

UNIFORM STABILIZATION IN WEIGHTED SOBOLEV SPACES FOR THE KDV EQUATION POSED ON THE HALF-LINE

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ABSTRACT. Studied here is the large-time behavior of solutions of the Korteweg-de Vries equation posed on the right half-line under the effect of a localized damping. Assuming as in [19] that the damping is active on a set $(a_0, +\infty)$ with $a_0 > 0$, we establish the exponential decay of the solutions in the weighted spaces $L^2((x+1)^m dx)$ for $m \in \mathbb{N}^*$ and $L^2(e^{2bx} dx)$ for $b > 0$ by a Lyapunov approach. The decay of the spatial derivatives of the solution is also derived.

1. Introduction. The Korteweg-de Vries (KdV) equation was first derived as a model for the propagation of small amplitude long water waves along a channel [8, 14, 15]. It has been intensively studied from various aspects for both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [12, 21]. It is now well known that the KdV equation is not only a good model for water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance weak nonlinear and dispersive effects.

The initial boundary value problems (IBVP) arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end [1, 2, 3, 28]. Such mathematical formulations have received considerable attention in the past, and a satisfactory theory of global well-posedness is available for initial and boundary conditions satisfying physically relevant smoothness and consistency assumptions (see e.g. [1, 4, 6, 7, 9, 10, 11] and the references therein).

The analysis of the long-time behavior of IBVP on the quarter-plane for KdV has also received considerable attention over recent years, and a review of some of the results related to the issues we address here can be found in [5, 7, 18]. For stabilization and controllability issues on the half line, we refer the reader to [19] and [26, 27, 28], respectively.

In this work, we are concerned with the asymptotic behavior of the solutions of the IBVP for the KdV equation posed on the positive half line under the presence

2000 *Mathematics Subject Classification.* Primary: 93D15, 35Q53; Secondary: 93C20.
Key words and phrases. Exponential Decay, Korteweg-de Vries equation, Stabilization.

of a localized damping represented by the function a ; that is,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, & x, t \in \mathbb{R}^+, \\ u(0, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x > 0. \end{cases} \tag{1}$$

Assuming $a(x) \geq 0$ a.e. and that $u(\cdot, t) \in H^3(\mathbb{R}^+)$, it follows from a simple computation that

$$\frac{dE}{dt} = - \int_0^\infty a(x)|u(x, t)|^2 dx - \frac{1}{2}|u_x(0, t)|^2 \tag{2}$$

where

$$E(t) = \frac{1}{2} \int_0^\infty |u(x, t)|^2 dx \tag{3}$$

is the total energy associated with (1). Then, we see that the term $a(x)u$ plays the role of a feedback damping mechanism and, consequently, it is natural to wonder whether the solutions of (1) tend to zero as $t \rightarrow \infty$ and under what rate they decay. When $a(x) > a_0 > 0$ almost everywhere in \mathbb{R}^+ , it is very simple to prove that $E(t)$ converges to zero as t tends to infinity. The problem of stabilization when the damping is effective only in a subset of the domain is much more subtle. The following result was obtained in [19].

Theorem 1.1. *Assume that the function $a = a(x)$ satisfies the following property*

$$a \in L^\infty(\mathbb{R}^+), \ a \geq 0 \text{ a.e. in } \mathbb{R}^+ \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } (x_0, +\infty) \tag{4}$$

for some numbers $a_0, x_0 > 0$. Then for all $R > 0$ there exist two numbers $C > 0$ and $\nu > 0$ such that for all $u_0 \in L^2(\mathbb{R}^+)$ with $\|u_0\|_{L^2(\mathbb{R}^+)} \leq R$, the solution u of (1) satisfies

$$\|u(t)\|_{L^2(\mathbb{R}^+)} \leq C e^{-\nu t} \|u_0\|_{L^2(\mathbb{R}^+)}. \tag{5}$$

Actually, Theorem 1.1 was proved in [19] under the additional hypothesis that

$$a(x) \geq a_0 \text{ a.e. in } (0, \delta) \tag{6}$$

for some $\delta > 0$, but (6) may be dropped by replacing the unique continuation property [19, Lemma 2.4] by [29, Theorem 1.6]. The exponential decay of $E(t)$ is obtained following the methods in [22, 24, 25] which combine multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions of KdV. For the stabilization of the Korteweg-de Vries equation on the torus (i.e. with periodic boundary conditions), we refer the reader to [17, 30] and the references therein.

Along this work we assume that the real-valued function $a = a(x)$ satisfies the condition (4) for some given positive numbers a_0, x_0 . In this paper we investigate the stability properties of (1) in the weighted spaces introduced by Kato in [13]. More precisely, for $b > 0$ and $m \in \mathbb{N}$, we prove that the solution u exponentially decays to 0 in L_b^2 and $L_{(x+1)^m dx}^2$ (if $u(0)$ belongs to one of these spaces), where

$$L_b^2 = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \int_0^\infty |u(x)|^2 e^{2bx} dx < \infty\},$$

$$L_{(x+1)^m dx}^2 = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \int_0^\infty |u(x)|^2 (x+1)^m dx < \infty\}.$$

The following weighted Sobolev spaces

$$H_b^s = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \partial_x^i u \in L_b^2 \text{ for } 0 \leq i \leq s; u(0) = 0 \text{ if } s \geq 1\}$$

and

$$H_{(x+1)^m dx}^s = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \partial_x^i u \in L_{(x+1)^m dx}^2 \text{ for } 0 \leq i \leq s; u(0) = 0 \text{ if } s \geq 1\},$$

endowed with their usual inner products, will be used thereafter. Note that $H_b^0 = L_b^2$ and that $H_{(x+1)^m dx}^0 = L_{(x+1)^m dx}^2$.

The exponential decay in $L_{(x+1)^m dx}^2$ is obtained by constructing a convenient Lyapunov function (which actually decreases strictly on the sequence of times $\{kT\}_{k \geq 0}$) by induction on m . For $u_0 \in L_{(x+1)^m dx}^2$, we also prove the following estimate

$$\|u(t)\|_{H_{(x+1)^m dx}^1} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L_{(x+1)^m dx}^2} \tag{7}$$

in two situations: (i) $m = 1$ and $\|u_0\|_{L_{(x+1)^m dx}^2}$ is arbitrarily large; (ii) $m \geq 2$ and $\|u_0\|_{L_{(x+1)^m dx}^2}$ is small enough. In the situation (ii), we first establish a similar estimate for the linearized system and next apply the contraction mapping principle in a space of functions fulfilling the exponential decay. Note that (7) combines the (global) Kato smoothing effect to the exponential decay.

The exponential decay in L_b^2 is established for any initial data $u_0 \in L_b^2$ under the additional assumption that $4b^3 + b < a_0$. Next, we can derive estimates of the form

$$\|u(t)\|_{H_b^s} \leq C \frac{e^{-\mu t}}{t^{s/2}} \|u_0\|_{L_b^2}$$

for any $s \geq 1$, revealing that $u(t)$ decays exponentially to 0 in strong norms.

It would be interesting to see if such results are still true when the function a has a smaller support. It seems reasonable to conjecture that similar positive results can be derived when the support of a contains a set of the form $\cup_{k \geq 1} [ka_0, ka_0 + b_0]$ where $0 < b_0 < a_0$, while a negative result probably holds when the support of a is a finite interval, as the L^2 norm of a soliton-like initial data may not be sufficiently dissipated over time. Such issues will be discussed elsewhere.

The plan of this paper is as follows. Section 2 is devoted to global well-posedness results in the weighted spaces L_b^2 and $L_{(x+1)^2 dx}^2$. In section 3, we prove the exponential decay in $L_{(x+1)^m dx}^2$ and L_b^2 , and establish the exponential decay of the derivatives as well.

2. Global well-posedness.

2.1. Global well-posedness in L_b^2 . Fix any $b > 0$. To begin with, we apply the classical semigroup theory to the linearized system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = 0, & x, t \in \mathbb{R}^+, \\ u(0, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x > 0. \end{cases} \tag{8}$$

Let us consider the operator

$$A : D(A) \subset L_b^2 \rightarrow L_b^2$$

with domain

$$D(A) = \{u \in L_b^2; \partial_x^i u \in L_b^2 \text{ for } 1 \leq i \leq 3 \text{ and } u(0) = 0\}$$

defined by

$$Au = -u_{xxx} - u_x - a(x)u.$$

Then, the following result holds.

Lemma 2.1. *The operator A defined above generates a continuous semigroup of operators $(S(t))_{t \geq 0}$ in L^2_b .*

Proof. We first introduce the new variable $v = e^{bx}u$ and consider the following (IBVP)

$$\begin{cases} v_t + (\partial_x - b)v + (\partial_x - b)^3v + a(x)v = 0, & x, t \in \mathbb{R}^+, \\ v(0, t) = 0, & t > 0, \\ v(x, 0) = v_0(x) = e^{bx}u_0(x), & x > 0. \end{cases} \tag{9}$$

Clearly, the operator $B : D(B) \subset L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ with domain

$$D(B) = \{u \in H^3(\mathbb{R}^+); u(0) = 0\}$$

defined by

$$Bv = -(\partial_x - b)v - (\partial_x - b)^3v - a(x)v$$

is densely defined and closed. So, we are done if we prove that for some real number λ the operator $B - \lambda$ and its adjoint $B^* - \lambda$ are both dissipative in $L^2(\mathbb{R}^+)$. It is readily seen that $B^* : D(B^*) \subset L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ is given by $B^*v = (\partial_x + b)v + (\partial_x + b)^3v - a(x)v$ with domain

$$D(B^*) = \{v \in H^3(\mathbb{R}^+); v(0) = v'(0) = 0\}.$$

Pick any $v \in D(B)$. After some integration by parts, we obtain that

$$(Bv, v)_{L^2} = -\frac{1}{2}v_x^2(0) - 3b \int_0^\infty v_x^2 dx + (b + b^3) \int_0^\infty v^2 dx - \int_0^\infty a(x)v^2 dx,$$

that is,

$$([B - (b^3 + b)]v, v)_{L^2} \leq 0.$$

Analogously, we deduce that for any $v \in D(B^*)$

$$(v, [B^* - (b^3 + b)]v)_{L^2} \leq 0$$

which completes the proof. □

The following linear estimates will be needed.

Lemma 2.2. *Let $u_0 \in L^2_b$ and $u = S(\cdot)u_0$. Then, for any $T > 0$*

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 dx \\ & - \frac{1}{2} \int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 e^{2bx} dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx \\ & + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \\ & + \int_0^T \int_0^\infty a(x)|u|^2 e^{2bx} dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0. \end{aligned} \tag{11}$$

As a consequence,

$$\|u\|_{L^\infty(0, T; L^2_b)} + \|u_x\|_{L^2(0, T; L^2_b)} \leq C \|u_0\|_{L^2_b}, \tag{12}$$

where $C = C(T)$ is a positive constant.

Proof. Pick any $u_0 \in D(A)$. Multiplying the equation in (1) by u and integrating over $(0, +\infty) \times (0, T)$, we obtain (10). Then, the identity may be extended to any initial state $u_0 \in L_b^2$ by a density argument. To derive (11) we first multiply the equation by $(e^{2bx} - 1)u$ and integrate by parts over $(0, +\infty) \times (0, T)$ to deduce that

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 (e^{2bx} - 1) dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 (e^{2bx} - 1) dx + \\ & + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt + \\ & + \int_0^T \int_0^\infty a(x) |u|^2 (e^{2bx} - 1) dx dt = 0. \end{aligned}$$

Adding the above equality and (10) hand to hand, we obtain (11) using the same density argument. Then, Gronwall inequality, (4) and (11) imply that

$$\|u\|_{L^\infty(0, T; L_b^2)} \leq C \|u_0\|_{L_b^2},$$

with $C = C(T) > 0$. This estimate together with (11) gives us

$$\|u_x\|_{L^2(0, T; L_b^2)} \leq C \|u_0\|_{L_b^2},$$

where $C = C(T)$ is a positive constant. □

The global well-posedness result reads as follows:

Theorem 2.3. *For any $u_0 \in L_b^2$ and any $T > 0$, there exists a unique solution $u \in C([0, T]; L_b^2) \cap L^2(0, T; H_b^1)$ of (1).*

Proof. By computations similar to those performed in the proof of Lemma 2.2, we obtain that for any $f \in C^1([0, T]; L_b^2)$ and any $u_0 \in D(A)$, the solution u of the system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = f, & x \in \mathbb{R}^+, t \in (0, T), \\ u(0, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^+, \end{cases}$$

fulfills

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L_b^2} + \left(\int_0^T \int_0^\infty |u_x|^2 e^{2bx} dx dt \right)^{\frac{1}{2}} \leq C \left(\|u_0\|_{L_b^2} + \int_0^T \|f\|_{L_b^2} dt \right) \quad (13)$$

for some constant $C = C(T)$ nondecreasing in T . A density argument yields that $u \in C([0, T]; L_b^2)$ when $f \in L^1(0, T; L_b^2)$ and $u_0 \in L_b^2$.

Let $u_0 \in L_b^2$ be given. To prove the existence of a solution of (1) we introduce the map Γ defined by

$$(\Gamma u)(t) = S(t)u_0 + \int_0^t S(t-s)N(u(s)) ds$$

where $N(u) = -uu_x$, and the space

$$F = C([0, T]; L_b^2) \cap L^2(0, T; H_b^1)$$

endowed with its natural norm. We shall prove that Γ has a fixed-point in some ball $B_R(0)$ of F . We need the following

CLAIM 1. If $u \in H_b^1$ then

$$\|u^2 e^{2bx}\|_{L^\infty(\mathbb{R}^+)} \leq (2 + 2b) \|u\|_{L_b^2} \|u\|_{H_b^1}.$$

From Cauchy-Schwarz inequality, we get for any $\bar{x} \in \mathbb{R}^+$

$$\begin{aligned} & u^2(\bar{x})e^{2b\bar{x}} \\ &= \int_0^{\bar{x}} [u^2 e^{2bx}]_x dx = \int_0^{\bar{x}} [2uu_x e^{2bx} + 2bu^2 e^{2bx}] dx \\ &\leq 2\left(\int_0^\infty u^2 e^{2bx} dx\right)^{\frac{1}{2}} \left(\int_0^\infty u_x^2 e^{2bx} dx\right)^{\frac{1}{2}} + 2b \int_0^\infty u^2 e^{2bx} dx \\ &\leq (2 + 2b) \|u\|_{L_b^2} \|u\|_{H_b^1} \end{aligned}$$

which guarantees that Claim 1 holds.

CLAIM 2. There exists a constant $K > 0$ such that for $0 < T \leq 1$

$$\|\Gamma(u) - \Gamma(v)\|_F \leq KT^{\frac{1}{4}} (\|u\|_F + \|v\|_F) \|u - v\|_F, \quad \forall u, v \in F.$$

According to the previous analysis,

$$\|\Gamma(u) - \Gamma(v)\|_F \leq C \|uu_x - vv_x\|_{L^1(0,T;L_b^2)}.$$

So, applying triangular inequality and Hölder inequality, we have

$$\begin{aligned} \|\Gamma(u) - \Gamma(v)\|_F &\leq C \{ \|u - v\|_{L^2(0,T;L^\infty(0,\infty))} \|u\|_{L^2(0,T;H_b^1)} + \\ &\quad + \|v\|_{L^2(0,T;L^\infty(0,\infty))} \|u - v\|_{L^2(0,T;H_b^1)} \}. \end{aligned} \quad (14)$$

Now, by Claim 1, we have

$$\|u\|_{L^2(0,T;L^\infty(0,\infty))} \leq CT^{\frac{1}{4}} \|u\|_{L^\infty(0,T;L_b^2)}^{\frac{1}{2}} \|u\|_{L^2(0,T;H_b^1)}^{\frac{1}{2}}. \quad (15)$$

Then, combining (14) and (15), we deduce that

$$\|\Gamma(u) - \Gamma(v)\|_F \leq CT^{\frac{1}{4}} \{ \|u\|_F + \|v\|_F \} \|u - v\|_F. \quad (16)$$

Let $T > 0$, $R > 0$ be numbers whose values will be specified later, and let $u \in B_R(0) \subset F$ be given. Then, by Claim 2 and Lemma 2.2, $\Gamma u \in F$ and

$$\|\Gamma u\|_F \leq C (\|u_0\|_{L_b^2} + T^{\frac{1}{4}} \|u\|_F^2).$$

Consequently, for $R = 2C\|u_0\|_{L_b^2}$ and $T > 0$ small enough, Γ maps $B_R(0)$ into itself. Moreover, we infer from (16) that this mapping contracts if T is small enough. Then, by the contraction mapping theorem, there exists a unique solution $u \in B_R(0) \subset F$ to the problem (1) for T small enough.

In order to prove that this solution is global, we need some a priori estimates. So, we proceed as in the proof of Lemma 2.2 to obtain for the solution u of (1)

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x,T)|^2 dx \\ & - \frac{1}{2} \int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0,t) dt = 0 \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 e^{2bx} dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^T u_x^2(0, t) dt \\ & + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \\ & + \int_0^T \int_0^\infty a(x) |u|^2 e^{2bx} dx dt - \frac{2b}{3} \int_0^T \int_0^\infty u^3 e^{2bx} dx dt = 0. \end{aligned} \tag{18}$$

First, observe that

$$\left| \int_0^\infty u^2 e^{2bx} dx \right| = \left| -\frac{1}{b} \int_0^\infty uu_x e^{2bx} dx \right| \leq \frac{1}{b} \left(\int_0^\infty u^2 e^{2bx} dx \right)^{\frac{1}{2}} \left(\int_0^\infty u_x^2 e^{2bx} dx \right)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{b^2} \int_0^\infty u_x^2 e^{2bx} dx.$$

Combined to Claim 1, this yields

$$\|u(x)e^{bx}\|_{L^\infty(\mathbb{R}^+)} \leq C \|u_x\|_{L_b^2}.$$

On the other hand, it follows from (17) that

$$\|u(t)\|_{L^2(\mathbb{R}^+)} \leq \|u_0\|_{L^2(\mathbb{R}^+)},$$

hence

$$\begin{aligned} \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt & \leq \int_0^T \|ue^{bx}\|_{L^\infty(\mathbb{R}^+)} \left(\int_0^\infty |u|^2 e^{2bx} dx \right) dt \\ & \leq C \int_0^T \|u_x\|_{L_b^2} \|u\|_{L_b^2} \|u\|_{L^2} dt \\ & \leq \delta \|u_x\|_{L^2(0, T; L_b^2)}^2 + C_\delta \|u\|_{L^2(0, T; L_b^2)}^2, \end{aligned}$$

where $\delta > 0$ is arbitrarily chosen and $C = C(b, \delta, \|u_0\|_{L^2(\mathbb{R}^+)})$ is a positive constant. Combining this inequality (with $\delta < 9/2$) to (18) results in

$$\|u(T)\|_{L_b^2}^2 \leq \|u_0\|_{L_b^2}^2 + C \int_0^T \|u\|_{L_b^2}^2 dt$$

where $C = C(b, \|u_0\|_{L^2(\mathbb{R}^+)})$ does not depend on T . It follows from Gronwall lemma that

$$\|u(T)\|_{L_b^2}^2 \leq \|u_0\|_{L_b^2}^2 e^{CT}$$

for all $T > 0$, which gives the global well-posedness. □

2.2. Global well-posedness in $L^2_{(x+1)^2 dx}$.

Definition 2.4. For $u_0 \in L^2_{(x+1)^2 dx}$ and $T > 0$, we denote by a *mild solution* of (1) any function $u \in C([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$ which solves (1), and such that for some $b > 0$ and some sequence $\{u_{n,0}\} \subset L_b^2$ we have

$$\begin{aligned} u_{n,0} & \rightarrow u_0 \text{ strongly in } L^2_{(x+1)^2 dx}, \\ u_n & \rightarrow u \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}), \\ u_n & \rightarrow u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}), \end{aligned}$$

u_n denoting the solution of (1) emanating from $u_{n,0}$ at $t = 0$.

Theorem 2.5. *For any $u_0 \in L^2_{(x+1)^2 dx}$ and any $T > 0$, there exists a unique mild solution $u \in C([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$ of (1).*

Proof. We prove the existence and the uniqueness in two steps.

STEP 1. EXISTENCE

Since the embedding $L^2_b \subset L^2_{(x+1)^2 dx}$ is dense, for any given $u_0 \in L^2_{(x+1)^2 dx}$ we may construct a sequence $\{u_{n,0}\} \subset L^2_b$ such that $u_{n,0} \rightarrow u_0$ in $L^2_{(x+1)^2 dx}$ as $n \rightarrow \infty$. For each n , let u_n denote the solution of (1) emanating from $u_{n,0}$ at $t = 0$, which is given by Theorem 2.3. Then $u_n \in C([0, T]; L^2_b) \cap L^2(0, T; H^1_b)$ and it solves

$$u_{n,t} + u_{n,x} + u_{n,xxx} + u_n u_{n,x} + a(x)u_n = 0, \tag{19}$$

$$u_n(0, t) = 0 \tag{20}$$

$$u_n(x, 0) = u_{n,0}(x). \tag{21}$$

Multiplying (19) by $(x + 1)^2 u_n$ and integrating by parts, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^\infty (x + 1)^2 |u_{n,0}(x)|^2 dx \\ &= \frac{1}{2} \int_0^\infty (x + 1)^2 |u_n(x, T)|^2 dx + 3 \int_0^T \int_0^\infty (x + 1) |u_{n,x}|^2 dx dt \\ &+ \frac{1}{2} \int_0^T \int_0^\infty |u_{n,x}(0, t)|^2 dt - \int_0^T \int_0^\infty (x + 1) |u_n|^2 dx dt \\ &- \frac{2}{3} \int_0^T \int_0^\infty (x + 1) u_n^3 dx dt + \int_0^T \int_0^\infty (x + 1)^2 u_n^2 a(x) dx. \end{aligned} \tag{22}$$

Scaling in (19) by u_n gives

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u_{n,0}(x)|^2 dx \\ &= \frac{1}{2} \int_0^\infty |u_n(x, T)|^2 dx + \frac{1}{2} \int_0^T \int_0^\infty |u_{n,x}(0, t)|^2 dt + \int_0^T \int_0^\infty a(x) |u_n(x, t)|^2 dx dt, \end{aligned}$$

hence

$$\|u_n\|_{L^2(\mathbb{R}^+)} \leq \|u_{n,0}\|_{L^2(\mathbb{R}^+)} \leq C \tag{23}$$

where $C = C(\|u_0\|_{L^2(\mathbb{R}^+)})$. It follows that

$$\begin{aligned} & \frac{2}{3} \int_0^\infty (x + 1) |u_n|^3 dx \\ & \leq \frac{2\sqrt{2}}{3} \|u_{n,x}\|_{L^{\frac{1}{2}}(\mathbb{R}^+)} \|u_n\|_{L^{\frac{3}{2}}(\mathbb{R}^+)} \|(x + 1)u_n\|_{L^2(\mathbb{R}^+)} \\ & \leq \int_0^\infty (x + 1) |u_{n,x}|^2 dx + C \int_0^\infty (x + 1)^2 |u_n|^2 dx \end{aligned} \tag{24}$$

which, combined to (22), gives

$$\begin{aligned} & \frac{1}{2} \int_0^\infty (x + 1)^2 |u_n(x, T)|^2 dx + 2 \int_0^T \int_0^\infty (x + 1) |u_{n,x}|^2 dx dt + \frac{1}{2} \int_0^T \int_0^\infty |u_{n,x}(0, t)|^2 dt \\ & \leq \frac{1}{2} \int_0^\infty (x + 1)^2 |u_{n,0}(x)|^2 dx + C \int_0^T \int_0^\infty (x + 1)^2 |u_n(x, t)|^2 dx dt. \end{aligned}$$

An application of Gronwall’s lemma yields

$$\begin{aligned} \|u_n\|_{L^\infty(0,T;L^2_{(x+1)^2 dx})} &\leq C(T, \|u_{n,0}\|_{L^2_{(x+1)^2 dx}}), \\ \|u_{n,x}\|_{L^2(0,T;H^1_{(x+1)^2 dx})} &\leq C(T, \|u_{n,0}\|_{L^2_{(x+1)^2 dx}}), \\ \|u_{n,x}(0, \cdot)\|_{L^2(0,T)} &\leq C(T, \|u_{n,0}\|_{L^2_{(x+1)^2 dx}}). \end{aligned}$$

Therefore, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}), \\ u_n \rightharpoonup u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}), \\ u_{n,x}(0, \cdot) \rightharpoonup u_x(0, \cdot) \text{ weakly in } L^2(0, T). \end{cases}$$

Note that, for all $L > 0$, $\{u_n\}$ is bounded in $L^2(0, T; H^1(0, L)) \cap H^1(0, T; H^{-2}(0, L))$, hence by Aubin’s lemma, we have (after extracting a subsequence if needed)

$$u_n \rightarrow u \text{ strongly in } L^2(0, T; L^2(0, L)) \text{ for all } L > 0.$$

This gives that $u_n u_{n,x} \rightarrow uu_x$ in the sense of distributions, hence the limit $u \in L^\infty(0, T; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$ is a solution of (1). Let us check that $u \in C([0, T]; L^2_{(x+1)^2 dx})$. Since $u \in C([0, T]; H^{-2}(\mathbb{R}^+)) \cap L^\infty(0, T; L^2_{(x+1)^2 dx})$, we have that $u \in C_w([0, T]; L^2_{(x+1)^2 dx})$ (see e.g. [20]), where $C_w([0, T]; L^2_{(x+1)^2 dx})$ denotes the space of sequentially weakly continuous functions from $[0, T]$ into $L^2_{(x+1)^2 dx}$.

We claim that $u \in L^3(0, T; L^3(\mathbb{R}^+))$. Indeed, from Moser estimate (see [31])

$$\|u\|_{L^\infty(\mathbb{R}^+)} \leq \sqrt{2} \|u_x\|_{L^2(\mathbb{R}^+)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^+)}^{\frac{1}{2}} \tag{25}$$

and Young inequality we get

$$\int_0^\infty |u|^3 dx \leq \|u\|_{L^\infty} \|u\|_{L^2}^2 \leq \sqrt{2} \|u_x\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{5}{2}} \leq \varepsilon \|u_x\|_{L^2}^2 + c_\varepsilon \|u\|_{L^2}^{\frac{10}{3}} \tag{26}$$

where $\varepsilon > 0$ is arbitrarily chosen and c_ε denotes some positive constant. Since $u \in C_w([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$, it follows that $u \in L^3(0, T; L^3(\mathbb{R}^+))$. On the other hand, $u(0, t) = 0$ for $t \in (0, T)$ and $u_x(0, \cdot) \in L^2(0, T)$. Scaling in (1) by $(x + 1)^2 u$ yields for all $t_1, t_2 \in (0, T)$

$$\begin{aligned} &\frac{1}{2} \int_0^\infty (x + 1)^2 |u(x, t_2)|^2 dx - \frac{1}{2} \int_0^\infty (x + 1)^2 |u(x, t_1)|^2 dx \\ &= -3 \int_{t_1}^{t_2} \int_0^\infty (x + 1) |u_x|^2 dx dt - \frac{1}{2} \int_{t_1}^{t_2} |u_x(0, t)|^2 dt \\ &+ \int_{t_1}^{t_2} \int_0^\infty (x + 1) |u|^2 dx dt + \frac{2}{3} \int_{t_1}^{t_2} \int_0^\infty (x + 1) u^3 dx dt \\ &- \int_{t_1}^{t_2} \int_0^\infty (x + 1)^2 a(x) |u|^2 dx dt. \end{aligned} \tag{27}$$

Therefore $\lim_{t_1 \rightarrow t_2} \left| \|u(t_2)\|_{L^2_{(x+1)^2 dx}}^2 - \|u(t_1)\|_{L^2_{(x+1)^2 dx}}^2 \right| = 0$. Combined to the fact that $u \in C_w([0, T]; L^2_{(x+1)^2 dx})$, this yields $u \in C([0, T], L^2_{(x+1)^2 dx})$.

STEP 2. UNIQUENESS

Here, C will denote a universal constant which may vary from line to line. Pick

$u_0 \in L^2_{(x+1)^2 dx}$, and let $u, v \in C([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$ be two mild solutions of (1). Pick two sequences $\{u_{n,0}\}, \{v_{n,0}\}$ in L^2_b for some $b > 0$ such that

$$u_{n,0} \rightarrow u_0 \text{ strongly in } L^2_{(x+1)^2 dx}, \tag{28}$$

$$u_n \rightarrow u \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}), \tag{29}$$

$$u_n \rightarrow u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}) \tag{30}$$

and also

$$v_{n,0} \rightarrow u_0 \text{ strongly in } L^2_{(x+1)^2 dx}, \tag{31}$$

$$v_n \rightarrow v \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}), \tag{32}$$

$$v_n \rightarrow v \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}). \tag{33}$$

We shall prove that $w = u - v$ vanishes on $\mathbb{R}^+ \times [0, T]$ by providing some estimate for $w_n = u_n - v_n$. Note first that w_n solves the system

$$w_{n,t} + w_{n,x} + w_{n,xxx} + aw_n = f_n = v_n v_{n,x} - u_n u_{n,x}, \tag{34}$$

$$w_n(0, t) = 0, \tag{35}$$

$$w_n(x, 0) = w_{n,0}(x) = u_{n,0}(x) - v_{n,0}(x). \tag{36}$$

Scaling in (34) by $(x + 1)w_n$ yields

$$\begin{aligned} & \frac{1}{2} \int_0^\infty (x + 1)|w_n(x, t)|^2 dx + \frac{3}{2} \int_0^t \int_0^\infty |w_{n,x}|^2 dx d\tau - \frac{1}{2} \int_0^t \int_0^\infty |w_n|^2 dx d\tau \\ & \leq \frac{1}{2} \int_0^\infty (x + 1)|w_{n,0}|^2 dx + \int_0^t \left(\int_0^\infty (x + 1)|w_n|^2 dx \right)^{\frac{1}{2}} \left(\int_0^\infty (x + 1)|f_n|^2 dx \right)^{\frac{1}{2}} d\tau \\ & \leq \frac{1}{2} \int_0^\infty (x + 1)|w_{n,0}|^2 dx + \frac{1}{4} \sup_{0 < \tau < t} \int_0^\infty (x + 1)|w_n(x, \tau)|^2 dx \\ & \quad + \left[\int_0^T \left(\int_0^\infty (x + 1)|f_n|^2 dx \right)^{\frac{1}{2}} d\tau \right]^2. \end{aligned}$$

Since $\|w_n(t)\|_{L^2(\mathbb{R}^+)} \leq \|w_n(t)\|_{L^2_{(x+1)dx}}$, this yields for $T < 1/10$

$$\begin{aligned} & \sup_{0 < t < T} \int_0^\infty (x + 1)|w_n(x, t)|^2 dx + \int_0^T \int_0^\infty |w_{n,x}|^2 dx dt \\ & \leq C \left[\int_0^\infty (x + 1)|w_{n,0}(x)|^2 dx + \left(\int_0^T \left(\int_0^\infty (x + 1)|f_n|^2 dx \right)^{\frac{1}{2}} d\tau \right)^2 \right]. \end{aligned} \tag{37}$$

It remains to estimate $\int_0^T \left(\int_0^\infty (x + 1)|f_n|^2 dx \right)^{\frac{1}{2}} dt$. We split f_n into

$$f_n = (v_n - u_n)v_{n,x} + u_n(v_{n,x} - u_{n,x}) = f_n^1 + f_n^2.$$

We have that

$$\begin{aligned} \int_0^T \left(\int_0^\infty (x + 1)|f_n^1|^2 dx \right)^{\frac{1}{2}} dt & = \int_0^T \left(\int_0^\infty (x + 1)|w_n|^2 |v_{n,x}|^2 dx \right)^{\frac{1}{2}} dt \\ & \leq \int_0^T \|w_n\|_{L^\infty(\mathbb{R}^+)} \left(\int_0^\infty (x + 1)|v_{n,x}|^2 dx \right)^{\frac{1}{2}} dt \\ & \leq \left(\int_0^T \|w_n\|_{L^\infty(\mathbb{R}^+)}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \int_0^\infty (x + 1)|v_{n,x}|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

By Sobolev embedding, we have that

$$\begin{aligned} \left(\int_0^T \|w_n\|_{L^\infty(\mathbb{R}^+)}^2 dt\right)^{\frac{1}{2}} &\leq \left(\int_0^T \|w_n\|_{H^1(\mathbb{R}^+)}^2 dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{T} \sup_{0 < t < T} \|w_n\|_{L^2(\mathbb{R}^+)} + \|w_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))}. \end{aligned}$$

Thus

$$\begin{aligned} &\int_0^T \left(\int_0^\infty (x+1)|f_n^1|^2 dx\right)^{\frac{1}{2}} dt \\ &\leq \|v_{n,x}\|_{L^2(0,T;L^2_{(x+1)dx})} \left(\sqrt{T} \sup_{0 < t < T} \|w_n\|_{L^2(\mathbb{R}^+)} + \|w_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))}\right). \end{aligned} \tag{38}$$

On the other hand, we have that

$$\begin{aligned} &\int_0^T \left(\int_0^\infty (x+1)|f_n^2|^2 dx\right)^{\frac{1}{2}} dt = \int_0^T \left(\int_0^\infty (x+1)|u_n|^2 |w_{n,x}|^2 dx\right)^{\frac{1}{2}} dt \\ &\leq \int_0^T \|(x+1)^{\frac{1}{2}} u_n\|_{L^\infty(\mathbb{R}^+)} \|w_{n,x}\|_{L^2(\mathbb{R}^+)} dt \\ &\leq C \int_0^T \left(\|(x+1)^{\frac{1}{2}} u_n\|_{L^2(\mathbb{R}^+)} + \|(x+1)^{\frac{1}{2}} u_{n,x}\|_{L^2(\mathbb{R}^+)}\right) \|w_{n,x}\|_{L^2(\mathbb{R}^+)} dt \\ &\leq C \left(\sqrt{T} \|(x+1)u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^+))} + \|(x+1)^{\frac{1}{2}} u_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))}\right) \|w_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))}. \end{aligned} \tag{39}$$

Gathering together (37), (38) and (39), we conclude that for $T < 1/10$

$$h_n(T) \leq K_n(T)h_n(T) + C\|w_{n,0}\|_{L^2_{(x+1)dx}}^2$$

where

$$h_n(t) := \sup_{0 < \tau < T} \int_0^\infty (x+1)|w_n(x, \tau)|^2 dx + \int_0^T \int_0^\infty |w_{n,x}|^2 dx dt \tag{40}$$

$$\begin{aligned} K_n(T) &\leq C \left(\int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt + T\|(x+1)u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^+))}^2 \right. \\ &\quad \left. + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \right) \end{aligned} \tag{41}$$

and C denotes a universal constant. The following claim is needed.

CLAIM 3.

$$\lim_{T \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt = 0, \quad \lim_{T \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt = 0.$$

Clearly, it is sufficient to prove the claim for the sequence $\{u_n\}$ only. From (27) applied with $u = u_n$ on $[0, T]$, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^\infty (x+1)^2 |u_n(x, T)|^2 dx + 3 \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \\ &\leq \frac{1}{2} \int_0^\infty (x+1)^2 |u_{n,0}|^2 dx + \int_0^T \int_0^\infty (x+1)|u_n|^2 dx dt \\ &\quad + \frac{2}{3} \int_0^T \int_0^\infty (x+1)|u_n|^3 dx dt. \end{aligned}$$

Combined to (23)-(24), this gives

$$\begin{aligned} & \|u_n(T)\|_{L^2_{(x+1)^{2dx}}}^2 \\ & + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \leq \|u_{n,0}\|_{L^2_{(x+1)^{2dx}}}^2 + C \int_0^T \|u_n\|_{L^2_{(x+1)^{2dx}}}^2 dt. \end{aligned} \tag{42}$$

It follows from Gronwall lemma that

$$\|u_n(t)\|_{L^2_{(x+1)^{2dx}}}^2 \leq \|u_{n,0}\|_{L^2_{(x+1)^{2dx}}}^2 e^{Ct} \tag{43}$$

Using (43) in (42) and taking the limit sup as $n \rightarrow \infty$ gives for a.e. T

$$\|u(T)\|_{L^2_{(x+1)^{2dx}}}^2 + \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dx dt \leq e^{CT} \|u_0\|_{L^2_{(x+1)^{2dx}}}^2$$

As u is continuous from \mathbb{R}^+ to $L^2_{(x+1)^{2dx}}$, we infer that

$$\lim_{T \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dx dt = 0.$$

The claim is proved. Therefore, we have that for $T > 0$ small enough and n large enough, $K_n(T) < \frac{1}{2}$, and hence

$$h_n(T) \leq 2C \|u_n(0)\|_{L^2_{(x+1)^{dx}}}^2.$$

This yields

$$\|u - v\|_{L^\infty(0,T;L^2_{(x+1)^{dx}})} \leq \liminf_{n \rightarrow \infty} h_n(T) \leq 2C \liminf_{n \rightarrow \infty} \|u_n(0)\|_{L^2_{(x+1)^{dx}}}^2 = 0$$

and $u = v$ for $0 < t < T$. This proves the uniqueness for T small enough. The general case follows by a classical argument. \square

Remark 1. 1. If we assume only that $u_0 \in L^2_{(x+1)^{dx}}$, then a proof similar to Step 1 gives the existence of a mild solution $u \in C([0, T]; L^2_{(x+1)^{dx}}) \cap L^2(0, T; H^1_{(x+1)^{dx}})$ of (1). The uniqueness of such a solution is open. The existence and uniqueness of a solution issuing from $u_0 \in L^2_{(x+1)^{dx}}$ in a class of functions involving a Bourgain norm has been given in [11].

2. If $u_0 \in L^2_{(x+1)^m dx}$ with $m \geq 3$, then $u \in C([0, T]; L^2_{(x+1)^m dx}) \cap L^2(0, T; H^1_{(x+1)^m dx})$ for all $T > 0$ (see below Theorem 3.1).

3. Asymptotic behavior.

3.1. Decay in $L^2_{(x+1)^m dx}$.

Theorem 3.1. *Assume that the function $a = a(x)$ satisfies (4). Then, for all $R > 0$ and $m \geq 1$, there exist numbers $C > 0$ and $\nu > 0$ such that*

$$\|u(t)\|_{L^2_{(x+1)^m dx}} \leq C e^{-\nu t} \|u_0\|_{L^2_{(x+1)^m dx}}$$

for any solution given by Theorem 2.5, whenever $\|u_0\|_{L^2_{(x+1)^m dx}} \leq R$.

Proof. The proof will be done by induction in m . We set

$$V_0(u) = E(u) = \frac{1}{2} \int_0^\infty u^2 dx \tag{44}$$

and define the Lyapunov function V_m for $m \geq 1$ in an inductive way

$$V_m(u) = \frac{1}{2} \int_0^\infty (x+1)^m u^2 dx + d_{m-1} V_{m-1}(u), \tag{45}$$

where $d_{m-1} > 0$ is chosen sufficiently large (see below).

Suppose first that $m = 1$ and put $V = V_1$. Multiplying the first equation in (1) by u and integrating by parts over $\mathbb{R}^+ \times (0, T)$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 dx \\ &= \frac{1}{2} \int_0^\infty |u_0(x)|^2 dx - \int_0^T \int_0^\infty a(x)|u|^2 dx dt - \frac{1}{2} \int_0^T u_x^2(0, t) dt. \end{aligned} \tag{46}$$

Now, multiplying the equation by xu , we deduce that

$$\begin{aligned} & \frac{1}{2} \int_0^\infty x|u(x, T)|^2 dx - \frac{1}{2} \int_0^\infty x|u_0(x)|^2 dx + \frac{3}{2} \int_0^T \int_0^\infty u_x^2 dx dt \\ & - \frac{1}{2} \int_0^T \int_0^\infty u^2 dx dt - \frac{1}{3} \int_0^T \int_0^\infty u^3 dx dt + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt = 0. \end{aligned} \tag{47}$$

Combining (46) and (47) it follows that

$$\begin{aligned} & V(u) - V(u_0) + (d_0 + 1) \left(\frac{1}{2} \int_0^T \int_0^\infty u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x)|u|^2 dx dt \right) \\ & + \frac{3}{2} \int_0^T \int_0^\infty u_x^2 dx dt - \frac{1}{2} \int_0^T \int_0^\infty u^2 dx dt - \frac{1}{3} \int_0^T \int_0^\infty u^3 dx dt \\ & + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt = 0. \end{aligned} \tag{48}$$

The next step is devoted to estimate the nonlinear term in the left hand side of (48). To do that, we first assume that $\|u_0\|_{L^2} \leq 1$.

By (26) we have that

$$\int_0^\infty |u|^3 dx \leq \varepsilon \|u_x\|_{L^2}^2 + c_\varepsilon \|u\|_{L^2}^{\frac{10}{3}}$$

for any $\varepsilon > 0$ and some constant $c_\varepsilon > 0$. Thus, if $\|u_0\|_{L^2} \leq 1$, we have $\|u\|_{L^2}^{\frac{10}{3}} \leq \|u\|_{L^2}^2$ and

$$\int_0^T \int_0^\infty |u|^3 dx dt \leq \varepsilon \int_0^T \int_0^\infty u_x^2 dx dt + c_\varepsilon \int_0^T \int_0^\infty u^2 dx dt. \tag{49}$$

Moreover, according to [19], there exists $c_1 > 0$, satisfying

$$\int_0^T \int_0^\infty u^2 dx dt \leq c_1 \left\{ \frac{1}{2} \int_0^T \int_0^\infty u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x)u^2 dx dt \right\}. \tag{50}$$

Now, combining (48)-(50) and taking $\varepsilon < \frac{1}{2}$ and $d_0 := 2c_1(\frac{1}{2} + \frac{c_\varepsilon}{3})$ we obtain

$$\begin{aligned} & V(u(T)) - V(u_0) + \frac{d_0 + 1}{2} \left(\frac{1}{2} \int_0^T \int_0^\infty u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x)|u|^2 dx dt \right) \\ & + \left(\frac{3}{2} - \frac{\varepsilon}{3} \right) \int_0^T \int_0^\infty u_x^2 dx dt + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt \leq 0 \end{aligned} \tag{51}$$

or

$$\begin{aligned} & V(u(T)) - V(u_0) \\ & \leq -\tilde{c} \left\{ \int_0^T \int_0^\infty u_x^2(0, t) dt + \int_0^T \int_0^\infty (x + 1)a(x)|u|^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt \right\} \end{aligned} \tag{52}$$

where $\tilde{c} > 0$. We aim to prove the existence of a constant $c > 0$ satisfying

$$V(u(T)) - V(u_0) \leq -cV(u_0) \tag{53}$$

Indeed, such an inequality gives at once the decay $V(u(t)) \leq ce^{-\nu t}V(u_0)$. To this end, we need to establish two claims.

CLAIM 4. There exists $c > 0$ such that

$$\int_0^T V(u)dt \leq c \left\{ \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty (x + 1)a(x)u^2 dxdt \right\}.$$

Since $u_0 \in L^2_{(x+1)dx} \subset L^2$, from (4) and (50) we get

$$\begin{aligned} \int_0^T V(u)dt &= \frac{1}{2} \int_0^T \int_0^\infty (x + 1)u^2 dxdt + \frac{d_0}{2} \int_0^T \int_0^\infty u^2 dxdt \\ &\leq \frac{c_1 d_0}{2} \left\{ \frac{1}{2} \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt \right\} \\ &+ \frac{1}{2} \int_0^T \int_0^{x_0} (x + 1)u^2 dxdt + \frac{1}{2} \int_0^T \int_{x_0}^\infty (x + 1)u^2 dxdt \\ &\leq \frac{c_1 d_0}{2} \left\{ \frac{1}{2} \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt \right\} \\ &+ \frac{1}{2}(x_0 + 1) \int_0^T \int_0^{x_0} u^2 dxdt + \frac{1}{2} \int_0^T \int_{x_0}^\infty (x + 1) \frac{a(x)}{a_0} u^2 dxdt \\ &\leq c \left\{ \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty (x + 1)a(x)u^2 dxdt \right\}. \end{aligned}$$

CLAIM 5.

$$V(u_0) \leq C \left(\int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty (x + 1)a(x)u^2 dxdt + \int_0^T \int_0^\infty u_x^2 dxdt \right) \tag{54}$$

where $C > 0$.

Multiplying the first equation in (1) by $(T - t)u$ and integrating by parts in $(0, \infty) \times (0, T)$, we obtain

$$\begin{aligned} \frac{T}{2} \int_0^\infty |u_0(x)|^2 dx &= \frac{1}{2} \int_0^T \int_0^\infty |u|^2 dxdt \\ &+ \int_0^T \int_0^\infty (T - t)a(x)|u|^2 dxdt + \frac{1}{2} \int_0^T (T - t)u_x^2(0, t)dt, \end{aligned} \tag{55}$$

and therefore, using (50)

$$\int_0^\infty |u_0(x)|^2 dx \leq C \left(\int_0^T \int_0^\infty a(x)|u|^2 dxdt + \int_0^T u_x^2(0, t)dt \right). \tag{56}$$

Now, multiplying by $(T - t)xu$, it follows that

$$\begin{aligned} &-\frac{T}{2} \int_0^\infty x|u_0(x)|^2 dx + \frac{1}{2} \int_0^T \int_0^\infty x|u|^2 dxdt + \frac{3}{2} \int_0^T \int_0^\infty (T - t)u_x^2 dxdt \\ &-\frac{1}{2} \int_0^T \int_0^\infty (T - t)u^2 dxdt + \int_0^T \int_0^\infty (T - t)xa(x)|u|^2 dxdt - \\ &-\frac{1}{3} \int_0^T \int_0^\infty (T - t)u^3 dxdt = 0. \end{aligned}$$

The identity above and (49) allow us to conclude that

$$\begin{aligned}
 & \int_0^\infty x|u_0(x)|^2 dx \\
 & \leq C \left\{ \int_0^T \int_0^\infty (x+1)|u|^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt \right. \\
 & \quad \left. + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt + \int_0^T \int_0^\infty |u|^3 dx dt \right\} \\
 & \leq C \left\{ \int_0^T V(u(t)) dt + \int_0^T \int_0^\infty xa(x)u^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt \right\}
 \end{aligned} \tag{57}$$

for some $C > 0$. Claim 5 follows from Claim 4 and (56)-(57). □

The previous computations give us (53) (and the exponential decay) when $\|u_0\|_{L^2} \leq 1$. The general case is proved as follows. Let $u_0 \in L^2_{(x+1)dx} \subset L^2$ be such that $\|u_0\|_{L^2} \leq R$. Since $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^+))$ and $\|u(t)\|_{L^2} \leq \alpha e^{-\beta t} \|u_0\|_{L^2}$, where $\alpha = \alpha(R)$ and $\beta = \beta(R)$ are positive constants, $\|u(T)\|_{L^2} \leq 1$ if we pick T satisfying $\alpha e^{-\beta T} R < 1$. Then, it follows from (48)-(26) and (53) that for some constants $\nu > 0, c > 0, C > 0$

$$V(u(t+T)) \leq ce^{-\nu t} V(u(T)) \leq c(T\|u_0\|_{L^2}^2 + T\|u_0\|_{L^2}^{\frac{10}{3}} + V(u_0))e^{-\nu t},$$

hence

$$V(u(t)) \leq Ce^{-\nu t} V(u_0),$$

where $C = C(R)$, which concludes the proof when $m = 1$.

Induction Hypothesis: There exist $c > 0$ and $\rho > 0$ such that if $V_{m-1}(u_0) \leq \rho$, we have

$$\begin{aligned}
 & V_m(u) - V_m(u_0) && (*)_m \\
 & \leq -c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt \right\} \\
 & V_m(u_0) && (**)_m \\
 & \leq c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt \right\}.
 \end{aligned}$$

By (52)-(54), the induction hypothesis is true for $m = 1$. Pick now an index $m \geq 2$ and assume that d_0, \dots, d_{m-2} have been constructed so that $(*)_k - (**)_k$ are fulfilled for $1 \leq k \leq m - 1$. We aim to prove that for a convenient choice of the constant d_{m-1} in (45), the properties $(*)_m - (**)_m$ hold true.

Let us investigate first $(*)_m$. We multiply the first equation in (1) by $(x+1)^m u$ to obtain

$$\begin{aligned}
 & V_m(u) - V_m(u_0) - d_{m-1}(V_{m-1}(u) - V_{m-1}(u_0)) \\
 & - \frac{m(m-1)(m-2)}{2} \int_0^T \int_0^\infty (x+1)^{m-3} u^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0,t) dt \\
 & + \frac{3m}{2} \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt - \frac{m}{2} \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt \\
 & - \frac{m}{3} \int_0^T \int_0^\infty (x+1)^{m-1} u^3 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt = 0.
 \end{aligned} \tag{58}$$

The next steps are devoted to estimate the terms in the above identity. First, combining (4) and (50) we infer the existence of a positive constant $c > 0$ such that

$$\begin{aligned}
 & \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt \\
 &= \int_0^T \int_0^{x_0} (x+1)^{m-1} u^2 dx dt + \int_0^T \int_{x_0}^\infty (x+1)^{m-1} u^2 dx dt \\
 &\leq (x_0+1)^{m-1} \int_0^T \int_0^\infty u^2 dx dt + \frac{1}{a_0} \int_0^T \int_0^\infty a(x)(x+1)^{m-1} u^2 dx dt \tag{59} \\
 &\leq c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} a(x) u^2 dx dt \right\} \\
 &\leq -c \{V_{m-1}(u) - V_{m-1}(u_0)\}
 \end{aligned}$$

where we used $(*)_{m-1}$. In the same way

$$\begin{aligned}
 & \int_0^T \int_0^\infty (x+1)^{m-3} u^2 dx dt \\
 &\leq \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt \leq -c \{V_{m-1}(u) - V_{m-1}(u_0)\} \tag{60}
 \end{aligned}$$

where $c > 0$ is a positive constant. Moreover, assuming $V_{m-1}(u_0) \leq \rho$ with $\rho > 0$ small enough (so that by exponential decay of $V_{m-1}(u(t))$ we have $\int_0^\infty (x+1)^{m-1} |u(x,t)|^2 dx \leq 1$ for all $t \geq 0$) and proceeding as in the case $m = 1$, we obtain the existence of $\varepsilon > 0$ and $c_\varepsilon > 0$ satisfying

$$\begin{aligned}
 & \int_0^T \int_0^\infty (x+1)^{m-1} |u|^3 dx dt \\
 &\leq \varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + c_\varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt. \tag{61}
 \end{aligned}$$

Indeed,

$$\begin{aligned}
 & \int_0^\infty (x+1)^{m-1} |u|^3 dx \leq \|u\|_{L^\infty} \int_0^\infty (x+1)^{m-1} u^2 dx \\
 &\leq \sqrt{2} \|u_x\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \int_0^\infty (x+1)^{m-1} u^2 dx \leq \varepsilon \int_0^\infty (x+1)^{m-1} u_x^2 dx \tag{62} \\
 &+ c_\varepsilon \int_0^\infty u^2 dx + c_\varepsilon \left(\int_0^\infty (x+1)^{m-1} u^2 dx \right)^2.
 \end{aligned}$$

Then, if we return to (58) and take $\varepsilon < 9/2$ and $d_{m-1} > 0$ large enough, from (59)-(61) it follows that

$$\begin{aligned}
 & V_m(u) - V_m(u_0) \\
 &\leq -c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt \right. \tag{63} \\
 & \left. + \int_0^T \int_0^\infty a(x)(x+1)^m u^2 dx dt \right\} + \frac{d_{m-1}}{2} (V_{m-1}(u) - V_{m-1}(u_0)).
 \end{aligned}$$

This yields $(*)_m$, by $(*)_{m-1}$. Let us now check $(**)_m$. It remains to estimate the terms in the right hand side of (63). We multiply the first equation in (1) by

$(T - t)(x + 1)^m u$ to obtain

$$\begin{aligned} \frac{T}{2} \int_0^\infty (x + 1)^m u_0^2 dx &= \frac{1}{2} \int_0^T \int_0^\infty (x + 1)^m u^2 dx dt \\ &- \frac{m(m - 1)(m - 2)}{2} \int_0^T \int_0^\infty (T - t)(x + 1)^{m-3} u^2 dx dt + \frac{1}{2} \int_0^T (T - t) u_x^2(0, t) dt \\ &+ \frac{3m}{2} \int_0^T \int_0^\infty (T - t)(x + 1)^{m-1} u_x^2 dx dt - \frac{m}{2} \int_0^T \int_0^\infty (T - t)(x + 1)^{m-1} u^2 dx dt \\ &- \frac{m}{3} \int_0^T \int_0^\infty (T - t)(x + 1)^{m-1} u^3 dx dt + \int_0^T \int_0^\infty (T - t)(x + 1)^m a(x) u^2 dx dt. \end{aligned}$$

Then, proceeding as above, we deduce that

$$\begin{aligned} &\int_0^T (x + 1)^m u_0^2 dx \\ &\leq c \left\{ \int_0^T \int_0^\infty (x + 1)^{m-1} u^2 dx dt + \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 dx dt \right. \\ &\quad \left. + \int_0^T \int_0^\infty (x + 1)^m a(x) u^2 dx dt \right\} \\ &\leq c \left\{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty (x + 1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty (x + 1)^m a(x) u^2 dx dt \right\}. \end{aligned}$$

Combined to $(**)_{m-1}$, this yields $(**)_{m-1}$. This completes the construction of the sequence $\{V_m\}_{m \geq 1}$ by induction.

Let us now check the exponential decay of V_m for $m \geq 2$. It follows from $(*)_m - (**)_m$ that

$$V_m(u) - V_m(u_0) \leq -c V_m(u_0)$$

where $c > 0$, which completes the proof when $V_{m-1}(u_0) \leq \rho$. The global result $(V_{m-1}(u_0) \leq R)$ is obtained as above for $m = 1$. \square

Corollary 1. *Let $a = a(x)$ fulfilling (4) and $a \in W^{2,\infty}(0, \infty)$. Then for any $R > 0$, there exist positive constants $c = c(R)$ and $\mu = \mu(R)$ such that*

$$\|u_x(t)\|_{L^2(\mathbb{R}^+)} \leq c \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)dx}} \tag{64}$$

for all $t > 0$ and all $u_0 \in L^2_{(x+1)dx}$ satisfying $\|u_0\|_{L^2_{(x+1)dx}} \leq R$.

Proof. Pick any $R > 0$ and any $u_0 \in L^2_{(x+1)dx}$ with $\|u_0\|_{L^2_{(x+1)dx}} \leq R$. By Theorem 3.1 there are some constants $C = C(R)$ and $\nu = \nu(R)$ such that

$$\|u(t)\|_{L^2_{(x+1)dx}} \leq C e^{-\nu t} \|u_0\|_{L^2_{(x+1)dx}}. \tag{65}$$

Using the multiplier $t(u^2 + 2u_{xx})$ we obtain after some integrations by parts that for all $0 < t_1 < t_2$

$$\begin{aligned} &t_2 \int_0^\infty u_x^2(x, t_2) dx + \int_{t_1}^{t_2} t u_x^2(0, t) dt + 2 \int_{t_1}^{t_2} \int_0^\infty t a(x) u_x^2 dx dt + \int_{t_1}^{t_2} t u_{xx}^2(0, t) dt \\ &= -\frac{1}{3} \int_{t_1}^{t_2} \int_0^\infty u^3 dx dt + \frac{t_2}{3} \int_0^\infty u^3(x, t_2) dx + \int_{t_1}^{t_2} \int_0^\infty t u^3 a(x) dx dt \\ &+ \int_{t_1}^{t_2} \int_0^\infty u_x^2 dx dt + \int_{t_1}^{t_2} \int_0^\infty t a''(x) u^2 dx dt. \end{aligned} \tag{66}$$

1. Let us assume first that $T > 1$. Applying (66) on the time interval $[T - 1, T]$, we infer that

$$\begin{aligned} & \int_0^\infty |u_x(x, T)|^2 dx \\ & \leq c \left(\int_{T-1}^T \int_0^\infty |u|^3 dx dt + \|u(T)\|_{L^3(\mathbb{R}^+)}^3 + \int_{T-1}^T \|u\|_{H^1(\mathbb{R}^+)}^2 dt \right). \end{aligned} \tag{67}$$

To estimate the cubic terms in (67), we use (26) to obtain

$$\begin{aligned} \int_0^\infty |u_x(x, T)|^2 dx & \leq \varepsilon \int_0^\infty |u_x(x, T)|^2 dx \\ & + c_\varepsilon (\|u(T)\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}} + \int_{T-1}^T (\|u\|_{H^1(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt). \end{aligned} \tag{68}$$

Note that by (65)

$$\|u(T)\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}} \leq (C e^{-\nu T} \|u_0\|_{L^2_{(x+1)dx}})^{\frac{10}{3}} \leq C^{\frac{10}{3}} R^{\frac{4}{3}} e^{-\nu T} \|u_0\|_{L^2_{(x+1)dx}}^2.$$

It follows from (48), (26), and (65) that

$$\begin{aligned} & \int_{T-1}^T (\|u\|_{H^1(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt \\ & \leq C \left(V_1(u(T-1)) + \int_{T-1}^T (\|u\|_{L^2(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt \right) \\ & \leq C e^{-\nu T} \|u_0\|_{L^2_{(x+1)dx}}^2 \end{aligned} \tag{69}$$

where $C = C(R, \nu)$. (64) for $T \geq 1$ follows from (68) and (69) by choosing $\varepsilon < 1$ and $\mu < \nu$.

2. Assume now that $T \leq 1$. Estimating again the cubic terms in (66) (with $[t_1, t_2] = [0, T]$) by using (26), we obtain

$$\begin{aligned} T \int_0^\infty u_x^2(x, T) dx & \leq \frac{T}{3} \left(\varepsilon \|u_x(T)\|_{L^2(\mathbb{R}^+)}^2 + C_\varepsilon \|u(T)\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}} \right) \\ & + C_\varepsilon \int_0^T (\|u\|_{H^1(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt. \end{aligned} \tag{70}$$

By (48), (26) and (65), we have that

$$\int_0^1 \int_0^\infty |u_x|^2 dx dt \leq C(R) \|u_0\|_{L^2_{(x+1)dx}}^2 \tag{71}$$

which, combined to (70) with $\varepsilon = 1$ and (65), gives

$$\|u_x(T)\|_{L^2(\mathbb{R}^+)}^2 \leq C(R) T^{-1} \|u_0\|_{L^2_{(x+1)dx}}^2$$

for all $T < 1$. This gives (64) for $T < 1$. □

Corollary 1 may be extended (locally) to the weighted space $L^2_{(x+1)^m dx}$ ($m \geq 2$) in following the method of proof of [23, Theorem 1.1].

Corollary 2. *Let $a = a(x)$ fulfilling (4) and $m \geq 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that*

$$\|u(t)\|_{H^1_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^m dx}}$$

for all $t > 0$ and all $u_0 \in L^2_{(x+1)^m dx}$ satisfying $\|u_0\|_{L^2_{(x+1)^m dx}} \leq \rho$.

Proof. We first prove estimates for the linearized problem

$$u_t + u_x + u_{xxx} + au = 0 \tag{72}$$

$$u(0, t) = 0 \tag{73}$$

$$u(x, 0) = u_0(x) \tag{74}$$

and next apply a perturbation argument to extend them to the nonlinear problem (1). Let us denote by $W(t)u_0 = u(t)$ the solution of (72)-(74). By computations similar to those performed in the proof of Theorem 3.1, we have that

$$\|W(t)u_0\|_{L^2_{(x+1)^m dx}} \leq C_0 e^{-\nu t} \|u_0\|_{L^2_{(x+1)^m dx}}.$$

We need the

CLAIM 6. Let $k \in \{0, \dots, 3\}$. Then there exists a constant $C_k > 0$ such that for any $u_0 \in H^k_{(x+1)^m dx}$,

$$\|W(t)u_0\|_{H^k_{(x+1)^m dx}} \leq C_k e^{-\nu t} \|u_0\|_{H^k_{(x+1)^m dx}}. \tag{75}$$

Indeed, if $u_0 \in H^3_{(x+1)^m dx}$, then $u_t(\cdot, 0) \in L^2_{(x+1)^{m-3} dx}$, and since $v = u_t$ solves (72)-(73), we also have that

$$\|u_t(\cdot, t)\|_{L^2_{(x+1)^{m-3} dx}} \leq C_0 e^{-\nu t} \|u_t(\cdot, 0)\|_{L^2_{(x+1)^{m-3} dx}}.$$

Using (72), this gives

$$\|W(t)u_0\|_{H^3_{(x+1)^m dx}} \leq C_3 e^{-\nu t} \|u_0\|_{H^3_{(x+1)^m dx}}.$$

This proves (75) for $k = 3$. The fact that (75) is valid for $k = 1, 2$ follows from a standard interpolation argument, for $H^k_{(x+1)^m dx} = [H^0_{(x+1)^m dx}, H^3_{(x+1)^m dx}]^{\frac{k}{3}}$.

Lemma 3.2. Pick any number $\mu \in (0, \nu)$. Then there exists some constant $C = C(\mu) > 0$ such that for any $u_0 \in L^2_{(x+1)^m dx}$

$$\|W(t)u_0\|_{H^1_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^m dx}}. \tag{76}$$

Proof. Let $u_0 \in L^2_{(x+1)^m dx}$ and set $u(t) = W(t)u_0$ for all $t \geq 0$. By scaling in (72) by $(x + 1)^m u$, we see that for some constant $C_K = C_K(T)$

$$\|u\|_{L^2(0,1; H^1_{(x+1)^m dx})} \leq C_K \|u_0\|_{L^2_{(x+1)^m dx}}.$$

This implies that $u(t) \in H^1_{(x+1)^m dx}$ for a.e. $t \in (0, 1)$ which, combined to (75), gives that $u(t) \in H^1_{(x+1)^m dx}$ for all $t > 0$. Pick any $T \in (0, 1]$. Note that, by (75),

$$\|u(T)\|_{H^1_{(x+1)^m dx}} \leq C_1 e^{-\nu(T-t)} \|u(t)\|_{H^1_{(x+1)^m dx}}, \quad \forall t \in (0, T). \tag{77}$$

Integrating with respect to t in (77) yields

$$[C_1^{-1} \|u(T)\|_{H^1_{(x+1)^m dx}}]^2 \int_0^T e^{2\nu(T-t)} dt \leq \int_0^T \|u(t)\|_{H^1_{(x+1)^m dx}}^2 dt,$$

and hence

$$\begin{aligned} \|u(T)\|_{H^1_{(x+1)^m dx}} &\leq C_K C_1 \sqrt{\frac{2\nu}{e^{2\nu T} - 1}} \|u_0\|_{L^2_{(x+1)^m dx}} \\ &\leq \frac{C_K C_1}{\sqrt{T}} \|u_0\|_{L^2_{(x+1)^m dx}} \end{aligned}$$

for $0 < T \leq 1$. Therefore

$$\|u(t)\|_{H^1_{(x+1)^{m dx}}} \leq C_K C_1 e^\nu \frac{e^{-\nu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^{m dx}}} \quad \forall t \in (0, 1). \tag{78}$$

(76) follows from (78) and (75), since $\mu < \nu$. □

Let us return to the proof of Corollary 2. Fix a number $\mu \in (0, \nu)$, where ν is as in (75), and let us introduce the space

$$F = \{u \in C(\mathbb{R}^+; H^1_{(x+1)^{m dx}}); \|e^{\mu t} u(t)\|_{L^\infty(\mathbb{R}^+; H^1_{(x+1)^{m dx}})} < \infty\}$$

endowed with its natural norm. Note that (1) may be recast in the following integral form

$$u(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s)) ds \tag{79}$$

where $N(u) = -uu_x$. We first show that (79) has a solution in F provided that $u_0 \in H^1_{(x+1)^{m dx}}$ with $\|u_0\|_{H^1_{(x+1)^{m dx}}}$ small enough. Let $u_0 \in H^1_{(x+1)^{m dx}}$ and $u \in F$ with $\|u_0\|_{H^1_{(x+1)^{m dx}}} \leq r_0$ and $\|u\|_F \leq R$, r_0 and R being chosen later. We introduce the map Γ defined by

$$(\Gamma u)(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s)) ds \quad \forall t \geq 0. \tag{80}$$

We shall prove that Γ has a fixed point in the closed ball $B_R(0) \subset F$ provided that $r_0 > 0$ is small enough.

For the forcing problem

$$\begin{cases} u_t + u_x + u_{xxx} + au = f \\ u(0, t) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

we have the following estimate

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|u(t)\|_{L^2_{(x+1)^{m dx}}}^2 + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt \\ & \leq C \left(\|u_0\|_{L^2_{(x+1)^{m dx}}}^2 + \|f\|_{L^1(0, T; L^2_{(x+1)^{m dx}})}^2 \right). \end{aligned}$$

Let us take $f = N(u) = -uu_x$. Observe that for all $x > 0$

$$\begin{aligned} (x+1)u^2(x) &= \left| \int_0^\infty \frac{d}{dx} [(x+1)u^2(x)] dx \right| \\ &\leq C \left(\int_0^\infty (x+1)^m |u|^2 dx + \int_0^\infty (x+1)^{m-1} |u_x|^2 dx \right) \end{aligned}$$

whenever $m \geq 2$. It follows that for some constant $K > 0$

$$\begin{aligned} \|uu_x\|_{L^2_{(x+1)^{m dx}}}^2 &\leq \|(x+1)u^2\|_{L^\infty(\mathbb{R}^+)} \int_0^\infty (x+1)^{m-1} |u_x|^2 dx \\ &\leq K \|u\|_{H^1_{(x+1)^{m dx}}}^4. \end{aligned}$$

Therefore, for any $T > 0$,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|(\Gamma u)(t)\|_{L^2_{(x+1)^m dx}}^2 + \int_0^T \int_0^\infty (x+1)^{m-1} |(\Gamma u)_x|^2 dx dt \\ & \leq C \left(\|u_0\|_{L^2_{(x+1)^m dx}}^2 + \left(\int_0^T \|u(t)\|_{H^1_{(x+1)^m dx}}^2 dt \right)^2 \right) < \infty. \end{aligned}$$

Thus $\Gamma u \in C(\mathbb{R}^+, L^2_{(x+1)^m dx}) \cap L^2_{loc}(\mathbb{R}^+; H^1_{(x+1)^m dx})$ with $(\Gamma u)(0) = u_0$. We claim that $\Gamma u \in F$. Indeed, by (75),

$$\|e^{\mu t} W(t)u_0\|_{H^1_{(x+1)^m dx}} \leq C_1 \|u_0\|_{H^1_{(x+1)^m dx}}$$

and for all $t \geq 0$

$$\begin{aligned} \|e^{\mu t} \int_0^t W(t-s)N(u(s))ds\|_{H^1_{(x+1)^m dx}} & \leq C e^{\mu t} \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{t-s}} \|N(u(s))\|_{L^2_{(x+1)^m dx}} ds \\ & \leq C \int_0^t \frac{e^{\mu s}}{\sqrt{t-s}} K(e^{-\mu s} \|u\|_F)^2 ds \\ & \leq CK \|u\|_F^2 \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{s}} ds \\ & \leq CK(2 + \mu^{-1}) \|u\|_F^2 \end{aligned}$$

where we used Lemma 3.2. Pick $R > 0$ such that $CK(2 + \mu^{-1})R \leq \frac{1}{2}$, and r_0 such that $C_1 r_0 = \frac{R}{2}$. Then, for $\|u_0\|_{H^1_{(x+1)^m dx}} \leq r_0$ and $\|u\|_F \leq R$, we obtain that

$$\|e^{\mu t} (\Gamma u)(t)\|_{H^1_{(x+1)^m dx}} \leq C_1 r_0 + CK(2 + \mu^{-1})R^2 \leq R, \quad t \geq 0.$$

Hence Γ maps the ball $B_R(0) \subset F$ into itself. Similar computations show that Γ contracts. By the contraction mapping theorem, Γ has a unique fixed point u in $B_R(0)$. Thus $\|u(t)\|_{H^1_{(x+1)^m dx}} \leq C e^{-\mu t} \|u_0\|_{H^1_{(x+1)^m dx}}$ provided that $\|u_0\|_{H^1_{(x+1)^m dx}} \leq r_0$ with r_0 small enough. Proceeding as in the proof of Lemma 3.2, we have that

$$\|u(t)\|_{H^1_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^m dx}} \quad \text{for } 0 < t < 1,$$

provided that $\|u_0\|_{L^2_{(x+1)^m dx}} \leq \rho_0$ with $\rho_0 < 1$ small enough. The proof is complete with a decay rate $\mu' < \mu$. □

Corollary 3. *Assume that $a(x)$ satisfies (4) and that $\partial_x^k a \in L^\infty(\mathbb{R}^+)$ for all $k \geq 0$. Pick any $u_0 \in L^2_{(x+1)^m dx}$. Then for all $\varepsilon > 0$, all $T > \varepsilon$, and all $k \in \{1, \dots, m\}$, there exists a constant $C = C(\varepsilon, T, k) > 0$ such that*

$$\int_\varepsilon^\infty (x+1)^{m-k} |\partial_x^k u(x, t)|^2 dx \leq C \|u_0\|_{L^2_{(x+1)^m dx}}^2 \quad \forall t \in [\varepsilon, T]. \tag{81}$$

Proof. The proof is very similar to the one in [16, Lemma 5.1] and so we only point out the small changes. First, it should be noticed that the presence in the KdV equation of the extra terms u_x and $a(x)u$ does not cause any serious trouble. On the other hand, choosing a cut-off function in x of the form $\eta(x) = \psi_0(x/\varepsilon)$ (instead of $\eta(x) = \psi_0(x - x_0 + 2)$ as in [16]) where $\psi_0 \in C^\infty(\mathbb{R}, [0, 1])$ satisfies $\psi_0(x) = 0$ for $x \leq 1/2$ and $\psi_0(x) = 1$ for $x \geq 1$, allows to overcome the fact that u is a solution of (1) on the half-line only. □

3.2. **Decay in L_b^2 .** This section is devoted to the exponential decay in L_b^2 . Our result reads as follows:

Theorem 3.3. *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$. Then, for all $R > 0$, there exist $C > 0$ and $\nu > 0$, such that*

$$\|u(t)\|_{L_b^2} \leq Ce^{-\nu t} \|u_0\|_{L_b^2} \quad t \geq 0$$

for any solution u given by Theorem 2.3.

Proof. We introduce the Lyapunov function

$$V(u) = \frac{1}{2} \int_0^\infty u^2 e^{2bx} dx + c_b \int_0^\infty u^2 dx, \tag{82}$$

where c_b is a positive constant that will be chosen later. Then, adding (17) and (18) hand by hand we obtain

$$\begin{aligned} & V(u) - V(u_0) \\ &= (4b^3 + b) \int_0^T \int_{x_0}^\infty u^2 e^{2bx} dx dt + (4b^3 + b) \int_0^T \int_0^{x_0} u^2 e^{2bx} dx dt \\ &\quad - 3b \int_0^\infty \int_0^\infty u_x^2 e^{2bx} dx dt + \frac{2b}{3} \int_0^T \int_0^\infty u^3 e^{2bx} dx dt \\ &\quad - (c_b + \frac{1}{2}) \int_0^T u_x^2(0, t) dt - \int_0^T \int_0^\infty a(x) |u|^2 (e^{2bx} + 2c_b) dx dt, \end{aligned} \tag{83}$$

where x_0 is the number introduced in (4). On the other hand, since $L_b^2 \subset L_{(x+1)dx}^2$, $\|u(t)\|_{L^2(0,\infty)}$ and $\|u_x(t)\|_{L^2(0,\infty)}$ decays to zero exponentially. Consequently, from Moser estimate we deduce that $\|u(t)\|_{L^\infty(0,\infty)} \rightarrow 0$. We may assume that $(2b/3)\|u(t)\|_{L^\infty} < \varepsilon = a_0 - (4b^3 + b)$ for all $t \geq 0$, by changing u_0 into $u(t_0)$ for t_0 large enough. Therefore

$$\begin{aligned} & \frac{2b}{3} \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt \\ & \leq \frac{2b}{3} \int_0^T \|u(t)\|_{L^\infty(0,\infty)} \left(\int_0^\infty |u|^2 e^{2bx} dx \right) dt \leq \varepsilon \int_0^T \int_0^\infty u^2 e^{2bx} dx dt. \end{aligned} \tag{84}$$

So, returning to (83), the following holds

$$\begin{aligned} & V(u) - V(u_0) - (4b^3 + b + \varepsilon) \int_0^T \int_0^{x_0} u^2 e^{2bx} dx dt + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt \\ & + (c_b + \frac{1}{2}) \int_0^T u_x^2(0, t) dt + 2c_b \int_0^T \int_0^\infty a(x) |u|^2 dx dt \leq 0. \end{aligned} \tag{85}$$

Moreover, according to [19] there exists $C > 0$ satisfying

$$\begin{aligned} & \int_0^T \int_0^{x_0} u^2 e^{2bx} dx dt \\ & \leq e^{2bx_0} \int_0^T \int_0^{x_0} u^2 dx dt \leq C \{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x) u^2 dx dt \} \end{aligned}$$

since $L_b^2 \subset L^2(\mathbb{R}^+)$. Then, choosing c_b sufficiently large, the above estimate and (85) give us that

$$\begin{aligned}
 V(u) - V(u_0) &\leq -C \left\{ \int_0^T \int_0^\infty u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x) u^2 dx dt \right. \\
 &\quad \left. + \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt \right\} \leq -C V(u_0),
 \end{aligned}
 \tag{86}$$

which allows to conclude that $V(u)$ decays exponentially. The last inequality is a consequence of the following results:

CLAIM 7. There exists a positive constant $C > 0$, such that

$$\int_0^T V(u(t)) dt \leq C \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt.$$

First, observe that

$$\left| \int_0^\infty u^2 e^{2bx} dx \right| = \left| -\frac{1}{b} \int_0^\infty uu_x e^{2bx} dx \right| \leq \frac{1}{b} \left(\int_0^\infty u^2 e^{2bx} dx \right)^{\frac{1}{2}} \left(\int_0^\infty u_x^2 e^{2bx} dx \right)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{b^2} \int_0^\infty u_x^2 e^{2bx} dx.
 \tag{87}$$

Then, from (4) and (87) we have

$$V(u(t)) \leq \left(\frac{1}{2} + c_b \right) \int_0^\infty u^2 e^{2bx} dx \leq \left(\frac{1}{2} + c_b \right) b^{-2} \int_0^\infty u_x^2 e^{2bx} dx$$

which gives us Claim 7.

CLAIM 8.

$$V(u_0) \leq C \left\{ \int_0^T \int_0^\infty u_x^2(0, t) dt + \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt + \int_0^T V(u(t)) dt \right\},$$

where C is a positive constant.

Multiplying the first equation in (1) by $(T - t)ue^{2bx}$ and integrating by parts in $(0, \infty) \times (0, T)$, we obtain

$$\begin{aligned}
 &-\frac{T}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^T \int_0^\infty |u|^2 e^{2bx} dx dt \\
 &+ 3b \int_0^T \int_0^\infty (T - t) u_x^2 e^{2bx} dx dt + \frac{1}{2} \int_0^T (T - t) u_x^2(0, t) dt \\
 &- (4b^3 + b) \int_0^T \int_0^\infty (T - t) u^2 e^{2bx} dx dt + \int_0^T \int_0^\infty (T - t) a(x) |u|^2 e^{2bx} dx dt \\
 &- \frac{2b}{3} \int_0^T \int_0^\infty (T - t) u^3 e^{2bx} dx dt = 0
 \end{aligned}
 \tag{88}$$

and therefore,

$$\begin{aligned}
 \int_0^\infty |u_0(x)|^2 e^{2bx} dx &\leq C \left(\int_0^T \int_0^\infty u_x^2(0, t) dt + \frac{1}{2} \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \right. \\
 &\quad \left. + \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt + \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt \right).
 \end{aligned}
 \tag{89}$$

Then, combining (87) and (84), we derive Claim 8. (86) follows at once. This proves the exponential decay when $\|u(t)\|_{L^\infty} \leq 3\varepsilon/(2b)$. The general case is obtained as in Theorem 3.1 □

Corollary 4. *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$. Then for any $R > 0$, there exist positive constants $c = c(R)$ and $\mu = \mu(R)$ such that*

$$\|u_x(t)\|_{L_b^2} \leq c \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L_b^2} \quad (90)$$

for all $t > 0$ and all $u_0 \in L_b^2$ satisfying $\|u_0\|_{L_b^2} \leq R$.

Corollary 5. *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$, and let $s \geq 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that*

$$\|u(t)\|_{H_b^s} \leq C \frac{e^{-\mu t}}{t^{\frac{s}{2}}} \|u_0\|_{L_b^2}$$

for all $t > 0$ and all $u_0 \in L_b^2$ satisfying $\|u_0\|_{L_b^2} \leq \rho$.

The proof of Corollary 4 (resp. 5) is very similar to the proof of Corollary 1 (resp. 2), so it is omitted.

Acknowledgments. This work was achieved while the first author was visiting Université Paris-Sud and the second author was visiting IMPA and UFRJ. LR was partially supported by the “Agence Nationale de la Recherche” (ANR) through grant JCJC06-137283. AFP was partially supported by the Cooperation Agreement Brazil-France and CNPq (Brazil).

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Received August 2009; revised December 2009.

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