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GLOBAL STABILIZATION OF A COUPLED SYSTEM OF TWO GENERALIZED KORTEWEG-DE VRIES TYPE EQUATIONS POSED ON A FINITE DOMAIN

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Abstract. The purpose of this work is to study the internal stabilization of a coupled system of two generalized Korteweg-de Vries equations under the effect of a localized damping term. The exponential stability, as well as, the global existence of weak solutions are investigated when the exponent in the nonlinear term ranges over the interval $[1, 4)$. To obtain the decay we use multiplier techniques combined with compactness arguments and reduce the problem to prove a unique continuation property for weak solutions. Here, the unique continuation is obtained via the usual Carleman estimate.

1. **Introduction.** We consider the initial-boundary problem for a coupled system of two generalized Korteweg-de Vries equations Vries equation in the domain $(0, L)$ under the presence of a localized damping represented by a function $b = b(x)$, that is,

$$\begin{aligned} &u_t + u_{xxx} + a_3 v_{xxx} + a(u)u_x + a_1 v v_x + a_2(uv)_x + b(x)u = 0 \\ &b_1 v_t + r v_x + v_{xxx} + b_2 a_3 u_{xxx} + a(v)v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x + b(x)v = 0, \end{aligned} \tag{1}$$

where $0 < x < L$, $t > 0$, with boundary conditions

$$u(0,t) = v(0,t) = u(L,t) = v(L,t) = u_x(L,t) = v_x(L,t) = 0, \quad t > 0 \tag{2}$$

and initial conditions

$$u(x,0) = u^0(x) \ \text{ and } \ v(x,0) = v^0(x), \quad 0 < x < L. \tag{3}$$

In (1),

r, a_1, a_2, a_3, b_1 and b_2 are real constants with $0 < a_3^2 b_2 < 1$ and $b_1, b_2 > 0$.

When $a(s) = s$ and $b \equiv 0$ system (1) was derived by Gear and Grimshaw in [8] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV

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equations with both linear and nonlinear coupling terms and has been object of intensive research in recent years (see, for instance, [1], [2], [4], [9], [11], [14], [15]).

This apparently complicated system appears as a special case of a broad class of nonlinear evolution equations, which can be solved by the inverse scattering method (see [1]). It can also be interpreted as a coupled nonlinear version of Korteweg-de Vries generalized equations of the form

$$\begin{cases} u_t + u_{xxx} + f(u,v)_x = 0 \\ v_t + v_{xxx} + g(u,v)_x = 0, \end{cases}$$

with f and g satisfying $f(u,v) = H_u(u,v)$ and $g(u,v) = H_v(u,v)$ for a smooth function H.

There is a large body of literature on dispersive models for wave problems, but most of the studies are concerned with initial-value problems or with periodic boundary conditions. However, the practical use of the wave systems and its relatives does note always involve such mathematical formulation. Instead, the initial boundary value problem often comes to the fore. Therefore, from a mathematical point of view, it is also of interest to study the mathematical properties of the Korteweg-de Vries family on a finite spatial interval. Moreover, it is important to point out that there are many fundamental differences between the initial-value problem and the initial- boundary-value problem. Most notably, the solution of an initial-value (or with periodic boundary conditions) equation has many conservation properties (e.g., the L^2-norm), while the initial-boundary-value problems often dissipate the energy at the boundary. Hence, we can use different mathematical frameworks to study these two set problems.

Along this work we assume that $a = a(s)$ and $b = b(x)$ are real-valued function that satisfy the conditions

$$\begin{cases} a(0) = 0, \quad |a^{(j)}(s)| \le c\,(1 + |s|^{p-j}), \forall\, s \in \mathbb{R}, \\ \text{where } c \text{ is a positive constant and } j = 0,1 \text{ if } 1 \le p < 2 \\ \text{and } j = 0,1,2 \text{ if } p \ge 2, \end{cases} \tag{4}$$

and

$$\begin{cases} b \in L^2(0,L) \text{ is a nonnegative function, such that} \\ b(x) \ge b_0 > 0 \text{ a. e. in } \omega, \text{ where } \omega \subset (0,L) \text{ is a nonempty open set.} \end{cases} \tag{5}$$

Therefore, the damping term is acting effectively in ω.

Let us consider the total energy associated to (1), in this case

$$E(t) = \frac{1}{2}\int_0^L (b_2 u^2 + b_1 v^2)dx. \tag{6}$$

Then, we can (formally) verify that

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_0^L (b_2u^2 + b_1v^2)dx = -[\frac{b_2}{2}u_x^2(0,t) + \frac{1}{2}v_x^2(0,t) + a_3b_2u_x(0,t)v_x(0,t)] \\ &-\int_0^L b(x)(b_2u^2 + v^2)dx = -\frac{1}{2}\left(\sqrt{b_2}u_x(0,t) + \sqrt{a_3^2b_2}v_x(0,t)\right)^2 \\ &-\frac{1}{2}\left(1 - a_3^2b_2\right)v_x^2(0,t) - \int_0^L b(x)(b_2u^2 + v^2)dx \le 0, \end{aligned} \tag{7}$$

for any $t > 0$. The inequality above shows that the term $b(x)(u + v)$ plays the role of a feedback damping mechanism and, consequently, we can investigate whether the solutions of (1)-(3) tend to zero as $t \to \infty$ and under what rate they decay. When $0 < a_3^2 b_2 < 1$, from (7) we can see that even when $b \equiv 0$ or $\omega = \emptyset$, the energy is dissipated through the extreme $x = 0$. However, the dissipation due to the boundary terms $u_x(0,t)$ and $v_x(0,t)$ is not strong enough to guarantee the decay of solutions of (1)-(3) for all values of L. In fact, in [11, 14, 15] it was proved that the decay of the solutions of the linearized system may fail for some critical values of the length L of the interval $(0, L)$. Their analysis was inspired in the results obtained by Rosier in [17] who discovered that, if the length L of the domain $(0, L)$ lies in a countable set of critical lengths of the form

$$\mathcal{E} = \Big\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2},\ k \text{ and } l \text{ are positive natural numbers} \Big\}, \tag{8}$$

the linear KdV equation possesses a solution with a constant L^2-norm. Therefore, these works suggests that, adding an extra damping term (like $b(x)u$ and $b(x)v$, for instance) the decay of solutions may be obtained for any $L > 0$.

When $b(x) \geq b_0 > 0$ almost everywhere in $(0, L)$, it is very simple to prove that the energy $E(t)$ decays exponentially as t tends to infinity. The problem of stabilization when the damping is effective only on a subset of $(0, L)$ is much more subtle. In this paper we are concerned with this problem. More precisely, our purpose is to prove that, for any $R > 0$, there exist constants $C = C(R)$ and $\alpha = \alpha(R)$ satisfying

$$E(t) \leq C(R) E(0) e^{-\alpha(R) t},\ \ \forall\ t > 0,$$

provided $E(0) \leq R$. This can be stated in the following equivalent form: Find $T > 0$ and $C > 0$ such that

$$\begin{aligned} E(0) \leq C \int_0^T \Big[\int_0^L b(x)(b_2 u^2 + v^2) dx + \frac{1}{2} \left(\sqrt{b_2} u_x(0,t) + \sqrt{a_3^2 b_2} v_x(0,t) \right)^2 \\ + \frac{1}{2} \left(1 - a_3^2 b_2 \right) v_x^2(0,t) \Big] dt \end{aligned} \tag{9}$$

holds for every finite energy solution of (1)-(3). Indeed, if the above inequality holds, from (7) we obtain $0 < \gamma < 1$ such that

$$E(kT) \leq \gamma^k E(0), \quad \forall\, k > 0.$$

Since $E(t) \leq E(kT)$ for $kT < t < (k+1)T$ we get

$$E(t) \leq \frac{1}{\gamma} E(0) e^{\frac{\ln \gamma}{T} t}$$

from which we obtain the main result of this paper:

Theorem 1.1. *Let $a = a(x)$ be a $\mathcal{C}^2$ function such that*

$$|a(x)| \leq C(1 + |x|^p),\ |a'(x)| \leq C(1 + |x|^{p-1}),\ |a''(x)| \leq C(1 + |x|^{p-2}), \quad \forall\, x \in \mathbb{R}$$

where C is a positive constant and $1 \leq p < 4$. Then, if b satisfies (5), system (1)-(3) is globally uniformly exponential stable.

This problem was first addressed in [12] for the scalar KdV equation assuming that the damping function $b = b(x)$ is active simultaneously in a neighborhood of both extremes of the interval $(0, L)$. Later on, in [3] the same analysis was developed for the case of the corresponding coupled system considered here. To obtain (9) the authors follow closely the multiplier techniques developed in [17] for the analysis of controllability properties of the scalar KdV equation. However, when using multipliers, the nonlinearity produces extra terms that in [3] were handled by the so-called "Compactness-Uniqueness Argument". Then, the problem of obtaining (9) is reduced to show that the solution which satisfies $b(x)u = b(x)v = 0$ a. e. and $u_x(0,t) = v_x(0,t) = 0$ for all time t, has to be the trivial one. This problem can be viewed as a unique continuation one since $b(x)u = b(x)v = 0$ implies that $(u,v) \equiv (0,0)$ in $\{b(x) > 0\} \times (0,T)$. When the damping term is active in a subset of the form $(0,\delta) \cup (\delta, L-\delta)$, $\delta > 0$, as in [3, 12], the unique continuation property was solved in two steps: first, by extending the solution as being zero outside the interval $(0,L)$, one gets a compactly supported (in space) solution of the Cauchy problem for the KdV equation on the whole line. Then, one applies the classical smoothing properties in [23] to show that the solution is smooth. This allows us to use the unique continuation property results in [7] on smooth solutions to conclude that $u = v \equiv 0$. The general case, that is, the case in which the damping function is active in any open subset of the domain was solved in [13]. To obtain the result they proceed as in [16] and prove that the solutions vanishing in any subset of the domain are necessarily smooth.

The problem we address here, as well as, the global well-posedness of strong and weak solutions was first studied in [20] for the scalar KdV equation and $1 \leq p < 4$. The critical case $p = 4$ was solved in [10] considering data u^0 such that $||u^0||_{L^2(0,L)}$ is small. In both works, the exponential decay is obtained following the analysis described above, i. e., combining multiplier and compactness arguments. The main task of our work is to extend the analysis developed in [10] and [20] for the coupled system (1)-(3). The main difficulty in this context comes from the structure of nonlinearities and the lack of regularity of the solutions we are dealing with. Indeed, as we pointed out before, the unique continuation property can not be applied directly. To overcome this problem we develop a Carleman inequality which allows us to prove directly the unique continuation of weak solution. Such inequality has some resemblance with those developed in [18, 19]. A possible way of attacking the problem of the unique continuation would be to proceed as in [16], i. e., showing that the solutions vanishing in any subinterval are necessarily smooth. We have not pursued this approach due to the difficulties introduced by the nonlinear terms.

We also prove the existence of weak solutions. Uniqueness remains wide open. We rely on the smoothing effects of Kato's type which are exhibited by solutions of the corresponding linear problem combined with the semigroups theory. These smoothing effects together with a contraction principle argument gives the local result. The global result follows from some a priori estimates.

The paper is organized as follows. In section 2 we prove the existence of solutions. Section 3 is devoted to obtain the Carleman estimate. In Section 4 we prove the Unique Continuation Property and the exponential decay. Finally, in Section 5 we present some closed related results.

2. **Existence of solutions.**

2.1. **The linear system.** In this section we study the existence of solutions of the linear system corresponding to (1)-(3):

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + b(x)u = 0, & \text{in } (0,L)\times(0,\infty) \\ b_1 v_t + r v_x + v_{xxx} + b_2 a_3 u_{xxx} + b(x)v = 0, & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = u_x(L,t) = 0, & t\in(0,\infty) \\ v(0,t) = v_x(L,t) = v_x(L,t) = 0, & t\in(0,\infty) \\ u(x,0) = u^0(x) \text{ and } v(x,0) = v^0(x), & x\in(0,L). \end{cases} \tag{10}$$

We introduce the Hilbert space

$$X = [L^2(0,L)]^2$$

endowed with the inner product

$$((u,v),(\varphi,\psi))_X = \frac{b_2}{b_1}\int_0^L u\varphi dx + \int_0^L v\psi dx$$

and consider the operator

$$\mathcal{A} : D(\mathcal{A}) \subset X \to X$$

where

$$D(\mathcal{A}) = \left\{(u,v)\in[H^3(0,L)]^2 \,:\, u(0) = v(0) = u(L) = v(L) = u_x(L) = v_x(L) = 0\right\}$$

and

$$\mathcal{A}(u,v) = \begin{pmatrix} -u_{xxx} - a_3 v_{xxx} \\ -\frac{r}{b_1}v_x - \frac{b_2 a_3}{b_1}u_{xxx} - \frac{1}{b_1}v_{xxx} \end{pmatrix}. \tag{11}$$

With the notation introduced above, system (10) can be now written as an abstract Cauchy problem in X. Setting $U = (u,v)$ we have

$$\frac{dU}{dt} = \mathcal{A}U$$
$$U(0) = U^0 = (u^0, v^0).$$

On the other hand, it is easy to see that the adjoint of the operator $\mathcal{A}$ is the operator $\mathcal{A}^*$ defined by

$$\mathcal{A}^*(\varphi,\psi) = \begin{pmatrix} \varphi_{xxx} + a_3\psi_{xxx} \\ \frac{r}{b_1}\psi_x + \frac{1}{b_1}\psi_{xxx} + \frac{b_2 a_3}{b_1}\varphi_{xxx} \end{pmatrix} \tag{12}$$

where

$$\mathcal{A}^* : D(\mathcal{A}^*) \subset X \to X$$

and

$$D(\mathcal{A}^*) = \left\{(\varphi,\psi)\in[H^3(0,L)]^2 \,:\, \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(0) = 0\right\}.$$

We are interested in the following property of these two operators:

Proposition 1. *The operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ are dissipative in X.*

Proof. Let $u, v \in D(\mathcal{A})$. Multiplying the first equation of (10) by u and integrating by parts in $(0,L)$ we obtain

$$\int_0^L (-u_{xxx} - a_3 v_{xxx})u dx = -\frac{1}{2}u_x^2(0) + a_3\int_0^L v_{xx}u_x dx. \tag{13}$$

On the other hand, multiplying the second equation of (10) by v the following holds

$$\int_0^L (-\frac{r}{b_1}v_x - \frac{1}{b_1}v_{xxx} - \frac{a_3b_2}{b_1}u_{xxx})vdx = -\frac{1}{2b_1}v_x^2(0) + \frac{a_3b_2}{b_1}\int_0^L u_{xx}v_xdx. \tag{14}$$

Then, adding (13) and (14) hand to hand it follows that

$$\begin{aligned}
&(\mathcal{A}(u,v),(u,v))_X \\
= \;& \frac{b_2}{b_1}\int_0^L (-u_{xxx} - a_3v_{xxx})udx + \int_0^L \left(-\frac{r}{b_1}v_x - \frac{b_2a_3}{b_1}u_{xxx} - \frac{1}{b_1}v_{xxx}\right)vdx \\
= \;& -\frac{b_2}{2b_1}u_x^2(0) + \frac{b_2a_3}{b_1}\int_0^L v_{xx}u_xdx - \frac{1}{2b_1}v_x^2(0) + \frac{a_3b_2}{b_1}\int_0^L u_{xx}v_xdx \\
= \;& -\frac{b_2}{2b_1}u_x^2(0) - \frac{1}{2b_1}v_x^2(0) + \frac{a_3b_2}{b_1}\int_0^L (u_xv_x)_xdx \qquad (15)\\
= \;& -\frac{b_2}{2b_1}u_x^2(0) - \frac{1}{2b_1}v_x^2(0) - \frac{a_3b_2}{b_1}u_x(0)v_x(0) \\
= \;& -\frac{1}{2b_1}\left(b_2u_x^2(0) + v_x^2(0) + 2a_3b_2u_x(0)v_x(0)\right) \\
= \;& -\frac{1}{2b_1}\left(\left(\sqrt{b_2}u_x(0) + \sqrt{a_3^2b_2}v_x(0)\right)^2 + \left(1 - a_3^2b_2\right)v_x^2(0)\right) \leq 0.
\end{aligned}$$

Hence, $\mathcal{A}$ is a dissipative operator in X. Analogously, we can deduce that

$$\begin{aligned}
&((\varphi,\psi),\mathcal{A}^*(\varphi,\psi))_X \\
= \;& -\frac{1}{2b_1}\left(\left(\sqrt{b_2}\varphi_x(L) + \sqrt{a_3^2b_2}\psi_x(L)\right)^2 + \left(1 - a_3^2b_2\right)\psi_x^2(L)\right) \leq 0.
\end{aligned}$$

Therefore $\mathcal{A}^*$ is also dissipative in X. □

As a consequence of the Proposition 1 we obtain the global well-posedness for (10):

Theorem 2.1. *Let $(u^0,v^0) \in X$. There exists a unique weak solution $(u,v) = S(\cdot)(u^0,v^0)$ of (10) such that*

$$(u,v) \in C([0,T];X).$$

Moreover, if $(u^0,v^0) \in D(\mathcal{A})$, then (10) has a unique (classical) solution (u,v) such that

$$(u,v) \in C([0,T];D(\mathcal{A})) \cap C^1(0,T;X).$$

Proof. Since $\mathcal{A}$ and $\mathcal{A}^*$ are both dissipative, $\mathcal{A}$ is a closed operator and the respective domains $D(\mathcal{A})$ and $D(\mathcal{A}^*)$ are dense and compactly embedded in X we conclude that $\mathcal{A}$ generates a C^0 semigroup of contractions on X which we denote by $\{S(t)\}_{t\geq 0}$. Classical existence results then give us the global well-posedness for (10). □

An additional regularity result for the weak solutions of (10) is proven in the next Theorem.

Theorem 2.2. *Let $(u^0,v^0) \in X$ and $(u,v) = S(\cdot)(u^0,v^0)$ the weak solution of (10). Then, $(u,v) \in L^2(0,T;[H^1(0,L)]^2) \cap H^1(0,T;[H^{-2}(0,L)]^2)$ and there exists a positive constant c_0 such that*

$$||(u,v)||_{L^2(0,T;[H^1(0,L)]^2)} \leq c_0||(u^0,v^0)||_X.$$

Proof. The proof will be omitted since we use similar arguments as the one used in Lemma 2.5. □

Using the previous results and some interpolation argument, we derive the global well-posedness result in each space $[H^s(0,L)]^2$, for $s \in [0,3]$.

Corollary 1. *For any $s \in [0,3]$ and any $(u^0, v^0) \in [H^s(0,L)]^2$, the solution (u,v) of (10) belongs to $C([0,T];[H^s(0,L)]^2)$.*

2.2. **The nonlinear system.** For $0 \le s \le 3$, let X_s denote the collection of all the functions $w \in H^s(0,L)$ satisfying the s-compatibility conditions

$$\begin{array}{l} w(0) = w(L) = 0 \text{ when } 1/2 < s \le 3/2 \\ w(0) = w(L) = w'(L) = 0 \text{ when } 3/2 < s \le 3. \end{array}$$

X_s is endowed with the Hilbertian norm $||w||_{H^s}$. For any $T > 0$ we introduce the space

$$Y_{s,T} = C([0,T];X_s) \cap L^2([0,T];H^{s+1}(0,L))$$

endowed with the norm

$$||w||_{Y_{s,T}} = ||w||_{C([0,T];H^s(0,L))} + ||w||_{L^2([0,T];H^{s+1}(0,L))}.$$

The following technical Lemmas proved in [20] will be used to obtain the results of this section. For the sake of completeness, we present a sketch of the proof.

Lemma 2.3. *Let $a = a(x)$ be a $\mathcal{C}^0$ function such that, for $0 \le p < 2$,*

$$|a(x)| \le C(1 + |x|^p), \quad \forall\, x \in \mathbb{R},$$

where C is a positive constant. Then, for any $T > 0$ and $u, v \in Y_{0,T}$,

$$\int_0^T \|a(u(.,t))v_x(.,t)\|_{L^2(0,L)}\, dt \le CT^{(2-p)/4} \|u\|^p_{Y_{0,T}} \|v\|_{Y_{0,T}} + CT^{1/2} \left(1 + \|u\|^p_{Y_{0,T}}\right) \|v\|_{Y_{0,T}},$$

where C is a positive constant. If $u, v \in Y_{0,T} \cap L^2(0,T;H^1_0(0,L))$, then

$$\int_0^T \|a(u(.,t))v_x(.,t)\|_{L^2(0,L)}\, dt \le CT^{(2-p)/4} \|u\|^p_{Y_{0,T}} \|v\|_{Y_{0,T}}.$$

Proof. We denote by C a universal positive constant.

Using the assumptions on the function a and the inequality

$$\|u(.,t)\|_{L^\infty(0,L)} \le C \left(\|u(.,t)\|_{L^2(0,L)} + \|u(.,t)\|^{\frac12}_{L^2(0,L)} \|u_x(.,t)\|^{\frac12}_{L^2(0,L)} \right),$$

we get

$$\begin{aligned} \|a(u(.,t))v_x(.,t)\|_{L^2(0,L)} &\le C \|u(.,t)\|^p_{L^\infty(0,L)} \|v_x(.,t)\|_{L^2(0,L)} + C \|v_x(.,t)\|_{L^2(0,L)} \\ &\le C \|u(.,t)\|^{p/2}_{L^2(0,L)} \|u_x(.,t)\|^{p/2}_{L^2(0,L)} \|v_x(.,t)\|_{L^2(0,L)} \\ &+ C \left(1 + \|u(.,t)\|^p_{L^2(0,L)}\right) \|v_x(.,t)\|_{L^2(0,L)}. \end{aligned}$$

The first term on right hand side of the above inequality, when integrated, can be bounded as follows:

$$\begin{aligned}&\int_0^T \|u(.,t)\|_{L^2(0,L)}^{p/2} \|u_x(.,t)\|_{L^2(0,L)}^{p/2} \|v_x(.,t)\|_{L^2(0,L)}\, dt\\ &\leq \sup_{0\leq t\leq T} \|u(.,t)\|_{L^2(0,L)}^{p/2} \left\{\int_0^T \|u_x(.,t)\|_{L^2(0,L)}^{2}\, dt\right\}^{p/4} \times\\ &\left\{\int_0^T \|v_x(.,t)\|_{L^2(0,L)}^{4/(4-p)}\, dt\right\}^{(4-p)/4} \leq T^{(2-p)/4} \|u\|_{Y_{0,T}}^p \|v\|_{Y_{0,T}},\end{aligned}$$

if $0 \leq p \leq 2$. Then,

$$\begin{aligned}&\int_0^T \|a(u(.,t))v_x(.,t)\|_{L^2(0,L)}\, dt\\ &\leq C2^{\frac{2}{p}}T^{(2-p)/4} \|u\|_{Y_{0,T}}^p \|v\|_{Y_{0,T}} + CT^{1/2}\left(1+\|u\|_{Y_{0,T}}^p\right)\|v\|_{Y_{0,T}},\end{aligned}$$

which completes the proof of the first inequality. If $u, v \in Y_{0,T} \cap L^2(0,T; H_0^1(0,L))$, we have

$$\|u(.,t)\|_{L^\infty(0,L)} \leq C\left(\|u(.,t)\|_{L^2(0,L)}^{\frac{1}{2}} \|u_x(.,t)\|_{L^2(0,L)}^{\frac{1}{2}}\right),$$

and the result is obtained in a similar way. □

The next technical Lemma reads as follows.

Lemma 2.4. *For any $T > 0$, $1 \leq p \leq 2$, $b \in L^2(0,L)$ and $u, v, w \in Y_{0,T}$,*

$$\int_0^T \|bu\|_{L^2(0,L)}\, dt \leq CT^{1/2} \|b\|_{L^2(0,L)} \|u\|_{Y_{0,T}}, \tag{16}$$

$$\int_0^T \|uw_x\|_{L^2(0,L)}\, dt \leq CT^{1/4} \|u\|_{Y_{0,T}} \|w\|_{Y_{0,T}}, \tag{17}$$

$$\int_0^T \left\|u\,|w|^{p-1} w_x\right\|_{L^2(0,L)} dt \leq CT^{(2-p)/4} \|u\|_{Y_{0,T}} \|w\|_{Y_{0,T}}^p, \tag{18}$$

$$\int_0^T \left\|u\,|v|^{p-1} w_x\right\|_{L^2(0,L)} dt \leq CT^{(2-p)/4} \|u\|_{Y_{0,T}} \|w\|_{Y_{0,T}} \|v\|_{Y_{0,T}}^{p-1}, \tag{19}$$

where C is a positive constant that depends only on L.

Proof. Estimates (17), (18) and (19) can be obtained following closely the arguments used in the previous Lemma (see also [20]). Therefore, we omit the proofs. To obtain (16) we use a direct computation

$$\begin{aligned}\int_0^T \left(\int_0^L |b(x)|^2 |u(x,t)|^2\, dx\right)^{\frac{1}{2}} dt &\leq \int_0^T \sup_{x\in(0,L)} |u(x,t)| \left(\int_0^L |b(x)|^2\, dx\right)^{\frac{1}{2}} dt\\ &\leq CT^{1/2} \|b\|_{L^2(0,L)} \|u\|_{Y_{0,T}}.\end{aligned}$$

The proof is complete. □

The following Lemma is devoted to global a priori estimates for the solutions of (1)-(3). We point out that the proof of Theorem 2.2 is obtained using the same approach.

Lemma 2.5. *Let $a = a(x)$ be a $\mathcal{C}^0$ function such that, for $0 \leq p < 4$,*

$$|a(x)| \leq C(1 + |x|^p), \quad \forall\, x \in \mathbb{R},$$

where C is a positive constant. Then, for any $T > 0$

$$\begin{aligned}
&\|(u(.,T), v(.,T))\|_X^2 - \left\|(u^0, v^0)\right\|_X^2 \\
&+\frac{1}{b_1}\int_0^T \left[\left(\sqrt{b_2}u_x(0,t) + \sqrt{a_3^2 b_2}v_x(0,t)\right)^2 + \left(1 - a_3^2 b_2\right) v_x^2(0,t)\right] dt \\
&+\frac{2}{b_1}\int_0^T \int_0^L b(x)\left(b_2 u^2 + v^2\right) dxdt = 0
\end{aligned}$$

and

$$\begin{aligned}
&||(u,v)||^2_{L^2(0,T;[H_0^1(0,L)]^2)} \\
&\quad \leq C\left\{(1+T)\left\|(u^0,v^0)\right\|_X^2 + T\left\|(u^0,v^0)\right\|_X^6 + T\left\|(u^0,v^0)\right\|_X^{\frac{8+2p}{4-p}}\right\}
\end{aligned}$$

where C is a positive constant.

Proof. We first introduce the functions

$$A(\varphi) := \int_0^\varphi a(s)ds \quad \text{and} \quad \widetilde{A}(\varphi) := \int_0^\varphi sa(s)ds.$$

The first identity is obtained multiplying the first equation of (1) by u, the second one by v and integrating over $(0,L) \times (0,T)$. Therefore, we observe that

$$\begin{aligned}
&\int_0^T \int_0^L uu_t dxdt = \frac{1}{2}\|u(.,T)\|^2_{L^2(I)} - \frac{1}{2}\left\|u^0\right\|^2_{L^2(I)} \\
&\int_0^T \int_0^L ua(u)u_x dxdt = \int_0^T \int_0^L \widetilde{A}_x(u)dxdt = 0 \\
&\int_0^T \int_0^L uu_{xxx}dxdt = -\frac{1}{2}\int_0^T \int_0^L \frac{\partial}{\partial x}u_x^2 dxdt = \frac{1}{2}\int_0^T u_x^2(0,t)dt \\
&\int_0^T \int_0^L uv_{xxx}dxdt = -\int_0^T \int_0^L u_x v_{xx}dxdt \\
&\int_0^T \int_0^L uvv_x dxdt = \frac{1}{2}\int_0^T \int_0^L u\frac{\partial}{\partial x}v^2 dxdt = -\frac{1}{2}\int_0^T \int_0^L u_x v^2 dxdt \\
&\int_0^T \int_0^L u(uv)_x dxdt = -\int_0^T \int_0^L u_x uv dxdt = \frac{1}{2}\int_0^T \int_0^L u^2 v_x dxdt.
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\int_0^T\int_0^L vv_t dxdt = \frac{1}{2}\|v(.,T)\|^2_{L^2(I)} - \frac{1}{2}\left\|v^0\right\|^2_{L^2(I)}\\
&\int_0^T\int_0^L va(v)v_x dxdt = \int_0^T\int_0^L \widetilde{A}_x(v)dxdt = 0\\
&\int_0^T\int_0^L vv_{xxx} dxdt = -\frac{1}{2}\int_0^T\int_0^L \frac{\partial}{\partial x}v_x^2 dxdt = \frac{1}{2}\int_0^T v_x^2(0,t)dt\\
&\int_0^T\int_0^L vu_{xxx} dxdt = -\int_0^T\int_0^L v_x u_{xx} dxdt\\
&\int_0^T\int_0^L vuu_x dxdt = \frac{1}{2}\int_0^T\int_0^L v\frac{\partial}{\partial x}u^2 dxdt = -\frac{1}{2}\int_0^T\int_0^L v_x u^2 dxdt\\
&\int_0^T\int_0^L v(uv)_x dxdt = -\int_0^T\int_0^L v_x uv dxdt = \frac{1}{2}\int_0^T\int_0^L v^2 u_x dxdt.
\end{aligned}$$

Since

$$\int_0^T\int_0^L u_x v_{xx} dxdt = -\int_0^T u_x(0,t)v_x(0,t)dt - \int_0^T\int_0^L v_x u_{xx} dxdt$$

we can multiply the first equation in (1) by b_2 and add the above identities hand to hand to obtain

$$\begin{aligned}
&\frac{b_2}{2}\|u(.,T)\|^2_{L^2(0,L)} + \frac{b_1}{2}\|v(.,T)\|^2_{L^2(0,L)} + \frac{b_2}{2}\int_0^T u_x^2(0,t)dt\\
&+b_2a_3\int_0^T u_x(0,t)v_x(0,t)dt + \frac{1}{2}\int_0^T v_x^2(0,t)dt + b_2\int_0^T\int_0^L b(x)u^2 dxdt\\
&+\int_0^T\int_0^L b(x)v^2 dxdt = \frac{b_2}{2}\left\|u^0\right\|^2_{L^2(0,L)} + \frac{b_1}{2}\left\|v^0\right\|^2_{L^2,(0,L)},
\end{aligned}$$

that is,

$$\begin{aligned}
&\|(u(.,T),v(.,T))\|^2_X - \left\|(u^0,v^0)\right\|^2_X\\
&+\frac{1}{b_1}\int_0^T\left[\left(\sqrt{b_2}u_x(0,t) + \sqrt{a_3^2b_2}v_x(0,t)\right)^2 + \left(1-a_3^2b_2\right)v_x^2(0,t)\right]dt \qquad (20)\\
&+\frac{2}{b_1}\int_0^T\int_0^L b(x)\left(b_2u^2+v^2\right)dxdt = 0.
\end{aligned}$$

To bound the solution in $L^2(0,T;[H^1(0,L)]^2)$ we proceed in the same way using the multipliers xu for the first equation and xv for the second one. We obtain

$$\begin{aligned}
&\int_0^T\int_0^L xuu_t dxdt = \frac{1}{2}\int_0^L xu^2(x,T)dx - \frac{1}{2}\int_0^L x(u^0)^2 dx\\
&\int_0^T\int_0^L xua(u)u_x dxdt = \int_0^T\int_0^L x\widetilde{A}_x(u)dxdt = -\int_0^T\int_0^L \widetilde{A}(u)dxdt\\
&\int_0^T\int_0^L xuu_{xxx}dxdt = \frac{3}{2}\int_0^T\int_0^L u_x^2 dxdt\\
&\int_0^T\int_0^L xuv_{xxx}dxdt = 2\int_0^T\int_0^L u_xv_x dxdt + \int_0^T\int_0^L xu_{xx}v_x dxdt\\
&\int_0^T\int_0^L xuvv_x dxdt = -\frac{1}{2}\int_0^T\int_0^L (uv^2 + xu_xv^2)dxdt\\
&\int_0^T\int_0^L xu(uv)_x dxdt = \frac{1}{2}\int_0^T\int_0^L xu^2v_x dxdt - \frac{1}{2}\int_0^T\int_0^L vu^2 dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T\int_0^L xvv_t dxdt = \frac{1}{2}\int_0^L xv^2(x,T)dx - \frac{1}{2}\int_0^L x(v^0)^2 dx\\
&\int_0^T\int_0^L xvv_x dxdt = -\frac{1}{2}\int_0^T\int_0^L v^2 dxdt\\
&\int_0^T\int_0^L xva(v)v_x dxdt = \int_0^T\int_0^L x\widetilde{A}_x(v)dxdt = -\int_0^T\int_0^L \widetilde{A}(v)dxdt\\
&\int_0^T\int_0^L xvv_{xxx}dxdt = \frac{3}{2}\int_0^T\int_0^L v_x^2 dxdt\\
&\int_0^T\int_0^L xvu_{xxx}dxdt = \int_0^T\int_0^L v_xu_x dxdt - \int_0^T\int_0^L xv_xu_{xx}dxdt\\
&\int_0^T\int_0^L xvuu_x dxdt = -\frac{1}{2}\int_0^T\int_0^L (vu^2 + xv_xu^2)dxdt\\
&\int_0^T\int_0^L xv(uv)_x dxdt = \frac{1}{2}\int_0^T\int_0^L xv^2u_x dxdt - \frac{1}{2}\int_0^T\int_0^L uv^2 dxdt.
\end{aligned}$$

As a consequence, the following holds

$$\begin{aligned}
&\frac{3b_2}{2}\int_0^T\int_0^L u_x^2 dxdt + 3a_3b_2\int_0^T\int_0^L u_xv_x dxdt + \frac{3}{2}\int_0^T\int_0^L v_x^2 dxdt\\
&+\frac{b_2}{2}\int_0^L xu^2(x,T)dx + \frac{b_1}{2}\int_0^L xv^2(x,T)dx + b_2\int_0^T\int_0^L b(x)xu^2 dxdt\\
&+\int_0^T\int_0^L b(x)xv^2 dxdt = \frac{b_2}{2}\int_0^L x(u^0)^2 dx + \frac{b_1}{2}\int_0^L x(v^0)^2 dx\\
&+a_1b_2\int_0^T\int_0^L uv^2 dxdt + a_2b_2\int_0^T\int_0^L vu^2 dxdt\frac{r}{2}\int_0^T\int_0^L v^2 dxdt\\
&+b_2\int_0^T\int_0^L \widetilde{A}(u)dxdt + \int_0^T\int_0^L \widetilde{A}(v)dxdt.
\end{aligned}$$

Then, from (20) we deduce that

$$\begin{aligned} 3b_2\int_0^T\int_0^L u_x^2dxdt + 6a_3b_2\int_0^T\int_0^L u_xv_xdxdt + 3\int_0^T\int_0^L v_x^2dxdt \\ \leq (b_1L+Tr)||(u^0,v^0)||_X^2 + 2a_1b_2\int_0^T\int_0^L uv^2dxdt + 2a_2b_2\int_0^T\int_0^L vu^2dxdt \\ +2b_2\int_0^T\int_0^L \widetilde{A}(u)dxdt + 2\int_0^T\int_0^L \widetilde{A}(v)dxdt. \end{aligned} \tag{21}$$

Moreover, choosing ε such that $0<\sqrt{a_3^2b_2}<\varepsilon<1$, we obtain

$$2a_3b_2u_xv_x = 2\left(\varepsilon\sqrt{b_2}u_x\right)\left(\frac{1}{\varepsilon}\sqrt{a_3^2b_2}\right) \geq -\varepsilon^2b_2u_x^2 - \frac{a_3^2b_2}{\varepsilon^2}v_x^2.$$

This inequality together with (21) allow us to deduce that

$$\begin{aligned} 3b_2(1-\varepsilon^2)\int_0^T\int_0^L u_x^2dxdt + 3\left(1-\frac{a_3^2b_2}{\varepsilon^2}\right)\int_0^T\int_0^L v_x^2dxdt \\ \leq (b_1L+Tr)||(u^0,v^0)||_X^2 + 2a_1b_2\int_0^T\int_0^L uv^2dxdt + 2a_2b_2\int_0^T\int_0^L vu^2dxdt \\ +2b_2\int_0^T\int_0^L \widetilde{A}(u)dxdt + 2\int_0^T\int_0^L \widetilde{A}(v)dxdt. \end{aligned} \tag{22}$$

The next steps are devoted to estimate the terms on the right hand side of (22).

From the assumptions on the function a, we have

$$\left|\widetilde{A}(\varphi)\right| \leq C\left(\frac{\varphi^2}{2}+\frac{|\varphi|^{p+2}}{p+2}\right),$$

for some constant $C>0$. Then, from (20) and the Gagliardo-Nirenberg and Young's inequalities, we obtain positive constants C and C' such that

$$\begin{aligned} \left|\int_0^T\int_0^L \widetilde{A}(u)dxdt\right| &\leq \frac{C}{2}\int_0^T\int_0^L |u|^2\,dxdt + \frac{C}{p+2}\int_0^T\int_0^L |u|^{p+2}\,dxdt \\ &\leq \frac{CTb_1}{2b_2}||(u^0,v^0)||_X^2 + \frac{C}{p+2}\int_0^T ||u||_{L^2(0,L)}^2||u||_{L^\infty(0,L)}^p dt \\ &\leq \frac{CTb_1}{2b_2}||(u^0,v^0)||_X^2 + \frac{2^{\frac{p}{2}}C}{p+2}\int_0^T ||u||_{L^2(0,L)}^{2+\frac{p}{2}}||u_x||_{L^2(0,L)}^{\frac{p}{2}} dt \\ &\leq \frac{CTb_1}{2b_2}||(u^0,v^0)||_X^2 + \frac{2^{\frac{p}{2}}CT^{1-\frac{p}{4}}}{(p+2)}\left(\frac{b_1}{b_2}\right)^{\frac{1}{2}}||(u^0,v^0)||_X^{2+\frac{p}{2}}\left(\int_0^T ||u_x||^2dt\right)^{\frac{p}{4}} \\ &\leq \frac{CTb_1}{2b_2}||(u^0,v^0)||_X^2 + \frac{2C'T}{\delta^{\frac{p}{4-p}}}||(u^0,v^0)||_X^{\frac{8+2p}{4-p}} + \frac{3\delta}{2}\int_0^T ||u_x||^2dt, \end{aligned} \tag{23}$$

for any $\delta>0$. Analogously, we get

$$\left|\int_0^T\int_0^L \widetilde{A}(v)dxdt\right| \leq \frac{\widetilde{C}T}{2b_2}||(u^0,v^0)||_X^2 + \frac{2\widetilde{C'}}{\delta^{\frac{p}{4-p}}}||(u^0,v^0)||_X^{\frac{8+2p}{4-p}} + \frac{3\delta}{2}\int_0^T ||v_x||^2dt, \tag{24}$$

for any $\delta > 0$, where $\widetilde{C}, \widetilde{C'} > 0$. Now, letting $p = 2$ we can proceed as in the previous estimate to obtain

$$\begin{aligned}
&\left| b_2 a_1 \int_0^T \int_0^L uv^2 dxdt \right| \leq \frac{b_2 |a_1|}{2} \int_0^T \int_0^L u^2 + v^4 dxdt \\
&\leq \frac{b_2 |a_1| T}{2} \left\|(u^0, v^0)\right\|_X^2 + \frac{C b_2 |a_1| T^{\frac{1}{2}}}{2} \left\|(u^0, v^0)\right\|_X^3 \left(\int_0^T \int_0^L v_x^2 dxdt \right)^{\frac{1}{2}} \\
&\leq \frac{b_2 |a_1| T}{2} \left\|(u^0, v^0)\right\|_X^2 + C^2 b_2^2 a_1^2 \frac{T}{\lambda} \left\|(u^0, v^0)\right\|_X^6 + \frac{3\lambda}{2} \int_0^T \int_0^L v_x^2 dxdt
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\left| a_2 \int_0^T \int_0^L vu^2 dxdt \right| \\
&\leq \frac{|a_2| T}{2} \left\|(u^0, v^0)\right\|_X^2 + C^2 a_2^2 \frac{T}{\lambda} \left\|(u^0, v^0)\right\|_X^6 + \frac{3\lambda}{2} \int_0^T \int_0^L u_x^2 dxdt.
\end{aligned} \tag{26}$$

Returning to (22) and taking estimates (23) up to (26) into account we deduce that

$$\begin{aligned}
&3b_2(1 - \varepsilon^2 - 2\lambda) \int_0^T \int_0^L u_x^2 dxdt + 3\left(1 - \frac{a_3^2 b_2}{\varepsilon^2} - 2\lambda\right) \int_0^T \int_0^L v_x^2 dxdt \\
&\leq C_1(1 + T) \left\|(u^0, v^0)\right\|_X^2 + \frac{C_2 T}{\lambda} \left\|(u^0, v^0)\right\|_X^6 + \frac{C_3 T}{\lambda^{\frac{p}{4-p}}} \left\|(u^0, v^0)\right\|_X^{\frac{8+2p}{4-p}},
\end{aligned} \tag{27}$$

where C_1, C_2 and C_3 are positive constants. For $\varepsilon > 0$ defined before, we can choose $\lambda > 0$ such that $1 - \varepsilon^2 - 2\lambda > 0$ and $1 - \frac{b_2 a_3^2}{\varepsilon^2} - 2\lambda > 0$, which completes the proof. □

In the sequel, we prove a well-posedness result constituting the basic ingredient for obtaining the main result of this section.

Lemma 2.6. *Let $a = a(x)$ be a $\mathcal{C}^1$ function such that*

$$|a(x)| \leq C(1 + |x|^p) \quad \text{and} \quad |a'(x)| \leq C(1 + |x|^{p-1}), \quad \forall x \in \mathbb{R},$$

where C is a positive constant and $1 \leq p < 2$. Then, for any $T > 0$ and $(u^0, v^0) \in X$ system (1)-(3) has a unique global solution.

Proof. By computations similar to those performed in the proof of Lemma 2.5, we obtain that for any $f = (f_1, f_2) \in C^1([0, T]; X)$ and any $U^0 = (u^0, v^0) \in D(\mathcal{A})$, the solution $U = (u, v)$ of the system

$$\begin{cases}
u_t + u_{xxx} + a_3 v_{xxx} + b(x)u = f_1, & \text{in } (0, L) \times (0, \infty) \\
b_1 v_t + r v_x + v_{xxx} + b_2 a_3 u_{xxx} + b(x)v = f_2, & \text{in } (0, L) \times (0, \infty) \\
u(0, t) = u(L, t) = u_x(L, t) = 0, & t \in (0, \infty) \\
v(0, t) = v_x(L, t) = v_x(L, t) = 0, & t \in (0, \infty) \\
u(x, 0) = u^0(x) \text{ and } v(x, 0) = v^0(x), & x \in (0, L)
\end{cases}$$

fulfills

$$\sup_{0 \leq t \leq T} ||U(t)||_X + (\int_0^T \int_0^L (u_x^2 + v_x^2) dxdt)^{\frac{1}{2}} \leq C \left(||u^0||_X + \int_0^T ||f||_X dt \right)$$

for some constant $C = C(T)$ nondecreasing in T. A density argument yields that $U \in C([0, T]; X)$ when $f \in L^1(0, T; X)$ and $U^0 \in X$. This will be done by applying the fixed point argument. Therefore, we use the variation of the constant formula to rewrite system in the integral form

In order to obtain the result we first prove the local existence.

$$U(t) = S(t)(u^0, v^0) - \int_0^t S(t-\tau)\left(a(u)u_x + a_1 vv_x + a_2(uv)_x + b(x)u, \frac{a(v)}{b_1}v_x \right.$$
$$\left. + \frac{a_2b_2}{b_1}uu_x + \frac{a_1b_2}{b_1}(uv)_x + \frac{b(x)}{b_1}v\right) d\tau,$$

where $\{S(t)\}_{t\geq 0}$ is the C^0 semigroup obtained in Theorem 2.1. Then, for positive constants R and θ, to be chosen later, we consider the set

$$B_{R,\theta} = \{U = (u,v) \in [Y_{0,\theta} \cap L^2(0,T;H_0^1(0,L))]^2 : ||U||_{Y^2_{0,\theta}} \leq R\}.$$

With the notation above, for $U^0 \in X$ fixed we introduce the map P on $B_{R,\theta}$ as follows

$$PU(t) = S(t)(u^0, v^0) - \int_0^t S(t-\tau)\left(a(u)u_x + a_1 vv_x + a_2(uv)_x + b(x)u, \frac{a(v)}{b_1}v_x \right.$$
$$\left. + \frac{a_2b_2}{b_1}uu_x + \frac{a_1b_2}{b_1}(uv)_x + \frac{b(x)}{b_1}v\right) d\tau.$$

First we have to prove that P maps $B_{R,\theta}$ into itself. Therefore, from Lemmas 2.3 and 2.4 we obtain positive constants C_0, C_1, C_2 and C_3, such that

$$\begin{aligned}
&\|\Gamma(u,v)\|_{Y^2_{0,\theta}} \leq C_0 \left\|(u^0,v^0)\right\|_X + C_1 \int_0^\theta \Big(\|a(u)u_x\|_{L^2(0,L)} + \|a_1 vv_x\|_{L^2(0,L)} \\
&+ \|a_2(uv)_x\|_{L^2(0,L)} + \|b(x)u\|_{L^2(0,L)} + \frac{1}{b_1}\|a(v)v_x\|_{L^2(0,L)} \\
&+ \frac{1}{b_1}\|b_2a_2uu_x\|_{L^2(0,L)} + + \frac{1}{b_1}\|b_2a_1(uv)_x\|_{L^2(0,L)} + \frac{1}{b_1}\|b(x)v\|_{L^2(0,L)}\Big) d\tau \\
&\leq C_0 \left\|(u^0,v^0)\right\|_X + C_1\theta^{(2-p)/4}\left(\|u\|^{p+1}_{Y_{0,\theta}} + \|v\|^{p+1}_{Y_{0,\theta}}\right) \qquad (28)\\
&+ C_2\theta^{1/2}\left(1 + \|b\|_{L^2(0,L)}\right)\left(\|u\|_{Y_{0,\theta}} + \|v\|_{Y_{0,\theta}}\right) + C_3\theta^{1/4}\left(\|u\|_{Y_{0,\theta}} + \|v\|_{Y_{0,\theta}}\right)^2 \\
&\leq C_0 \left\|(u^0,v^0)\right\|_X + C_1\theta^{(2-p)/4}R^p\left(\|u\|_{Y_{0,\theta}} + \|v\|_{Y_{0,\theta}}\right) \\
&+ C_2\theta^{1/2}\left(1 + \|b\|_{L^2(0,L)}\right)\left(\|u\|_{Y_{0,\theta}} + \|v\|_{Y_{0,\theta}}\right) + 2C_3\theta^{1/4}R\left(\|u\|_{Y_{0,\theta}} + \|v\|_{Y_{0,\theta}}\right),
\end{aligned}$$

for any $(u,v) \in B_{R,\theta}$. We should also prove that P is a contraction from $B_{R,\theta}$ into itself for some R and T. Then, for any (u,v) and (ϕ,φ) in $B_{R,\theta}$ we note that

$$\begin{aligned}
&P(u,v) - P(\phi,\varphi) \\
&= -\int_0^t S(t-\tau)\left(a(u)(u-\phi)_x + (a(u)-a(\phi))\phi_x + a_1 v(v-\varphi)_x + a_1(v-\varphi)\varphi_x\right. \\
&+ a_2(u-\phi)_x v + a_2(u-\phi)v_x + a_2\phi_x(v-\varphi) + a_2\phi(v-\varphi)_x + b(x)(u-\phi), \\
&\frac{a(v)}{b_1}(v-\varphi)_x + (a(v)-a(\varphi))\frac{\varphi_x}{b_1} + \frac{a_2b_2}{b_1}u(u-\phi)_x + \frac{a_2b_2}{b_1}(u-\phi)\phi_x \\
&+ \tfrac{a_1b_2}{b_1}(u-\phi)_x v + \frac{a_1b_2}{b_1}(u-\phi)v_x + \frac{a_1b_2}{b_1}\phi_x(v-\varphi) + \frac{a_1b_2}{b_1}\phi(v-\varphi)_x \\
&\left.+ \tfrac{b(x)}{b_1}(v-\varphi)\right)d\tau.
\end{aligned}$$

Since

$$\int_0^\theta \|(a(u)-a(\phi))\phi_x\|_{L^2(0,L)}\,d\tau \le \int_0^\theta \left\|(1+|u|^{p-1}+|\phi|^{p-1})(u-\phi)\phi_x\right\|_{L^2(0,L)}d\tau$$

from Lemmas 2.3 and 2.4 we obtain

$$\begin{aligned}
&\|P(u,v)-P(\phi,\varphi)\|_{Y^2_{0,\theta}}\\
&\le C\theta^{(2-p)/4}\|u-\phi\|_{Y_{0,\theta}}\left(\|u\|^p_{Y_{0,\theta}}+\|u\|^{p-1}_{Y_{0,\theta}}\|\phi\|_{Y_{0,\theta}}+\|\phi\|^p_{Y_{0,\theta}}\right)\\
&+C\theta^{1/2}\|u-\phi\|_{Y_{0,\theta}}\left(1+\|b\|_{L^2(0,L)}\right)\\
&+C\theta^{1/4}\|u-\phi\|_{Y_{0,\theta}}\left(\|u\|_{L^2(0,L)}+2\|\phi\|_{L^2(0,L)}+4\|v\|_{L^2(0,L)}\right)\\
&+C\theta^{1/4}\|v-\varphi\|_{Y_{0,\theta}}\left(\|v\|_{L^2(0,L)}+2\|\varphi\|_{L^2(0,L)}+4\|\phi\|_{L^2(0,L)}\right)\\
&+C\theta^{1/2}\|v-\varphi\|_{Y_{0,\theta}}\left(1+\|b\|_{L^2(0,L)}\right)\\
&+C\theta^{(2-p)/4}\|v-\varphi\|_{Y_{0,\theta}}\left(\|v\|^p_{Y_{0,\theta}}+\|v\|^{p-1}_{Y_{0,\theta}}\|\varphi\|_{Y_{0,\theta}}+\|\varphi\|^p_{Y_{0,\theta}}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|P(u,v)-P(\phi,\varphi)\|_{Y^2_{0,\theta}} \le C_1\theta^{(2-p)/4}R^p\left(\|u-\phi\|_{Y_{0,\theta}}+\|v-\varphi\|_{Y_{0,\theta}}\right)\\
&+C_2\theta^{1/2}\left(1+\|b\|_{L^2(0,L)}\right)\left(\|u-\phi\|_{Y_{0,\theta}}+\|v-\varphi\|_{Y_{0,\theta}}\right)\\
&+C_3\theta^{1/4}R\left(\|u-\phi\|_{Y_{0,\theta}}+\|v-\varphi\|_{Y_{0,\theta}}\right)\\
&\le C_2\theta^{1/2}\left(1+\|b\|_{L^2(0,L)}\right)\|(u,v)-(\phi,\varphi)\|_{Y^2_{0,\theta}}\\
&+C_1\theta^{(2-p)/4}R^p\|(u,v)-(\phi,\varphi)\|_{Y^2_{0,\theta}}+C_3\theta^{1/4}R\|(u,v)-(\phi,\varphi)\|_{Y^2_{0,\theta}}.
\end{aligned} \tag{29}$$

Estimate (28) shows that P sends $Y^2_{0,\theta}$ into itself if we choose $R=2C_0\left\|(u^0,v^0)\right\|$ and θ (possibly small) satisfying

$$C_1\theta^{(2-p)/4}R^p+C_2\theta^{1/2}\left(1+\|b\|_{L^2(0,L)}\right)+C_3\theta^{1/4}R\le\frac{1}{2}.$$

Indeed, in this case we obtain

$$\|P(u,v)\|_{Y^2_{0,\theta}}\le R.$$

With this choice of θ and R from estimate (29) it follows that

$$\|P(u,v)-P(\phi,\varphi)\|_{Y^2_{0,\theta}}\le\frac{1}{2}\|(u,v)-(\phi,\varphi)\|_{Y^2_{0,\theta}},$$

which shows that P is a contraction in $Y^2_{0,\theta}$.

This concludes the proof of the local existence and uniqueness. The global existence comes from the a priori bound of the solutions. □

The same arguments as those in the proof of Lemma 2.6 and Corollary 1 lead to the following local well-posedness result.

Corollary 2. *Let $b\in H^1(0,L)$, let $a=a(x)$ be a $\mathcal{C}^2$ function such that*

$$|a(x)|\le C(1+|x|^p),\ |a'(x)|\le C(1+|x|^{p-1}),\ |a''(x)|\le C(1+|x|^{p-2}),\quad \forall\, x\in\mathbb{R},$$

where C is a positive constant and $p\ge 2$. Then, for any $(u^0,v^0)\in[H^1_0(0,L)]^2$ there exists a $T^*>0$, depending only on $||(u^0,v^0)||_{[H^1(0,L)]^2}$, such that system (1)-(3) admits a unique solution $(u,v)\in L^\infty(0,T^*;[H^1_0(0,L)]^2)$.

Proof. The ideas involved in the proof follow closely the previous arguments and those presented in the proof of Lemmas 2.11 and 2.12 in [20]. The extension of such results for the model under consideration was proved in [4], Proposition 5.3 (see also Remark 5.5 in [6]). Therefore, we omit the details.

We note that in order to apply the fixed point argument we first rewrite the system in the following integral form

$$U(t) = T(t)(u^0, v^0)$$
$$- \int_0^t T(t-\tau)\left(a(u)u_x + a_1 v v_x + a_2(uv)_x, \frac{a(v)}{b_1}v_x + \frac{a_2 b_2}{b_1}u u_x + \frac{a_1 b_2}{b_1}(uv)_x\right) d\tau,$$

where $\{T(t)\}_{t\geq 0}$ denotes the C^0 semigroup property generated by the linear part of the system.

To obtain the global well-posedness one needs to establish the corresponding global a priori estimate in the space $H^1(0,L)$, which is not available. $\square$

Using Lemma 2.6 we prove the main result of this section:

Theorem 2.7. *Let $a = a(x)$ be a $\mathcal{C}^2$ function such that*

$$|a(x)| \leq C(1+|x|^p),\ |a'(x)| \leq C(1+|x|^{p-1}),\ |a''(x)| \leq C(1+|x|^{p-2}), \quad \forall\, x \in \mathbb{R},$$

where C is a positive constant and $1 \leq p < 4$. Then, for any $(u^0, v^0) \in X$, system (1)-(3) admits at least one solution $(u,v) \in \mathcal{C}_w(\mathbb{R}; X) \cap L^2_{loc}(\mathbb{R}^+; [H^1(0,L)]^2)$.

Proof. We consider the sequence of functions $\{a_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R};\mathbb{R})$, such that

$$\begin{aligned} &a_n(\mu) \to a(\mu) \quad \text{uniformly on each compact set of } \mathbb{R}, \\ &\left|a_n^j(\mu)\right| \leq C\left(1+|\mu|^{p-j}\right), \quad \forall\, n \geq 0, \quad \forall\, \mu \in \mathbb{R}, \quad j = 0,1,2, \text{ where } C > 0. \end{aligned} \tag{30}$$

Observe that

$$|a_n(\mu)| \leq C(1+|\mu|^p) \text{ and } |a_n'(\mu)| \leq C(n)(1+|\mu|^{p-1}).$$

For each n, Lemma 2.6 guarantees the existence of a unique function $U = (u,v) \in \mathcal{C}(\mathbb{R}^+; X) \cap L^2(\mathbb{R}; [H_0^1(0,L)]^2)$ which solves

$$\begin{cases} u_{n,t} + a_n(u_n)u_{n,x} + u_{n,xxx} + a_3 v_{n,xxx} + a_1 v_n v_{n,x} + a_2(u_n v_n)_x \\ \qquad\qquad + b(x)u_n = 0, \\ b_1 v_{n,t} + r v_{n,x} + a_n(v_n)v_{n,x} + b_2 a_3 u_{n,xxx} + v_{n,xxx} + b_2 a_2 u_n u_{n,x} \\ \qquad\qquad + b_2 a_1 (u_n v_n)_x + b(x)v_n = 0, \\ u_n(0,t) = u_n(L,t) = u_{n,x}(L,t) = 0, \\ v_n(0,t) = v_n(L,t) = v_{n,x}(L,t) = 0, \\ u_n(x,0) = u_0(x),\ \ v_n(x,0) = v_0(x), \end{cases} \tag{31}$$

where $0 < x < L$ and $t > 0$. Moreover, from Lemma 2.5 we get

$$\begin{aligned} &\|(u_n(.,T), v_n(.,T))\|_X^2 \\ &+\frac{1}{b_1}\int_0^T [(\sqrt{b_2}u_{n,x}(0,t) + \sqrt{a_3^2 b_2}v_{n,x}(0,t))^2 + \left(1-a_3^2 b_2\right) v_{n,x}^2(0,t)]dt \\ &+\frac{2}{b_1}\int_0^T\int_0^L b(x)\left(b_2 u_n^2 + v_n^2\right) dxdt = \|(u_0,v_0)\|_X^2 \end{aligned}$$

and

$$\int_0^T \int_0^L u_{n,x}^2 dxdt + \int_0^T \int_0^L v_{n,x}^2 dxdt$$
$$\leq C\left\{(1+T)\left\|(u^0,v^0)\right\|_X^2 + T\left\|(u^0,v^0)\right\|_X^6 + T\left\|(u^0,v^0)\right\|_X^{\frac{8+2p}{4-p}}\right\},$$

for any $T \geq 0$, where $C > 0$. The estimates above show that the sequence $\{U_n\}_{n\in\mathbb{N}} = \{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+; X) \cap L^2_{loc}(\mathbb{R}^+; [H_0^1(0,L)]^2)$. Therefore, there exists a function $U = (u,v)$ and a subsequence, still denoted by the same index n, such that

$$U_n \to U \quad \text{weakly }^* \quad \text{in} \quad L^\infty(\mathbb{R}^+; X) \tag{32}$$
$$U_n \to U \quad \text{weakly} \quad \text{in} \quad L^2_{loc}(\mathbb{R}^+; [H_0^1(0,L)]^2). \tag{33}$$

The goal is to pass the limit in (31) to prove that $U = (u,v)$ is a weak solution of the problem. The main difficult is the study of the nonlinear terms. In order to do that we introduce the functions

$$A_n(\varphi) := \int_0^\varphi a_n(s)ds \quad \text{and} \quad \widetilde{A}_n(\varphi) := \int_0^\varphi s a_n(s)ds \tag{34}$$

and prove that

$$(A_n(u_n), A_n(v_n)) \to (A(u), A(v)) \quad \text{in} \quad [\mathcal{D}'((0,L)\times(0,+\infty))]^2, \qquad \text{as } n \to \infty. \tag{35}$$

To obtain the result, need that the following holds:

Claim 1. For any $T > 0$ and $\alpha \in (1, \frac{6}{p+1}]$, the sequence $\{(A_n(u_n), A_n(v_n))\}_{n\in\mathbb{N}}$ is bounded in the space $[L^\alpha((0,T)\times(0,L))]^2$.

Indeed, from (30)

$$|A_n(\varphi)| \leq \int_0^\varphi |a_n(s)|\, ds \leq \int_0^\varphi C_1(1+|s|^p)dv \leq C_1(1+|\varphi|^{p+1}),$$

which give that

$$|A_n(\varphi)|^\alpha \leq C_2(1+|\varphi|^{\alpha(p+1)}),$$

for some positive constants C_1 and C_2 that does not depend on n. Then, since $\frac{\alpha(p+1)-2}{2} \leq 2$ we can combine Lemma 2.5 and Gagliardo-Nirenberg's inequality to obtain

$$\int_0^T \int_0^L |A_n(u_n)|^\alpha \, dxdt \leq C_2\left(TL + \int_0^T \int_0^L |u_n|^{\alpha(p+1)}\, dxdt\right)$$
$$\leq C_2\left(TL + \int_0^T \|u_n\|_{L^\infty(0,L)}^{\alpha(p+1)-2} \|u_n\|_{L^2(0,L)}^2\, dt\right)$$
$$\leq C_2\left(TL + C_3\int_0^T \|u_n\|_{L^2(0,L)}^{\frac{\alpha(p+1)-2}{2}} \|u_{n,x}\|_{L^2(0,L)}^{\frac{\alpha(p+1)-2}{2}} \|u_n\|_{L^2(0,L)}^2\, dt\right)$$
$$\leq C_2\left(TL + C_3\left\|u^0\right\|^{\frac{\alpha(p+1)}{2}+1} \int_0^T \|u_{n,x}\|_{L^2(0,L)}^{\frac{\alpha(p+1)-2}{2}}\, dt\right) \leq C_4,$$

for some $C_3 > 0$ and $C_4 = C_4(T, ||u^0||_X) > 0$. Analogously, we have

$$\int_0^T \int_0^L |A_n(v_n)|^\alpha \, dxdt \quad \leq \quad C_4.$$

This completes the proof of the Claim 1.

CLAIM 2. For any $T > 0$ and α as in Step 1, the sequence $\{U_{n,t}\}_{n\in\mathbb{N}} = \{(u_{n,t}, v_{n,t})\}_{n\in\mathbb{N}}$ is bounded in $L^\alpha(0,T;[H^{-2}(0,L)]^2)$.

The estimates obtained for $\{U_n\}_{n\in\mathbb{N}}$ guarantees that the terms $v_n v_{n,x}$, $(u_n v_n)_x$, $u_n u_{n,x}$ and $(u_n v_n)_x$ that appears in (31) are bounded in $L^\alpha(0,T;[H^{-2}(0,L)]^2)$. Indeed, observe that $\alpha \le 2$, $L^1(0,L) \subset H^{-2}(0,L)$ and

$$\|u_n v_{n,x}\|_{L^2(0,T;L^1(0,L))} \le \|u_n\|_{L^\infty(0,T;L^2(0,L))} \|v_n\|_{L^2(0,T;H_0^1(0,L))}.$$

The same result is valid for the linear terms. On the other hand, due to the embedding $L^\alpha(0,L) \hookrightarrow H^{-1}(0,L)$ and Claim 1 we conclude that

$$\partial_x(A_n(u_n), A_n(v_n)) = (a(u_n)u_{n,x}, a(v_n)v_{n,x}) \text{ is bounded in } [L^2(0,T;[H^{-2}(0,L)]^2)]^2.$$

Now, noting that

$$u_{n,t} = -(a(u_n)u_{n,x} + u_{n,xxx} + a_3 v_{n,xxx} + a_1 v_n v_{n,x} + a_2(u_n v_n)_x + b(x)u_n),$$
$$b_1 v_{n,t} =$$
$$-(r v_{n,x} + a(v_n)v_{n,x} + b_2 a_3 u_{n,xxx} + v_{n,xxx} + b_2 a_2 u_n u_{n,x} + b_2 a_1 (u_n v_n)_x + b(x) v_n)$$

we obtain the result.

CLAIM 3. (*See Ergoroff Theorem*) Let Ω be an open subset in $\mathbb{R}^N$. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^p(\Omega)$, with $1 < p < \infty$, such that $f_n \rightharpoonup f$ and $f_n(x) \to g(x)$ a.e., as $n \to \infty$, then $f(x) = g(x)$ a. e.

Now, we can complete the proof. Since $\{U_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;[H_0^1(0,L)]^2)$ and $\{U_{n,t}\}_{n\in\mathbb{N}}$ in $L^2(0,T;[H^{-2}(0,L)]^2)$, from Corollary 4 in [21], we obtain a subsequence, still denoted by the same index, such that

$$U_n \to U \quad \text{in} \quad [L^2(0,T;L^2(0,L))]^2, \quad \text{strongly and a. e.} \tag{36}$$

Then, from (30) and (36), we have

$$(A_n(u_n(x,t)), A_n(v_n(x,t))) \to (A(u(x,t)), A(v(x,t))) \text{ a. e. for } (x,t) \in (0,L)\times\mathbb{R}^+.$$

Moreover, from Claim 1, we can pass to a subsequence (if necessary) to obtain a function $g = (g_1, g_2) \in L^\alpha(0,T;[L^\alpha(0,L)]^2)$ for which

$$(A_n(u_n(x,t)), A_n(v_n(x,t))) \rightharpoonup (g_1, g_2) \quad \text{in } [L^\alpha(0,T;L^\alpha(0,L))]^2.$$

Consequently, Claim 3 allows us to conclude that $(A(u(x,t)), A(v(x,t))) = (g_1, g_2)$ and then

$$(A_n(u_n(x,t)), A_n(v_n(x,t))) \to (A(u(x,t)), A(v(x,t))) \quad \text{in} \quad [\mathcal{D}'((0,L)\times(0,+\infty))]^2.$$

Taking the spatial derivative we deduce that

$$(a_n(u_n)u_{n,x}, a_n(v_n)v_{n,x}) \to (a(u)u_x, a(v)v_x) \quad \text{in} \quad [\mathcal{D}'((0,L)\times(0,+\infty))]^2.$$

Finally, putting the convergences above together we can pass the weak limit in the system (31). However, to conclude that U is a weak solution it remains to prove U satisfies $U(x,0) = u^0(x)$ and $U \in \mathcal{C}_w([0,T];X)$. Since $\{U_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;X)$ and $\{U_{n,t}\}_{n\in\mathbb{N}}$ in $L^\alpha(0,T;[H^{-2}(0,L)]^2)$, with $\alpha > 1$, we can apply again Corollary 4 in [21] to obtain a subsequence $\{U_n\}_{n\in\mathbb{N}}$ satisfying

$$U_n \to U \quad \text{in} \quad \mathcal{C}([0,T];[H^{-1}(0,L)]^2), \quad \text{for any } T > 0. \tag{37}$$

In particular,

$$U^0(x) = U_n(x,0) \to U(x,0).$$

As $U \in L^\infty(0,T;X) \cap \mathcal{C}([0,T];[H^{-1}(0,L)]^2)$, from Lemma 1.4 in [22] we deduce that $U \in \mathcal{C}_w([0,T];X)$. □

3. **A Carleman estimate.** The proof of our main result is based in the so called "Compactness-Uniqueness Argument". The key is to establish a unique continuation property for weak solution of the linearized system

$$b_1 U_t + B_1 U_x + A U_{xxx} = 0 \tag{38}$$

where

$$A = \begin{pmatrix} b_1 & b_1 a_3 \\ b_2 a_3 & 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

with $f_i = f_i(x,t)$, $i = 1,2,3,4$, being real-valued functions in a suitable space. Since the matrix A has two real eigenvalues there exists a diagonal matrix $P \in \mathcal{M}_{2\times 2}(\mathbb{R})$ such that $A = P^{-1}DP$, where D is a diagonal matrix whose diagonal elements are the eigenvalues of A. Hence, by means of the following change of variable

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} := P^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$$

system (38) can be written as

$$\begin{aligned} b_1 \widetilde{u}_t + \widetilde{f}_1 \widetilde{u}_x + \lambda_1 \widetilde{u}_{xxx} + \widetilde{f}_2 \widetilde{v}_x = 0 \\ b_1 \widetilde{v}_t + \widetilde{f}_4 \widetilde{v}_x + \lambda_2 \widetilde{v}_{xxx} + \widetilde{f}_3 \widetilde{u}_x = 0, \end{aligned} \tag{39}$$

where λ_1 and λ_2 are the eigenvalues of A and $\widetilde{f}_i = \widetilde{f}_i(x,t)$, $i = 1,2,3,4$, are real-valued functions. System (38) is simpler to deal with since the coupling terms are of first order. Moreover, if the unique continuation property holds for (39) it also remains valid for (38). We also remark that the above system is equivalent to the equation

$$\mathcal{L}U = 0, \tag{40}$$

where $\mathcal{L}$ has the form

$$\mathcal{L} = \begin{pmatrix} L_1 & \widetilde{f}_2 \frac{\partial}{\partial x} \\ \widetilde{f}_3 \frac{\partial}{\partial x} & L_2 \end{pmatrix}$$

with L_1 and L_2 given by

$$L_1 = b_1 \frac{\partial}{\partial t} + \lambda_1 \frac{\partial^3}{\partial x^3} + \widetilde{f}_1 \frac{\partial}{\partial x} \text{ and } L_2 = b_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^3}{\partial x^3} + \widetilde{f}_4 \frac{\partial}{\partial x}.$$

For the sake of simplicity, from now onwards we will drop the notation $\widetilde{u}, \widetilde{v}$ and use the notation u, v in the system (39)-(40).

The following Carleman estimate plays the main role for proving the unique continuation property for (39)-(40). In order to establish the result we introduce the space

$$V = \{q \in L^2(0,T;H^3(0,L)) \cap H^1(0,T;H^1(0,L)) : \\ q(t,0) = q_x(t,L) = q_{xx}(t,L) = q(t,L) = 0\}$$

endowed with the norm

$$||z||_V = \left\{ ||z||^2_{L^2(0,T;H^3(0,L))} + ||z_t||^2_{L^2(0,T;H^1(0,L))} \right\}^{1/2}$$

and observe that

$$V \subset \mathcal{C}([0,T],H^2(0,L)). \tag{41}$$

With the notation introduced above we can state the main result of this section. The proof is obtained following the arguments developed in [18, 19].

Theorem 3.1. *Let T, L and R be positive numbers and $\widetilde{f}_i \in V$, such that $||\widetilde{f}_i||_V \leq R$, for $1 \leq i \leq 4$. Then, there exist a smooth positive function ψ on $[0, L]$ and two constants $c > 0$ and $s_0 > 0$ such that for any $\Phi \in V \times V$ and $s \geq s_0$, we have*

$$\int_0^T \int_0^L \left\{ \frac{s^5}{t^5(T-t)^5}|\Phi|^2 + \frac{s^3}{t^3(T-t)^3}|\Phi_x|^2 + \frac{s}{t(T-t)}|\Phi_{xx}|^2 \right\} e^{-2s\frac{\psi(x)}{t(T-t)}}\,dxdt \\ \leq c\int_0^T \int_0^L |\mathcal{L}\Phi|^2 e^{-2s\frac{\psi(x)}{t(T-t)}}\,dxdt.$$

Proof. Let $R > 0$, and $\widetilde{f}_i \in V$, such that $||\widetilde{f}_i||_V \leq R$, for $1 \leq i \leq 4$. Let $\psi = \psi(x)$ be a positive function (to be specified later) of class C^3 in $[0, L]$ and let $\varphi(t, x) = \frac{\psi(x)}{t(T-t)}$.

For $p, q \in V$ and $s > 0$, we set $u = e^{-s\varphi}p$, $v = e^{-s\varphi}q$ and

$$W = e^{-s\varphi}\mathcal{L}(e^{s\varphi}u, e^{s\varphi}v).$$

Thus,

$$W = \begin{pmatrix} e^{-s\varphi}L_1(e^{s\varphi}u) + e^{-s\varphi}\widetilde{f}_2\frac{\partial}{\partial x}(e^{s\varphi}v) \\ e^{-s\varphi}\widetilde{f}_3\frac{\partial}{\partial x}(e^{s\varphi}u) + e^{-s\varphi}L_2(e^{s\varphi}v) \end{pmatrix}.$$

Now, we introduce $w_1 = e^{-s\varphi}L_1(e^{s\varphi}u)$ and $w_2 = e^{-s\varphi}L_2(e^{s\varphi}v)$. Then,

$$\begin{aligned} w_1 &= e^{-s\varphi}(b_1\frac{\partial}{\partial t} + \lambda_1\frac{\partial^3}{\partial x^3} + \widetilde{f}_1\frac{\partial}{\partial x})(e^{s\varphi}u) \\ &= e^{-s\varphi}b_1\frac{\partial}{\partial t}(e^{s\varphi}u) + e^{-s\varphi}\lambda_1\frac{\partial^3}{\partial x^3}(e^{s\varphi}u) + e^{-s\varphi}\widetilde{f}_1\frac{\partial}{\partial x}(e^{s\varphi}u), \end{aligned}$$

where

$$e^{-s\varphi}b_1\tfrac{\partial}{\partial t}(e^{s\varphi}u) = e^{-s\varphi}b_1(e^{s\varphi}s\varphi_t u + e^{s\varphi}u_t) = b_1 s\varphi_t u + b_1 u_t,$$

$$\begin{aligned} &e^{-s\varphi}\lambda_1\frac{\partial^3}{\partial x^3}(e^{s\varphi}u) = e^{-s\varphi}\lambda_1\frac{\partial^2}{\partial x^2}(e^{s\varphi}s\varphi_x u + e^{s\varphi}u_x) \\ &= e^{-s\varphi}\lambda_1\frac{\partial}{\partial x}(e^{s\varphi}s^2\varphi_x^2 u + e^{s\varphi}s\varphi_{xx}u + e^{s\varphi}s\varphi_x u_x + e^{s\varphi}s\varphi_x u_x + e^{s\varphi}u_{xx}) \\ &= e^{-s\varphi}\lambda_1(e^{s\varphi}s^3\varphi_x^3 u + 2e^{s\varphi}s^2\varphi_x\varphi_{xx}u + e^{s\varphi}s^2\varphi_x^2 u_x + e^{s\varphi}s^2\varphi_x\varphi_{xx}u \\ &+e^{s\varphi}s\varphi_{xxx}u + e^{s\varphi}s\varphi_{xx}u_x + 2e^{s\varphi}s^2\varphi_x^2 u_x + 2e^{s\varphi}s\varphi_{xx}u_x + 2e^{s\varphi}s\varphi_x u_{xx} \\ &+e^{s\varphi}s\varphi_x u_{xx} + e^{s\varphi}u_{xxx}) = \lambda_1 s^3\varphi_x^3 u + 3\lambda_1 s^2\varphi_x\varphi_{xx}u + 3\lambda_1 s^2\varphi_x^2 u_x + \lambda_1 s\varphi_{xxx}u \\ &+3\lambda_1 s\varphi_{xx}u_x + 3\lambda_1 s\varphi_x u_{xx} + \lambda_1 u_{xxx}, \\ &e^{-s\varphi}\widetilde{f}_1\frac{\partial}{\partial x}(e^{s\varphi}u) = e^{-s\varphi}\widetilde{f}_1(e^{s\varphi}s\varphi_x u + e^{s\varphi}u_x) = s\widetilde{f}_1\varphi_x u + \widetilde{f}_1 u_x. \end{aligned}$$

Combining the above identities, we get

$$\begin{aligned} w_1 &= b_1 s\varphi_t u + b_1 u_t + \lambda_1 s^3\varphi_x^3 u + 3\lambda_1 s^2\varphi_x\varphi_{xx}u + 3\lambda_1 s^2\varphi_x^2 u_x + \lambda_1 s\varphi_{xxx}u \\ &\quad +3\lambda_1 s\varphi_{xx}u_x + 3\lambda_1 s\varphi_x u_{xx} + \lambda_1 u_{xxx} + s\widetilde{f}_1\varphi_x u + \widetilde{f}_1 u_x \\ &= [s(b_1\varphi_t + \widetilde{f}_1\varphi_x + \lambda_1\varphi_{xxx}) + 3\lambda_1 s^2\varphi_x\varphi_{xx} + \lambda_1(s\varphi_x)^3]u \qquad (42) \\ &\quad +[\widetilde{f}_1 + 3\lambda_1 s\varphi_{xx} + 3\lambda_1(s\varphi_x)^2]u_x + (3\lambda_1 s\varphi_x)u_{xx} + \lambda_1 u_{xxx} + b_1 u_t \\ &= Au + Bu_x + Cu_{xx} + \lambda_1 u_{xxx} + b_1 u_t \end{aligned}$$

where

$$\begin{aligned} A &= s(b_1\varphi_t + \widetilde{f}_1\varphi_x + \lambda_1\varphi_{xxx}) + 3\lambda_1 s^2\varphi_x\varphi_{xx} + \lambda_1(s\varphi_x)^3 \\ B &= \widetilde{f}_1 + 3\lambda_1 s\varphi_{xx} + 3\lambda_1(s\varphi_x)^2 \\ C &= 3\lambda_1 s\varphi_x. \end{aligned} \tag{43}$$

We also readily get

$$\begin{aligned} w_2 &= [s(b_1\varphi_t + \widetilde{f}_4\varphi_x + \lambda_2\varphi_{xxx}) + 3\lambda_2 s^2\varphi_x\varphi_{xx} + \lambda_2(s\varphi_x)^3]v \\ &\quad + [\widetilde{f}_4 + 3\lambda_2 s\varphi_{xx} + 3\lambda_2(s\varphi_x)^2]v_x + (3\lambda_2 s\varphi_x)v_{xx} + \lambda_2 v_{xxx} + b_1 v_t \\ &= Ev + Fv_x + Gv_{xx} + \lambda_2 v_{xxx} + b_1 v_t, \end{aligned} \tag{44}$$

where

$$\begin{aligned} E &= s(b_1\varphi_t + \widetilde{f}_4\varphi_x + \lambda_2\varphi_{xxx}) + 3\lambda_2 s^2\varphi_x\varphi_{xx} + \lambda_2(s\varphi_x)^3 \\ F &= \widetilde{f}_4 + 3\lambda_2 s\varphi_{xx} + 3\lambda_2(s\varphi_x)^2 \\ G &= 3\lambda_2 s\varphi_x. \end{aligned} \tag{45}$$

At that point we observe that the arguments we are going to develop can be applied to both w_1 and w_2 leading to the same result. Therefore, we will only deal with w_1. We set

$$\begin{aligned} \widetilde{A} &= s(b_1\varphi_t + \lambda_1\varphi_{xxx}) + \lambda_1(s\varphi_x)^3 \\ \widetilde{B} &= (3-\delta_1)\lambda_1 s\varphi_{xx} + 3\lambda_1(s\varphi_x)^2 \\ M_1(u) &= b_1u_t + \lambda_1 u_{xxx} + \widetilde{B}u_x \\ M_2(u) &= \widetilde{A}u + Cu_{xx}, \end{aligned}$$

where $\delta_1 \in (0,1)$ is a small number to be chosen later, we have

$$M_1(u) + M_2(u) = w_1 - (s\widetilde{f}_1\varphi_x + 3\lambda_1 s^2\varphi_x\varphi_{xx})u - (\widetilde{f}_1 + \delta_1\lambda_1 s\varphi_{xx})u_x.$$

Then, if $|\widetilde{f}_1| \le \delta_1\lambda_1 s|\varphi_{xx}|$ for all $(x,t) \in Q = (0,L)\times(0,T)$,

$$\begin{aligned} &\|M_1(u) + M_2(u)\|^2_{L^2(Q)} \le 3(\|w_1\|^2_{L^2(Q)} \\ &+ s^2\left\|(\widetilde{f}_1 + 3\lambda_1 s\varphi_{xx})\varphi_x u\right\|^2_{L^2(Q)} + \left\|(\widetilde{f}_1 + \delta_1\lambda_1 s\varphi_{xx})u_x\right\|^2_{L^2(Q)}) \\ &\le 3\left(\|w_1\|^2_{L^2(Q)} + (3+\delta_1)^2\lambda_1^2 s^4\|\varphi_{xx}\varphi_x u\|^2_{L^2(Q)} + 4\delta_1^2\lambda_1^2 s^2\|\varphi_{xx}u_x\|^2_{L^2(Q)}\right). \end{aligned} \tag{46}$$

This is possible if s is large enough and $|\psi''(x)| > 0$ on $[0,L]$, since $||\widetilde{f}_1||_{L^\infty} \le K||\widetilde{f}_1||_V \le KR$ for some $K > 0$. On the other hand,

$$\begin{aligned} &\|M_1(u) + M_2(u)\|^2_{L^2(Q)} \\ &= \|M_1(u)\|^2_{L^2(Q)} + \|M_2(u)\|^2_{L^2(Q)} + 2\int_0^T\int_0^L M_1(u)M_2(u)\,dxdt, \end{aligned} \tag{47}$$

therefore, the next steps are devoted to estimate the last term of the above identity.

From now on, for the sake of brevity, we introduce the notation

$$\int\int f := \int_0^T\int_0^L f(t,x)\,dxdt \quad \text{and} \quad \int f := \int_0^T f(t,L)\,dt:$$

$$
\begin{aligned}
2\int\int M_1(u)M_2(u) &= 2\int\int M_1(u)(\widetilde{A}u + Cu_{xx}) = 2\int\int M_1(u)\widetilde{A}u \\
&+2\int\int(b_1u_t + \lambda_1 u_{xxx} + \widetilde{B}u_x)Cu_{xx} = \int\int 2M_1(u)\widetilde{A}u \\
&+\int\int 2b_1u_tCu_{xx} + \int\int 2(\lambda_1 u_{xxx} + \widetilde{B}u_x)Cu_{xx} = I_1 + I_2 + I_3.
\end{aligned} \tag{48}
$$

Now, performing integration by parts with respect to t and x we get

$$
\begin{aligned}
I_1 &= \int\int(b_1u_t + \lambda_1 u_{xxx} + \widetilde{B}u_x)2\widetilde{A}u \\
&= \int\int b_1\frac{\partial}{\partial t}(u^2)\widetilde{A} + \int\int 2\lambda_1 uu_{xxx}\widetilde{A} + \int\int\frac{\partial}{\partial x}(u^2)\widetilde{A}\widetilde{B} \\
&= -\int\int b_1u^2\widetilde{A}_t - \int\int 2\lambda_1 u_{xx}u_x\widetilde{A} - \int\int 2\lambda_1 u_{xx}u\widetilde{A}_x - \int\int u^2(\widetilde{A}\widetilde{B})_x \\
&= -\int\int b_1u^2\widetilde{A}_t - \int\lambda_1 u_x^2\widetilde{A} + \int\int 3\lambda_1 u_x^2\widetilde{A}_x + \int\int 2\lambda_1 u_xu\widetilde{A}_{xx} - \int\int u^2(\widetilde{A}\widetilde{B})_x \\
&= -\int\int b_1u^2\widetilde{A}_t - \int\lambda_1 u_x^2\widetilde{A} + \int\int 3\lambda_1 u_x^2\widetilde{A}_x - \int\int\lambda_1 u^2\widetilde{A}_{xxx} - \int\int u^2(\widetilde{A}\widetilde{B})_x \\
&= -\int\int(b_1\widetilde{A}_t + \lambda_1\widetilde{A}_{xxx} + (\widetilde{A}\widetilde{B})_x)u^2 + 3\lambda_1\int\int u_x^2\widetilde{A}_x - \lambda_1\int u_x^2\widetilde{A}
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int\int 2\lambda_1 u_{xxx}u_{xx}C + \int\int 2\widetilde{B}Cu_xu_{xx} \\
&= \int\lambda_1 Cu_{xx}^2 - \int\int\lambda_1 C_xu_{xx}^2 + \int\widetilde{B}Cu_x^2 - \int\int(\widetilde{B}C)_xu_x^2
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= -\int\int 2b_1(Cu_t)_xu_x = -\int\int 2b_1C_xu_tu_x - \int\int 2b_1Cu_{tx}u_x \\
&= -\int\int 2b_1Cu_{tx}u_x + \int\int 2C_x(Au + Bu_x + Cu_{xx} + \lambda_1 u_{xxx} - w_1)u_x \\
&= \int\int b_1C_tu_x^2 - \int\int(C_xA)_xu^2 + \int\int 2C_xBu_x^2 + \int C_xCu_x^2 - \int\int(C_xC)_xu_x^2 \\
&+\int 2C_x\lambda_1 u_{xx}u_x - \int\int 2\lambda_1 C_{xx}u_xu_{xx} - \int\int 2\lambda_1 C_xu_{xx}^2 - \int\int 2C_xw_1u_x \\
&= -\int\int(C_xA)_xu^2 + \int\int(b_1C_t + 2C_xB - (CC_x)_x + \lambda_1 C_{xxx})u_x^2 \\
&\int(C_xCu_x^2 + 2\lambda_1 C_xu_xu_{xx} - \lambda_1 C_{xx}u_x^2) - \int\int 2\lambda_1 C_xu_{xx}^2 - \int\int 2C_xw_1u_x.
\end{aligned}
$$

Combining (48) and the previous computations, it follows that

$$
\begin{aligned}
2\int\int M_1(u)M_2(u) &= -\int\int(b_1\widetilde{A}_t + \lambda_1\widetilde{A}_{xxx} + (\widetilde{A}\widetilde{B})_x + (C_xA)_x)u^2 \\
&+\int\int(3\lambda_1\widetilde{A}_x + b_1C_t + 2C_xB - (CC_x)_x + \lambda_1 C_{xxx} - (\widetilde{B}C)_x)u_x^2 \\
&-\int\int 3\lambda_1 C_xu_{xx}^2 - \int\int 2C_xw_1u_x + \int(-\lambda_1\widetilde{A} + C_xC - \lambda_1 C_{xx} + \widetilde{B}C)u_x^2
\end{aligned} \tag{49}
$$

$$+\int \lambda_1 C u_{xx}^2 + \int 2\lambda_1 C_x u_x u_{xx}$$
$$= \int\int \widetilde{D}u^2 + \int\int \widetilde{E}u_x^2 - \int\int 3\lambda_1 C_x u_{xx}^2 - \int\int 2C_x w_1 u_x$$
$$+\int \widetilde{F}u_x^2 + \int \lambda_1 C u_{xx}^2 + \int 2\lambda_1 C_x u_x u_{xx},$$

where

$$\begin{aligned}
\widetilde{D} &= -b_1\widetilde{A}_t - \lambda_1\widetilde{A}_{xxx} - (\widetilde{A}\widetilde{B})_x - (C_xA)_x \\
\widetilde{E} &= 3\lambda_1\widetilde{A}_x + b_1C_t + 2C_xB - (CC_x)_x + \lambda_1 C_{xxx} - (\widetilde{B}C)_x \\
\widetilde{F} &= -\lambda_1\widetilde{A} + C_xC - \lambda_1C_{xx} + \widetilde{B}C.
\end{aligned}$$

Now, observe that,

$$2\left|\int\int w_1 C_x u_x\right| \leq \epsilon \int\int C_x^2 u_x^2 + \frac{1}{\epsilon}\int\int w_1^2, \tag{50}$$

for any $\varepsilon > 0$, and

$$2\left|\int \lambda_1 C_x u_x u_{xx}\right| \leq \lambda_1 \int u_{xx}^2 + \lambda_1 \int C_x^2 u_x^2. \tag{51}$$

Then, combining (46), (47) and (49)-(51) we deduce that

$$\int\int \widetilde{D}u^2 + \int\int \widetilde{E}u_x^2 - \int\int 3\lambda_1 C_x u_{xx}^2 + \int \widetilde{F}u_x^2 + \int \lambda_1 C u_{xx}^2$$
$$+\|M_1(u)\|_{L^2(Q)}^2 + \|M_2(u)\|_{L^2(Q)}^2 = 2\int\int w_1 C_x u_x - 2\int \lambda_1 C_x u_x u_{xx}$$
$$+\|M_1(u)+M_2(u)\|_{L^2(Q)}^2 \leq \epsilon \int\int C_x^2 u_x^2 + \frac{1}{\epsilon}\int\int w_1^2 + \lambda_1\int u_{xx}^2 + \lambda_1 \int C_x^2 u_x^2$$
$$+3\int\int w_1^2 + 3(3+\delta_1)^2\lambda_1^2 s^4 \int\int \varphi_{xx}^2\varphi_x^2 u^2 + 12\,\delta_1^2\lambda_1^2 s^2 \int\int \varphi_{xx}^2 u_x^2.$$

Hence

$$\begin{aligned}
&\int\int \left\{\widetilde{D} - 3(3+\delta_1)^2\lambda_1^2 s^4 \varphi_x^2\varphi_{xx}^2\right\} u^2 \\
&+ \int\int \left\{\widetilde{E} - \varepsilon C_x^2 - 12\delta_1^2\lambda_1^2 s^2 \varphi_{xx}^2\right\} u_x^2 + \int\int (-3\lambda_1 C_x)u_{xx}^2 \\
&+ \int (\widetilde{F} - \lambda_1 C_x^2)u_x^2 + \int (\lambda_1 C - \lambda_1)u_{xx}^2 \leq (3 + \frac{1}{\varepsilon})\int\int w_1^2.
\end{aligned} \tag{52}$$

The function ψ and the constants δ_1, ε and s_o (defined in the statement of the Theorem) are chosen in such way that the functions in the brackets on the left hand side of (52) are positive. On the other hand, the functions $\widetilde{f}_1$ and $\widetilde{f}_{1x}$ that appears in $A, B, \widetilde{D}$ and $\widetilde{E}$ are uniformly bounded since

$$\left\|\widetilde{f}_1\right\|_{L^\infty((0,L)\times(0,T))} + \left\|\widetilde{f}_{1x}\right\|_{L^\infty((0,L)\times(0,T))} \leq K\left\|\widetilde{f}_1\right\|_V \leq KR.$$

Then,

$$\begin{aligned}
\widetilde{D} &= -(\widetilde{A}\widetilde{B})_x + O(\frac{s^4}{t^4(T-t)^4}) = -(3\lambda_1^2 s^5(\varphi_x)^5)_x + O(\frac{s^4}{t^4(T-t)^4}) \\
&= -15\lambda_1^2 s^5\varphi_x^4\varphi_{xx} + O(\frac{s^4}{t^4(T-t)^4}) = -15\lambda_1^2 s^5 \frac{\psi'(x)^4\psi''(x)}{t^5(T-t)^5} + O(\frac{s^4}{t^4(T-t)^4}),
\end{aligned}$$

as $s \to \infty$. Moreover, if s is large enough,

$$|\psi'(x)| > 0 \text{ and } \psi''(x) < 0 \text{ for } x \in [0, L],$$

we get

$$\widetilde{D} - 3(3+\delta_1)^2\lambda_1^2 s^4 \varphi_x^2 \varphi_{xx}^2 \geq K_1 \frac{s^5}{t^5(T-t)^5}, \tag{53}$$

for some constant $K_1 > 0$. On the other hand,

$$\begin{aligned} \widetilde{E} &= 9\lambda_1^2 s^3 \varphi_x^2 \varphi_{xx} + 6\lambda_1^2 s\varphi_{xx}(3s\varphi_{xx} + 3s^2\varphi_x^2) - (9\lambda_1^2 s^2 \varphi_x \varphi_{xx})_x \\ &\quad - \left\{((3-\delta_1)s\varphi_{xx} + 3s^2\varphi_x^2)3s\lambda_1^2\varphi_x\right\}_x + O(\frac{s}{t^2(T-t)^2}) \\ &= 3\delta_1 s^2 \lambda_1^2 \varphi_{xx}^2 + (3\delta_1 - 18)s^2\lambda_1^2\varphi_x\varphi_{xxx} + O(\frac{s}{t^2(T-t)^2}), \end{aligned}$$

and, therefore,

$$\begin{aligned} \widetilde{E} - \epsilon C_x^2 - 12\delta_1^2\lambda_1^2 s^2 \varphi_{xx}^2 &= \lambda_1^2 s^2(3\delta_1 - 9\epsilon - 12\delta_1^2)\frac{\psi''(x)^2}{t^2(T-t)^2} \\ &+ (3\delta_1 - 18)s^2\lambda_1^2 \frac{\psi'(x)\psi'''(x)}{t^2(T-t)^2} + O(\frac{s}{t^2(T-t)^2}). \end{aligned}$$

Now, choosing $\delta_1 = 10^{-1}$ and $\epsilon = 10^{-2}$, we get $3\delta_1 - 9\epsilon - 12\delta_1^2 > 0$. Then, if

$$\psi''(x) \neq 0 \text{ and } \psi'(x)\psi'''(x) \leq 0 \text{ for } x \in [0, L],$$

we obtain a positive constant $K_2 > 0$, such that, for s sufficiently large

$$\widetilde{E} - \epsilon C_x^2 - 12\delta_1^2\lambda_1^2\varphi_{xx}^2 \geq K_2 \frac{s^2}{t^2(T-t)^2}. \tag{54}$$

Finally,

$$- 3\lambda_1 C_x = -9\lambda_1^2 s \frac{\psi''(x)}{t(T-t)} \geq K_3 \frac{s}{t(T-t)} \tag{55}$$

$$\widetilde{F} - \lambda_1 C_x^2 = -\lambda_1^2 s^3 \varphi_x^3 + 9\lambda_1^2 s^3 \varphi_x^3 + O(\frac{s^2}{t^2(T-t)^2}) \geq K_4 \frac{s^3}{t^3(T-t)^3} \tag{56}$$

$$\lambda_1 C - \lambda_1 = 3\lambda_1^2 s\varphi_x - \lambda_1 \geq K_5 \frac{s}{t(T-t)}, \tag{57}$$

for some positive constants K_3, K_4 and K_5, provided s is sufficiently large and

$$\psi''(x) < 0 \text{ and } \psi'(x) > 0 \text{ for } x \in [0, L].$$

To summarize, the function ψ has to fulfill the following conditions:

$$\psi \in C^3([0, L]), \quad \psi > 0, \quad \psi' > 0, \quad \psi'' < 0 \text{ and } \psi'\psi''' \leq 0 \text{ in } [0, L].$$

$\psi(x) = 1 + 4L^2 + x(3L - x)$ is convenient. Thus, for s sufficiently large, we infer from (53)-(57)

$$\int\int \left\{ \frac{s^5}{t^5(T-t)^5} u^2 + \frac{s^2}{t^2(T-t)^2} u_x^2 + \frac{s}{t(T-t)} u_{xx}^2 \right\} \leq K_6 \int\int w_1^2 \tag{58}$$

for some positive constant $K_6 > 0$. Now if we take into account that

$$\begin{aligned}\int\int \frac{s^3}{t^3(T-t)^3}u_x^2 &= -\int\int \frac{s^3}{t^3(T-t)^3}uu_{xx}\\ &\leq \frac{1}{2}\left\{\int\int \frac{s^5}{t^5(T-t)^5}u^2 + \int\int \frac{s}{t(T-t)}u_{xx}^2\right\}\\ &\leq \frac{K_6}{2}\int\int w_1^2,\end{aligned}$$

we can improve the above estimate as follows

$$\int\int \left\{\frac{s^5}{t^5(T-t)^5}u^2 + \frac{s^3}{t^3(T-t)^3}u_x^2 + \frac{s}{t(T-t)}u_{xx}^2\right\} \leq \frac{3}{2}K_6\int\int w_1^2, \tag{59}$$

for s large enough.

As we pointed before, similar computations give us a similar estimate for (44). Therefore, to conclude the proof we proceed as follows. We replace u by $e^{-s\varphi}p$ in (59) to obtain

$$\begin{aligned}\int\int \left\{\frac{s^5}{t^5(T-t)^5}p^2 + \frac{s^3}{t^3(T-t)^3}p_x^2 + \frac{s}{t(T-t)}p_{xx}^2\right\} e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt&\\ \leq c\int\int (L_1(p))^2 e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt&,\end{aligned}$$

where $c = c(L,T,R)$. Next, we set $v = e^{-s\varphi}q$:

$$\begin{aligned}\int\int \left\{\frac{s^5}{t^5(T-t)^5}q^2 + \frac{s^3}{t^3(T-t)^3}q_x^2 + \frac{s}{t(T-t)}q_{xx}^2\right\} e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt&\\ \leq \widetilde{c}\int\int (L_2(q))^2 e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt&.\end{aligned}$$

Letting $\Phi = \begin{pmatrix} p \\ q \end{pmatrix}$ and adding the last two inequalities hand to hand we deduce that

$$\begin{aligned}\int\int \left\{\frac{s^5}{t^5(T-t)^5}|\Phi|^2 + \frac{s^3}{t^3(T-t)^3}|\Phi_x|^2 + \frac{s}{t(T-t)}|\Phi_{xx}|^2\right\} e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt&\\ \leq C_o\int\int (L_1(p)^2 + L_2(q)^2) e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt&,\end{aligned} \tag{60}$$

where $C_o = C_o(L,T,R)$. On the other hand,

$$\begin{aligned}&C_o\int\int \left|\widetilde{f}_2 q_x\right|^2 e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt \leq C_o\left\|\widetilde{f}_2\right\|^2_{L^\infty(Q)}\int\int |q_x|^2 e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt\\ &\leq \frac{1}{4}\int\int \left\{\frac{s^5}{t^5(T-t)^5}|\Phi|^2 + \frac{s^3}{t^3(T-t)^3}|\Phi_x|^2 + \frac{s}{t(T-t)}|\Phi_{xx}|^2\right\} e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt\end{aligned}$$

and

$$\begin{aligned}&C_o\int\int \left|\widetilde{f}_3 p_x\right|^2 e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt \leq C_o\left\|\widetilde{f}_3\right\|^2_{L^\infty(Q)}\int\int |p_x|^2 e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt\\ &\leq \frac{1}{4}\int\int \left\{\frac{s^5}{t^5(T-t)^5}|\Phi|^2 + \frac{s^3}{t^3(T-t)^3}|\Phi_x|^2 + \frac{s}{t(T-t)}|\Phi_{xx}|^2\right\} e^{\frac{-2s\psi(x)}{t(T-t)}}\,dxdt,\end{aligned}$$

for s large enough.

The above computations combined with (59) allows to obtain the following estimates

$$\frac{1}{2}\int\int\left\{\frac{s^5}{t^5(T-t)^5}|\Phi|^2+\frac{s^3}{t^3(T-t)^3}|\Phi_x|^2+\frac{s}{t(T-t)}|\Phi_{xx}|^2\right\}e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt$$
$$\leq C_o\int\int(L_1(p)^2+L_2(q)^2-2\left|\widetilde{f}_2q_x\right|^2)e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt \quad (61)$$

and

$$\frac{1}{2}\int\int\left\{\frac{s^5}{t^5(T-t)^5}|\Phi|^2+\frac{s^3}{t^3(T-t)^3}|\Phi_x|^2+\frac{s}{t(T-t)}|\Phi_{xx}|^2\right\}e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt$$
$$\leq C_o\int\int(L_1(p)^2+L_2(q)^2-2\left|\widetilde{f}_3p_x\right|^2)e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt. \quad (62)$$

Since

$$|L_1(p)|^2\leq 2\left|L_1(p)+\widetilde{f}_2q_x\right|^2+2\left|\widetilde{f}_2q_x\right|^2 \text{ and}$$
$$|L_2(q)|^2\leq 2\left|L_2(q)+\widetilde{f}_3p_x\right|^2+2\left|\widetilde{f}_3p_x\right|^2,$$

we can add (61) and (62) to conclude that

$$\int\int\left\{\frac{s^5}{t^5(T-t)^5}|\Phi|^2+\frac{s^3}{t^3(T-t)^3}|\Phi_x|^2+\frac{s}{t(T-t)}|\Phi_{xx}|^2\right\}e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt$$
$$\leq C_o\int\int(L_1(p)^2+L_2(q)^2-2|\widetilde{f}_2q_x|^2-2|\widetilde{f}_3p_x|^2)e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt$$
$$\leq 2C_o\int\int((L_1(p)+\widetilde{f}_2q_x)^2+(L_2(q)+\widetilde{f}_3p_x)^2)e^{\frac{-2s\psi(x)}{t(T-t)}}dxdt.$$

The proof is now complete. □

4. **Exponential stabilization.**

In this section we prove the uniform exponential decay of the total energy $E(t)$. We will require the so-called "Compactness-Uniqueness Argument" which reduces the problem to prove a Unique Continuation Property for weak solutions. As the weak solution of (1)-(3) may fail to be unique, we will say that the solution is exponential stable if the following property holds.

Definition 4.1. System (1)-(3) is said to be locally uniformly exponentially stable in X if for any $R>0$ there exist positive constants C and α such that for any $u^0=(u^0,v^0)$ with $E(0)\leq R$ and for any weak solution $U=(u,v)$ of (1)-(3), the following holds

$$E(t)\leq C\,E(0)e^{-\alpha t}, \quad \forall\, t>0.$$

If the constant α is independent of R, the system (1)-(3) is said to be globally uniformly exponentially stable in X.

As a consequence of Theorem 3.1 we obtain a unique continuation result for (1). In order to prove the result the following technical lemma will be needed.

Lemma 4.2. *If V is a Banach space and $g\in L^p(0,T;V)$, with $1\leq p\leq\infty$, then for any $h>0$ the function $g^{[h]}=g^{[h]}(x,t)$ given by*

$$g^{[h]}(x,t)=\frac{1}{h}\int_t^{t+h}g(x,s)ds$$

satisfies

(i) $g^{[h]} \in W^{1,p}(0,T-h;V)$,
(ii) $||g^{[h]}||_{L^p(0,T-h;V)} \leq ||g||_{L^p(0,T;V)}$,
(iii) $g^{[h]} \to g$ *in* $L^p(0,T';V)$ *as* $h \to 0$, *for* $p < \infty$ *and* $T' < T$.

Proof. See Proposition 1.4.29 in [5] (see also [21]). □

The following result is the first step to establish the unique continuation result.

Proposition 2. *Let T and l be positive numbers. If $U = (u,v) \in$ is such that $U \in L^\infty(0,T;[H^1(0,l)]^2)$ and solves*

$$\begin{array}{ll} u_t + a(u)u_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2(uv)_x = 0 & \textit{in } (0,l)\times(0,T), \\ b_1 v_t + r v_x + a(v)v_x + b_2 a_3 u_{xxx} + v_{xxx} & \\ \qquad\qquad + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0 & \textit{in } (0,l)\times(0,T), \\ u(0,t) = v(0,t) = 0 & \textit{for}\, t \in (0,T), \\ u \equiv v \equiv 0 & \textit{in } (l',l)\times(0,T) \end{array}$$

with $a \in C^0(\mathbb{R},\mathbb{R})$ and $0 < l' < l$, then $u \equiv v \equiv 0$ in $(0,l)\times(0,T)$.

Proof. We first observe that the above system can be written as

$$\begin{aligned} b_1 U_t + AU_{xxx} + [B(U)+R]U_x + C(U)U_x = 0 \quad & \textit{in} \quad (0,l)\times(0,T), \\ U(0,t) = 0 \quad & \textit{for} \qquad t \in (0,T), \\ U \equiv 0 \quad & \textit{in} \quad (l',l)\times(0,T), \end{aligned}$$

where $U = (u,v)$, A is given in (38),

$$B(U) = \begin{pmatrix} b_1 a_2 v & b_1(a_2 u + a_1 v) \\ b_2(a_2 u + a_1 v) & b_2 a_1 u \end{pmatrix}, \ R = \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} \text{ and}$$
$$C(U) = \begin{pmatrix} b_1 a(u) & 0 \\ 0 & a(v) \end{pmatrix}.$$

Now, for any U we consider the function

$$U^{[h]} = (u^{[h]}, v^{[h]})$$

given by Lemma 4.2.

Then, for $T' < T$ and h small enough $U^{[h]} \in W^{1,\infty}(0,T';[H_0^1(0,l)]^2)$ and solves

$$\begin{aligned} b_1 U_t^{[h]} + AU_{xxx}^{[h]} + ([B(U)+R]U_x)^{[h]} + (C(U)U_x)^{[h]} = 0 \quad & \text{in} \quad (0,l)\times(0,T'), \\ U^{[h]}(0,t) = 0 \quad & \text{for} \qquad t \in (0,T'), \\ U^{[h]} \equiv 0 \quad & \text{in} \quad (l',l)\times(0,T'). \end{aligned}$$

Since $u, v \in L^\infty(0,T;H^1(0,l))$, then

$$a(u)u_x, a(v)v_x, (uv)_x, uu_x, vv_x \in L^\infty(0,T;L^2(0,l)).$$

Consequently, we get $U_{xxx}^{[h]} \in L^\infty(0,T';X)$ and so $U^{[h]} \in L^\infty(0,T';[H^3(0,l)]^2)$.

On the other hand, returning to (38) we can define $f_1 = b_1 a_2 v$, $f_2 = b_1(a_2 u + a_1 v)$, $f_3 = b_2(a_2 u + a_1 v)$ and $f_4 = b_2 a_1 u + r$. Then, we have $B(U)+R = B_1$. Furthermore, we can use the change of variable $P^{-1}U := \widetilde{U}$, where P is the diagonalization of the matrix of A, to obtain (39). Note that the functions $\widetilde{f}_i = \widetilde{f}_i(x,t)$ introduced in (39) are such that $\widetilde{f}_i \in L^\infty_{loc}(0,L)$ since $f_i \in L^\infty_{loc}(0,L)$, $i = 1,2,3,4$.

For the sake of simplicity, from now onwards we will drop the notation $\widetilde{U}$ and use the notation U. Then, letting $P = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$ the analysis presented above allows us to apply Theorem 3.1 to obtain,

$$\begin{aligned}
&\int_0^{T'}\int_0^l \left\{ \tfrac{s^5}{t^5(T'-t)^5}|U^{[h]}|^2 + \tfrac{s^3}{t^3(T'-t)^3}|U_x^{[h]}|^2 + \tfrac{s}{t(T'-t)}|U_{xx}^{[h]}|^2 \right\} e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&\le C_o \int_0^{T'}\int_0^l \left(\left|[P^{-1}C(PU)[PU]_x]^{[h]}\right|^2 + \left|[P^{-1}B(PU)[PU]_x]^{[h]}\right|^2 \right) e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&\le C_oC_1 \int_0^{T'}\int_0^l \left(\left|(a(cu+dv)u_x)^{[h]}\right|^2 + \left|(a(eu+fv)v_x)^{[h]}\right|^2 \right) e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&+ C_oC_1 \int_0^{T'}\int_0^l \left(\left|(a(eu+fv)u_x)^{[h]}\right|^2 + \left|(a(cu+dv)v_x)^{[h]}\right|^2 \right) e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&+ C_oC_1 \int_0^{T'}\int_0^l \left(|(uu_x)^{[h]}|^2 + |(vv_x)^{[h]}|^2 + |(vu_x)^{[h]}|^2 + |(uv_x)^{[h]}|^2 \right) e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&= \textstyle\sum_{i=1}^8 I_i,
\end{aligned} \tag{63}$$

for any $s \ge s_0$, where C_o and C_1 are positive constants.

The next steps are devoted to estimate the terms on the right hand side of (63). Let us focus on I_1. First observe that

$$\begin{aligned}
I_1 &= \int_0^{T'}\int_0^l \left|(a(cu+dv)u_x)^{[h]}\right|^2 e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&\le 2\int_0^{T'}\int_0^l \left(\left|(a(cu+dv)u_x)^{[h]} - a(cu+dv)u_x^{[h]}\right|^2 + \left|a(cu+dv)u_x^{[h]}\right|^2 \right) e^{-2s\frac{\psi(x)}{t(T'-t)}} \\
&= I_{11} + I_{12}.
\end{aligned}$$

Since $a(cu+dv) \in L^\infty(0,T;L^\infty(0,l))$, we get

$$I_{12} \le C\int_0^{T'}\int_0^l \left|u_x^{[h]}\right|^2 e^{-2s\frac{\psi(x)}{t(T'-t)}}.$$

Then, comparing the power s in both sides of (63) we conclude that the term I_{12} can be dropped by increasing the constants C_o and s_o in a convenient way. Moreover, the fact that $a(cu+dv)u_x \in L^2(0,T;L^2(0,l))$ together with Lemma 4.2 allow us to deduce that, as $h \to 0$,

$$\begin{aligned}
(a(cu+dv)u_x)^{[h]} &\to a(cu+dv)u_x \quad \text{in} \quad L^2(0,T';L^2(0,l)),\\
a(cu+dv)u_x^{[h]} &\to a(cu+dv)u_x \quad \text{in} \quad L^2(0,T';L^2(0,l)).
\end{aligned}$$

Now, fixing $s = s_0$ and observing that $e^{-2s_o\frac{\psi(x)}{t(T'-t)}} \le 1$, from the above convergences the following holds

$$I_{11} \to 0, \quad as \quad h \to 0.$$

A similar analysis can be done for the terms I_i, $i = 2, \dots, 8$.

Consequently, as $h \to 0$,

$$\int_0^{T'}\int_0^l \left\{ \frac{s^5}{t^5(T'-t)^5}|U^{[h]}|^2 + \frac{s^3}{t^3(T'-t)^3}|U_x^{[h]}|^2 + \frac{s}{t(T'-t)}|U_{xx}^{[h]}|^2 \right\} e^{-2s\frac{\psi(x)}{t(T'-t)}}$$

converges to zero. This completes the proof. Indeed, from Lemma 4.2 we have that $U^{[h]} \to U$ in $L^2(0,T';X)$ which guarantees that $U \equiv 0$ in $(0,l)\times(0,T')$. Moreover, since T' can be taken arbitrarily close to T, we obtain $U \equiv 0$ in $(0,l)\times(0,T)$. □

The unique continuation result reads as follows:

Corollary 3. *Let $\omega \subset (0,L)$ be a nonempty open subset. If $U=(u,v)$ is such that $U \in L^\infty(0,T;[H^1(0,L)]^2)$ and solves*

$$\begin{array}{ll} u_t + a(u)u_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2(uv)_x = 0 & \textit{in}\,(0,l)\times(0,T), \\ b_1 v_t + r v_x + a(v)v_x + b_2 a_3 u_{xxx} + v_{xxx} & \\ \qquad\qquad + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0 & \textit{in}\,(0,l)\times(0,T), \\ u(0,t) = v(0,t) = 0 & \textit{for } t \in (0,T), \\ u \equiv v \equiv 0 & \textit{in } \omega \times (0,T) \end{array}$$

with $a \in C^0(\mathbb{R},\mathbb{R})$, then $u \equiv v \equiv 0$ in $(0,L)\times(0,T)$.

Proof. The arguments used to obtain the result are known (see, for instance, [20]).

Without loss of generality we may assume that $\omega = (l_1, l_2)$ with $0 \le l_1 < l_2 \le L$. Pick $l = (l_1+l_2)/2$. Applying Proposition 2 to the function $U(x,t)$ on $(0,l)\times(0,T)$ and then to the function $U(L-x, T-t)$ on $(0, L-l)\times(0,T)$, we conclude that $U \equiv 0$ on $(0,L)\times(0,T)$. □

We first prove a local uniform result.

Proposition 3. *Let $a = a(x)$ be a $\mathcal{C}^2$ function such that*

$$|a(x)| \le C(1+|x|^p),\ |a'(x)| \le C(1+|x|^{p-1}),\ |a''(x)| \le C(1+|x|^{p-2}), \quad \forall x \in \mathbb{R}$$

where C is a positive constant and $1 \le p < 4$. Then, if b satisfies (5), system (1)-(3) is locally uniformly stable.

Proof. As we pointed out in Section 1, to obtain the exponential decay of $E(t)$ we claim that the following inequality holds

$$\begin{aligned} E(0) \le C \int_0^T \Big[\int_0^L b(x)(b_2 u^2 + v^2)dx + \\ \frac{1}{2}\left(\sqrt{b_2}u_x(0,t) + \sqrt{a_3^2 b_2}v_x(0,t)\right)^2 + \frac{1}{2}\left(1 - a_3^2 b_2\right) v_x^2(0,t)\Big] dt \end{aligned} \tag{64}$$

for every finite energy solution of (1)-(3), where $C = C(R,T)$ is a positive constant. To prove (64) we first multiply the first equation of (1) by $(T-t)b_2 u$ and add with the second one multiplied by $(T-t)v$. Performing integration by parts we get

$$\begin{aligned} &\frac{b_2}{2}\int_0^T\int_0^L u^2 dxdt + \frac{b_1}{2}\int_0^T\int_0^L v^2 dxdt + \frac{b_2}{2}\int_0^T (T-t)u_x^2(0,t)dt \\ &+\frac{1}{2}\int_0^T (T-t)v_x^2(0,t)dt + b_2 a_3 \int_0^T (T-t)u_x(0,t)v_x(0,t)dt \\ &+b_2\int_0^T\int_0^L (T-t)b(x)u^2 dxdt + \int_0^T\int_0^L (T-t)b(x)v^2 dxdt \\ &= \frac{Tb_2}{2}\int_0^L (u^0)^2 dx + \frac{Tb_1}{2}\int_0^L (v^0)^2 dx, \end{aligned}$$

that is,

$$\begin{aligned}&\frac{b_2}{2}\int_0^L (u^0)^2dx+\frac{b_1}{2}\int_0^L (v^0)^2dx\\&\leq \frac{1}{T}\left(\frac{b_2}{2}\int_0^T\int_0^L u^2dxdt+\frac{b_1}{2}\int_0^T\int_0^L v^2dxdt\right)\\&+\frac{1}{2}\int_0^T\Big[(\sqrt{b_2}u_x(0,t)+\sqrt{a_3^2b_2}v_x(0,t))^2+\left(1-a_3^2b_2\right)v_x^2(0,t)\Big]dt\\&+\int_0^T\int_0^L b(x)(b_2u^2+v^2)dxdt.\end{aligned}\tag{65}$$

Then, to obtain (64) we have to prove that, for any $T>0$ and $R>0$, there exists a constant $C(R,T)>0$ satisfying

$$\begin{aligned}&\frac{b_2}{b_1}\int_0^T\int_0^L u^2dxdt+\int_0^T\int_0^L v^2dxdt\\&\leq C(R,T)\Big(\int_0^T\Big[(\sqrt{b_2}u_x(0,t)+\sqrt{a_3^2b_2}v_x(0,t))^2+\left(1-a_3^2b_2\right)v_x^2(0,t)\Big]dt\\&+\int_0^T\int_0^L 2b(x)(b_2u^2+v^2)dxdt\Big)\end{aligned}\tag{66}$$

for any weak solution U of (1)-(3), whenever $||u^0||_X\leq R$.

We argue by contradiction. Suppose that (66) is not true. Then, there exists a sequence of functions $\{U_n\}_{n\in\mathbb{N}}=\{(u_n,v_n)\}_{n\in\mathbb{N}}\in C_w([0,T];X)\cap L^2(0,T;[H_0^1(0,L)]^2)$, such that

$$||(u_n(.,0),v_n(.,0))||_X\leq R,\tag{67}$$

solution of

$$\begin{cases}b_1U_{n,t}+AU_{n,xxx}+RU_{n,x}+B(U_n)U_{n,x}+C(U_n)U_n+D(x)U_n=0,\\U_n(0,t)=U_n(L,t)=U_{n,x}(L,t)=0,\\U_n(x,0)=U_{n,0}(x),\end{cases}\tag{68}$$

where $x\in(0,L)$, $t>0$ and A,R,B,C were introduced in the proof of Corollary 2 and D is as diagonal matrix whose diagonal elements are damping functions $b_1b(x)$ and $b(x)$. Moreover,

$$\lim_{n\to\infty}\frac{\frac{b_2}{b_1}||u_n||^2_{L^2(0,T;L^2(0,L))}+||v_n||^2_{L^2(0,T;L^2(0,L))}}{I_n}=\infty,\tag{69}$$

where

$$\begin{aligned}I_n=&\int_0^T\left[(\sqrt{b_2}u_{n,x}(0,t)+\sqrt{a_3^2b_2}v_{n,x}(0,t))^2+\left(1-a_3^2b_2\right)v_{n,x}^2(0,t)\right]dt\\&+2\int_0^T\int_0^L b(x)(b_2u_n^2+v_n^2)dxdt.\end{aligned}$$

Let

$$\sigma_n=\left(\frac{b_2}{b_1}||u_n||^2_{L^2(0,T;L^2(0,L))}+||v_n||^2_{L^2(0,T;L^2(0,L))}\right)^{\frac{1}{2}}=||U_n||_{L^2(0,T;X)}\tag{70}$$

and consider

$$Z_n(x,t) = \frac{1}{\sigma_n} U_n(x,t) = \begin{pmatrix} y_n(x,t) \\ w_n(x,t) \end{pmatrix}. \tag{71}$$

For each $n \in \mathbb{N}$, Z_n satisfies

$$\begin{cases} b_1 Z_{n,t} + A Z_{n,xxx} + R Z_{n,x} + \sigma_n B(Z_n) Z_{n,x} + C(\sigma_n Z_n) Z_n + D(x) Z_n = 0, \\ Z_n(0,t) = Z_n(L,t) = Z_{n,x}(L,t) = 0, \\ Z_n(x,0) = Z_{n,0}(x) = \frac{U_{n,0}(x)}{\sigma_n}, \end{cases} \tag{72}$$

with $0 < x < L$ and $t > 0$,

$$||Z_n||^2_{L^2(0,T;X)} = 1 \tag{73}$$

and

$$\begin{aligned} &\int_0^T [(\sqrt{b_2} y_{n,x}(0,t) + \sqrt{a_3^2 b_2} w_{n,x}(0,t))^2 + \left(1 - a_3^2 b_2\right) w_{n,x}^2(0,t)] dt \\ &+ \int_0^T \int_0^L 2b(x)(b_2 y_n^2 + w_n^2) dx dt \to 0, \end{aligned} \tag{74}$$

as $n \to \infty$. Observe that the energy dissipation law and (67) guarantee that σ_n is bounded. Then, extracting a subsequence, still denoted by the same index, we can assume that

$$\sigma_n \to \sigma \geq 0.$$

Moreover, combining (65), (73) and (74) we deduce that $||Z_{n,0}||_X$ is bounded. Then, following the arguments used in the proof of Theorem 2.7, we can prove that there exists a function $Z = (y, w)$ such that

$$\begin{aligned} &Z_n \rightharpoonup Z \quad \text{in} \quad L^\infty(0,T;X) \quad \text{weak } * \,; \\ &Z_n \rightharpoonup Z \quad \text{in} \quad L^2(0,T;[H^1(0,L)]^2) \quad \text{weak} \,; \\ &Z_n \to Z \quad \text{in} \quad L^2(0,T;X) \quad \text{a. e.} \,; \\ &Z_n \to Z \quad \text{in} \quad C([0,T];[H^{-1}(0,L)]^2); \\ &C(\sigma_n Z_n) Z_{n,x} \to C(\sigma Z) Z_x \quad \text{in} \quad D'((0,L) \times (0,T)). \end{aligned} \tag{75}$$

The last convergence follows from the fact that

$$|a(\sigma_n \mu)| \leq C(1 + |\sigma_n|^p \, |\mu|^p) \leq C'(1 + |\mu|^p),$$

where C' is a positive constant. Consequently, by (74) and (75) we obtain

$$||Z||_{L^2(0,T;X)} = 1 \tag{76}$$

and

$$\begin{aligned} &\int_0^T \left[(\sqrt{b_2} y_x(0,t) + \sqrt{a_3^2 b_2} w_x(0,t))^2 + \left(1 - a_3^2 b_2\right) w_x^2(0,t)\right] dt \\ &+ \int_0^T \int_0^L 2b(x)(b_2 y^2 + w^2) dx dt \leq 0. \end{aligned} \tag{77}$$

Due to the statements above we conclude that Z fulfills

$$\begin{cases} b_1 Z_t + A Z_{xxx} + R Z_x + \sigma B(Z) Z_x + C(\sigma Z) Z + D(x) Z = 0 \\ Z(0,t) = Z(L,t) = 0, \end{cases} \tag{78}$$

in $D'((0,L) \times (0,T))$ and

$$Z \equiv 0 \quad \text{on} \quad \omega \times (0,T). \tag{79}$$

Now, we can apply the unique continuation property given by Corollary 3 to conclude that $Z \equiv 0$ in $(0,L)\times(0,T)$, which contradicts (76). Consequently, (66) holds and the result follows.

However, to apply the Corollary 3 we need to show that $Z \in L^\infty(0,T;[H^1(0,L)]^2)$. Indeed, the boundary conditions follow from the second convergence in (75). So, we claim that the following holds:

For $0 < t_1 < t_2 < T$, *there exists a subinterval* $(t_1', t_2') \subset (t_1, t_2)$ *such that* $Z \in L^\infty(t_1', t_2'; [H^1(0,L)]^2)$.

Indeed, according to (37), for each weak solution W_n we can obtain a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty(\mathbb{R})$ satisfying (30). Moreover if W_n is a solution of

$$\begin{cases} b_1 W_{n,t} + A_n W_{n,xxx} + R W_{n,x} + \sigma_n B(W_n) W_{n,x} + C(\sigma_n W_n) W_n + D(x) W_n = 0, \\ W_n(0,t) = W_n(L,t) = W_{n,x}(L,t) = 0, \\ W_n(x,0) = Z_{n,0}(x) \end{cases} \tag{80}$$

we get

$$W_n - Z_n \to 0 \quad \text{in} \quad C([0,T];[H^{-1}(0,L)]^2), \quad \text{as} \quad n \to \infty. \tag{81}$$

Since

$$||W_n||_{L^2(0,T;[H^1(0,L)]^2)} \le C,$$

for some $C > 0$, we can pick a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset (t_1, (t_1+t_2)/2)$, such that, as $n \to \infty$, $\alpha_n \to \alpha$ and $||W_n(\alpha_n)||_{[H^1(0,L)]^2} \le C'$, where $C' > 0$. On the other hand, from (75) and (81) it follows that

$$W_n(\alpha_n + \cdot) \to Z(\alpha + \cdot) \quad \text{in} \quad C([0,\varepsilon];[H^{-1}(0,L)]^2), \quad \text{as} \quad n \to \infty,$$

for any $\varepsilon < (t_2 - t_1)/2$. Consequently, from Corollary 2, if ε is sufficiently small, we obtain $C'' > 0$ satisfying

$$||W_n(\alpha_n + \cdot)||_{L^\infty(0,\varepsilon;[H^1(0,L)]^2)} \le C'',$$

which allows to conclude that $Z \in L^\infty(\alpha, \alpha+\varepsilon; [H^1(0,L)]^2)$.

Now we can complete the proof. Let $t_1 \in (0,T)$ and $t_2 \in (t_1,T)$. According to the statement above, there exists some interval (t_1', t_2'), such that $Z \in L^\infty(t_1', t_2'; [H^1(0,L)]^2)$. Then, Lemma 3 guarantees that $Z \equiv 0$ in $(0,L)\times(t_1',t_2')$. As t_2 is arbitrarily close to t_1, from the continuity of Z in $H^{-1}(0,L)$ we obtain $Z(0,t) = 0$. □

Now we can prove the main result of this paper.

Proof of Theorem 1.1. The arguments used to obtain the result are known (see, for instance, [20]).

Proposition 3 guarantees the existence of a constant $\alpha > 0$, such that if $E(0) < 1$, the corresponding solution fulfill

$$E(t) \le C' E(0) e^{-\alpha t}, \quad \forall\, t > 0.$$

Moreover, for a given $R > 0$ we obtain positive constants $C = C(R)$ and $\beta = \beta(R)$ such that

$$E(t) \le C E(0) e^{-\beta t}, \quad \forall\, t > 0,$$

whenever $E(0) < R$. Then, setting $T_R := \beta^{-1} \ln(RC)$, we get

$$E(t) \le C' E(T_R) e^{-\alpha(t - T_R)}, \quad \forall\, t > T_R,$$

which give us that

$$E(t) \le C'CE(0)e^{\alpha T_R}e^{-\alpha t}, \quad \forall\, t > 0.$$

This completes the proof. □

5. **Further comments.**

5.1. **The critical case $\mathbf{a(s) = s^4}$.** If the procedure of the previous Section is carried out, then we can address the existence of weak solutions, as well as, the exponential decay of the total energy $E(t)$ assuming that $||(u^0, v^0)||_X << 1$. We follow the arguments in [10].

5.1.1. *Exponential Decay.* We first remark that the energy dissipation law, as well as, (65) remains valid when $a(u) = u^4$. Therefore, we claim that, for any $T > 0$ and $R > 0$, there exists a constant $C = C(R, T) > 0$, such that

$$\begin{aligned}
&\int_0^T \int_0^L u^2 dxdt + \frac{b_1}{b_2} \int_0^T \int_0^L v^2 dxdt \\
&\le C(R,T)\Big(\int_0^T [(\sqrt{b_2}u_x(0,t) + \sqrt{a_3^2 b_2} v_x(0,t))^2 + \left(1 - a_3^2 b_2\right) v_x^2(0,t)]dt \qquad (82)\\
&+ \int_0^T \int_0^L 2b(x)(b_2 u^2 + v^2) dxdt\Big),
\end{aligned}$$

for any solution solution of (1)-(3), whenever $||(u^0, v^0)||_X^2 \le R^2$. Now, the idea is to use the "Compactness-Uniqueness Argument". Therefore, the following estimates will be needed.

First estimate. Multiplying the first equation in (1) by xu, the second by xv and performing integration by parts we get

$$\int_0^T \int_0^L (u_x^2 + v_x^2) dxdt \le C \left[||(u^0, v^0)||_X^2 + \int_0^T \int_0^L (u^6 + v^6) dxdt \right], \qquad (83)$$

where $C = C(T, L)$ is a positive. On the other hand, the Gagliardo-Nirenberg inequality and (7) imply

$$\begin{aligned}
\int_0^T \int_0^L u^6 dxdt &\le C \int_0^T \|u(t)\|_{L^2(0,L)}^4 \|u_x(t)\|_{L^2(0,L)}^2 dt \\
&\le C \,\|(u^0, v^0)\|_X^4 \int_0^T \|u_x(t)\|_{L^2(0,L)}^2 dt
\end{aligned} \qquad (84)$$

for some constant $C > 0$. Similarly, we estimate $\int_0^T \int_0^L v^6 dxdt$. Then, returning to (83) we deduce that

$$(1 - C||(u^0, v^0)||_X^4)||(u, v)||_{L^2(0,T;[H_0^1(0,L)]^2)}^2 \le C \,||(u^0, v^0)||_X^2. \qquad (85)$$

Second estimate. To obtain a bound for u_t we have to pay some attention to the nonlinear term $u^4 u_x = \dfrac{1}{5}\partial_x(u^5)$. First, observe that the argument used in (84) gives

$$\begin{aligned}
\int_0^T \int_0^L |u^5|^{6/5} dxdt &\le c\,\|u^0\|^4 \int_0^T \|u_x(t)\|_{L^2(0,L)}^2 dt \\
&\le c\,\|(u^0, v^0)\|_X^4 \int_0^T \|u_x(t)\|_{L^2(0,L)}^2 dt.
\end{aligned}$$

Therefore, from the (7) and (85) we get

$$\{u^5\} \text{ is bounded in } L^{\frac{6}{5}}((0,T)\times(0,L)).$$

On the other hand, since $L^{\frac{6}{5}}(0,L) \hookrightarrow H^{-1}(0,L)$ we conclude that

$$\{u^4u_x\} = \{\frac{1}{5}\partial_x(u^5)\} \text{ is bounded in } L^{\frac{6}{5}}(0,T;H^{-2}(0,L)).$$

Similarly, we estimate v^4v_x.

Third estimate. Now, we can obtain a bound for (u_t, v_t). Indeed, since

$$\begin{aligned} u_t &= -(u_{xxx} + a_3v_{xxx} + u^4u_x + a_1vv_x + a_2(uv)_x + a(x)u) \\ b_1v_t &= -(v_x + v_{xxx} + b_2a_3u_{xxx} + v^4v_x + b_2a_2uu_x + b_2a_1(uv)_x + a(x)v), \end{aligned}$$

the previous estimates allows to conclude that

$$(u_t, v_t) \text{ is bounded in } L^{\frac{6}{5}}(0,T;[H^{-2}(0,L)]^2). \tag{86}$$

Fourth estimate. Taking the arguments above into account, the main difficult in this case is to pass to the limit in the nonlinear term $w_n^4w_{n,x}$ when $\{w_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H_0^1(0,L))\cap L^\infty(0,T;L^2(0,L))$. Therefore, we claim that the following hold: *There exists* $s>0$ *such that* $\{w_n\}_{n\in\mathbb{N}}$ *is bounded in* $L^4(0,T;H^s(0,L))$, *the embedding* $H^s(0,L)\hookrightarrow L^4(0,L)$ *being compact.*

In fact, by interpolation we can deduce that $\{w_n\}$ is bounded in

$$[L^q(0,T;L^2(0,L)), L^2(0,T;H_0^1(0,L))]_\theta = L^p(0,T;[L^2(0,L),H_0^1(0,L)]_\theta),$$

where $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$ and $0<\theta<1$. Thus, choosing $q=\infty$, $\theta = 1/2$, so that $p=4$, the claim holds with $s=1/2$, i.e.,

$$[L^2(0,L), H_0^1(0,L)]_{\frac{1}{2}} = H^{\frac{1}{2}}(0,L).$$

Furthermore, the embedding $H^{\frac{1}{2}}(0,L)\hookrightarrow L^4(0,L)$ is compact.

Thus, from the statements above and classical compactness results ([21], Corollary 4) we can extract a subsequence of $\{w_n\}_{n\in\mathbb{N}}$, still denoted by the same index n, such that

$$w_n \to w \text{ strongly in } L^4(0,T;L^4(0,L)). \tag{87}$$

Then, arguing as in previous section, we deduce that $E(t)$ decays to zero exponentially.

5.1.2. *Existence of Weak Solutions.*

Definition 5.1. For $(u^0,v^0)\in X$ and $T>0$, we denote by a weak solution of (1)-(3) any function $u\in C_w([0,T];X)\cap L^2(0,T;[H^1(0,L)]^2)$ which solves (1)-(3), and such that, as $p\to 4$,

$$\begin{aligned} &u_p \to u \text{ weakly} * \text{ in } L^\infty(0,T;X), \\ &u_p \to u \text{ weakly in } L^2(0,T;[H^1(0,L)]^2), \end{aligned}$$

u_p denoting a solution of (1)-(3) (as given by Theorem 2.7) for $a(x)=x^p$ and $2<p<4$.

We follow the same steps of the previous estimates and for the sake of simplicity we drop the notation u_p and and use the notation u.

First estimate. Using the multipliers xu and xv the corresponding solution fulfill

$$\int_0^T\int_0^L (u_x^2+v_x^2)dxdt \leq C\left[||(u^0,v^0)||_X^2+\int_0^T\int_0^L (u^{p+2}+v^{p+2})dxdt\right], \tag{88}$$

where $C=C(T,L)$ is a positive constant. Then, from Gagliardo-Nirenberg inequality we obtain

$$\begin{aligned}\int_0^T\int_0^L u^{p+2}dxdt &\leq C\int_0^T \|u(t)\|_{L^2(0,L)}^{(1-\frac{2}{p+2})(p+2)}\|u_x(t)\|_{L^2(0,L)}^2dt\\ &\leq C\,\|(u^0,v^0)\|_X^p\int_0^T \|u_x(t)\|_{L^2(0,L)}^2dt\end{aligned} \tag{89}$$

for some constant $C>0$ that does not depend on p. Similarly, we estimate $\int_0^T\int_0^L v^{p+2}dxdt$. The above estimate and (88) give us that

$$(1-||(u^0,v^0)||_X^p)||(u,v)||^2_{L^2(0,T;[H_0^1(0,L)]^2)} \leq C\,||(u^0,v^0)||_X^2. \tag{90}$$

Second estimate. This estimate is devoted to bound the term u^pu. Arguing as in the previous subsection and using (89)-(90), we deduce that

$$\{u^{p+1}\} \text{ is bounded in } L^{\frac{p+2}{p+1}}((0,T)\times(0,L)),$$

with a bound uniform in p. Therefore,

$$\{(p+1)u^pu_x\}=\{\partial_x(u^{p+1})\} \text{ is bounded in } L^{\frac{p+2}{p+1}}(0,T;H^{-2}(0,L)),$$

i.e.,

$$\{u^pu_x\}=\{\partial_x(u^{p+1})\} \text{ is bounded in } L^{\frac{p+2}{p+1}}(0,T;H^{-2}(0,L))\subset L^{\frac{6}{5}}(0,T;H^{-2}(0,L)),$$

since p is intended to go to 4 and $\frac{p+2}{p+1}>\frac{6}{5}$. Similarly, we can estimate v^pv_x.

Third estimate. Combining the equations in (1) and the previous estimates, we deduce that

$$(u_t,v_t) \text{ is bounded in } L^{\frac{p+2}{p+1}}(0,T;[H^{-2}(0,L)]^2)\subset L^{\frac{6}{5}}(0,T;H^{-2}(0,L)),$$

with a bound uniform in p.

Fourth estimate. To deal with to the nonlinear term we claim that the following hold: *There exists $s>0$ such that $\{u_p\}$ is bounded in $L^4(0,T;H^s(0,L))$, the embedding $H^s(0,L)\hookrightarrow L^4(0,L)$ being compact.*

We can argue as before and use interpolation. Indeed, since $\frac{1}{p}=\frac{1-\theta}{q}+\frac{\theta}{2}$, choosing $\theta=\frac{1}{2}$ and $q=\infty$ we obtain the claim with $s=\frac{1}{2}$.

Due to the statement above and classical compactness results ([21], Corollary 4) we can extract a subsequence of $\{u_p\}$ (still denoted by $\{u_p\}$) and a function $u\in L^\infty(0,T;X)\cap L^2(0,T;[H^1(0,L)]^2)$, such that $u_p\to u$, as $p\to 4$, in the sense described above.

5.2. **Extension of the previous results.** The previous results lead one to consider the same problem for other systems of dispersive equations. For example, a coupled system represented by the following equations

$$\begin{cases} u_t + u_{xxx} + (u^p v^{p+1})_x = 0 \\ v_t + v_x + v_{xxx} + (u^{p+1} v^p)_x = 0. \end{cases}$$

with boundary conditions (2) and $p \in [1, 4)$. The methods developed in this paper allow showing that the same exponential decay property holds when the damping potential $b = b(x)$ is effective in any non-empty subinterval, but the details remains to be done.

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