

# Homogeneous Lyapunov function for homogeneous continuous vector field

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**Abstract:** The goal of this article is to provide a construction of a *homogeneous* Lyapunov function  $\bar{V}$  associated with a system of differential equations  $\dot{x} = f(x)$ ,  $x \in \mathbb{R}^n$  ( $n \geq 1$ ), under the hypotheses: (1)  $f \in C(\mathbb{R}^n, \mathbb{R}^n)$  vanishes at  $x = 0$  and is homogeneous; (2) the zero solution of this system is locally asymptotically stable. Moreover, the Lyapunov function  $\bar{V}(x)$  tends to infinity with  $\|x\|$ , and belongs to  $C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \cap C^p(\mathbb{R}^n, \mathbb{R})$ , with  $p \in \mathbb{N}^*$  as large as wanted. As application to the theory of homogeneous systems, we present two well known results of robustness, in a slightly extended form, and with simpler proofs.

**Keywords:** Local asymptotic stability; Lyapunov function; homogeneity; non-differentiable feedback.

## 1. Introduction

Kurzweil, in [9], proved the converse of Lyapunov's second theorem in the quite general framework of a system of differential equations  $\dot{x} = f(x, t)$ , where  $f \in C(G \times [0, +\infty), \mathbb{R}^n)$ ,  $G$  being an open set in  $\mathbb{R}^n$ , and  $C(G \times [0, +\infty), \mathbb{R}^n)$  being the set of continuous maps from  $G \times [0, +\infty)$  into  $\mathbb{R}^n$ . In the particular case of autonomous systems on  $\mathbb{R}^n$ , he obtains the following result:

**Theorem 1 (Kurzweil).** *If  $f \in C(\mathbb{R}^n, \mathbb{R}^n)$  is such that  $f(0) = 0$ , the trivial solution of the equation  $\dot{x} = f(x)$  is strongly stable in  $\mathbb{R}^n$  (see below for a definition) if and only if there exists a (so-called Lyapunov) function  $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$  such that  $V(0) = 0$ ,  $V(x) > 0$  for all  $x \neq 0$ ,  $V(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$  and  $\nabla V(x) \cdot f(x) < 0 \forall x \neq 0$ .*

An important class of functions  $f$  for which one would like to study the differential system  $\dot{x} = f(x)$  is the class of homogeneous functions. A function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  (resp.  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ) is said to be *homogeneous* if there exist  $(r_1, \dots, r_n) \in ((0, +\infty))^n$  and  $\tau \in \mathbb{R}$  such that  $\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n \setminus \{0\}$ ,  $\forall t > 0$ ,  $g(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^\tau g(x)$  (resp.  $\forall i, \forall x \neq 0, \forall t > 0, g_i(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^{\tau+r_i} g_i(x)$ ).

It is natural to ask if homogeneity of  $V$  may be imposed when  $f$  is assumed to be homogeneous. Many authors have replied in the affirmative when  $f$  is in addition assumed to be of class  $C^1$  (see [4, Theorem 57.4; 10, Theorem 36; 7, Proposition p.1246]). Extending this result to the much more general framework where only continuity of  $f$  is supposed necessitates proceeding differently. To convince ourselves of this, let us examine in detail the hypotheses used by Hahn (see [4]) in proving his theorem.

(i) First to define his Lyapunov function, Hahn needs uniqueness of the trajectory for a given initial data. Indeed, he puts  $V(x) := \int_0^{+\infty} |p(\tau, x)|^\alpha d\tau$  where  $\tau \mapsto p(\tau, x)$  is the solution of  $\dot{x} = f(x)$  starting from  $x$  at  $t = 0$ , and  $\alpha$  a suitable exponent.

(ii) Then to prove that  $V$  is of class  $C^1$ , he must assume  $\partial f / \partial x_i$  exists and is bounded for each  $i \in \{1, \dots, n\}$ . This implies that  $\tau \geq 0$ .

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We remark that in his definition of homogeneity the variable  $t$  ranges over the whole set  $\mathbb{R}$ , instead of the interval  $(0, +\infty)$ . In the restricted case where  $r_i = 1$  for all  $i$ , he can only treat values for  $\tau$  of the form  $p/(2q+1)$ ,  $p, q \in \mathbb{N}$ .

These hypotheses will not be required in the following theorem, which is the principal result of this paper.

**Theorem 2.** *Let  $f$  be a function satisfying:*

- (a)  $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ ,  $f(0) = 0$ ;  
 (b)  $f$  is homogeneous: there exist  $(r_1, \dots, r_n) \in ((0, +\infty))^n$  and  $\tau \in \mathbb{R}$  such that:

$$\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n \setminus \{0\}, \forall t > 0 \quad f_i(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^{\tau+r_i}f_i(x_1, \dots, x_n); \quad (1)$$

- (c) the trivial solution  $x = 0$  of system  $\dot{x} = f(x)$  is locally asymptotically stable.

Let  $p$  be a positive integer, and  $k$  be a real number larger than  $p \cdot \max_{1 \leq i \leq n} r_i$ . Then there exists a function  $\bar{V}: \mathbb{R}^n \rightarrow \mathbb{R}$  such that:

- (i)  $\bar{V} \in C^p(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ ;  
 (ii)  $\bar{V}(0) = 0$ ,  $\bar{V}(x) > 0$  for all  $x \neq 0$  and  $\bar{V}(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ ;  
 (iii)  $\bar{V}$  is homogeneous:  $\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n \setminus \{0\}$ ,  $\forall t > 0 \quad \bar{V}(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^k \bar{V}(x)$ ;  
 (iv)  $\forall x \neq 0 \quad \nabla \bar{V}(x) \cdot f(x) < 0$ .

**Remarks.** (a) Using the homogeneity of  $f$  we shall prove that the zero solution of the system  $\dot{x} = f(x)$  is strongly stable in  $\mathbb{R}^n$  if and only if it is locally asymptotically stable. Thus, with the minimal hypothesis concerning the stability at the origin we will obtain what we need to start our construction of  $\bar{V}$  (namely a Lyapunov function  $V$ ) by using Kurzweil's theorem.

(b) In addition to the fact that the Lyapunov function  $\bar{V}$  is homogeneous (with the same  $r_i$  as for  $f$ ) it can be as smooth as we want, except that, with our method, we cannot assert that  $\bar{V} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ . (A  $C^\infty \bar{V}$  would necessarily be a polynomial: indeed, doing  $s \rightarrow 0$  in (7) (see below), we see that  $\partial^\alpha \bar{V} = 0$  for  $|\alpha|$  large enough.) Furthermore  $\bar{V}(x)$  tends to infinity as  $\|x\|$  tends to infinity.

(c) Extension of the existence of a homogeneous Lyapunov function to the framework of continuous (homogeneous) vector valued functions is not devoid of interest. Indeed, many smooth nonlinear systems  $\dot{x} = g(x, u)$  can be stabilized by only *continuous* feedback laws  $x \mapsto u(x)$  (e.g. [2,3,8]). In that case  $f(x) := g(x, u(x))$  is only continuous.

We now recall how Kurzweil defines the different notions of stability in the continuous framework, without requiring uniqueness of the trajectory for a given initial data. Let  $G$  be an open subset of  $\mathbb{R}^n$  which contains the origin,  $F$  the closed set  $\mathbb{R}^n \setminus G$ , and  $\omega$  the function defined on  $G$  by:

$$\omega(x) = \begin{cases} \max\left(\|x\|, \frac{1}{d(x, F)} - \frac{2}{d(0, F)}\right) & \text{if } F \text{ is nonempty,} \\ \|x\| & \text{if } F \text{ is empty.} \end{cases}$$

Let  $f: G \rightarrow \mathbb{R}^n$  be a continuous function such that  $f(0) = 0$ . The zero solution of the equation

$$\dot{x} = f(x), \quad x \in G, \quad (2)$$

is said to be:

(a) *locally stable* if for every  $\varepsilon > 0$  there exists a  $\delta = \delta(\varepsilon) > 0$  such that, for any solution  $x(t)$  of the equation (2) with  $\omega(x(0)) < \delta$ , defined for  $0 \leq t \leq T$ ,  $0 < T < +\infty$ , there is a solution  $y(t)$  of the equation (2), defined for all  $t \geq 0$ , such that

$$x(t) = y(t) \quad \text{for } 0 \leq t \leq T \quad \text{and} \quad \omega(y(t)) < \varepsilon \quad \text{for } t \geq 0;$$

(b) *locally asymptotically stable* if it is locally stable and if, in addition, there exists a  $\delta_0, 0 < \delta_0 < \delta(1)$ , such that if  $x(t)$  is a solution of the equation (2) with  $\omega(x(0)) < \delta_0$ , defined for all  $t \geq 0$ , then  $x(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ;

(c) *strongly stable in G* if there exist two functions  $B : (0, +\infty) \rightarrow (0, +\infty)$  and  $T : (0, +\infty)^2 \rightarrow (0, +\infty)$ , with  $B$  increasing and  $\lim_{\beta \rightarrow 0} B(\beta) = 0$ , such that for all  $\beta > 0$  and  $\varepsilon > 0$ , for every solution  $x(t)$  of (2) defined on an interval  $[0, t_1)$ , where  $0 < t_1 \leq +\infty$ , such that  $\omega(x(0)) \leq \beta$ , there exists a solution  $y(t)$  of (2) defined on  $[0, +\infty)$  such that  $y(t) = x(t)$  for  $0 \leq t < t_1$ ,  $\omega(y(t)) < B(\beta)$  for  $t \geq 0$ , and  $\omega(y(t)) < \varepsilon$  for  $t \geq T(\beta, \varepsilon)$ .

**Remarks.** (a) By the extension theorem (see [5, Th. 3.1, Chap. II]), one proves easily that *the zero solution of (2) is locally stable if and only if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every solution  $x(t)$  of (2) with  $\omega(x(0)) < \delta$ , defined for  $0 \leq t < T$  ( $0 < T \leq +\infty$ ), we have  $\omega(x(t)) < \varepsilon \forall t \in [0, T)$ .*

(b) In the definition of local stability and local asymptotic stability,  $\omega$  may everywhere be replaced by  $\| \cdot \|$ . Indeed,  $\omega(x) = \|x\|$  for all  $x$  such that  $\|x\| < \frac{1}{2}d(0, F)$ .

The *region of asymptotic stability A* will be the set of points  $x_0 \in G$  for which the following is true: if  $x(t)$  is a solution of (2) with  $x(0) = x_0$ , defined for  $0 \leq t < T$  ( $0 < T \leq +\infty$ ), then there exists a solution  $y(t)$  defined for all  $t \geq 0$  such that  $y(t) = x(t)$  for  $0 \leq t < T$  and  $y(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . It is clear that every solution of  $\dot{x} = f(x), x(t) \in G$  coming from a point of  $A$  at  $t_0 = 0$ , will remain in  $A$  for all times greater than 0.

Let us assume that the zero solution of (2) is locally asymptotically stable. In this case Kurzweil proved also (in [9, pp. 69–71]) that  $A$  is an *open set* containing 0, in which the zero solution of  $\dot{x} = f(x)$  ( $x \in A$ ) is *strongly stable*.

Before giving the proof of Theorem 2, we recall the main properties of homogeneous functions which are used here. Suppose  $\bar{V} : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function which is homogeneous, more precisely

$$\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n \setminus \{0\} \quad \forall t > 0 \quad \bar{V}(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^k \bar{V}(x_1, \dots, x_n) \tag{3}$$

where  $r_1, \dots, r_n$  are some positive real numbers, and  $k$  is a non-negative real number.

The first property is that  $\bar{V}$  is entirely defined by its values on  $S^{n-1} := \{x \mid \|x\| = 1\}$ . Indeed the map  $\phi : (0, +\infty) \times S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$  given by  $\phi(t, (y_1, \dots, y_n)) = (t^{r_1}y_1, \dots, t^{r_n}y_n)$  is surjective. As this function will appear again, we present some of its properties in the following lemma.

**Lemma 1.** (1) *The map*

$$\phi : (0, +\infty) \times S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad (t, (y_1, \dots, y_n)) \mapsto (t^{r_1}y_1, \dots, t^{r_n}y_n)$$

*is a bijection, and its inverse function  $\phi^{-1}$ , which we write  $\psi = (\psi_0, \psi_1, \dots, \psi_n)$ , is of class  $C^\infty$ .*

(2) *The function  $\psi_0$  satisfies*

$$\lim_{x \rightarrow 0, x \neq 0} \psi_0(x) = 0 \quad \text{and} \quad \lim_{\|x\| \rightarrow +\infty} \psi_0(x) = +\infty.$$

We give a sketch of the proof of this lemma. Since the map  $t \mapsto \sum_{i=1}^n x_i^2 / t^{2r_i}$  from  $(0, +\infty)$  into itself (for  $x \in \mathbb{R}^n \setminus \{0\}$  fixed) is decreasing and onto, it follows that  $\phi$  is bijective. Moreover the implicit function theorem applied to  $g(t, x) := \sum_{i=1}^n x_i^2 / t^{2r_i} - 1$  shows that  $\psi_0$ , and thus  $\psi_i(x) = x_i / \psi_0(x)^{r_i}$  for  $i = 1, \dots, n$ , are  $C^\infty$  maps. The remaining properties of  $\psi_0$  are easy to prove.

The second property is that the partial derivatives  $\partial \bar{V} / \partial x_i$  are also homogeneous. More precisely

$$\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n \setminus \{0\}, \quad \forall t > 0 \quad \frac{\partial \bar{V}}{\partial x_i}(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^{k-r_i} \frac{\partial \bar{V}}{\partial x_i}(x_1, \dots, x_n). \tag{4}$$

Indeed, by differentiating each member of equation (3) with respect to  $x_i$ , we get that

$$\forall x \neq 0, \forall t > 0 \quad t^{r_i} \frac{\partial \bar{V}}{\partial x_i}(t^{r_1} x_1, \dots, t^{r_n} x_n) = t^k \frac{\partial \bar{V}}{\partial x_i}(x_1, \dots, x_n).$$

This property implies that  $\nabla \bar{V}(x) \cdot f(x) < 0$  on  $\mathbb{R}^n \setminus \{0\}$  whenever  $\nabla \bar{V}(x) \cdot f(x) < 0$  holds on  $S^{n-1}$ . Indeed, using (1) we get that for all  $y \in S^{n-1}$  and  $t > 0$ ,

$$\nabla \bar{V}(t^{r_1} y_1, \dots, t^{r_n} y_n) \cdot f(t^{r_1} y_1, \dots, t^{r_n} y_n) = t^{\tau+k} \nabla \bar{V}(y) \cdot f(y) \quad (5)$$

and, since the map  $\phi$  is onto, we obtain  $\nabla \bar{V}(x) \cdot f(x) < 0$  for all  $x \neq 0$ .

For the convenience of the reader, we now describe the organization of the paper.

In Section 2 we give the proof of Theorem 2. First we establish a preliminary proposition, which claims that the zero solution of  $\dot{x} = f(x)$  is strongly stable in  $\mathbb{R}^n$ . To accomplish this we remark that the set  $A$ , which is a neighborhood of the origin, is invariant under the action of the dilation group, and then is all of  $\mathbb{R}^n$ . We will denote by  $V$  the (non-homogeneous) Lyapunov function associated with  $f$ , given by Theorem 1 (Kurzweil). Afterwards we construct, from  $V$  (and another function  $a$ ) a function  $\bar{V}$  satisfying all the properties (i)–(iv) of Theorem 2. The expression of  $\bar{V}$  is given in Proposition 2.

In Section 3 we study two problems of robustness for homogeneous stable systems, namely: is the asymptotic stability of the origin preserved after adding an integrator or some perturbing term? Two results, recently obtained by Coron and Praly (see [1, Proposition 3]) and Hermes (see [6, Theorem 1]) are easily proved by means of Theorem 2, and slightly generalized.

## 2. Proof of Theorem 2

We first prove the following result.

**Proposition 1.** *Under the assumptions of Theorem 2, the region of asymptotic stability is all of  $\mathbb{R}^n$ , and so the trivial solution of (2) is strongly stable in  $\mathbb{R}^n$ .*

**Proof.** Let  $A$  be the region of asymptotic stability,  $A$  is a subset of  $G = \mathbb{R}^n$ . Let  $x_0$  be in  $\mathbb{R}^n$ , and let  $x(t)$  be a solution of (2), defined on  $[0, T)$ , where  $0 < T \leq +\infty$ , such that  $x(0) = x_0$ . Let  $\varepsilon$  be a positive real number such that  $(\varepsilon^{r_1} x_1(0), \dots, \varepsilon^{r_n} x_n(0)) \in A$ . (Such an  $\varepsilon$  exists since  $A$  is an open set which contains 0.) For  $i = 1, \dots, n$  and  $t \in [T/\varepsilon^\tau)$  we set  $\tilde{x}_i(t) = \varepsilon^{r_i} x_i(\varepsilon^\tau t)$ . Then  $\tilde{x}(t) := (\tilde{x}_1(t), \dots, \tilde{x}_n(t))$  is a solution of (2). Indeed, we have

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= \varepsilon^{r_i+\tau} \dot{x}_i(\varepsilon^\tau t) \\ &= \varepsilon^{r_i+\tau} f_i(x_1(\varepsilon^\tau t), \dots, x_n(\varepsilon^\tau t)) \quad (\text{since } x \text{ is a solution of (2)}) \\ &= f_i(\varepsilon^{r_1} x_1(\varepsilon^\tau t), \dots, \varepsilon^{r_n} x_n(\varepsilon^\tau t)) \quad (\text{since } f \text{ is homogeneous}) \\ &= f_i(\tilde{x}_1(t), \dots, \tilde{x}_n(t)). \end{aligned}$$

Moreover  $\tilde{x}(0) \in A$ , and so there exists a solution  $\tilde{y}$  of (2), defined on  $[0, +\infty)$ , which extends  $\tilde{x}$ , and which tends to 0 as  $t$  tends to infinity. It follows that the function  $y$  defined by  $y_i(t) = (1/\varepsilon)^{r_i} \tilde{y}_i((1/\varepsilon)^\tau t)$  for  $t \geq 0$ , is a solution of (2) such that

$$y(t) = x(t) \text{ for } 0 \leq t < T \quad \text{and} \quad y(t) \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Thus,  $x_0 \in A$ , and the proof of  $A = \mathbb{R}^n$  is complete. Since we know the zero solution of  $\dot{x} = f(x)$ ,  $x \in A$ , is strongly stable in  $A$  (see [9, proof of Theorem 12]), we can apply Theorem 1, thereby obtaining a Lyapunov function  $V$  associated with  $f$  which is of class  $C^\infty$  on  $\mathbb{R}^n$  and which tends to infinity with  $\|x\|$ . Now, a candidate for the Lyapunov function  $\bar{V}$  of Theorem 2 is proposed by the following proposition.

**Proposition 2.** Let  $f$  and  $p$  be as in Theorem 2. Let  $a \in C^\infty(\mathbb{R}, \mathbb{R})$  be such that

$$a = \begin{cases} 0 & \text{on } (-\infty, 1], \\ 1 & \text{on } [2, \infty), \end{cases} \quad \text{and } a' \geq 0 \text{ on } \mathbb{R}.$$

Let  $k$  be a positive integer. Then the function

$$\bar{V}(x) := \begin{cases} \int_0^{+\infty} \frac{1}{t^{k+1}} (a \circ V)(t^{r_1}x_1, \dots, t^{r_n}x_n) dt & \text{if } x \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } x = 0, \end{cases}$$

is well defined, of class  $C^\infty$  on  $\mathbb{R}^n \setminus \{0\}$ , and satisfies

- (a)  $\nabla \bar{V}(x) \cdot f(x) < 0$ ,
- (b)  $\bar{V}(s^{r_1}x_1, \dots, s^{r_n}x_n) = s^k \bar{V}(x_1, \dots, x_n)$ ,

for all  $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$  and  $s > 0$ .

Moreover, if  $k > p \cdot \max\{r_i \mid 1 \leq i \leq n\}$ , then  $\bar{V}$  is  $C^p$  at 0.

**Proof.** Since  $V(x)$  tends to infinity with  $\|x\|$  and vanishes at 0, the function  $\bar{V}$  is well defined. Moreover, we may find two numbers  $l > 0$  and  $L > 0$  such that

$$\begin{aligned} V(t^{r_1}x_1, \dots, t^{r_n}x_n) &\leq 1 && \text{for all } \|x\| \in \left[\frac{1}{2}, 2\right], t \leq l, \\ V(t^{r_1}x_1, \dots, t^{r_n}x_n) &\geq 2 && \text{for all } \|x\| \in \left[\frac{1}{2}, 2\right], t \geq L. \end{aligned}$$

Then, for all  $x \in \mathbb{R}^n$  such that  $\|x\| \in (\frac{1}{2}, 2)$ ,

$$\bar{V}(x) = \int_l^L \frac{1}{t^{k+1}} (a \circ V)(t^{r_1}x_1, \dots, t^{r_n}x_n) dt + \frac{1}{kL^k}.$$

Clearly,  $\bar{V}$  is  $C^\infty$  on  $\{x \mid \|x\| \in (\frac{1}{2}, 2)\}$ , and we may write, on this set

$$\frac{\partial \bar{V}}{\partial x_i}(x) = \int_l^L \frac{t^{r_i}}{t^{k+1}} a'(V(t^{r_1}x_1, \dots, t^{r_n}x_n)) \frac{\partial V}{\partial x_i}(t^{r_1}x_1, \dots, t^{r_n}x_n) dt. \tag{6}$$

Hence, by (1),

$$\sum_{i=1}^n f_i \frac{\partial \bar{V}}{\partial x_i} = \int_l^L \frac{1}{t^{\tau+k+1}} a'(V(t^{r_1}x_1, \dots, t^{r_n}x_n)) \left[ \sum_{i=1}^n \left( f_i \frac{\partial V}{\partial x_i} \right) (t^{r_1}x_1, \dots, t^{r_n}x_n) \right] dt,$$

and since  $a'(s) > 0$  for some  $s \in (1, 2)$ ,  $\nabla \bar{V}(x) \cdot f(x) < 0$  holds for  $\frac{1}{2} < \|x\| < 2$ .

By an obvious change of variable of integration, we obtain at once (b). Hence,  $\bar{V}$  is  $C^\infty$  on  $\mathbb{R}^n \setminus \{0\}$  and (a) is true. (See (5).)

Now, let  $k > p \cdot \max\{r_i \mid 1 \leq i \leq n\}$ . For a multiindex  $\alpha = (\alpha_1, \dots, \alpha_n)$  let  $\partial^\alpha$  denote  $\partial^{\alpha_1 + \dots + \alpha_n} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ . Then, by a straightforward induction argument using (4), we get, for  $\tau > 0$  and  $y \in S^{n-1}$ ,

$$\partial^\alpha \bar{V}(s^{r_1}y_1, \dots, s^{r_n}y_n) = s^{k-\alpha \cdot r} \partial^\alpha \bar{V}(y_1, \dots, y_n). \tag{7}$$

Hence, by Lemma 1, for  $0 \leq \alpha_1 + \dots + \alpha_n \leq p$ ,  $\partial^\alpha \bar{V}(x) \rightarrow 0$  as  $\|x\| \rightarrow 0$ . It follows that  $\bar{V}$  is  $C^p$  at the origin of  $\mathbb{R}^n$ . This completes the proof of Proposition 2, as well as Theorem 2.

### 3. Robustness of stable homogeneous systems

The theorem we just proved will be used now to give simpler proofs of two results of control theory. By the way we shall weaken somewhat their hypotheses: the smoothness of  $f$  in [1, Proposition 3] and

the uniqueness of solutions in [6, Theorem 1] will no longer be required. The first result claims that a by means of homogeneous feedback stabilizable homogeneous system gives rise, after adding an integrator, to a stabilizable system.

**Proposition 3.** *Let  $f = (f_i)_{i=1,n} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  be a  $C^0$  map such that*

$$\forall i = 1, \dots, n, \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n, \forall t \geq 0, \forall u \in \mathbb{R}$$

$$f_i(t^{r_1}x_1, \dots, t^{r_n}x_n, t^{r_{n+1}}u) = t^{\tau+r_i}f_i(x_1, \dots, x_n, u)$$

for some  $r_i > 0$ ,  $1 \leq i \leq n+1$ , and some  $\tau \in (-\min_i\{r_i\}, \infty)$ . Assume that the system  $\dot{x} = f(x, u)$  is locally asymptotically stabilizable with a continuous feedback law  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$u(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^{r_{n+1}}u(x_1, \dots, x_n) \quad \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n, \forall t \geq 0.$$

Under these conditions, the system  $\dot{x} = f(x, y)$ ,  $\dot{y} = v$  is globally asymptotically stabilizable with a continuous feedback law  $v : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n, \forall y \in \mathbb{R}, \forall t \geq 0 \quad v(t^{r_1}x_1, \dots, t^{r_n}x_n, t^{r_{n+1}}y) = t^{\tau+r_{n+1}}v(x_1, \dots, x_n, y).$$

Proposition 3 is proved in [1] under the extra assumption that  $f$  is of class  $C^1$ .

The following lemma, which provides a smooth homogeneous stabilizing control law  $\bar{u}$  and a homogeneous Lyapunov function associated with the system  $\dot{x} = f(x, \bar{u}(x))$  is a key technical step in the proof of Proposition 3. We shall only simplify the proof of this lemma, the proof of Proposition 3 being the same as in [1]. Note that in this case no smoothness assumption on  $f$  is needed.

**Lemma 2** (Lemma 3 in [1]). *Under the assumptions of Proposition 3, there exist*

(i) *a stabilizing control law  $\bar{u} : \mathbb{R}^n \rightarrow \mathbb{R}$  in  $C^1(\mathbb{R}^n \setminus \{0\}) \cap C^0(\mathbb{R}^n)$  which satisfies*

$$\bar{u}(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^{r_{n+1}}\bar{u}(x_1, \dots, x_n) \quad \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n, \forall t \geq 0;$$

(ii) *a  $C^1$  function  $\bar{V} : \mathbb{R}^n \rightarrow \mathbb{R}$  which is positive definite (see [4, Def. 24.3]), radially unbounded (see [4, Def. 24.5]) and satisfies*

$$\bar{V}(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^k\bar{V}(x_1, \dots, x_n) \quad \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n, \forall t \geq 0$$

where  $k$  is a real number satisfying:  $k > r_i$ ,  $\forall i \in \{1, \dots, n\}$ ;  
such that

$$\nabla \bar{V}(x) \cdot f(x, \bar{u}(x)) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (8)$$

**Proof.** Using Theorem 2, we know there exists, for the system  $\dot{x} = f(x, u(x))$ ,  $x \in \mathbb{R}^n$ , a Lyapunov function  $\bar{V}$  of class  $C^1$ , which is homogeneous (with the same  $r_i$ , for  $1 \leq i \leq n$ , as for  $f$ ). The restriction of  $u$  to  $S^{n-1}$  is easily approximated by a smooth function  $\bar{u}$  satisfying  $\nabla \bar{V}(x) \cdot f(x, \bar{u}(x)) < 0$  for all  $x \in S^{n-1}$ . Then it is sufficient to extend  $\bar{u}$  to  $\mathbb{R}^n$  as a homogeneous function, i.e. to set

$$\bar{u}(x) = \begin{cases} (\psi_0(x))^{r_{n+1}}\bar{u}(\psi_1(x), \dots, \psi_n(x)) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for getting (i) and (ii). ((8) follows from (5).)

The second result, which claims that a stable homogeneous system remains stable after adding higher order perturbing terms, comes from [6], and is proved by the author of this article under the extra assumption that the solution of  $\dot{x} = f(x)$ ,  $x(0) = x_0$  is unique on  $[0, +\infty)$  for any  $x_0 \in \mathbb{R}^n$ .

**Theorem 3** (Theorem 1 in [6]). *Let  $f$ ,  $\tau$ ,  $r_1, \dots, r_n$  be as in Theorem 2 and  $g$  be a continuous vector valued function defined on  $\mathbb{R}^n$  such that, for all  $i \in \{1, \dots, n\}$ ,  $g_i(t^{r_1}x_1, \dots, t^{r_n}x_n)/t^{\tau+r_i} \rightarrow 0$  uniformly on  $S^{n-1}$  as*

$t \rightarrow 0$ . Then if the zero solution of  $\dot{x} = f(x)$  is locally asymptotically stable, the same is true for the zero solution of  $\dot{x} = f(x) + g(x)$ .

**Proof.** We denote, as Hermes,  $\delta_t^r x = (t^{r_1} x_1, \dots, t^{r_n} x_n)$  for  $t > 0$  and  $x \neq 0$ . Let  $\bar{V}$  be a Lyapunov function for the system  $\dot{x} = f(x)$ , which is of class  $C^1$  and homogeneous (see Theorem 2). Thus, for some  $k \in \mathbb{R}$  we have  $\forall t > 0, \forall x \neq 0 \bar{V}(\delta_t^r x) = t^k \bar{V}(x)$ . So, for  $x \in S^{n-1}$  and  $t \in (0, 1)$  we get (see (5))

$$f(\delta_t^r x) \cdot \nabla \bar{V}(\delta_t^r x) = t^{k+\tau} f(x) \cdot \nabla \bar{V}(x).$$

Let  $\lambda$  be the negative number  $\max_{x \in S^{n-1}} f(x) \cdot \nabla \bar{V}(x)$ . Since  $g(\delta_t^r x) \cdot \nabla \bar{V}(\delta_t^r x) = o(t^{k+\tau})$  uniformly on  $S^{n-1}$  as  $t \rightarrow 0$ , there exists  $t_0$  in  $(0, 1)$  such that, for  $0 < t < t_0$  and  $x \in S^{n-1}$ , we have

$$|g(\delta_t^r x) \cdot \nabla \bar{V}(\delta_t^r x)| \leq \frac{1}{2} t^{k+\tau} |\lambda|.$$

Therefore, for  $0 < t < t_0$  and  $x \in S^{n-1}$ , we have  $(f(\delta_t^r x) + g(\delta_t^r x)) \cdot \nabla \bar{V}(\delta_t^r x) \leq \frac{1}{2} t^{k+\tau} \lambda < 0$ . Now the set  $\{\delta_t^r x, 0 < t < t_0, x \in S^{n-1}\} \cup \{0\}$  is a neighborhood of the origin in  $\mathbb{R}^n$ , since by Lemma 1, 2) there exists  $r_0 > 0$  such that  $0 \leq \|x\| < r_0$  implies  $0 < \psi_0(x) < t_0$ . Thus the function  $\bar{V}$  is, in a neighborhood of 0, a Lyapunov function for  $f+g$ . It follows that the trivial solution of the system  $\dot{x} = (f+g)(x)$  is locally asymptotically stable (see [5, Th. 8.2, Ch. III]).

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## References

- [1] J.M. Coron and L. Praly, Adding an integrator for the stabilization problem, *Systems Control Lett.* **17** (1991) 89–104.
- [2] W.P. Dayawansa and C.F. Martin, Asymptotic stabilization of two dimensional real analytic systems, *Systems Control Lett.* **12** (1989) 205–212.
- [3] W.P. Dayawansa, C.F. Martin and G. Knowles, Asymptotic stabilization of a class of smooth two-dimensional systems, *SIAM J. Control Optim.* **28** (1990) 1321–1349.
- [4] W. Hahn, *Stability of Motion* (Springer-Verlag, Berlin, 1967).
- [5] P. Hartman, *Ordinary Differential Equations*, Second Edition (Birkhauser, Boston, MA, 1982).
- [6] H. Hermes, Homogeneous coordinates and continuous asymptotically stabilizing feedback controls, in: S. Elaydi, Ed., *Differential Equations Stability and Controls*, Lecture Notes in Applied Math. Vol. 109 (Marcel Dekker, New York, 1991) 249–260.
- [7] M. Kawski, Stabilization and nilpotent approximations, *Proc. 27th IEEE Conference on Decision & Control II* (1988), 1244–1248.
- [8] M. Kawski, Stabilization of nonlinear systems in the plane, *Systems Control Lett.* **12** (1989) 169–175.
- [9] J. Kurzweil, On the inversion of Lyapunov's second theorem on stability of motion, *Ann. Math. Soc. Transl. Ser. 2* **24** (1956) 19–77.
- [10] V.I. Zubov, *Methods of A.M. Lyapunov and their Applications* (Noordhoff, Leiden, 1964).