of the cubic Schrödinger equation with various boundary conditions. The internal stabilization is investigated in section 4. In particular, the proof of

Theorem 1.5 is presented in this section. As for Theorem 1.6, its proof is similar to the one in [18, Theorem 1.1] and is, therefore, omitted.

Finally, the readers are referred to [19, 22, 23, 24, 25, 28] and references therein for the study of control and stabilization of another important nonlinear dispersive wave equation, the Korteweg–de Vries equation.

2. Linear systems. We first consider the associated linear open loop control system of the Schrödinger equation posed on $(-\pi, \pi)$ with the periodic boundary conditions:

$$(2.1) \quad \begin{cases} iv_t + v_{xx} = iGh, & v(x, 0) = v_0(x), \\ v(-\pi, t) = v(\pi, t), & v_x(-\pi, t) = v_x(\pi, t), \end{cases}$$

where the operator G is defined by (1.6) and h is the applied control function.

Let A denote the operator

$$(2.2) \quad Aw = iw''$$

on the domain $\mathcal{D}(A) = H_p^2(-\pi, \pi)$. It generates a strongly continuous group $W(t)$ in the space $L^2(-\pi, \pi)$; the eigenfunctions are simply the orthonormal Fourier basis functions in $L^2(-\pi, \pi)$

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k = 0, \pm 1, \pm 2, \dots$$

We have the following exact controllability result for the system (2.1).

THEOREM 2.1. *Let $T > 0$ and $s \geq 0$ be given. For any $v_0, v_1 \in H_p^s(-\pi, \pi)$, there exists a control $h \in L^2([0, T]; H_p^s(-\pi, \pi))$ such that the system (2.1) admits a unique solution*

$$v \in C([0, T]; H_p^s(-\pi, \pi))$$

satisfying

$$v(x, T) \equiv v_1(x).$$

Proof. The system (2.1) can be rewritten as an abstract control system in the space $H_p^s(-\pi, \pi)$,

$$(2.3) \quad \frac{d}{dt}v(t) = Av(t) + Gh, \quad v(0) = v_0.$$

By the standard semigroup theory, for any $s \geq 0$, $T > 0$, $v_0 \in H_p^s(-\pi, \pi)$, and $h \in L^2(0, T; H_p^s(-\pi, \pi))$, (2.2) admits a unique solution $v \in C([0, T]; H_p^s(-\pi, \pi))$. It is familiar that the operator A , as defined in (2.2), has eigenvalues

$$\lambda_k = -ik^2$$

with the corresponding eigenfunctions ϕ_k for $-\infty < k < \infty$. Relative to the basis $\{\phi_k\}_{-\infty}^{\infty}$, the initial state v_0 and the terminal state v_1 have the expansions, convergent in $H_p^s(-\pi, \pi)$,

$$(2.4) \quad v_j = \sum_{k=-\infty}^{\infty} v_{k,j} \phi_k, \quad v_{k,j} = \int_{-\pi}^{\pi} v_j(x) \overline{\phi_k(x)} dx \quad \text{for } j = 0, 1$$

and the solution v has the expansion

$$v(x, t) = \sum_{k=-\infty}^{+\infty} v_{k,0} e^{\lambda_k t} \phi_k(x) + \sum_{k=-\infty}^{+\infty} \int_0^t e^{\lambda_k(t-\tau)} (Gh)_k(\tau) d\tau \phi_k(x),$$

where

$$(Gh)_k(t) = \int_{-\pi}^{\pi} g(x) h(x, t) \overline{\phi_k(x)} dx, \quad k = 0, \pm 1, \pm 2, \dots$$

In order to find an appropriate control input h such that $v(x, T) = v_1(x)$, it suffices to solve the following moment problem:

$$(2.5) \quad v_{k,1} - v_{k,0} e^{\lambda_k T} = \int_0^T e^{\lambda_k(T-\tau)} (Gh)_k(\tau) d\tau$$

for $k = 0, \pm 1, \dots$

Defining $p_k(t) = e^{\lambda_k t}$, $\mathcal{P} \equiv \{p_k \mid 0 \leq k < \infty\}$ may be seen, from the result in [26], to form a Riesz basis for its closed span, \mathcal{P}_T , in $L^2(0, T)$. We let $\mathcal{Q} \equiv \{q_k \mid 0 \leq k < \infty\}$ be the unique dual Riesz basis for \mathcal{P} in \mathcal{P}_T , which fulfills

$$(2.6) \quad \int_0^T q_j(t) \overline{p_k(t)} dt = \delta_{kj}, \quad 0 \leq j, k < \infty.$$

We take the control h in (2.1) to have the form

$$(2.7) \quad h(x, t) = \sum_{j=-\infty}^{+\infty} h_j q_j(t) (G\phi_j)(x),$$

where $q_{-j} = q_j$ for $j \geq 0$ and the coefficients h_j are to be determined so that, among other things, the series (2.7) is appropriately convergent. Substituting (2.7) into (2.5) yields, using the biorthogonality (2.6),

$$(2.8) \quad v_{0,1} - v_{0,0} = \sum_{j=-\infty}^{+\infty} h_j \int_0^T e^{\lambda_0 t} q_j(t) \int_{-\pi}^{\pi} G(G\phi_j)(x) \overline{\phi_0(x)} dx dt = h_0 \int_{-\pi}^{\pi} |(G\phi_0)(x)|^2 dx$$

and for $-\infty < k < \infty, k \neq 0$,

$$(2.9) \quad \begin{aligned} v_{k,1} - v_{k,0} e^{\lambda_k T} &= e^{\lambda_k T} \sum_{j=-\infty}^{+\infty} h_j \int_0^T e^{\overline{\lambda_k t}} q_j(t) \int_{-\pi}^{\pi} G(G\phi_j)(x) \overline{\phi_k(x)} dx dt \\ &= h_k e^{\lambda_k T} \int_{-\pi}^{\pi} |(G\phi_k)(x)|^2 dx + h_{-k} e^{\lambda_k T} \int_{-\pi}^{\pi} G\phi_{-k}(x) \overline{G\phi_k(x)} dx \end{aligned}$$

as G is a self-adjoint operator in $L^2(-\pi, \pi)$. Since

$$\begin{aligned} \int_{-\pi}^{\pi} |G\phi_k(x)|^2 dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g^2(x) dx := a, \\ \int_{-\pi}^{\pi} G\phi_{-k}(x) \overline{G\phi_k(x)} dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g^2(x) e^{-2ikx} dx := b_k \end{aligned}$$

for $-\infty < k < \infty$, (2.8)–(2.9) may be rewritten as

$$(2.10) \quad c_0 = h_0 a$$

$$(2.11) \quad \begin{cases} c_k = ah_k + b_k h_{-k} \\ c_{-k} = b_{-k} h_k + ah_{-k} \end{cases} \quad k = 1, 2, \dots,$$

where $c_k = v_{k,1}e^{-\lambda_k T} - v_{k,0}$. As $\beta_k := a^2 - b_k b_{-k} \neq 0$ for any $k > 0$ and

$$\lim_{k \rightarrow \infty} |\beta_k| = a^2,$$

there exists a $\gamma > 0$ such that

$$|\beta_k| > \gamma \quad \forall k > 0.$$

Thus, it follows from (2.10)–(2.11) by the Gram’s rule that

$$(2.12) \quad h_0 = a^{-1}c_0, \quad h_k = \beta_k^{-1}(ac_k - c_{-k}b_k), \quad k = \pm 1, \pm 2, \dots$$

It remains to show that h defined by (2.7) and (2.12) belongs to $L^2([0, T]; H_p^s(-\pi, \pi))$ provided that $v_0, v_1 \in H_p^s(-\pi, \pi)$. To this end, let us write

$$(2.13) \quad G\phi_j(x) = \sum_{k=-\infty}^{+\infty} a_{jk}\phi_k(x),$$

where

$$a_{jk} = \int_{-\pi}^{\pi} G\phi_j(x)\overline{\phi_k(x)}dx, \quad -\infty < j, k < \infty.$$

Thus,

$$h(x, t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} h_j a_{jk} q_j(t) \phi_k(x)$$

and

$$(2.14) \quad \begin{aligned} \|h\|_{L^2([0, T]; H_p^s(-\pi, \pi))}^2 &= \int_0^T \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} \left| \sum_{j=-\infty}^{+\infty} a_{jk} h_j q_j(t) \right|^2 dt \\ &= \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} \int_0^T \left| \sum_{j=-\infty}^{+\infty} a_{jk} h_j q_j(t) \right|^2 dt \\ &\leq c \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} \sum_{j=-\infty}^{+\infty} |h_j|^2 |a_{jk}|^2 \\ &\leq c \sum_{j=-\infty}^{+\infty} |h_j|^2 \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} |a_{jk}|^2, \end{aligned}$$

where the constant c comes from the Riesz basis property of \mathcal{Q} in \mathcal{P}_T . However,

$$|a_{jk}| = |(G\phi_j, \phi_k)_{L^2(-\pi, \pi)}| = |(g\phi_j, \phi_k)_{L^2(-\pi, \pi)}| = \left| \frac{1}{\sqrt{2\pi}} g_{k-j} \right|,$$

where

$$g = \sum_{m=-\infty}^{+\infty} g_m \phi_m.$$

Hence,

$$|a_{jk}|^2 \leq c|g_{k-j}|^2$$

and

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} |a_{jk}|^2 &\leq c \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} |g_{k-j}|^2 \\ &\leq c \sum_{k=-\infty}^{+\infty} (1 + |k + j|)^{2s} |g_k|^2 \\ &\leq c(1 + |j|)^{2s} \sum_{k=-\infty}^{+\infty} (1 + |k|)^{2s} |g_k|^2 \\ &\leq c(1 + |j|)^{2s} \|g\|_s^2. \end{aligned}$$

We have by (2.14)

$$\begin{aligned} \|h\|_{L^2([0,T]; H_p^s(-\pi, \pi))}^2 &\leq c \left(\sum_{j=-\infty}^{+\infty} (1 + |j|)^{2s} |h_j|^2 \right) \|g\|_s^2 \\ &\leq c \left(\sum_{j=-\infty}^{+\infty} (1 + |j|)^{2s} \frac{|e^{-\lambda_j T} v_{j,1} - v_{j,0}|^2}{|\beta_j|^2} \right) \|g\|_s^2 \\ &\leq c \sup_{j \neq 0} |\beta_j|^{-2} \|g\|_s^2 \sum_{j=-\infty}^{+\infty} (1 + |j|)^{2s} (|v_{j,1}|^2 + |v_{j,0}|^2) \\ &\leq c \sup_{j \neq 0} \frac{1}{|\beta_j|^2} \|g\|_s^2 (\|v_1\|_s^2 + \|v_0\|_s^2). \end{aligned}$$

With this the proof is complete. \square

COROLLARY 2.1. Equations (2.5), (2.7), and (2.12) define, for $s \geq 0$, a bounded operator Φ :

$$h = \Phi(v_0, v_1), \quad \forall v_0, v_1 \in H_p^s(-\pi, \pi)$$

from $H_p^s(-\pi, \pi) \times H_p^s(-\pi, \pi)$ to $L^2([0, T]; H_p^s(-\pi, \pi))$ such that

$$(2.15) \quad W(T)v_0 + \int_0^T W(T - \tau)(G(\Phi(v_0, v_1))) (\cdot, \tau) d\tau = v_1$$

for any $(v_0, v_1) \in H_p^s(-\pi, \pi) \times H_p^s(-\pi, \pi)$ and

$$(2.16) \quad \|\Phi(v_0, v_1)\|_{L^2([0,T]; H_p^s(-\pi, \pi))} \leq c(\|v_0\|_s + \|v_1\|_s),$$

where c depends only on T and g .

The following observation, while simple, is important to study the control properties of the linear Schrödinger equation posed on the finite interval with either Dirichlet boundary conditions or Neumann boundary conditions.

COROLLARY 2.2. *Assume that the control structure function g is an even function. If both the initial state v_0 and the terminal state v_1 are even (odd) functions of the x variable, then the control input h constructed in the proof of Theorem 2.1 is also an even (odd) function of the x variable and so is the corresponding solution v .*

Now we consider the linear Schrödinger equation posed on the finite interval $(0, \pi)$

$$(2.17) \quad iu_t + u_{xx} = iQh(x, t), \quad u(x, 0) = u_0(x), \quad x \in (0, \pi), \quad t \geq 0$$

with either the Dirichlet boundary conditions

$$(2.18) \quad u(0, t) = 0, \quad u(\pi, t) = 0$$

or the Neumann boundary conditions

$$(2.19) \quad u_x(0, t) = 0, \quad u_x(\pi, t) = 0,$$

where again $Qh(x, t) = q(x)h(x, t)$, $q(x)$ is a given smooth function supported in a subinterval of $(0, \pi)$.

THEOREM 2.2. *Let $s \geq 0$ and $T > 0$ be given. For any $u_0, u_1 \in H_{odd}^s(0, \pi)$, there exists*

$$h \in L^2(0, T; H^s(0, \pi))$$

such that (2.17)–(2.18) admits a unique solution $u \in C([0, T]; H^s(0, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x), \quad x \in (0, \pi).$$

THEOREM 2.3. *Let $s \geq 0$ and $T > 0$ be given. For any $u_0, u_1 \in H_{even}^s(0, \pi)$, there exists*

$$h \in L^2(0, T; H^s(0, \pi))$$

such that (2.17)–(2.19) admits a unique solution $u \in C([0, T]; H^s(0, \pi))$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x), \quad x \in (0, \pi).$$

We provide just the proof of Theorem 2.2. The proof of Theorem 2.3 is similar and is, therefore, omitted.

Proof of Theorem 2.2. Note that if $u_0, u_1 \in H_{odd}^s(0, \pi)$, let v_0 and v_1 be the odd extension of u_0 and u_1 , respectively, from $(0, \pi)$ to $(-\pi, \pi)$, then both v_0 and v_1 belong to the space $H_p^s(-\pi, \pi)$. Let g be the even extension of q from $(0, \pi)$ to $(-\pi, \pi)$ and consider the control system (2.1)–(2.2). According to Corollary 2.2, the corresponding control input $h(x, t)$ is also an odd (or even) function in the x variable. Consequently, the corresponding solution $v(x, t)$ is an odd function in the x variable, which, when restricted to the interval $(0, \pi)$, is a solution of the IBVP (2.17)–(2.18). The controllability results regarding (2.17)–(2.18) as described in Theorem 2.3, thus, follow from Theorem 2.1. The proof is complete. \square

3. Exact controllability for NLS. In this section, we intend to extend the controllability results obtained for the linear Schrödinger equation to the nonlinear Schrödinger equation.

Consideration is first given to the system described by the nonlinear Schrödinger equation posed on the interval $(-\pi, \pi)$ with the periodic boundary conditions:

$$(3.1) \quad \begin{cases} iu_t + u_{xx} + \lambda|u|^2u = iGh, & x \in (-\pi, \pi), t \in (0, T), \\ u(x, 0) = u_0(x), & u(-\pi, t) = u(\pi, t), \quad u_x(-\pi, t) = u_x(\pi, t). \end{cases}$$

According to Bourgain [2, 3], for given $s \geq 0$, $u_0 \in H_p^s(-\pi, \pi)$, and $h \in L_{loc}^1(\mathbb{R}; H_p^s(-\pi, \pi))$, (3.1) admits a unique solution $u \in C(\mathbb{R}; H_p^s(-\pi, \pi))$. Our main concern is the exact controllability of (3.1) as a distributive control system.

Recall that $W(t)$ is the C^0 -group generated by the operator A , defined by (2.2), on the space $L^2(-\pi, \pi)$, with which, the system (3.1) has the following equivalent integral equation form:

$$(3.2) \quad u(t) = W(t)u_0 + \int_0^t W(t-\tau)(Gh)(\tau)d\tau + i\lambda \int_0^t W(t-\tau)(|u|^2u)(\tau)d\tau.$$

As it has been pointed out in the introduction, a smoothing property is needed for the operator from f to v :

$$v(t) = \int_0^t W(t-\tau)f(\tau)d\tau.$$

This needed smoothing property was provided in Bourgain's work [2, 3] where he dealt with the Cauchy problem for the periodic Schrödinger equation.

Let b and s be two given real numbers. For a function $w : (-\pi, \pi) \times \mathbb{R}$, define the quantity

$$\Lambda_{b,s}(w) = \left(\sum_{n=-\infty}^{\infty} (1+|n|)^{2s} \int_{-\infty}^{\infty} |\hat{w}(n, \lambda)|^2 (1+|\lambda+n^2|)^{2b} d\lambda \right)^{1/2},$$

where $\hat{w}(n, \lambda)$ denotes the Fourier transform of $w(x, t)$ with respect to both the x and t variables. Following Bourgain [2, 3], we introduce the following space:

$$(3.3) \quad X^{b,s} = \{w \in L^2(\mathbb{R}; H_p^s(-\pi, \pi)); \Lambda_{b,s}(w) < \infty\}$$

with the norm $\|\cdot\|_{X^{b,s}} := \Lambda_{b,s}(\cdot)$. For any given $T > 0$, $X_T^{b,s}$ denotes the restriction space of $X^{b,s}$ on the time interval $(0, T)$ with the associated quotient norm. It is clear that $X_T^{b,s}$ is a Hilbert space,

$$X_T^{0,s} = L^2(0, T; H_p^s(-\pi, \pi))$$

and

$$X_T^{b_1,s} \subset X_T^{b_2,s}$$

if $b_1 > b_2$.

Before we proceed to show the exact controllability results, we present the following technical lemmas due to Bourgain [2, 3] which will play important roles in the proof of Theorem 1.1.

LEMMA 3.1. *Let $T > 0$, $s \geq 0$, and $0 \leq b \leq 1$ be given. There exists a constant $C > 0$ depending only on s and b such that*

$$\|W(t)\phi\|_{X_T^{b,s}} \leq C\|\phi\|_s$$

for any $\phi \in H_p^s(-\pi, \pi)$.

LEMMA 3.2. *Let $T > 0$, $s \geq 0$, and $b \in (\frac{3}{8}, \frac{5}{8})$ and $\frac{5}{8} > b' > \max\{\frac{1}{2}, b\}$ be given. There exists a constant $C > 0$ depending only on s , b , and b' such that*

$$\left\| \int_0^t W(t-\tau)f(\tau)d\tau \right\|_{X_T^{b,s}} \leq C\|f\|_{X_T^{b'-1,s}}$$

for any $f \in X_T^{b'-1,s}$.

LEMMA 3.3. *Let $s \geq 0$ and $b \in (\frac{3}{8}, \frac{5}{8})$ and $\frac{5}{8} > b' > \max\{\frac{1}{2}, b\}$ be given. There exist some constants $C > 0$ and $\alpha > 0$ depending only on s , b , and b' such that*

$$\|u\bar{v}w\|_{X_T^{b'-1,s}} \leq CT^\alpha\|u\|_{X_T^{b,s}}\|v\|_{X_T^{b,s}}\|w\|_{X_T^{b,s}}$$

for any $T > 0$ and $u, v, w \in X_T^{b,s}$.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Define

$$(3.4) \quad \omega(T, u) := -i\lambda \int_0^T W(T-\tau) (|u|^2u) (\tau)d\tau.$$

According to Corollary 2.1, for given $u_0, u_1 \in H_p^s(-\pi, \pi)$, if one chooses

$$(3.5) \quad h = \Phi(u_0, u_1 + \omega(T, u))$$

in (3.2), then

$$(3.6) \quad \begin{aligned} u(t) &= W(t)u_0 + \int_0^t W(t-\tau)(G\Phi(u_0, u_1 + \omega(T, u)))(\tau)d\tau \\ &+ i\lambda \int_0^t W(t-\tau) (|u|^2u) (\tau)d\tau \end{aligned}$$

and

$$(3.7) \quad u(0) = u_0, \quad u(T) = u_1$$

by virtue of the definition of the operator Φ (cf. Corollary 2.1). This suggests that we consider the map

$$(3.8) \quad \begin{aligned} \Gamma(u) &= W(t)u_0 + \int_0^t W(t-\tau)(G(\Phi(u_0, u_1 + \omega(T, u))))(\tau)d\tau \\ &+ i\lambda \int_0^t W(t-\tau) (|u|^2u) (\tau)d\tau. \end{aligned}$$

If the map Γ is shown to be a contraction in an appropriate space, then its fixed point u is a solution of (1.2)–(1.3) with $h = \Phi(u_0, u_1 + \omega(T, u))$ and satisfies $u(x, T) \equiv u_1(x)$. We show this is the case in the space $X_T^{b,s}$.

Note that $X_T^{0,s} = L^2(0, T; H_p^s(-\pi, \pi))$ and $X_T^{b_1,s}$ is continuously imbedded into $X_T^{b_2,s}$ if $b_2 < b_1$. Applying Lemma 3.1, Lemma 3.2, and Lemma 3.3 with $\frac{5}{8} > b' > b > \frac{1}{2}$ to (3.8) yields

$$\|\Gamma(u)\|_{X_T^{b,s}} \leq c\|u_0\|_s + c\|G(\Phi(u_0, u_1 + \omega(T, u)))\|_{X_T^{b'-1,s}} + c\||u|^2u\|_{X_T^{b'-1,s}}.$$

It follows from the definitions of the operator Φ that

$$\begin{aligned} \|G(\Phi(u_0, u_1 + \omega(T, u)))\|_{X_T^{b'-1,s}} &\leq c \|G\Phi(u_0, u_1 + \omega(T, u))\|_{L^2(0,T;H_p^s)} \\ &\leq c(\|u_0\|_s + \|u_1\|_s + \|\omega(T, u)\|_s). \end{aligned}$$

Using Lemmas 3.1, 3.2, and 3.3, it follows that (note that $\lambda = 1$ or -1)

$$\begin{aligned} \|\omega(T, u)\|_s &= \left\| \int_0^T W(T - \tau)(|u|^2u)(\tau)d\tau \right\|_s \\ &\leq \sup_{t \in [0,T]} \left\| \int_0^t W(t - \tau)(|u|^2u)(\tau)d\tau \right\|_s \\ &\leq \left\| \int_0^t W(t - \tau)(|u|^2u)(\tau)d\tau \right\|_{X_T^{b,s}} \\ &\leq c\||u|^2u\|_{X_T^{b'-1,s}} \\ &\leq c\|u\|_{X_T^{b,s}}^3 \end{aligned}$$

Consequently,

$$(3.9) \quad \|\Gamma(u)\|_{X_T^{b,s}} \leq c(\|u_0\|_s + \|u_1\|_s) + c\|u\|_{X_T^{b,s}}^3.$$

For $M > 0$, let S_M be a bounded subset of X_s :

$$S_M = \left\{ v \in X_T^{b,s}, \|v\|_{X_T^{b,s}} \leq M \right\}.$$

Then, for any $u \in S_M$,

$$\|\Gamma(u)\|_{X_T^{b,s}} \leq c\|u_0\|_s + c\|u_1\|_s + cM^3.$$

We choose $\delta > 0$ and $M > 0$ such that

$$(3.10) \quad 2c\delta + cM^3 \leq M, \quad cM^2 < 1/2.$$

Then,

$$\|\Gamma(u)\|_{X_T^{b,s}} \leq M,$$

for any $u \in S_M$, if $\|u_0\|_s \leq \delta$ and $\|u_1\|_s \leq \delta$. In addition, for any $u, v \in S_M$, since

$$\begin{aligned} \Gamma(u) - \Gamma(v) &= \int_0^t W(t - \tau)(G\Phi(0, \omega(T, u) - \omega(T, v)))(\tau)d\tau \\ &\quad + i\lambda \int_0^t W(t - \tau) (|u|^2(u - v) + v\bar{u}(u - v) + (\bar{u} - \bar{v})v^2) (\tau)d\tau \end{aligned}$$

and

$$\omega(T, u) - \omega(T, v) = \int_0^T W(T - \tau) (|u|^2(u - v) + v\bar{u}(u - v) + (\bar{u} - \bar{v})v^2) (\tau)d\tau,$$

a similar argument shows that

$$\begin{aligned} \|\Gamma(u) - \Gamma(v)\|_{X_T^{b,s}} &\leq c(\|u\|_{X_T^{b,s}} + \|v\|_{X_T^{b,s}})^2 \|u - v\|_{X_T^{b,s}} \\ &\leq cM^2 \|u - v\|_{X_T^{b,s}} \\ &\leq \frac{1}{2} \|u - v\|_{X_T^{b,s}}. \end{aligned}$$

Thus, the map Γ is a contraction on S_M provided that δ and M are chosen according to (3.10) and $\|u_0\|_s \leq \delta, \|u_1\|_s \leq \delta$. As a result, its fixed point $u \in S_M$ is the unique solution of the integral equation (3.8). The proof is complete. \square

Next our attention is turned to the system described by the nonlinear Schrödinger equation posed on the finite interval $(0, \pi)$ with the Dirichlet boundary conditions:

$$(3.11) \quad \begin{cases} iv_t + v_{xx} + \lambda|v|^2v = Qh_1, & x \in (0, \pi), t \geq 0, \\ v(x, 0) = v_0(x), & v(0, t) = 0, \quad v(\pi, t) = 0. \end{cases}$$

It has been shown in [1] that for given $s \geq 0$ and $v_0 \in H_{odd}^s(0, \pi)$ and $h_1 \in L_{loc}^1(\mathbb{R}; H^s(0, \pi))$, (3.11) admits a unique solution $v \in C(\mathbb{R}; H^s(0, \pi))$. Moreover, if we let g be the even extension of q from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$, and u_0 be the odd extension of v_0 from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$. If $u = u(x, t)$ is the odd extension solution $v(x, t)$ of (3.11) from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$ with respect to the x -variable, then $u \in C(\mathbb{R}; H_p^s(-\pi, \pi))$ and solves (3.1) with h being the odd extension of h_1 from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$ with respect to the x -variable. On the other hand, if g is an even function, $u_0 \in H_p^s(-\pi, \pi)$ is an odd function and h is also an odd function with respect to the x -variable in (3.1), then the corresponding solution u of (3.1) is also an odd function. If we let $v(x, t)$ be its restriction on the interval $(0, \pi)$, then v solves (3.11) with v_0, q and h_1 being the restrictions of u_0, g , and h on the interval $(0, \pi)$, respectively.

Proof of Theorem 1.2. For given v_0 and $v_1 \in H_{odd}^s(0, \pi)$, let u_0 and u_1 be their odd extensions from $(0, \pi)$ to $(-\pi, \pi)$. We have $u_0, u_1 \in H_p^s(-\pi, \pi)$. In addition, let g be the even extension of q from $(0, \pi)$ to $(-\pi, \pi)$. It is sufficient to show that there exists a control input $h \in L^2(0, T; H_p^s(-\pi, \pi))$, which is odd with respect to the x -variable, such that (3.1) admits a solution $u \in C([0, T]; H_p^s(-\pi, \pi))$ which is odd with respect to the x -variable and satisfies

$$u(x, T) = u_1(x).$$

Indeed, if this is the case, let v be the restriction of u to $(0, \pi)$ with respect to the x -variable. Then $v \in C([0, T]; H^s(0, \pi))$ solves (3.11) and satisfies

$$v(x, T) = v_1(x).$$

To this end, as in the proof of Theorem 1.2, consider the map

$$\begin{aligned} \Gamma(u) &= W(t)u_0 + \int_0^t W(t - \tau)(G(\Phi(u_0, u_1 + \omega(T, u))))(\tau)d\tau \\ &\quad + i\lambda \int_0^t W(t - \tau) (|u|^2u) (\tau)d\tau \end{aligned}$$

for any

$$u \in S_{M,o} = \left\{ v \in X_T^{b,s}; v \text{ is odd with respect to } x\text{-variable, } \|v\|_{X_T^{b,s}} \leq M \right\}$$

where, we recall that

$$\omega(T, u) := i\lambda \int_0^T W(T - \tau) (|u|^2 u) (\tau) d\tau.$$

Note that $\omega(T, u)$ is an odd function of x if u is an odd function of x . Thus, by Corollary 2.2, $G(\Phi(u_0, u_1 + \omega(T, u)))$ is an odd function of x , and consequently, $\Gamma(u)$ is an odd function of x for any $u \in S_{M,o}$. Then the same argument as in the proof of Theorem 1.1 shows that Γ has a fixed point in the set $S_{M,o}$ as long as $\|u_0\|_s + \|u_1\|_s$ is small enough and M is chosen accordingly. The proof is complete. \square

Now we consider the system described by the nonlinear Schrödinger equation posed on the finite interval $(0, \pi)$ with the Neumann boundary conditions:

$$(3.12) \quad \begin{cases} iw_t + w_{xx} + \lambda|w|^2 w = Qh_1, & x \in (0, \pi), t \geq 0, \\ w(x, 0) = w_0(x), & w_x(0, t) = 0, \quad w_x(\pi, t) = 0. \end{cases}$$

It has been shown in [1] that for given $s \geq 0$ and $w_0 \in H_{even}^s(0, \pi)$ and $h_1 \in L_{loc}^1(\mathbb{R}; H^s(0, \pi))$, (3.12) admits a unique solution $v \in C(\mathbb{R}; H^s(0, \pi))$. Moreover, if we let g be the even extension of q from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$, and u_0 be the even extension of w_0 from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$, if $u = u(x, t)$ denotes the even extension solution of $w(x, t)$ from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$ with respect to the x -variable, then $u \in C(\mathbb{R}; H_p^s(-\pi, \pi))$ and solves (3.1) with h being the even extension of h_1 from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$ with respect to the x -variable. On the other hand, if g is an even function, $u_0 \in H_p^s(-\pi, \pi)$ is an even function and h is also an even function with respect to the x -variable in (3.1), then the corresponding solution u of (3.1) is also an even function. If we let $w(x, t)$ be its restriction on the interval $(0, \pi)$, then v solves (3.12) with w_0, q , and h_1 being the restrictions of u_0, g and h to the interval $(0, \pi)$, respectively. This leads us to the proof of Theorem 1.3.

Proof of Theorem 1.3. It is exactly the same as the one of Theorem 1.2, except that all the odd extensions become even extensions. The proof is complete. \square

Finally, we consider the boundary control of the nonlinear Schrödinger equation posed on the interval $(0, \pi)$ with the Dirichlet boundary conditions:

$$(3.13) \quad \begin{cases} iu_t + u_{xx} + \lambda|u|^2 u = 0, & x \in (0, \pi), t \geq 0, \\ u(x, 0) = u_0(x), & v(0, t) = h(t), \quad u(\pi, t) = 0 \end{cases}$$

or with the Neumann boundary conditions:

$$(3.14) \quad \begin{cases} iw_t + w_{xx} + \lambda|w|^2 w = 0, & x \in (0, \pi), t \geq 0, \\ w(x, 0) = w_0(x), & w_x(0, t) = h(t), \quad w_x(\pi, t) = 0. \end{cases}$$

Proof of Theorem 1.4. We prove only part (a). The proof of part (b) is similar. Consider the nonlinear Schrödinger equation posed on the finite interval $(-\pi, \pi)$ with Dirichlet boundary conditions:

$$(3.15) \quad \begin{cases} iw_t + w_{xx} + \lambda|w|^2 w = ig(x)\mu(x, t), & x \in (-\pi, \pi), t \in \mathbb{R}, \\ w(x, 0) = \tilde{u}_0(x), & x \in (-\pi, \pi), \\ w(-\pi, t) = 0, & w(\pi, t) = 0, \quad t \in \mathbb{R}. \end{cases}$$

where g is supported in the interval $(-\pi, 0)$ and $\mu(x, t)$ is a control input, and \tilde{u}_0 is an extension of $u_0 \in H^s(0, \pi)$ to the space $H^s(-\pi, \pi)$ satisfying $\tilde{u}_0(-\pi) = 0$. For given $u_1 \in H^s(0, \pi)$ let \tilde{u}_1 be its extension to the space $H^s(-\pi, \pi)$. According to Theorem 1.2, one can find $\mu \in L^2(0, T; H^s(-\pi, \pi))$ such that (3.15) admits a unique solution $w \in C([0, T]; H^s(-\pi, \pi))$ such that

$$w(x, 0) = \tilde{u}_0(x), \quad w(x, T) = \tilde{u}_1(x).$$

Let $u = u(x, t)$ be the restriction of $w(x, t)$ to the interval $(0, \pi)$. Then $u \in C([0, T]; H^s(0, \pi))$ solves (3.13) with $h(t) := w(0, t)$, and satisfies

$$u(x, 0) = u_0(x), \quad u(x, T) = u_1(x), \quad x \in (0, \pi).$$

The proof is complete. \square

4. Stabilization. In this section we study long-time behavior of the closed-loop system

$$(4.1) \quad \begin{cases} iu_t + u_{xx} + \lambda|u|^2u = -ig^2u, & x \in (-\pi, \pi), t \in \mathbb{R}, \\ (u(x, 0) = u_0(x), & x \in (-\pi, \pi), \\ u(-\pi, t) = u(\pi, t), & u_x(-\pi, t) = u_x(\pi, t). \end{cases}$$

We first consider the associated linear system:

$$(4.2) \quad \begin{cases} iu_t + u_{xx} = -ig^2u, & x \in (-\pi, \pi), t \in \mathbb{R}, \\ u(x, 0) = u_0(x), & x \in (-\pi, \pi), \\ u(-\pi, t) = u(\pi, t), & u_x(-\pi, t) = u_x(\pi, t). \end{cases}$$

For given $s \geq 0$, define an operator A_g in the space $H^s(-\pi, \pi)$ by

$$A_g = i\partial_x^2 - g^2$$

with domain $\mathcal{D}(A_g) = H_p^{s+2}(-\pi, \pi)$. By the standard semigroup theory, it generates a continuous group $(W_g(t))_{t \in \mathbb{R}}$ of operators on $\mathcal{H} := H_p^s(-\pi, \pi)$ and for any given initial data $u_0 \in H^s(-\pi, \pi)$, the corresponding solution u of (4.2) can be expressed as

$$u(t) = W_g(t)u_0.$$

Moreover, the semigroup $(W_g(t))_{t \in \mathbb{R}^+}$ is exponentially stable in \mathcal{H} .

PROPOSITION 4.1. *There exist positive constants $C > 0$ and $\nu > 0$ such that*

$$(4.3) \quad \|W_g(t)u_0\|_s \leq Ce^{-\nu t}\|u_0\|_s \quad \forall t \geq 0.$$

Proof. When $s = 0$, $g^2u = GG^*u$ and, thus, the exponential stability of $(W_g(t))_{t \in \mathbb{R}^+}$ is a direct consequence of Theorem 2.1 according to [12]. To prove (4.3) when $s = 2$, we pick $u_0 \in H_p^2(-\pi, \pi)$ and set $v := u_t$. Then v solves the system

$$(4.4) \quad \begin{cases} v_t = iv_{xx} - g^2(x)v, & v(x, 0) = v_0(x) := i\frac{d^2u_0}{dx^2} - g^2(x)u_0(x), \\ v(-\pi, t) = v(\pi, t), & v_x(-\pi, t) = v_x(\pi, t). \end{cases}$$

By the property (4.3) established when $s = 0$, we have

$$\|v(t)\|_0 \leq Ce^{-\nu t}\|u_0\|_0, \quad \|v(t)\|_0 \leq Ce^{-\nu t}\|v_0\|_0.$$

Since $iu_{xx} = v + g^2u$, we conclude that

$$\|u(t)\|_2 \leq Ce^{-\nu t} \|u_0\|_2 \quad \forall t \geq 0.$$

An easy induction yields (4.3) for any $s \in 2\mathbb{N}$. The proposition follows by a classical interpolation argument. The proof is complete. \square

Now we turn our attention to the stability properties of the nonlinear system (4.1) which can be rewritten in equivalent its integral form

$$(4.5) \quad u(t) = W_g(t)u_0 + i\lambda \int_0^t W_g(t - \tau) (|u|^2u) (\tau) d\tau.$$

At this point, we need to establish Lemmas 3.1 and 3.2 with the semigroup $W(t)$ substituted by the semigroup $W_g(t)$.

LEMMA 4.1. *Let $T > 0$, $s \geq 0$, and $0 \leq b \leq 1$ be given. There exists a constant $C > 0$ depending only on s and b such that*

$$\|W_g(t)\phi\|_{X_T^{b,s}} \leq C\|\phi\|_s$$

for any $\phi \in H_p^s(-\pi, \pi)$.

Proof. An application of Duhamel formula gives

$$(4.6) \quad W_g(t)\phi = W(t)\phi - \int_0^t W(t - \tau) (g^2W_g(\tau)\phi) d\tau.$$

It follows that

$$\begin{aligned} \|W_g(t)\phi\|_{X_T^{b,s}} &\leq \|W(t)\phi\|_{X_T^{b,s}} + \left\| \int_0^t W(t - \tau) (g^2W_g(\tau)\phi) d\tau \right\|_{X_T^{b,s}} \\ &\leq C\|\phi\|_s + C \|g^2W_g(t)\phi\|_{X_T^{b'-1,s}} \\ &\leq C\|\phi\|_s + C\|W_g(t)\phi\|_{L^2(0,T;H_p^s(-\pi,\pi))} \quad (\text{as } b' - 1 < 0) \\ &\leq C\|\phi\|_s, \end{aligned}$$

as desired. \square

LEMMA 4.2. *Let $T > 0$, $s \geq 0$, $b \in (\frac{1}{2}, \frac{5}{8})$, and $b' \in (b, \frac{5}{8})$ be given. There exists a constant $C > 0$ depending only on s , b , and b' such that*

$$\left\| \int_0^t W_g(t - \tau)f(\tau)d\tau \right\|_{X_T^{b,s}} \leq C\|f\|_{X_T^{b'-1,s}}$$

for any $f \in X_T^{b'-1,s}$.

Proof. It follows from (4.6) that

$$\int_0^t W_g(t - \tau)f(\tau)d\tau = \int_0^t W(t - \tau)f(\tau)d\tau - \int_0^t W(t - \tau)g^2 \left(\int_0^\tau W_g(\tau - s)f(s)ds \right) d\tau,$$

and hence, using Lemma 3.2,

$$\begin{aligned} \left\| \int_0^t W_g(t - \tau)f(\tau)d\tau \right\|_{X_T^{b,s}} &\leq C\|f\|_{X_T^{b'-1,s}} + C \left\| g^2 \int_0^t W_g(t - s)f(s)ds \right\|_{X_T^{b'-1,s}} \\ &\leq C\|f\|_{X_T^{b'-1,s}} + CT^\alpha \|g\|_{X_T^{b,s}}^2 \left\| \int_0^t W_g(t - s)f(s)ds \right\|_{X_T^{b,s}} \end{aligned}$$

for some constant $\alpha > 0$, by virtue of Lemma 3.3. The result follows at once if T is small enough, say $T < T_0$. For $T \geq T_0$, the result follows from Lemma 4.1 and an easy induction argument. \square

Now we are in position to prove Theorem 1.5

Proof of Theorem 1.5. For given $s \geq 0$, there exist some positive constants C, ν such that

$$\|W_g(t)u_0\|_s \leq Ce^{-\nu t}\|u_0\|_s \quad \forall t \geq 0$$

according to Proposition 4.1. Choose $T > 0$ such that

$$Ce^{-\nu T} < \frac{1}{4}$$

and fix a number $b \in (\frac{1}{2}, \frac{5}{4})$. We seek a solution u to the integral equation (4.5) as a fixed point of the map

$$\Gamma(u) = W_g(t)u_0 + i\lambda \int_0^t W_g(t - \tau) (|u|^2u) (\tau)d\tau$$

in some ball S_M of the space $X_T^{b,s}$. This will be done provided that $\|u_0\|_s \leq \delta$ where δ is a small number to be determined. Furthermore, to ensure the exponential stability, δ and M will be chosen in such a way that $\|u(T)\|_s \leq \|u_0\|_s/2$. Pick for the moment any $\delta > 0$ and $M > 0$, and let $u_0 \in \mathcal{H}$ be such that $\|u_0\|_s \leq \delta$. By computations similar to the ones displayed in the proof of Theorem 1.1 with $W_g(t)$ substituted to $W(t)$, we arrive to

$$\|\Gamma(u)\|_{X_T^{b,s}} \leq c\|u_0\|_s + cM^3 \quad \forall u \in S_M$$

and

$$\|\Gamma(u) - \Gamma(v)\|_{X_T^{b,s}} \leq cM^2\|u - v\|_{X_T^{b,s}}$$

for some constant $c > 0$ independent of δ, M , and t . On the other hand, using the estimate of $\|\omega(T, u)\|_s$ in the proof of Theorem 1.1, we obtain

$$\begin{aligned} \|\Gamma(u)(T)\|_s &\leq \|W_g(T)u_0\|_s + \left\| \int_0^T W_g(T - t)(|u|^2u)(t)dt \right\|_s \\ &\leq \frac{1}{4}\|u_0\|_s + cM^3. \end{aligned}$$

Pick $\delta = 4cM^3$ where $M > 0$ is chosen so that

$$(4c^2 + c) M^3 \leq M \quad \text{and} \quad cM^2 \leq \frac{1}{2}.$$

Then we have

$$\begin{aligned} \|\Gamma(u)\|_{X_T^{b,s}} &\leq M \quad \forall u \in S_M \\ \|\Gamma(u) - \Gamma(v)\|_{X_T^{b,s}} &\leq \frac{1}{2}\|u - v\|_{X_T^{b,s}} \quad \forall u, v \in S_M. \end{aligned}$$

Therefore, Γ is a contraction in S_M . Furthermore, its unique fixed point $u \in S_M$ fulfills

$$\|u(T)\|_s = \|\Gamma(u)(T)\|_s \leq \frac{\delta}{2}.$$

Assume now that $0 < \|u_0\|_s < \delta$. Changing δ into $\delta' := \|u_0\|_s$ and M into $M' := (\delta'/\delta)^{\frac{1}{3}}M$, we infer that $\|u(T)\|_s \leq \|u_0\|_s/2$, and an obvious induction yields $\|u(nT)\|_s \leq 2^{-n}\|u_0\|_s$ for any $n \geq 0$. As $X_T^{b,s} \subset C([0, T]; H_p^s(-\pi, \pi))$ for $b > 1/2$, and $\|u\|_{X_T^{b,s}} \leq M = (\delta/(4c))^{\frac{1}{3}}$, we infer by the semigroup property that there exist some constants $C' > 0, \nu' > 0$ such that

$$\|u(t)\|_s \leq C' e^{-\nu' t} \|u_0\|_s.$$

The proof is complete. \square

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