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CONTROL AND STABILIZATION OF THE NONLINEAR SCHRÖDINGER EQUATION ON RECTANGLES

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This paper studies the local exact controllability and the local stabilization of the semilinear Schrödinger equation posed on a product of n intervals ($n \geq 1$). Both internal and boundary controls are considered, and the results are given with periodic (resp. Dirichlet or Neumann) boundary conditions. In the case of internal control, we obtain local controllability results which are sharp as far as the localization of the control region and the smoothness of the state space are concerned. It is also proved that for the linear Schrödinger equation with Dirichlet control, the exact controllability holds in $H^{-1}(\Omega)$ whenever the control region contains a neighborhood of a vertex.

Keywords: Schrödinger equation; Bourgain spaces; exact boundary controllability; exact internal controllability; exponential stabilization.

AMS Subject Classification: 93B05, 93D15, 35Q53

1. Introduction

The control of the Schrödinger equation has received a lot of attention in the last decades. (See e.g. Ref. 52 for an excellent review of the contributions up to 2003). Significant progresses have been made for the linear Schrödinger equation on its controllability and stabilizability properties (see Refs. 21, 24, 31, 36–38, 40, 42 and 43 for control issues, and Refs. 3, 11, 12, 27, 39, 51 for Carleman estimates and their applications to inverse problems). For the control of the so-called *bilinear* Schrödinger equation, in which the bilinear term is linear in both the control and the state function, see e.g. Refs. 1, 2, 4−7, 10, 20, 41 and the references therein.

By contrast, the study of the nonlinear Schrödinger equation is still at its early stage. Recently, Illner, Lange and Teismann[19,20] considered the internal controllability of the nonlinear Schrödinger equation posed on a finite interval with periodic boundary conditions:

$$iu_t + u_{xx} + f(u) = ia(x)h(x,t). \tag{1.1}$$

In (1.1), a denotes a smooth real function which is strictly supported in $\mathbb{T}$, the one-dimensional torus. They showed that the system (1.1) is locally exactly controllable in the space $H^1(\mathbb{T})$. Their approach was based on the well-known Hilbert Uniqueness Method (HUM) and Schauder fixed point theorem. Later, Lange and Teismann[25] considered internal control for the nonlinear Schrödinger equation (1.1) posed on a finite interval with the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{1.2}$$

and established local exact controllability of the system (1.1)−(1.2) in the space $H_0^1(0,\pi)$ around a special ground state of the system. Their approach was mainly based upon HUM and the implicit function theorem. Dehman, Gérard and Lebeau[13] studied the internal control and stabilization of a class of defocusing nonlinear Schrödinger equations posed on a two-dimensional compact Riemannian manifold M without boundary

$$iu_t + \Delta u + f(u) = ia(x)h(x,t).$$

They demonstrated, in particular, that the system is (semiglobally) exactly controllable and stabilizable in the space $H^1(M)$ assuming that the Geometric Control Condition and some unique continuation properties are satisfied.

Recently, Rosier and Zhang[46] proved that the cubic Schrödinger equation on the torus $\mathbb{T}$ with a localized control

$$iu_t + u_{xx} + \lambda|u|^2u = ia(x)h(x,t), \quad x \in \mathbb{T}, \tag{1.3}$$

is locally exactly controllable in $H^s(\mathbb{T})$ for all $s \geq 0$ (hence, in $L^2(\mathbb{T})$). Inspired by the work of Russell−Zhang,[48] the method of proof combined the momentum approach and Bourgain analysis. In the same paper, the local stabilization by the feedback law $h = a(x)u(x,t)$ was established by applying the contraction mapping theorem in some Bourgain space. Finally, similar results were obtained with Dirichlet (resp. Neumann) homogeneous boundary conditions thanks to an extension argument. More recently, Laurent[28] has shown that the system (1.3) is semiglobally exactly controllable and stabilizable. The same result has also been derived by Laurent[29] for certain manifolds of dimension 3, including $\mathbb{T}^3$, S^3 and $S^2 \times S^1$. The propagation of compactness and regularity proved in Refs. 28 and 29 plays a crucial role in the derivation of the stabilization results in these papers. See also Ref. 30 for another application of these ideas to the semiglobal stabilization of the periodic Korteweg−de Vries equation.

In addition, Rosier and Zhang[47] considered the following nonlinear Schrödinger equation

$$iu_t + \Delta u + \lambda |u|^2 u = 0$$

posed on a bounded domain Ω in $\mathbb{R}^n$ with either the Dirichlet boundary conditions or the Neumann boundary conditions. They showed that if

$$s > \frac{n}{2},$$

or

$$0 \leq s < \frac{n}{2} \quad \text{with } 1 \leq n < 2 + 2s,$$

or

$$s = 0, 1 \quad \text{with } n = 2,$$

then the systems with control inputs acting on the whole boundary of Ω are locally exactly controllable in the classical Sobolev space $H^s(\Omega)$ around any smooth solution of the Schrödinger equation.

The aim of this paper is to extend the results of Rosier and Zhang[46] to any dimension. More precisely, we shall assume that the spatial variable lives in the rectangle

$$\Omega = (0, l_1) \times \cdots \times (0, l_n).$$

We shall investigate the control properties of the semilinear Schrödinger equation

$$iu_t + \Delta u + \lambda |u|^\alpha u = ia(x)h(x,t), \tag{1.4}$$

where $\lambda \in \mathbb{R}$ and $\alpha \in 2\mathbb{N}^*$, by combining new linear controllability results in the spaces $H^s(\Omega)$ with Bourgain analysis. Let us briefly review the results proved in this paper.

The internal controllability of the linear Schrödinger equation on $\mathbb{T}^n$

$$iu_t + \Delta u = ia(x)h(x,t), \quad x \in \mathbb{T}^n, \quad t \in (0,T) \tag{1.5}$$

is established in $H^s(\mathbb{T}^n)$ for any $s \geq 0$ and any function $a \not\equiv 0$. (Note that the Geometric Control Condition is not required.) It is derived from a well-known result in $L^2(\mathbb{T}^n)$, due to Jaffard[21] when $n = 2$ and Komornik[23] for any $n \geq 2$, by an argument allowing to shift the (state and control) space from $L^2(\mathbb{T}^n)$ to $H^s(\mathbb{T}^n)$. In particular, the exact controllability in $H^s(\mathbb{T}^n)$ will require a control input $h \in L^2(0,T; H^s(\mathbb{T}^n))$. Similar results with Dirichlet or Neumann homogeneous boundary conditions are deduced by using the extension argument from Rosier and Zhang.[46]

The boundary controllability of the linear Schrödinger equation is considered both with Dirichlet control

$$u = 1_{\Gamma_0} h(x,t) \tag{1.6}$$

and with Neumann control

$$\frac{\partial u}{\partial \nu} = 1_{\Gamma_0} h(x,t). \tag{1.7}$$

In (1.6) and in (1.7), Γ_0 denotes an open set in $\partial\Omega$. For the Dirichlet control, we shall prove that in *any* dimension $n \geq 2$ the exact controllability holds in $H^{-1}(\Omega)$ whenever Γ_0 is a neighborhood of a vertex of Ω. The observability inequality for this (arbitrarily small) control region is actually derived from the corresponding observability inequality for internal control by multiplier techniques.

For the Neumann control, the exact controllability in $L^2(\Omega)$ is obtained in any dimension when Γ_0 is a side of Ω. Finally, the results with Dirichlet (resp. Neumann) boundary controls are extended to any Sobolev space $H^s(\Omega)$ with $s < 1/2$ (resp. $s < 1$) by considering control inputs more regular in time, namely $h \in H^{\frac{s+1}{2}}(0,T;L^2(\partial\Omega))$ (resp. $h \in H^{\frac{s}{2}}(0,T;L^2(\partial\Omega))$).

The extension of the above exact controllability results to the semilinear Schrödinger equation

$$iu_t + \Delta u + \lambda |u|^\alpha u = ia(x)h(x,t) \tag{1.8}$$

is performed on the basis of Bourgain analysis. The needed linear and multilinear estimates are combined with a fixed-point argument to produce local exact controllability results. Sharp results (for the support of the control input) are given for the internal control. Boundary controllability results are derived from those established for the linear equation with the aid of estimates in Bourgain spaces of solutions of boundary-value problems with boundary terms given by HUM.

Finally, the local exponential stabilization with an internal feedback law is proved by following the same approach as in Rosier and Zhang.[46]

The paper is organized as follows. The controllability results for the linear Schrödinger equation are collected in Sec. 2. Section 3 is devoted to the controllability of the semilinear equations. Section 4 deals with the internal stabilization issue. Multilinear estimates for nonlinearities of the form $u^{\alpha_1}\overline{u}^{\alpha_2}$ are established in the Appendix.

2. Linear Systems

2.1. *Internal control*

We first consider the linear open loop control system for the Schrödinger equation posed on $\mathbb{T}^n := (-\pi,\pi)^n$ with periodic boundary conditions:

$$iu_t + \Delta u = iGh := ia(x)h(x,t), \quad u(x,0) = u_0(x), \tag{2.1}$$

where $a \in C^\infty(\mathbb{T}^n)$ is a given smooth real-valued function and $h = h(x,t)$ is the control input.

We denote by $H^s(\mathbb{T}^n)$ the Sobolev space of the functions u defined on the torus $\mathbb{T}^n$ (i.e. defined on $\mathbb{R}^n$ and periodic of period 2π with respect to each variable x_i) for which the H^s norm

$$\|u\|_s = \|(1-\Delta)^{s/2}u\|_{L^2(\mathbb{T}^n)}$$

is finite.

We first establish an internal observability inequality for the solution $v(t) = W(t)v_0$ of

$$\begin{cases} iv_t + \Delta v = 0 \quad (x,t) \in \mathbb{T}^n \times \mathbb{R}, \\ v(0) = v_0. \end{cases} \tag{2.2}$$

Proposition 2.1. (Observability inequality in $H^{-s}(\mathbb{T}^n)$) *Let $a \in C^\infty(\mathbb{T}^n)$ with $a \neq 0$ and $T > 0$. Then for any $s \geq 0$ there exists a constant $c > 0$ such that for any solution v of* (2.2) *with $v_0 \in H^{-s}(\mathbb{T}^n)$, it holds*

$$\|v_0\|_{-s}^2 \leq c \int_0^T \|av(t)\|_{-s}^2 dt. \tag{2.3}$$

Proof. We proceed in several steps.

Step 1. Assume that $s = 0$, and let

$$\omega = \{x \in (-\pi,\pi)^n;\ |a(x)| > \|a\|_{L^\infty(\mathbb{T}^n)}/2\}.$$

Then, by Lemma 8.9 in Ref. 24, there exists some positive constant c such that for any square-summable sequence $(c_k)_{k\in\mathbb{Z}^n\setminus\{0\}}$ we have

$$\sum_{k\neq 0} |c_k|^2 \leq c \int_0^T \int_\omega \left| \sum_{k\neq 0} c_k e^{i(k\cdot x - |k|^2 t)} \right|^2 dxdt. \tag{2.4}$$

The result is still valid when the set of indices is changed into $\mathbb{Z}^n$ by Proposition 8.4 in Ref. 24. This yields (2.3) when $s = 0$.

Step 2. We prove the weaker inequality

$$\|v_0\|_{-s}^2 \leq c\left(\int_0^T \|av(t)\|_{-s}^2 dt + \|v_0\|_{-s-1}^2 \right) \tag{2.5}$$

by contradiction. If (2.5) is false, then there exists a sequence $\{v_j\}$ of solutions of (2.2) in $C([0,T]; H^{-s}(\mathbb{T}^n))$ such that

$$1 = \|v_j(0)\|_{-s}^2 \geq j\left(\int_0^T \|av_j(t)\|_{-s}^2 dt + \|v_j(0)\|_{-s-1}^2 \right). \tag{2.6}$$

Since v_j is bounded in $L^\infty([0,T]; H^{-s}(\mathbb{T}^n))$ and $(v_j)_t$ is bounded in $L^\infty([0,T]; H^{-s-2}(\mathbb{T}^n))$ by (2.2), we infer from Aubin's lemma that, for a subsequence again denoted by $\{v_j\}$, we have for $j \to \infty$

$$\begin{cases} v_j \to v \quad \text{in } L^\infty([0,T]; H^{-s}(\mathbb{T}^n)) \quad \text{weak } * \\ v_j \to v \quad \text{in } C([0,T]; H^r(\mathbb{T}^n)) \quad \forall\, r < -s \end{cases}$$

where $v \in C_w([0,T]; H^{-s}(\mathbb{T}^n))$ is a solution of (2.2). ($C_w([0,T]; H^{-s}(\mathbb{T}^n)$ denotes the space of weakly sequentially continuous functions from $[0,T]$ to $H^{-s}(\mathbb{T}^n)$.) In particular, $v_j(0) \to v(0)$ in $H^r(\mathbb{T}^n)$ for any $r < -s$. Since $v_j(0) \to 0$ in $H^{-s-1}(\mathbb{T}^n)$ by (2.6), we conclude that $v \equiv 0$. Let $w_j = (1-\Delta)^{-s/2} v_j$. Then $w_j \in L^\infty([0,T]; L^2(\mathbb{T}^n))$ and

$$\begin{cases} w_j \to 0 & \text{in } L^\infty([0,T]; L^2(\mathbb{T}^n)) \quad \text{weak } * \\ w_j \to 0 & \text{in } C([0,T]; H^r(\mathbb{T}^n)) \quad \forall\, r < 0. \end{cases}$$

Let us split aw_j into

$$aw_j = (1-\Delta)^{-s/2}(av_j) - (1-\Delta)^{-s/2}[a, (1-\Delta)^{s/2}]w_j.$$

As the pseudodifferential operator $[a, (1-\Delta)^{s/2}]$ maps continuously $H^r(\mathbb{T}^n)$ into $H^{r-s+1}(\mathbb{T}^n)$, we have that

$$(1-\Delta)^{-s/2}[a, (1-\Delta)^{s/2}]w_j \to 0 \quad \text{in } C([0,T]; H^r(\mathbb{T}^n)) \quad \text{for any } r < 1. \tag{2.7}$$

Therefore, using (2.6) and (2.7), we obtain that

$$aw_j \to 0 \quad \text{in } L^2([0,T]; L^2(\mathbb{T}^n)).$$

Clearly, w_j also satisfies the linear Schrödinger equation (2.2), so we infer from the observability inequality (2.3) established for $s = 0$ that

$$w_j(0) \to 0 \quad \text{in } L^2(\mathbb{T}^n).$$

It follows that $v_j(0) = (1-\Delta)^{s/2} w_j(0) \to 0$ in $H^{-s}(\mathbb{T}^n)$, contradicting the fact that $\|v_j(0)\|_{-s} = 1$ for all j.

Step 3. We prove (2.3) by contradiction. If (2.3) is false, there exists a sequence $\{v_j\}$ of solutions of (2.2) in $C([0,T]; H^{-s}(\mathbb{T}^n))$ such that

$$1 = \|v_j(0)\|_{-s}^2 \geq j \int_0^T \|av_j(t)\|_{-s}^2 dt \quad \forall\, j \geq 0. \tag{2.8}$$

Extracting a subsequence if needed, we may assume that

$$v_j \to v \quad \text{in } L^\infty([0,T]; H^{-s}(\mathbb{T}^n)) \quad \text{weak } * \tag{2.9}$$

$$v_j \to v \quad \text{in } C([0,T]; H^r(\mathbb{T}^n)) \quad \forall\, r < -s \tag{2.10}$$

for some solution $v \in C_w([0,T]; H^{-s}(\mathbb{T}^n))$ of (2.2) (see Lemma 8.1 in Ref. 35). Clearly, $av_j \to av$ in $L^\infty([0,T]; H^{-s}(\mathbb{T}^n))$ weak $*$ which, combined with (2.8), yields $av \equiv 0$. An application of Holmgren theorem (see e.g. Theorem 8.6.5 in Ref. 18) gives $v \equiv 0$. On the other hand, (2.10) gives $v_j(0) \to 0$ in $H^{-s-1}(\mathbb{T}^n)$. It then follows from (2.5) that $v_j(0) \to 0$ in $H^{-s}(\mathbb{T}^n)$, and this contradicts (2.8). □

Applying HUM[34] with $L^2(\mathbb{T}^n)$ as pivot space, we infer from Proposition 2.1 the following internal controllability of the linear Schrödinger equation in $H^s(\mathbb{T}^n)$.

Theorem 2.1. *Let $T > 0$ and $s \geq 0$ be given. Then for any $(u_0, u_1) \in H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n)$ there exists a control $h \in L^2([0,T]; H^s(\mathbb{T}^n))$ such that the system (2.1) admits*

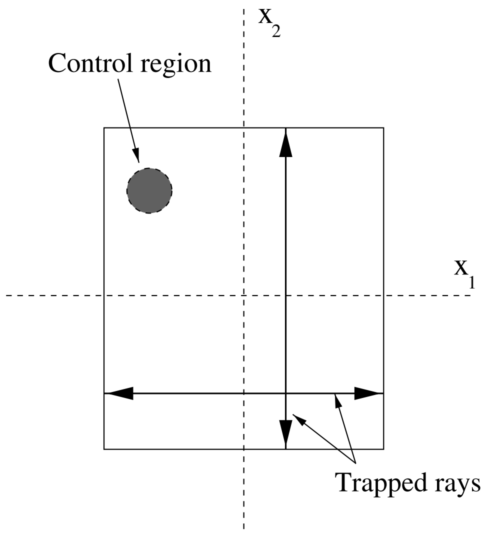

Fig. 1. Internal control of the Schrödinger equation.

a unique solution $u \in C([0,T]; H^s(\mathbb{T}^n))$ *satisfying* $u(T) = u_1$. *Moreover, we can define a bounded operator*

$$\Phi : H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n) \to L^2([0,T]; H^s(\mathbb{T}^n))$$

such that for any $(u_0, u_1) \in H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n)$ *it holds*

$$W(T)u_0 + \int_0^T W(T-\tau)(G(\Phi(u_0,u_1)))(\cdot,\tau)d\tau = u_1. \tag{2.11}$$

The (small) control region is represented in Fig. 1. Trapped rays are drawn to mean that the wave equation fails to be controllable with such control regions.

2.2. *Boundary control*

In this section $\Omega = (0,\pi)^n$, and Γ_0 denotes an open set in $\partial\Omega$.

2.2.1. *Dirichlet boundary control*

We first adopt the following definition.

Definition 2.1. The open set $\Gamma_0 \subset \partial\Omega$ is called a *Dirichlet control domain* if given any u_0, $u_1 \in H^{-1}(\Omega)$ and any time $T > 0$, one may find a control $h \in L^2(0,T; L^2(\Gamma_0))$ such that the solution $u = u(x,t)$ of

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = 1_{\Gamma_0} h(x,t) & \text{on } \partial\Omega \times (0,T) \\ u(0) = u_0 \end{cases} \tag{2.12}$$

satisfies $u(T) = u_1$.

The following result provides Dirichlet control domains which are arbitrarily small in *any* dimension $n \geq 2$. Note that the wave equation fails to be controllable with such control domains.

Theorem 2.2. *Let* $\Omega = (0,\pi)^n$, *and let* $\Gamma_0 \subset \partial\Omega$ *be any open set containing a vertex of* $\partial\Omega$. *Then* Γ_0 *is a Dirichlet control domain.*

By Dolecki–Russell test of controllability (or HUM), Theorem 2.2 is a direct consequence of the following boundary observability result for the system

$$\begin{cases} iv_t + \Delta v = 0 & \text{in } \Omega \times (0,T) \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v(0) = v_0. \end{cases} \tag{2.13}$$

Proposition 2.2. *Assume that the* (*open*) *control region* $\Gamma_0 \subset \partial\Omega$ *contains a vertex of* $\partial\Omega$. *Then for every* $T > 0$, *there exists a constant* $c > 0$ *such that*

$$\|\nabla v_0\|^2_{L^2(\Omega)} \leq c \int_0^T \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt \tag{2.14}$$

for any solution v *of* (2.13) *with* $v_0 \in H_0^1(\Omega)$.

Proof. We proceed in several steps.

Step 1. First, we prove an observability inequality in $H_0^1(\Omega)$ with an internal observation in an arbitrary subdomain of Ω.

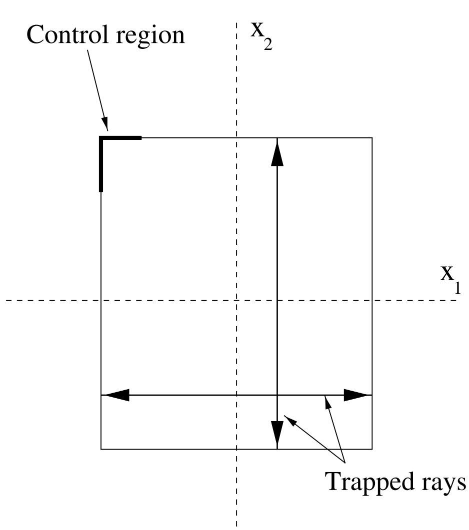

Fig. 2. Boundary control of the Schrödinger equation.

Lemma 2.1. *Let $\omega \subset \Omega$ be an arbitrary nonempty open set. Then there exists a constant $c > 0$ such that*

$$\|\nabla v_0\|^2_{L^2(\Omega)} \leq c \int_0^T \int_\omega |\nabla v(x,t)|^2 dxdt \tag{2.15}$$

for every solution v of (2.13) with $v_0 \in H_0^1(\Omega)$.

Proof of Lemma 2.1. Extend v to $(-\pi,\pi)^n \times (0,T)$ in such a way that v is an odd function of x_i for each $i = 1,\ldots,n$, and extend the initial state v_0 in a similar way. Then v solves (2.2). Writing $v_0 = \sum_{k\in\mathbb{Z}^n} c_k e^{ik\cdot x}$, we have that

$$\nabla v(x,t) = \sum_{k\in\mathbb{Z}^n} ic_k e^{i(k\cdot x - |k|^2 t)} k.$$

It follows then from (2.4) that

$$\begin{aligned}\|\nabla v_0\|^2_{L^2(\mathbb{T}^n)} &= \sum_{j=1}^n \sum_{k\in\mathbb{Z}^n} |k_j|^2 |c_k|^2 \\ &\leq c \sum_{j=1}^n \int_0^T \int_\omega \left| \sum_{k\in\mathbb{Z}^n} c_k e^{i(k\cdot x - |k|^2 t)} k_j \right|^2 dxdt \\ &\leq c \int_0^T \int_\omega |\nabla v|^2 dxdt.\end{aligned}$$

The lemma is proved. □

Step 2. We use the multiplier method to reduce the boundary observation inequality to an internal observation inequality. Without loss of generality, we may assume that Γ_0 is a (small) neighborhood of the vertex $M = (\pi,\ldots,\pi)$ defined as

$$\Gamma_0 = \{x \in \partial\Omega;\ x_1 + \cdots + x_n > n\pi - \varepsilon\},$$

where ε is a (possibly small) positive number. The following lemma is needed.

Lemma 2.2. *There exists a non-negative function $\theta \in C^3(\mathbb{R}^n)$ which is null on $\{x \in \mathbb{R}^n;\ x_1 \leq 0\}$ and strictly convex on $(0,+\infty)^n \cap B_1(0)$.*

Proof of Lemma 2.2. Set $y^+ = \max(y,0)$ for all $y \in \mathbb{R}$. Let

$$\theta(x_1,\ldots,x_n) = (x_1^+)^4 \left(1 + \delta \sum_{j=2}^n (x_j^+)^4\right),$$

where $\delta > 0$ is a small number whose value will be specified later. Clearly, θ is a non-negative function of class C^3 on $\mathbb{R}^n$, which vanishes on the set $\{x_1 \leq 0\}$. To prove that θ is strictly convex on $(0,+\infty)^n \cap B_1(0)$, it is sufficient to check that the Hessian matrix

$$H(x) = \left(\frac{\partial^2\theta}{\partial x_i \partial x_j}(x)\right) \tag{2.16}$$

is positive definite for every $x \in (0,+\infty)^n \cap B_1(0)$. Simple computations give that for any $\xi \in \mathbb{R}^n$,

$$\xi^T H(x)\xi = 12x_1^2\left(1+\delta\sum_{j=2}^n x_j^4\right)\xi_1^2 + 12\delta x_1^4\sum_{j=2}^n x_j^2\xi_j^2 + 32\delta x_1^3\xi_1\sum_{j=2}^n x_j^3\xi_j.$$

From Young inequality, we obtain that

$$32|x_1^3x_j^3\xi_1\xi_j| \le 26x_1^2x_j^4\xi_1^2 + 10x_1^4x_j^2\xi_j^2,$$

therefore

$$\xi^T H(x)\xi \ge (12-26(n-1)\delta)x_1^2\xi_1^2 + 2\delta x_1^4\sum_{j=2}^n x_j^2\xi_j^2 \ge c|\xi|^2 \tag{2.17}$$

if $x \in (0,+\infty)^n \cap B_1(0)$ and $\delta < (6/13)(n-1)^{-1}$. □

At this position, we need an identity from Ref. 37.

Lemma 2.3. (Lemma 2.2 in Ref. 37) *For any $q \in H^2(\Omega,\mathbb{R}^n)$ and any solution v of (2.13) issued from $v_0 \in H_0^1(\Omega)$, the following holds*

$$\begin{aligned}\frac{1}{2}\int_0^T\int_{\partial\Omega}(q\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt &= \frac{1}{2}\mathrm{Im}\int_\Omega (vq\cdot\nabla\bar{v})dx|_0^T + \frac{1}{2}\mathrm{Re}\int_0^T\int_\Omega (v\nabla(\mathrm{div}\ q)\cdot\nabla\bar{v})dxdt \\ &\quad + \mathrm{Re}\int_0^T\int_\Omega\sum_{j,k=1}^n \frac{\partial q_k}{\partial x_j}\frac{\partial\bar{v}}{\partial x_k}\frac{\partial v}{\partial x_j}dxdt.\end{aligned} \tag{2.18}$$

Let

$$\omega = \{x\in\Omega;\ x_1+\cdots+x_n > n\pi-\varepsilon\}.$$

We readily infer from Lemma 2.2 that there exists a convex function $\theta \in C^3(\overline{\Omega})$ which is strictly convex on ω and null on $\overline{\Omega\backslash\omega}$. Using (2.18) with $q=\nabla\theta$ we obtain

$$\begin{aligned}\int_0^T\int_\omega \nabla\bar{v}(x)^T H(x)\nabla v(x)dxdt \\ \le c\int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt + C_\delta\int_\Omega |v_0|^2dx + \delta\int_\Omega|\nabla v_0|^2dx,\end{aligned} \tag{2.19}$$

where $\delta>0$ is a small number and $H(x)$ denotes the Hessian matrix given in (2.16). In (2.19), we used the fact that both quantities $\|v(t)\|_{L^2(\Omega)}$ and $\|\nabla v(t)\|_{L^2(\Omega)}$ are conserved. Using Lemma 2.1 and the fact that the Hessian matrix $H(x) = (\partial^2\theta/\partial x_i\partial x_j)(x)$ is positive definite on ω, we obtain

$$\|\nabla v_0\|_{L^2(\Omega)}^2 \le c\int_0^T\int_{\Gamma_0}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma dt + C_\delta\int_\Omega |v_0|^2dx \tag{2.20}$$

for a convenient choice of δ. The proof of the estimate

$$\|v_0\|^2_{L^2(\Omega)} \le c \int_0^T \int_{\Gamma_0} \left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt \tag{2.21}$$

is classical (see e.g. pp. 27 and 28 in Ref. 37). Then (2.14) follows from (2.20) and (2.21). This completes the proof of Proposition 2.2 and Theorem 2.2. □

Remark 2.1. (1) Theorem 2.2 is stated for a square $\Omega = (0,\pi)^n$, but it is valid (with the same proof) for any rectangle $\Omega = (0,l_1) \times \cdots \times (0,l_n)$.

(2) Using a frequential criterion and number theoretic arguments, Ramdani *et al.*[43] proved that when $n = 2$, $\Gamma_0 \subset \partial\Omega$ is a Dirichlet control domain if and only if Γ_0 has both a horizontal and a vertical component. It is, however, unclear whether the approach in Ref. 43 can yield a similar result for $n \ge 3$.

(3) Using Theorem 2.1 on a rectangle $\tilde{\Omega} = (-1,\pi) \times (0,\pi)^{n-1}$ with a control input supported in $\tilde{\Omega}\backslash\Omega$, and next taking the restriction to Ω, we infer that the linear Schrödinger equation is controllable in $L^2(\Omega)$ with a control supported on a side. (This fact can also be deduced from the Carleman inequalities established in Ref. 39.) This suggests that the condition for a domain to be a Dirichlet control domain is less restrictive when the state space is smoothed.

We now aim to extend Theorem 2.2 to a control result in a space $H^s(\Omega)$, with $s \ge -1$. We define $H^s_D(\Omega) = D(A_D^{\frac{s}{2}})$, where A_D is the Dirichlet Laplacian; i.e. $A_D u = -\Delta u$ with domain $D(A_D) = H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega)$. First we need to replace the characteristic function 1_{Γ_0} by a smooth controller function $g \in L^\infty(\partial\Omega)$. We adopt the following

Definition 2.2. Let $g \in L^\infty(\partial\Omega)$. We say that g is a *smooth Dirichlet controller* if

(i) there exists a constant $C > 0$ such that

$$\|\nabla v_0\|^2_{L^2(\Omega)} \le C \int_0^T \int_{\partial\Omega} g(x) \left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt \tag{2.22}$$

for any solution v of (2.13) emanating from $v_0 \in H^1_0(\Omega)$ at $t = 0$;

(ii) for any face F of $\partial\Omega$, $g_F = g_{|_F} \in C^\infty(F)$ and for all $k \ge 0$

$$\frac{\partial^{2k+1} g_F}{\partial \nu^{2k+1}} = 0 \quad \text{on } \partial F, \tag{2.23}$$

where ν denotes the unit outward normal vector to ∂F.

Note that for any nonempty open set $\Gamma_0 \subset \partial\Omega$ containing a vertex of $\partial\Omega$ one can construct a smooth Dirichlet controller g supported in Γ_0. Consider for example a small neighborhood $\Gamma_0 = [0,\varepsilon)^n \cap \partial\Omega$ of 0 in $\partial\Omega$. A smooth Dirichlet controller g supported in Γ_0 is given by

$$g(x_1,\ldots,x_n) = \prod_{i=1}^n \rho(x_i),$$

where $\rho \in C^\infty(\mathbb{R})$ fulfills

$$\rho(s) = \begin{cases} 1 & \text{if } s \leq \dfrac{\varepsilon}{4}, \\ 0 & \text{if } s \geq \dfrac{\varepsilon}{2}. \end{cases}$$

Note also that $g \in C^0(\partial\Omega)$ and that the set $\{x \in \partial\Omega; g(x) > 0\}$ is an open neighborhood of 0 in $\partial\Omega$.

Let g be a smooth Dirichlet controller, and let S denote the bounded operator $H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $Sv_0 = u(T)$, where $u = u(x,t)$ solves

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = g(x)h(x,t) & \text{on } \partial\Omega \times (0,T) \\ u(0) = 0 \end{cases} \tag{2.24}$$

with $h(x,t) = (\partial v/\partial \nu)(x,t)$, $v = W_D(t)v_0$ denoting the solution of

$$\begin{cases} iv_t + \Delta v = 0 & \text{in } \Omega \times (0,T) \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v(0) = v_0. \end{cases} \tag{2.25}$$

Applying HUM, we infer from the observability inequality (2.22) that S is invertible. We shall prove that a similar result holds in more regular spaces.

Theorem 2.3. *Pick any number $s \in [-1, \frac{1}{2})$. Then S is an isomorphism from $H_D^{s+2}(\Omega)$ onto $H_D^s(\Omega)$. More precisely, for any $T > 0$ and any $u_T \in H_D^s(\Omega)$, if we set $h(x,t) = (\partial v/\partial \nu)(x,t)$ where v denotes the solution of (2.25) with $v_0 = S^{-1}u_T$, then $v_0 \in H_D^{s+2}(\Omega)$, $h \in H^{\frac{s+1}{2}}(0,T;L^2(\partial\Omega))$, and the solution u of (2.24) satisfies $u \in C([0,T];H_D^s(\Omega))$ and $u(T) = u_T$.*

Proof. Step 1. Let us first check that S^{-1} is a bounded operator from $H_D^s(\Omega)$ into $H_D^{s+2}(\Omega)$ for $s \in [-1, \frac{1}{2})$. The result is already known for $s = -1$. Assume first that $-1 < s < 0$, and pick any $u_T \in H_D^s(\Omega)$ decomposed as

$$u_T(x) = \sum_{p \in (\mathbb{N}^*)^n} u_{T,p} \sin(p_1 x_1) \cdots \sin(p_n x_n),$$

with $\sum_{p \in (\mathbb{N}^*)^n} |p|^{2s} |u_{T,p}|^2 < \infty$. Let $v_0 = S^{-1}(u_T) \in H_D^1(\Omega)$ decomposed as

$$v_0(x) = \sum_{p \in (\mathbb{N}^*)^n} v_p \sin(p_1 x_1) \cdots \sin(p_n x_n), \tag{2.26}$$

and let v denote the solution of (2.25). The control given by HUM driving (2.24) from 0 to u_T reads

$$h(x,t) := \frac{\partial v}{\partial \nu} = \sum_{p \in (\mathbb{N}^*)^n} v_p e^{-i|p|^2 t} \frac{\partial}{\partial \nu} (\sin(p_1 x_1) \cdots \sin(p_n x_n)). \tag{2.27}$$

Let us write the solution $u = u(x,t)$ of (2.24) in the form

$$u(x,t) = \sum_{p\in(\mathbb{N}^*)^n} u_p(t)\sin(p_1x_1)\cdots\sin(p_nx_n). \tag{2.28}$$

The moments $\{u_p(t)\}_{p\in(\mathbb{N}^*)^n}$ can be computed from the control input h by using duality. Scaling in (2.24) by $\overline{w}$, where $w = W_D(t)w_0$ is a smooth solution, we obtain

$$i\int_\Omega u(x,t)\overline{w(x,t)}dx = \int_0^t\int_{\partial\Omega} g(x)h(x,\tilde{t})\overline{\frac{\partial w}{\partial\nu}}\,d\sigma(x)d\tilde{t}.$$

Pick any $q \in (\mathbb{N}^*)^n$ and choose $w_0(x) = \sin(q_1x_1)\cdots\sin(q_nx_n)$. We obtain from (2.27) that

$$\begin{aligned}
&\left(\frac{\pi}{2}\right)^n ie^{i|q|^2t}u_q(t)\\
&\quad= \int_0^t\int_{\partial\Omega} g(x)h(x,\tilde{t})e^{i|q|^2\tilde{t}}\frac{\partial}{\partial\nu}(\sin(q_1x_1)\cdots\sin(q_nx_n))d\sigma(x)d\tilde{t}\\
&\qquad+\sum_{p\in(\mathbb{N}^*)^n} v_p\left(\int_0^t e^{i(|q|^2-|p|^2)\tilde{t}}d\tilde{t}\right)\\
&\qquad\times\int_{\partial\Omega} g(x)\frac{\partial}{\partial\nu}(\sin(p_1x_1)\cdots\sin(p_nx_n))\\
&\qquad\times\frac{\partial}{\partial\nu}(\sin(q_1x_1)\cdots\sin(q_nx_n))d\sigma(x).
\end{aligned}\tag{2.29}$$

It follows that for $t = T$

$$S(v_0) = u_T = u(T) = \sum_{q\in(\mathbb{N}^*)^n}\left(\sum_{p\in(\mathbb{N}^*)^n} a_{q,p}v_p\right)\sin(q_1x_1)\cdots\sin(q_nx_n) \tag{2.30}$$

with

$$\begin{aligned}
a_{q,p} = {}& -\left(\frac{2}{\pi}\right)^n\frac{e^{-i|p|^2T}-e^{-i|q|^2T}}{|q|^2-|p|^2}\int_{\partial\Omega} g(x)\frac{\partial}{\partial\nu}(\sin(p_1x_1)\cdots\sin(p_nx_n))\\
&\times\frac{\partial}{\partial\nu}(\sin(q_1x_1)\cdots\sin(q_nx_n))d\sigma(x).
\end{aligned}\tag{2.31}$$

In (2.31), we used the convention that

$$\frac{e^{-i|p|^2t}-e^{-i|q|^2t}}{|q|^2-|p|^2} = ite^{-i|q|^2t}\quad\text{for } |p| =| q|. \tag{2.32}$$

Introduce the operator D^σ defined by

$$D^\sigma\left(\sum_{p\in(\mathbb{N}^*)^n} c_p\sin(p_1x_1)\cdots\sin(p_nx_n)\right) = \sum_{p\in(\mathbb{N}^*)^n}|p|^\sigma c_p\sin(p_1x_1)\cdots\sin(p_nx_n).$$

In what follows, $\sum_p$ and $\sum_q$ will stand for $\sum_{p\in(\mathbb{N}^*)^n}$ and $\sum_{q\in(\mathbb{N}^*)^n}$, respectively. We aim to prove that $v_0 \in H_D^{s+2}(\Omega)$ for $u_T \in H_D^s(\Omega)$. For v_0 given by (2.26), let

$$\|v_0\|_s^2 = \sum_p |p|^{2s}|v_p|^2.$$

C denoting a constant varying from line to line, we have that

$$\begin{aligned}\|v_0\|_{s+2} &\le \|D^{s+1}v_0\|_1 \\ &\le C\|S(D^{s+1}v_0)\|_{-1} \\ &\le C\big(\|D^{s+1}(Sv_0)\|_{-1} + \|[S,D^{s+1}]v_0\|_{-1}\big) \\ &\le C\big(\|u_T\|_s + \|[S,D^{s+1}]v_0\|_{-1}\big).\end{aligned} \tag{2.33}$$

Clearly

$$[S,D^{s+1}]v_0 = \sum_q \left(\sum_p a_{q,p}(|p|^{s+1} - |q|^{s+1})v_p\right)\sin(q_1x_1)\cdots\sin(q_nx_n),$$

hence

$$\|[S,D^{s+1}]v_0\|_{-1}^2 = \sum_q |q|^{-2}\left[\sum_p a_{q,p}(|p|^{s+1} - |q|^{s+1})v_p\right]^2.$$

Writing $\partial\Omega = \cup_{0\le l<2^n-1}F_l$, where the F_l's denote the faces of Ω, the integral term in (2.31) may be written $\sum_{0\le l<2^n-1} I_{F_l}$, with

$$I_{F_l} := \int_{F_l} g(x)\frac{\partial}{\partial\nu}(\sin(p_1x_1)\cdots\sin(p_nx_n))\frac{\partial}{\partial\nu}(\sin(q_1x_1)\cdots\sin(q_nx_n))d\sigma(x).$$

Let us estimate I_{F_l} for $F_0 := \{x\in\partial\Omega; x_n = 0\} = [\,0,\pi]^{n-1}\times\{0\}$. Then

$$\begin{aligned}|I_{F_0}| &= p_nq_n\left|\int_{[0,\pi]^{n-1}} g(x_1,\ldots,x_{n-1},0)\left[\prod_{j=1}^{n-1}\sin(p_jx_j)\sin(q_jx_j)\right]dx_1\cdots dx_{n-1}\right| \\ &= p_nq_n\left|\int_{[0,\pi]^{n-1}} g(x_1,\ldots,x_{n-1},0)\right. \\ &\qquad \left.\times\left[\prod_{j=1}^{n-1}\frac{1}{2}(\cos(p_j-q_j)x_j - \cos(p_j+q_j)x_j)\right]dx_1\cdots dx_{n-1}\right|.\end{aligned}$$

Using (2.23) and integrations by parts, we see that for every $k\in\mathbb{N}$, we have for some constant $C_k > 0$

$$|I_{F_0}| \le C_kp_nq_n\prod_{j=1}^{n-1}\langle p_j - q_j\rangle^{-k}, \tag{2.34}$$

where $\langle y\rangle := (1+|y|^2)^{\frac{1}{2}}$.

The corresponding contribution in $\|[S, D^{s+1}]v_0\|_{-1}^2$ is therefore estimated by

$$A_{F_0} = \sum_q |q|^{-2}\left(\sum_p p_n q_n \left(\prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right)\langle |q|^2 - |p|^2\rangle^{-1} \big||p|^{s+1} - |q|^{s+1}\big| |v_p|\right)^2.$$

Since

$$\frac{\big||p|^{s+1} - |q|^{s+1}\big|}{\langle |q|^2 - |p|^2\rangle} \le C\frac{\big||p| - |q|\big|(|p|^s + |q|^s)}{\langle |q|^2 - |p|^2\rangle} \le C\frac{|p|^s + |q|^s}{|p| + |q|}$$

we have by Cauchy–Schwarz

$$\begin{aligned} A_{F_0} &\le C\sum_q \left[\sum_p p_n \left(\prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right)\frac{|p|^s + |q|^s}{|p| + |q|}|v_p|\right]^2 \\ &\le C\sum_q \left(\sum_p \frac{|p|^{2s} + |q|^{2s}}{(|p| + |q|)^2}\prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right) \\ &\quad \times \left(\sum_p p_n^2 |v_p|^2 \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right). \end{aligned} \tag{2.35}$$

Pick any $k > 1$. Then, as $s < 0$,

$$\begin{aligned} &\sum_{q_n}\sum_p \frac{|p|^{2s} + |q|^{2s}}{(|p| + |q|)^2}\prod_{j=1}^{n-1}\langle p_j - q_j\rangle^{-k} \\ &\quad \le \sum_{q_n}\sum_{p_n}\frac{p_n^{2s} + q_n^{2s}}{(p_n + q_n)^2}\sum_{p_1,\dots,p_{n-1}}\prod_{j=1}^{n-1}\langle p_j - q_j\rangle^{-k} < \infty. \end{aligned}$$

Therefore

$$\begin{aligned} A_{F_0} &\le C\sum_{q_1,\dots,q_{n-1}}\sum_p p_n^2|v_p|^2\prod_{j=1}^{n-1}\langle p_j - q_j\rangle^{-k} \\ &\le C\sum_p |p|^2|v_p|^2 \sum_{q_1,\dots,q_{n-1}}\prod_{j=1}^{n-1}\langle p_j - q_j\rangle^{-k} \\ &\le C\sum_p |p|^2|v_p|^2. \end{aligned}$$

The estimate for another face F_l is similar. We conclude that

$$\|[S, D^{s+1}]v_0\|_{-1}^2 \le C\|v_0\|_1^2 \le C\|u_T\|_{-1}^2$$

hence, with (2.33), $v_0 \in H_D^{s+2}(\Omega)$. Let us now assume that $u_T \in H_D^s(\Omega)$ with $0 \le s < \frac{1}{2}$. The proof is carried out as above when $-1 < s < 0$, except for the estimate of A_{F_0} in (2.35). We know from the lines above that $v_0 \in H_D^\sigma(\Omega)$ for any $\sigma < 2$. Then, by Cauchy−Schwarz inequality,

$$
\begin{aligned}
A_{F_0} &\leq C \sum_q \left(\sum_p p_n \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \frac{|p|^s + |q|^s}{|p| + |q|} |v_p| \right)^2 \\
&\leq C \sum_q \left(\sum_p \frac{|p|^{2s} + |q|^{2s}}{(|p| + |q|)^2} |p|^{-1} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right) \\
&\quad \times \left(\sum_p p_n^2 |p| |v_p|^2 \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right). \qquad (2.36)
\end{aligned}
$$

Note that

$$
\sum_{q_n} \left(\sum_p \frac{|p|^{2s} + |q|^{2s}}{(|p| + |q|)^2} |p|^{-1} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right) \leq C(S_1 + S_2 + S_3),
$$

where

$$
\begin{aligned}
S_1 &= \sum_{q_n} \left(\sum_p \frac{|p|^{2s-1}}{(|p| + |q|)^2} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right), \\
S_2 &= \sum_{q_n} \left(\sum_p \frac{q_n^{2s} |p|^{-1}}{(|p| + |q|)^2} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right), \\
S_3 &= \sum_{q_n} \left(\sum_p \frac{|q'|^{2s} |p|^{-1}}{(|p| + |q|)^2} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right), \quad \text{where } q = (q', q_n).
\end{aligned}
$$

Since $2s - 1 < 0$,

$$
S_1 \leq \sum_{q_n} \left(\sum_p \frac{p_n^{2s-1}}{(p_n + q_n)^2} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right) \leq \text{const.} < \infty.
$$

Also,

$$
\begin{aligned}
S_2 &\leq \sum_{q_n} \left(\sum_p \frac{q_n^{2s} p_n^{-1}}{(p_n + q_n)^2} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right) \\
&\leq C \sum_{p_n} \sum_{q_n} \frac{q_n^{2s} p_n^{-1}}{(p_n + q_n)^2} \\
&\leq C \sum_{p_n} \left(\frac{1}{p_n (p_n + 1)^2} + p_n^{2s-3} + \int_1^{\infty} \frac{x^{2s}}{p_n (p_n + x)^2} dx \right) \\
&\leq C \left(1 + \sum_{p_n \geq 1} p_n^{2s-2} \int_0^{+\infty} \frac{y^{2s}}{(1 + y)^2} dy \right) \\
&\leq \text{const.} < \infty.
\end{aligned}
$$

In the third line, we noticed that the function $x > 0 \mapsto x^{2s}(p_n + x)^{-2}$ is decreasing for $s \leq 0$, which leads to the bound by the first and third terms, while for $s > 0$ this function is first increasing up to $x = sp_n/(1 - s)$ and then decreasing, which leads to the bound by the second and third terms. Finally,

$$S_3 \leq |q'|^{2s} \sum_{q_n} \sum_{p} \frac{p_n^{-1}}{(p_n + q_n)^2} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \leq C|q'|^{2s}.$$

It follows that

$$A_{F_0} \leq C \sum_{q_1,\ldots,q_{n-1}} \sum_{p} p_n^2 |p||v_p|^2 |q'|^{2s} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k}.$$

Note that

$$\sum_{q_1,\ldots,q_{n-1}} |q'|^{2s} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \leq C|p'|^{2s}$$

since, for $k > 2s + 1$,

$$\sum_{q_j} q_j^{2s} \langle p_j - q_j \rangle^{-k} \leq C p_j^{2s}.$$

(Split the sum into one for $q_j \leq 2p_j$, and another one for $q_j > 2p_j$.) Therefore, since $0 \leq s < 1/2$,

$$A_{F_0} \leq C \sum_{p} |p|^{3+2s} |v_p|^2 = \|v_0\|_{s+\frac{3}{2}}^2 \leq C\|u_T\|_{s-\frac{1}{2}}^2. \tag{2.37}$$

Thus, we have proved that S^{-1} is bounded from $H_D^s(\Omega)$ into $H_D^{s+2}(\Omega)$ for $-1 \leq s < \frac{1}{2}$. Note that, for $v_0 \in H_D^{s+2}(\Omega)$, $h \in H^{\frac{s+1}{2}}(\mathbb{T}; L^2(\partial\Omega))$ by (2.27).

Step 2. Since S is an isomorphism from $H_D^1(\Omega)$ onto $H_D^{-1}(\Omega)$, it remains to prove that S maps $H_D^{s+2}(\Omega)$ into $H_D^s(\Omega)$. The proof of Theorem 2.3 will thus be complete with the following result.

Proposition 2.3. *Let $s \in [-1, \frac{1}{2})$ and $T > 0$. For any $v_0 \in H_D^{s+2}(\Omega)$, let $u = \Gamma v_0$ denote the solution of (2.24) associated with $h = \partial v/\partial\nu$, where $v(t) = W_D(t)v_0$. Then Γ is a bounded operator from $H_D^{s+2}(\Omega)$ into $C([0,T]; H_D^s(\Omega))$.*

Proof of Proposition 2.3. It is well known that for any $h \in L^2(0,T; L^2(\partial\Omega))$, there exists a unique solution $u \in C([0,T]; H^{-1}(\Omega))$ in the transposition sense of (2.24) (see e.g. Ref. 37). The result is therefore true for $s = -1$. Let us now assume that $s \in (-1, \frac{1}{2})$. From Step 1, we know that u is given by

$$u(t) = -\left(\frac{2}{\pi}\right)^n \sum_{q\in(\mathbb{N}^*)^n} \left(\sum_{p\in(\mathbb{N}^*)^n} v_p \frac{e^{-i|p|^2 t} - e^{-i|q|^2 t}}{|q|^2 - |p|^2} I(g,p,q) \right) \sin(q_1 x_1) \cdots \sin(q_n x_n), \tag{2.38}$$

where

$$I(g,p,q) = \int_{\partial\Omega} g(x)\frac{\partial}{\partial\nu}(\sin(p_1x_1)\cdots\sin(p_nx_n)) \times \frac{\partial}{\partial\nu}(\sin(q_1x_1)\cdots\sin(q_nx_n))d\sigma(x). \tag{2.39}$$

Again $I(g,p,q) = \sum_{0\le l<2^n-1} I_{F_l}$, where the F_l's denote the faces of Ω and I_{F_l} is given in (2.29). We have that

$$\begin{aligned}\|\Gamma v_0\|_{L^\infty(0,T;H^s_D(\Omega))} &= \|D^{s+1}(\Gamma v_0)\|_{L^\infty(0,T;H^{-1}_D(\Omega))}\\ &\le \|\Gamma(D^{s+1}v_0)\|_{L^\infty(0,T;H^{-1}_D(\Omega))}\\ &\quad + \|[\Gamma, D^{s+1}]v_0\|_{L^\infty(0,T;H^{-1}_D(\Omega))}.\end{aligned}$$

Since

$$\|\Gamma(D^{s+1}v_0)\|_{L^\infty(0,T;H^{-1}_D(\Omega))} \le C\|D^{s+1}v_0\|_1 \le C\|v_0\|_{s+2},$$

it remains to estimate the commutator $[\Gamma, D^{s+1}]v_0$. Clearly

$$\begin{aligned}&([\Gamma, D^{s+1}]v_0)(t)\\ &= -\left(\frac{2}{\pi}\right)^n \sum_q \left(\sum_{p;|p|\ne|q|} v_p \frac{|p|^{s+1}-|q|^{s+1}}{|q|^2-|p|^2}(e^{-i|p|^2t}-e^{-i|q|^2t})I(g,p,q)\right)\\ &\quad\times\prod_{j=1}^n \sin(q_jx_j).\end{aligned}$$

The contribution in $\|([\Gamma, D^{s+1}]v_0)(t)\|^2_{-1}$ due to $F_0 = \{x\in\partial\Omega;\ x_n = 0\}$ is estimated with (2.34) by

$$\begin{aligned}B_{F_0} &\le C\sum_q |q|^{-2}\left(\sum_{p;|p|\ne|q|} |v_p|\frac{|p|^s+|q|^s}{|p|+|q|}|I_{F_0}|\right)^2\\ &\le C\sum_q\left(\sum_{p;|p|\ne|q|}|v_p|\frac{|p|^s+|q|^s}{|p|+|q|}p_n\prod_{j=1}^{n-1}\langle p_j-q_j\rangle^{-k}\right)^2.\end{aligned}$$

Therefore, using the estimation of the R.H.S. of (2.36) in (2.37), we conclude that for $s<1/2$

$$B_{F_0} \le C\|v_0\|^2_{s+\frac{3}{2}},$$

the constant C being uniform in $t\in[0,T]$. Therefore

$$\|[\Gamma, D^{s+1}]v_0\|_{L^\infty(0,T;H^{-1}_D(\Omega))} \le C\|v_0\|_{s+2}.$$

Thus, we have proved that

$$\|u\|_{L^\infty(0,T;H^s_D(\Omega))} \le C\|v_0\|_{H^{s+2}_D(\Omega)}. \tag{2.40}$$

Since $u \in C([0,T]; H_D^{-1}(\Omega))$, we conclude that $u \in C_w([0,T]; H_D^s(\Omega))$. If we pick $\tilde{s} \in (s, 1/2)$ and $\tilde{v}_0 \in H_D^{\tilde{s}+2}(\Omega)$, the corresponding solution $\tilde{u}$ belongs to $C_w([0,T]; H_D^{\tilde{s}}(\Omega))$, hence to $C([0,T]; H_D^s(\Omega))$, the embedding $H_D^{\tilde{s}}(\Omega) \subset H_D^s(\Omega)$ being compact. It follows from (2.40) combined with the density of $H_D^{\tilde{s}+2}(\Omega)$ in $H_D^{s+2}(\Omega)$ that $u \in C([0,T]; H_D^s(\Omega))$ for $v_0 \in H_D^{s+2}(\Omega)$. In particular, $u(T) \in H_D^s(\Omega)$, so that S is an isomorphism from $H_D^{s+2}(\Omega)$ onto $H_D^s(\Omega)$. This completes the proof of Proposition 2.3 and Theorem 2.3. □

2.2.2. *Neumann boundary control*

We adopt the following definition.

Definition 2.3. The open set $\Gamma_0 \subset \partial\Omega$ is called a *Neumann control domain* if given any u_0, $u_1 \in L^2(\Omega)$ and any time $T > 0$, one may find a control $h \in L^2(0,T; L^2(\Gamma_0))$ such that the solution $u = u(x,t)$ of

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = 1_{\Gamma_0} h(x,t) & \text{on } \partial\Omega \times (0,T) \\ u(0) = u_0 \end{cases} \tag{2.41}$$

satisfies $u(T) = u_1$.

The following result provides Neumann control domains in *any* dimension $n \geq 2$.

Proposition 2.4. *Let $\Omega = (0,\pi)^n$, and let $\Gamma_0 \subset \partial\Omega$ be a side of Ω. Then Γ_0 is a Neumann control domain.*

Proof. Assume e.g. that $\Gamma_0 = \{0\} \times (0,\pi)^{n-1}$. By Dolecki–Russell criterion, we only have to check the following observability inequality

$$\|v_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_{\Gamma_0} |v(x,t)|^2 d\sigma dt, \tag{2.42}$$

where v_0 is any function in $L^2(\Omega)$ and $v = v(x,t)$ solves

$$\begin{cases} iv_t + \Delta v = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ v(0) = v_0. \end{cases} \tag{2.43}$$

Expanding v_0 as

$$v_0(x) = \sum_{k \in \mathbb{N}^n} c_k \cos(k_1 x_1) \cdots \cos(k_n x_n),$$

then the corresponding solution $v(x,t)$ reads

$$v(x,t) = \sum_{k \in \mathbb{N}^n} c_k e^{-i|k|^2 t} \cos(k_1 x_1) \cdots \cos(k_n x_n).$$

It follows that

$$\begin{aligned}
&\int_0^T\int_{\Gamma_0} |v(x,t)|^2 d\sigma dt \\
&\quad = \int_0^T\int_{(0,\pi)^{n-1}} \left|\sum_{k\in\mathbb{N}^n} c_k e^{-i|k|^2 t}\cos(k_2x_2)\cdots\cos(k_nx_n)\right|^2 dx_2\cdots dx_n dt \\
&\quad \sim \sum_{k_2,\ldots,k_n\geq 0}\int_0^T \left|\sum_{k_1\geq 0} c_k e^{-ik_1^2 t}\right|^2 dt \sim \sum_{k\in\mathbb{N}^n}|c_k|^2 \sim \|v_0\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used the orthogonality of the functions $\cos(k_2x_2)\cdots\cos(k_nx_n)$ in $L^2(\Gamma_0)$ and Ingham's lemma. □

We now aim to extend Proposition 2.4 to a control result in a space $H^s(\Omega)$, $s>0$. We define $H^s_N(\Omega) = D(A_N^{\frac{s}{2}})$, where A_N is the Neumann Laplacian (i.e. $A_N u = u - \Delta u$ with $D(A_N) = \{u\in H^2(\Omega),\ \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega\}\subset L^2(\Omega)$). A result similar to Theorem 2.3 may be obtained along the same lines. We limit ourselves to giving a weaker result with a very short proof.

Theorem 2.4. *Let Γ_0 be a Neumann control domain, $T=2\pi$, $s\in[0,1)$ and $u_0,u_1\in H^s_N(\Omega)$. Then there exists a control input $h\in H^{\frac{s}{2}}(\mathbb{T};L^2(\partial\Omega))$ such that the solution u of (2.41) satisfies $u(T)=u_1$.*

Proof. Without loss of generality, we may assume that $u_0=0$. A direct computation shows that for any (smooth) solution u of (2.41) emanating from $u_0=0$ and any (smooth) solution v of (2.43), the following holds

$$i\int_\Omega u(x,T)\overline{v(x,T)}dx = -\int_0^T\int_{\partial\Omega} 1_{\Gamma_0} h(x,t)\bar{v} d\sigma dt. \tag{2.44}$$

As usual, for any $h\in L^2(0,T;L^2(\partial\Omega))$, the solution $u\in C([0,T];L^2(\Omega))$ of (2.41) is defined by

$$i(u(t),v(t))_{L^2(\Omega)} = -(h,1_{\Gamma_0}v)_{L^2(0,t;L^2(\partial\Omega))},\quad \forall\, t\in[0,T],\quad \forall\, v_0\in L^2(\Omega) \tag{2.45}$$

where $v(t)$ solves (2.43).

Claim 1. *If $v_0\in H^{-s}_N(\Omega)$ for some $s\in\mathbb{R}$, then $v\in H^{-\frac{s}{2}}(\mathbb{T};L^2(\partial\Omega))$.*

Indeed, if we write $v_0 = \sum_{k\in\mathbb{N}^n} c_k\cos(k_1x_1)\cdots\cos(k_nx_n)$ and

$$v(x,t) = \sum_{k\in\mathbb{N}^n} c_k e^{-i|k|^2t}\cos(k_1x_1)\cdots\cos(k_nx_n)$$

then we have that

$$\|v\|^2_{H^{-\frac{s}{2}}(\mathbb{T},L^2(\partial\Omega))} \sim \sum_k (1+|k|^2)^{-s}|c_k|^2 \sim \|v_0\|^2_{H^{-s}_N(\Omega)}. \tag{2.46}$$

We may rewrite (2.44) in the form

$$i\langle u(T),v(T)\rangle_{H^s_N,H^{-s}_N} = -\langle h,1_{\Gamma_0}v\rangle_{H^{\frac{s}{2}}(\mathbb{T};L^2(\partial\Omega)),H^{-\frac{s}{2}}(\mathbb{T};L^2(\partial\Omega))}. \tag{2.47}$$

Note that $u \in C([0,T]; H_N^s(\Omega))$ if $0 \le s < 1$. It remains to establish the following:

Claim 2. (Observability inequality) *The following estimate holds for the solutions of* (2.43):

$$\|1_{\Gamma_0} v\|^2_{H^{-\frac{s}{2}}(\mathbb{T};L^2(\partial\Omega))} \ge \text{const. } \|v_0\|^2_{H_N^{-s}(\Omega)}. \tag{2.48}$$

If (2.48) is not true, one can construct a sequence $\{v_j\}$ such that

$$j\|1_{\Gamma_0} v_j\|^2_{H^{-\frac{s}{2}}(\mathbb{T};L^2(\partial\Omega))} < \|v_j(0)\|^2_{H_N^{-s}(\Omega)} = 1. \tag{2.49}$$

Let $w_j = (1-\partial_t^2)_p^{-\frac{s}{4}} v_j$, where for any $\sigma \in \mathbb{R}$

$$(1-\partial_t^2)_p^{\sigma} \sum_{l\in\mathbb{Z}} c_l e^{ilt} = \sum_{l\in\mathbb{Z}} (1+|l|^2)^{\sigma} c_l e^{ilt}.$$

Then w_j solves (2.43) with $w_j(0)$ substituted to v_0, and from (2.49) we obtain

$$1_{\Gamma_0} w_j \to 0 \quad \text{in } L^2(\mathbb{T}; L^2(\partial\Omega)). \tag{2.50}$$

As Γ_0 is a Neumann control domain, we infer that $w_j(0) \to 0$ in $L^2(\Omega)$, hence

$$w_j \to 0 \quad \text{in } L^2(\mathbb{T}; L^2(\partial\Omega)).$$

This gives

$$v_j \to 0 \quad \text{in } H^{-\frac{s}{2}}(\mathbb{T}; L^2(\partial\Omega)).$$

Using (2.46), we infer that $v_j(0) \to 0$ in $H_N^{-s}(\Omega)$, which contradicts (2.49). This completes the proof of Theorem 2.4. □

3. Nonlinear Systems

3.1. *Internal control*

In this section we consider the following nonlinear control system

$$\begin{cases} iu_t + \Delta u + N(u) = iGh = ia(x)h(x,t), & x \in \mathbb{T}^n, \quad t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{3.1}$$

where $a \in C^\infty(\mathbb{T}^n)$, and the nonlinearity $N(u)$ reads

$$N(u) = \lambda u^{\alpha_1} \overline{u}^{\alpha_2}, \quad \alpha_1 + \alpha_2 =: \alpha + 1 \ge 2, \tag{3.2}$$

with $\lambda \in \mathbb{R}$, and $\alpha, \alpha_1, \alpha_2 \in \mathbb{N}$. Note that for any $\alpha = 2\beta \in 2\mathbb{N}^*$, $|u|^\alpha u = u^{\beta+1}\overline{u}^\beta$.

We introduce the number

$$s_{\alpha,n} = \begin{cases} \dfrac{n}{2} - 1 & \text{if } \alpha = 1, \\ \dfrac{n}{2} - \dfrac{3}{4} - \dfrac{1}{4(n-1)} & \text{if } \alpha = 2, \\ \dfrac{n}{2} - \dfrac{2}{\alpha} & \text{if } \alpha \ge 3. \end{cases} \tag{3.3}$$

Thus $s_{\alpha,n} = s_c := \frac{n}{2} - \frac{2}{\alpha}$ (the critical Sobolev exponent obtained by scaling in NLS) for $\alpha \geq 3$, while $s_{\alpha,n} > s_c$ for $\alpha = 1, 2$ (except for $n = \alpha = 2$ where $s_{2,2} = s_c = 0$).

By Corollary 3.1 (see below), the system (3.1) is locally well-posed in the space $H^s(\mathbb{T}^n)$ for $\alpha \geq 1$ and $s > s_{\alpha,n}$ with $\phi \in H^s(\mathbb{T}^n)$ and $h \in L^2_{\mathrm{loc}}(\mathbb{R}, H^s(\mathbb{T}^n))$.

Our main concern is its exact controllability in the space $H^s(\mathbb{T}^n)$.

Theorem 3.1. *For given $n \geq 2$, $\alpha_1, \alpha_2 \in \mathbb{N}$ with $\alpha_1 + \alpha_2 =: \alpha + 1 \geq 2$, and $a \not\equiv 0$, the system (3.1) is locally exactly controllable in the space $H^s(\mathbb{T}^n)$ for any $s > s_{\alpha,n}$. More precisely, for any given $T > 0$, there exists a number $\delta > 0$ depending on α, n, T and λ such that if ϕ, $\psi \in H^s(\mathbb{T}^n)$ satisfy*

$$\|\phi\|_s \leq \delta, \quad \|\psi\|_s \leq \delta,$$

then one can choose a control input $h \in L^2(0, T; H^s(\mathbb{T}^n))$ such that the system (3.1) admits a solution $u \in C([0, T]; H^s(\mathbb{T}^n))$ satisfying

$$u(x, 0) = \phi(x), \quad u(x, T) = \psi(x).$$

The system (3.1) can be rewritten in its equivalent integral form

$$u(t) = W(t)\phi + i \int_0^t W(t-\tau)(N(u)(\tau))d\tau + \int_0^t W(t-\tau)[Gh](\tau)d\tau. \tag{3.4}$$

To prove Theorem 3.1, a smoothing property is needed for the operator from f to u, where

$$u(t) = \int_0^t W(t-\tau)f(\tau)d\tau.$$

This needed smoothing property was provided in Bourgain's work (see Refs. 8 and 9) where he dealt with the Cauchy problem for the periodic Schrödinger equation.

For given $s, b \in \mathbb{R}$, the Bourgain space $X_{s,b}$ is the space of functions $u : \mathbb{T}^n \times \mathbb{R} \to \mathbb{C}$ for which the norm

$$\|u\|_{X_{s,b}} = \|W(-t)u(\cdot, t)\|_{H^b_t(H^s_x)}$$

is finite. Decomposing u as

$$u(x, t) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}} \hat{u}(k, \tau) e^{i(k \cdot x + \tau t)} d\tau$$

we have that

$$\|u\|^2_{X_{s,b}} = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}} \langle \tau + |k|^2 \rangle^{2b} \langle k \rangle^{2s} |\hat{u}(k, \tau)|^2 d\tau,$$

where we remind that $\langle y \rangle := (1 + |y|^2)^{\frac{1}{2}}$. For given $T > 0$, $X^T_{s,b}$ is the restriction norm space

$$X^T_{s,b} = \{u_{|\mathbb{T}^n \times (0,T)};\ u \in X_{s,b}\}$$

with the restriction norm

$$\|u\|_{X^T_{s,b}} = \inf\{\|\tilde{u}\|_{X_{s,b}}; \tilde{u} \in X_{s,b},\ \tilde{u}_{|\mathbb{T}^n\times(0,T)} = u\}.$$

Before we proceed to show the exact controllability results, we present the following two technical lemmas (see e.g. Ref. 50) which play important roles in the proof of Theorem 3.1.

Lemma 3.1. *For given $T > 0$ and $s, b \in \mathbb{R}$, there exists a constant $C > 0$ such that*

$$\|W(t)\phi\|_{X^T_{s,b}} \le C\|\phi\|_s$$

for any $\phi \in H^s(\mathbb{T}^n)$.

Lemma 3.2. *For given $T > 0$, $b > 1/2$ and $s \in \mathbb{R}$, there exists a constant $C > 0$ such that*

$$\left\|\int_0^t W(t-\tau)f(\tau)d\tau\right\|_{X^T_{s,b}} \le C\|f\|_{X^T_{s,b-1}}$$

for any $f \in X^T_{s,b-1}$.

The following multilinear estimate is crucial when applying the contraction mapping theorem.

Proposition 3.1. *Let $n \ge 2$, $\alpha \in \mathbb{N}^*$ and $s > s_{\alpha,n}$. Then there exist some numbers $b \in (0, \frac{1}{2})$ and $C > 0$ such that*

$$\left\|\prod_{i=1}^{\alpha+1} \tilde{u}_i\right\|_{X_{s,-b}} \le C \prod_{i=1}^{\alpha+1} \|u_i\|_{X_{s,b}} \quad \forall\, u_1, \dots, u_{\alpha+1} \in X_{s,b}, \tag{3.5}$$

where $\tilde{u}_i$ denotes u_i or $\overline{u_i}$.

Corollary 3.1. *Let $n \ge 2$, $\alpha \in \mathbb{N}^*$ and $s > s_{\alpha,n}$. Pick $u_0 \in H^s(\mathbb{T}^n)$ and $h \in X_{s,0} = L^2(\mathbb{R}; H^s(\mathbb{T}^n))$. Then there exist two numbers $b > \frac{1}{2}$ and $T = T(\|u_0\|_{H^s(\mathbb{T}^n)}, \|h\|_{X_{s,0}})$ so that the initial value problem (3.1) admits a unique solution $u \in X^T_{s,b}$.*

Remark 3.1. Proposition 3.1, which is proved in the Appendix for the sake of completeness, is essentially due to Bourgain. It was proved in Ref. 9 when $\alpha = n = 2$, and in Ref. 8 in Besov-type spaces when $s > s_b$, where

$$s_b = \begin{cases} s_c & \text{if } n = 2, \\ \max\left(s_c, \dfrac{3}{4}\right) & \text{if } n = 3, \\ \max\left(s_c, \dfrac{3n}{n+4}\right) & \text{if } n \ge 4. \end{cases} \tag{3.6}$$

Notice that $s_b > s_c$ only for $(\alpha, n) \in \{(2,3), (2,4), (2,5), (3,4)\}$. The corresponding values of s_b, s_c and $s_{\alpha,n}$ are reported in Table 1. We notice that $s_{\alpha,n} < s_b$ for

Table 1. s_b, $s_{\alpha,n}$ and s_c for $(\alpha, n) \in \{(2,3), (2,4), (2,5), (3,4)\}$.

(α, n)	$(2,3)$	$(2,4)$	$(2,5)$	$(3,4)$
s_b	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{3}{2}$
$s_{\alpha,n}$	$\frac{5}{8}$	$\frac{7}{6}$	$\frac{27}{16}$	$\frac{4}{3}$
s_c	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{4}{3}$

$(\alpha, n) \in \{(2,3), (2,4), (3,4)\}$. On the other hand, $s_b = s_c < s_{\alpha,n}$ for $\alpha = 2$ and $n \geq 6$. Sharp results for the local well-posedness of NLS on $\mathbb{T}^n$ are also given in Ref. 22 for $\alpha = n = 1$, and in Ref. 17 for $(\alpha_1, \alpha_2) = (0, 2)$ and $2 \leq n \leq 4$.

It follows at once from Proposition 3.1 that for any $T > 0$, any $s > s_{\alpha,n}$, and some $b > 1/2$, $b' > b - 1$ we have

$$\|N(v) - N(w)\|_{X^T_{s,b'}} \leq C(\|v\|^{\alpha}_{X^T_{s,b}} + \|w\|^{\alpha}_{X^T_{s,b}})\|v - w\|_{X^T_{s,b}} \quad \forall\, v, w \in X^T_{s,b}.$$

We are now in a position to give a proof of Theorem 3.1.

Proof of Theorem 3.1. Set

$$\omega(v, T) = i \int_0^T W(T - \tau) N(v)(\tau) d\tau.$$

By Theorem 2.1, if we choose

$$h = \Phi(\phi, \psi - \omega(v, T)),$$

then

$$\begin{aligned} W(t)\phi &+ \int_0^t W(t - \tau)(iN(v)) + G\Phi(\phi, \psi - \omega(v, T))(\tau) d\tau \\ &= \begin{cases} \phi(x) \quad \text{in } \mathbb{T}^n & \text{when } t = 0, \\ \psi(x) - \omega(v, T) + \omega(v, T) = \psi(x) \quad \text{in } \mathbb{T}^n & \text{when } t = T. \end{cases} \end{aligned}$$

It suggests to us to consider the nonlinear map:

$$\Gamma(v) = W(t)\phi + i \int_0^t W(t - \tau)(iN(v)) + G\Phi(\phi, \psi - \omega(v, T))(\tau) d\tau.$$

The proof would be complete if we can show that this map Γ has a fixed point in the space $X^T_{s,b}$, with $b \in (\frac{1}{2}, 1)$.

To this end, note that by using Lemmas 3.1, 3.2 and Proposition 3.1, there exist a number $b \in (\frac{1}{2}, 1)$ and some constants C_j, $j = 1, 2, 3$ such that

$$\|\Gamma(v)\|_{X^T_{s,b}} \leq C_1(\|\phi\|_s + \|\psi\|_s + \|\omega(v, T)\|_s) + C_2\|v\|^{\alpha+1}_{X^T_{s,b}}$$

for any $v \in X^T_{s,b}$ and

$$\begin{aligned} \|\Gamma(v_1) - \Gamma(v_2)\|_{X^T_{s,b}} \leq\ & C_1\|\omega(v_1, T) - \omega(v_2, T)\|_s \\ & + C_3(\|v_1\|^{\alpha}_{X^T_{s,b}} + \|v_2\|^{\alpha}_{X^T_{s,b}})\|v_1 - v_2\|_{X^T_{s,b}} \end{aligned}$$

for any $v_1, v_2 \in X_{s,b}^T$. Note that there exists a constant $C_4 > 0$ such that

$$\begin{aligned}\|\omega(v,T)\|_s &\leq \left\| \int_0^t W(t-\tau)N(v)(\tau)d\tau \right\|_{C([0,T];H^s(\mathbb{T}^n))} \\ &\leq \text{const.} \left\| \int_0^t W(t-\tau)N(v)(\tau)d\tau \right\|_{X_{s,b}^T} \\ &\leq C_4 \|v\|_{X_{s,b}^T}^{\alpha+1}.\end{aligned}$$

Similarly

$$\|\omega(v_1,T) - \omega(v_2,T)\|_s \leq C_5(\|v_1\|_{X_{s,b}^T}^{\alpha} + \|v_2\|_{X_{s,b}^T}^{\alpha})\|v_1 - v_2\|_{X_{s,b}^T}.$$

As a result, by increasing the constants C_2 and C_3, we obtain

$$\|\Gamma(v)\|_{X_{s,b}^T} \leq C_1(\|\phi\|_s + \|\psi\|_s) + C_2\|v\|_{X_{s,b}^T}^{\alpha+1}$$

for any $v \in X_{s,b}^T$ and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{X_{s.b}^T} \leq C_3(\|v_1\|_{X_{s,b}^T}^{\alpha} + \|v_2\|_{X_{s,b}^T}^{\alpha})\|v_1 - v_2\|_{X_{s,b}^T}$$

for any $v_1, v_2 \in X_{s,b}^T$. Pick $\delta > 0$, ϕ, $\psi \in H^s(\mathbb{T}^n)$ with $\|\phi\|_s + \|\psi\|_s \leq \delta$, and set $M = 2C_1\delta$. If $\|v\|_{X_{s,b}^T} \leq M$ and

$$\|v_j\|_{X_{s,b}^T} \leq M, \quad j = 1, 2,$$

then

$$\begin{aligned}\|\Gamma(v)\|_{X_{s,b}^T} &\leq C_1\delta + C_2M^{\alpha+1} \\ &\leq 2C_1\delta = M\end{aligned}$$

as long as

$$C_2M^{\alpha} \leq \frac{1}{2}\cdot$$

Choose $\delta > 0$ so that $M = 2C_1\delta$ fulfills

$$C_2M^{\alpha} \leq \frac{1}{2} \quad \text{and} \quad C_3M^{\alpha} \leq \frac{1}{4},$$

and let B_M be the ball in the space $X_{s,b}^T$ centered at the origin of radius M. For given $\phi, \psi \in H^s(\mathbb{T}^n)$ with $\|\phi\|_s + \|\psi\|_s \leq \delta$, we have

$$\|\Gamma(v)\|_{X_{s,b}^T} \leq M$$

for any $v \in B_M$ and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{X_{s,b}^T} \leq \frac{1}{2}\|v_1 - v_2\|_{X_{s,b}^T}$$

for any $v_1, v_2 \in B_M$. That is to say, Γ is a contraction in the ball B_M. The proof is complete. □

Let us now consider the Schrödinger equation posed on a cube $\Omega = (0,\pi)^n$

$$iu_t + \Delta u + N(u) = ia(x)h(x,t), \quad x \in \Omega, \quad t \in (0,T) \tag{3.7}$$

with either the homogeneous Dirichlet boundary conditions

$$u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T) \tag{3.8}$$

or the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T). \tag{3.9}$$

The nonlinearity $N(u)$ is still as in (3.2).

It is remarkable that internal control results with Dirichlet (resp. Neumann) homogeneous boundary conditions can be deduced from those already proved for periodic boundary conditions.

Corollary 3.2. *For given $n \geq 2$, $\alpha_1, \alpha_2 \in \mathbb{N}$ with $\alpha_1 + \alpha_2 =: \alpha + 1 \geq 2$ and α even, and $a \not\equiv 0$, the system (3.7)–(3.8) is locally exactly controllable in the space $H^s_D(\Omega)$ for any $s > s_{\alpha,n}$. More precisely, for any given $T > 0$, there exists a number $\delta > 0$ depending on α, n, T and λ such that if $\phi, \psi \in H^s_D(\Omega)$ satisfy*

$$\|\phi\|_{H^s_D(\Omega)} \leq \delta, \quad \|\psi\|_{H^s_D(\Omega)} \leq \delta,$$

then one can choose a control input $h \in L^2(0,T; H^s_D(\Omega))$ such that the system (3.7)–(3.8) admits a solution $u \in C([0,T]; H^s_D(\Omega))$ satisfying

$$u(x,0) = \phi(x), \quad u(x,T) = \psi(x).$$

Corollary 3.3. *For given $n \geq 2$, $\alpha_1, \alpha_2 \in \mathbb{N}$ with $\alpha_1 + \alpha_2 =: \alpha + 1 \geq 2$ and $a \not\equiv 0$, the system (3.7)–(3.9) is locally exactly controllable in the space $H^s_N(\Omega)$ for any $s > s_{\alpha,n}$. More precisely, for any given $T > 0$, there exists a number $\delta > 0$ depending on α, n, T and λ such that if ϕ, $\psi \in H^s_N(\Omega)$ satisfy*

$$\|\phi\|_{H^s_N(\Omega)} \leq \delta, \quad \|\psi\|_{H^s_N(\Omega)} \leq \delta,$$

then one can choose a control input $h \in L^2(0,T; H^s_N(\Omega))$ such that the system (3.7)–(3.9) admits a solution $u \in C([0,T]; H^s_N(\Omega))$ satisfying

$$u(x,0) = \phi(x), \quad u(x,T) = \psi(x).$$

We shall say that a function from $(-\pi,\pi)^n$ to $\mathbb{C}$ is *odd* (resp. *even*), if it is odd with respect to each coordinate x_i, $1 \leq i \leq n$. The proof relies on the basic, but crucial observation that the functions in $H^s_D(\Omega)$ (resp. $H^s_N(\Omega)$) coincide with the restrictions to Ω of the functions in $H^s(\mathbb{T}^n)$ which are odd (resp. even). The issue is therefore reduced to an extension of Theorem 3.1 in the framework of odd (resp. even) functions in $H^s(\mathbb{T}^n)$. Extending the function a in (3.7) to $\mathbb{T}^n$ as an even function, we notice that the control input h in Theorem 2.1 can be chosen odd (resp. even) if the functions ϕ, ψ are odd (resp. even). Indeed, the observability inequality holds as well in the subspaces

$$
\begin{aligned}
H^s_{\text{odd}}(\mathbb{T}^n) &= \{u \in H^s_p(\mathbb{T}^n); u(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) = -u(x) \\
&\qquad \forall x \in \mathbb{T}^n,\ \forall i\}, \\
H^s_{\text{even}}(\mathbb{T}^n) &= \{u \in H^s_p(\mathbb{T}^n); u(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) = u(x) \\
&\qquad \forall x \in \mathbb{T}^n,\ \forall i\}
\end{aligned}
$$

of $H^s(\mathbb{T}^n)$ for $s \leq 0$. On the other hand, since u and $N(u)$ are simultaneously odd (resp. even), we see that the contraction mapping theorem can be applied in a space of odd (resp. even) trajectories to derive the result in Corollary 3.2 (resp. Corollary 3.3). Full details are provided in Ref. 46 for $n = 1$.

3.2. *Boundary control*

In this section we consider the Schrödinger equation posed on a rectangle $\Omega = (0, l_1) \times \cdots \times (0, l_n)$

$$
iu_t + \Delta u + N(u) = 0, \quad x \in \Omega, \quad t \in (0, T) \tag{3.10}
$$

with either the Dirichlet boundary conditions

$$
u(x, t) = 1_{\Gamma_0} h(x, t) \quad (x, t) \in \partial\Omega \times (0, T) \tag{3.11}
$$

or the Neumann boundary conditions

$$
\frac{\partial u}{\partial \nu}(x, t) = 1_{\Gamma_0} h(x, t) \quad (x, t) \in \partial\Omega \times (0, T). \tag{3.12}
$$

When we consider a smooth Dirichlet controller g, then the boundary condition (3.11) will be replaced by

$$
u(x, t) = g(x) h(x, t) \quad (x, t) \in \partial\Omega \times (0, T). \tag{3.13}
$$

$N(u)$ still stands for the nonlinear term in NLS. We first give a result (with a small control region) providing precise information on the smoothness of the control input and of the trajectories when $N(u)$ is *weakly* nonlinear. To simplify the exposition, we assume here that

$$
\Omega = (0, \pi)^n.
$$

We denote by $u = W_D(t)u_0$ the solution of (2.12) for $h = 0$. For given $s, b \in \mathbb{R}$, $X_{s,b}(\Omega)$ denotes the Bourgain space of functions $u : \Omega \times \mathbb{R} \to \mathbb{C}$ for which the norm

$$
\|u\|_{X_{s,b}(\Omega)} = c\|W_D(-t)u(\cdot, t)\|_{H^b(\mathbb{R}; H^s_D(\Omega))}
$$

is finite. Decomposing u as

$$
u(x, t) = \sum_{k \in (\mathbb{N}^*)^n} \int_{\mathbb{R}} \hat{u}(k, \tau) e^{i\tau t} \sin(k_1 x_1) \cdots \sin(k_n x_n) d\tau
$$

we can choose the constant c so that

$$\|u\|^2_{X_{s,b}(\Omega)} = \sum_{k\in(\mathbb{N}^*)^n} \int_{\mathbb{R}} \langle \tau + |k|^2\rangle^{2b} \langle k\rangle^{2s} |\hat{u}(k,\tau)|^2 d\tau < \infty.$$

The restriction norm space $X^T_{s,b}(\Omega)$ is defined in the usual way (see above the definition of $X^T_{s,b}$). For $u \in H^s_D(\Omega)$ given, we denote by $\tilde{u}$ its odd extension to $\mathbb{T}^n = (-\pi,\pi)^n$; i.e. $\tilde{u}_{|(0,\pi)^n} = u$, and $\tilde{u}$ is odd with respect to each coordinate x_i. Note that $\tilde{u} \in H^s(\mathbb{T}^n)$ and $\|\tilde{u}\|_s \sim \|u\|_{H^s_D(\Omega)}$. Defining $\tilde{u}(\cdot,t)$ from $u(\cdot,t)$ in a similar way, we observe that

$$\|\tilde{u}\|_{X^T_{s,b}} \sim \|u\|_{X^T_{s,b}(\Omega)}.$$

It is then clear that Lemmas 3.1 and 3.2 hold true with $W_D(t)$, $H^s_D(\Omega)$ and $X^T_{s,b}(\Omega)$ substituted to $W(t)$, $H^s(\mathbb{T}^n)$ and $X^T_{s,b}$, respectively. We shall assume that the nonlinear term $N(u)$ satisfies the following multilinear estimate

$$\|N(u) - N(v)\|_{X_{s,b'}(\Omega)} \le c(u,v)\|u - v\|_{X_{s,b}(\Omega)} \tag{3.14}$$

where $s \in \mathbb{R}$, $-1/2 < b' < b \le b' + 1$ and $c(u,v) \to 0$ as $u \to 0,\ v \to 0$ in $X_{s,b}(\Omega)$.

Theorem 2.3 can be extended to a semilinear context as follows.

Theorem 3.2. *Let g be a smooth Dirichlet controller, and let the nonlinearity $N(u)$ satisfy (3.2) and (3.14) with $s \in [-1,\frac{1}{2})$, $b > 0$ and $s + 2b < \frac{1}{2}$. Pick any $T > 0$. Then there exists $\delta > 0$ such that for any $u_0, u_T \in H^s_D(\Omega)$ satisfying*

$$\|u_0\|_{H^s_D(\Omega)} \le \delta, \quad \|u_T\|_{H^s_D(\Omega)} \le \delta$$

one may find a control input $h \in H^{\frac{s+1}{2}}(\mathbb{T}; L^2(\partial\Omega))$ and a solution $u \in C([0,T]; H^s_D(\Omega)) \cap X^T_{s,b}$ of (3.10) and (3.13) such that $u(0) = u_0$ and $u(T) = u_T$.

Proof. For $u_T \in H^s_D(\Omega)$, let h be the control given by HUM which steers (2.24) from 0 to u_T, namely $h = \partial v/\partial\nu$ with $v = W_D(t)v_0$ and $v_0 = S^{-1}u_T \in H^{s+2}_D(\Omega)$ (cf. Theorem 2.3). Recall that $h \in H^{\frac{s+1}{2}}(\mathbb{T}; L^2(\partial\Omega))$ by (2.27). We set $u = \Lambda u_T = \Gamma S^{-1}u_T$. The regularity of u is depicted in the following proposition. □

Proposition 3.2. *Assume that $-1 \le s < 1/2$ and $s + 2b < 1/2$. Then Λ maps continuously $H^s_D(\Omega)$ into $C([0,T]; H^s_D(\Omega)) \cap X^T_{s,b}(\Omega)$.*

Proof of Proposition 3.2. It follows from Proposition 2.3 and Theorem 2.3 that Λ maps continuously $H^s_D(\Omega)$ into $C([0,T]; H^s_D(\Omega))$. Let us turn our attention to the Bourgain space $X^T_{s,b}(\Omega)$.

Step 1. We prove several claims used thereafter.

Claim 3. *For any $\gamma > 1/2$, it holds*

$$\sup_{\lambda\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \langle \lambda^2 - k^2\rangle^{-\gamma} < \infty.$$

In what follows, C denotes a constant independent of λ and k which may vary from line to line. Pick $\lambda \in \mathbb{R}^+$. For $0 \le \lambda \le 1$

$$\langle \lambda^2 - k^2\rangle^{-\gamma} \le \langle k^2\rangle^{-\gamma} + \langle 1-k^2\rangle^{-\gamma}$$

and the result is then obvious. For $\lambda > 1$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \langle \lambda^2 - k^2\rangle^{-\gamma} &\le C\left(\int_0^{\lambda-1} |\lambda^2 - x^2|^{-\gamma}dx + \int_{\lambda+1}^{\infty} |x^2-\lambda^2|^{-\gamma}dx + 1\right)\\
&= C\lambda^{1-2\gamma}\left(\int_0^{1-\lambda^{-1}} |1-y^2|^{-\gamma}dy + \int_{1+\lambda^{-1}}^{+\infty} |y^2-1|^{-\gamma}dy + 1\right)\\
&\le C\lambda^{1-2\gamma}\left(\int_0^{1-\lambda^{-1}} |1-y|^{-\gamma}dy + \int_{1+\lambda^{-1}}^{2} |y-1|^{-\gamma}dy + 1\right)\\
&\le \begin{cases} C\lambda^{1-2\gamma}(\lambda^{-1+\gamma}+1) & \text{if } \gamma \ne 1;\\ C\lambda^{-1}(\ln\lambda + 1) & \text{if } \gamma = 1\end{cases}
\end{aligned}$$

and the claim follows.

Claim 4. *If* $s \ge -1,\ 0 < \delta < 1,\ s + 2\delta < 1/2,$ *and* $k > 1 + 2(s+1),$ *then for some constant* $C > 0$

$$S(p) := \sum_{q;|q|\ne|p|} \frac{q_n^{2s+2}}{||q|^2 - |p|^2|^{2(1-\delta)}} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \le C\langle p\rangle^{2s+2}.$$

Write $S(p) = S^1(p) + S^2(p)$, where the sum $S^1(p)$ is restricted to the $q = (q', q_n)$ with $|q'| \ge |p|$ and $|q| \ne |p|$. Noticing that $|q|^2 - |p|^2 = q_n^2 + |q'|^2 - |p|^2 \ge q_n^2$ for such q, we obtain that

$$S^1(p) \le \sum_{q_n} q_n^{2s+4\delta-2} \sum_{q'} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \le C \le C\langle p\rangle^{2s+2}.$$

To bound $S^2(p)$, we fix any $q' \in (\mathbb{N}^*)^{n-1}$ with $|q'| < |p|$ and set

$$\lambda = \sqrt{|p|^2 - |q'|^2} \ge 1.$$

We have that

$$\begin{aligned}
\sum_{q_n;|q_n^2-\lambda^2|\ge 1} \frac{q_n^{2s+2}}{|q_n^2 - \lambda^2|^{2(1-\delta)}} &\le C\left(\int_{|x^2-\lambda^2|\ge 1} \frac{x^{2s+2}}{|x^2-\lambda^2|^{2(1-\delta)}}dx + \lambda^{2s+2}\right)\\
&\le C\left(\lambda^{2s+4\delta-1}\int_{|y^2-1|\ge\lambda^{-2}} \frac{y^{2s+2}}{|y^2-1|^{2(1-\delta)}}dy + \lambda^{2s+2}\right)\\
&\le C(\lambda^{2s+4\delta-1}\cdot\lambda^{2-4\delta}\cdot\ln\lambda + \lambda^{2s+2})\\
&\le C\left(p_n^{2s+2} + \sum_{j=1}^{n-1} \langle p_j^2 - q_j^2\rangle^{s+1}\right).
\end{aligned}$$

It follows that

$$\begin{aligned} S^2(p) &\le C\sum_{q'}\left(p_n^{2s+2}+\sum_{j=1}^{n-1}\langle p_j^2-q_j^2\rangle^{s+1}\right)\prod_{l=1}^{n-1}\langle p_l-q_l\rangle^{-k} \\ &\le C\left(p_n^{2s+2}+\sum_{j=1}^{n-1}\sum_{q_j\ge 1}\langle p_j^2-q_j^2\rangle^{s+1}\langle p_j-q_j\rangle^{-k}\right) \\ &\le C\left(p_n^{2s+2}+\sum_{j=1}^{n-1}\sum_{q_j\ge 1}\langle p_j+q_j\rangle^{s+1}\langle q_j-p_j\rangle^{-(k-s-1)}\right). \end{aligned}$$

To complete the proof of Claim 4, we need the following:

Claim 5. *Let $\sigma \ge 0$ and $k > \sigma + 1$. Then there exists a constant $C > 0$ such that*

$$\sum_{m\ge 1}\langle m+n\rangle^{\sigma}\langle m-n\rangle^{-k}\le Cn^{\sigma}\quad \forall\, n\ge 1.$$

Split the sum into $\Sigma_1+\Sigma_2$ where $\Sigma_1=\sum_{1\le m\le 3n}\langle m+n\rangle^{\sigma}\langle m-n\rangle^{-k}$. Note that

$$\Sigma_1\le\langle 4n\rangle^{\sigma}\sum_{l\in\mathbb{Z}}\langle l\rangle^{-k}\le C\langle n\rangle^{\sigma}$$

since $k>1$. On the other hand, noticing that $m-n>(m+n)/2$ for $m>3n$, we have that

$$\Sigma_2\le\sum_{m>3n}\langle 2(m-n)\rangle^{\sigma}\langle m-n\rangle^{-k}\le C\sum_{m>3n}\langle m-n\rangle^{-(k-\sigma)}\le C.$$

Claim 5 is proved. Pick $k>1+2(s+1)\ge 1$. It follows from Claim 5 that

$$\sum_{q_j}\langle p_j+q_j\rangle^{s+1}\langle p_j-q_j\rangle^{-(k-s-1)}\le Cp_j^{s+1}.$$

Since $s+1\ge 0$ and $p_j\ge 1$, we conclude that

$$S(p)\le C(p_n^{2s+2}+\langle p'\rangle^{s+1})\le C\langle p\rangle^{2s+2}.$$

This completes the proof of Claim 4.

Step 2. Assume that $s<0$ and $s+2b<1/2$, and pick any $u_T\in H_D^s(\Omega)$ and any $\eta\in C_0^\infty(\mathbb{R})$ with $\eta(t)=1$ for $0\le t\le T$. Let $v_0=S^{-1}u_T\in H_D^{s+2}(\Omega)$ be decomposed as in (2.26). Let us prove that $u=\Lambda u_T\in X_{s,b}^T$. It is sufficient to prove that

$$\|\eta(t)u\|_{X_{s,b}}\le C\|v_0\|_{H_D^{s+2}(\Omega)}.$$

Recall that u is given by (2.38) and (2.39), and that $u(t)$ may be defined this way for all $t\in\mathbb{R}$. Again, we can limit ourselves to proving that $u_{F_0}\in X_{s,b}^T$, where u_{F_0} is the contribution due to $F_0=\{x\in\partial\Omega;\ x_n=0\}$ in u. u_{F_0} is decomposed as

$$u_{F_0}=\sum_{q\in(\mathbb{N}^*)^n}u_q(t)\sin(q_1x_1)\cdots\sin(q_nx_n),$$

where

$$u_q(t) = -\left(\frac{2}{\pi}\right)^n \sum_{p\in(\mathbb{N}^*)^n} v_p \frac{e^{-i|p|^2t} - e^{-i|q|^2t}}{|q|^2 - |p|^2} I_{F_0}$$

with the convention (2.32). $\hat{.}$ denoting time Fourier transform, an application of the elementary property

$$\widehat{e^{irt}\eta(t)}(\tau) = \hat{\eta}(\tau - r)$$

yields

$$\widehat{\eta u_q}(\tau) = -\left(\frac{2}{\pi}\right)^n \left(\sum_{p;|p|\neq|q|} v_p \frac{\hat{\eta}(\tau+|p|^2) - \hat{\eta}(\tau+|q|^2)}{|q|^2 - |p|^2} I_{F_0} \right.$$
$$\left. + \sum_{p;|p|=|q|} iv_p \widehat{t\eta(t)}(\tau+|q|^2) I_{F_0}\right).$$

For a function w decomposed as

$$w(x,t) = \sum_{q\in(\mathbb{N}^*)^n} w_q(t)\sin(q_1x_1)\cdots\sin(q_nx_n)$$

we recall that

$$\|w\|^2_{X_{s,b}(\Omega)} = \sum_{q\in(\mathbb{N}^*)^n} \int d\tau \langle\tau+|q|^2\rangle^{2b}\langle q\rangle^{2s}|\hat{w}_q(\tau)|^2.$$

Therefore, it is sufficient to check that

$$I := \sum_{q\in(\mathbb{N}^*)^n} \int d\tau \langle q\rangle^{2s}\langle\tau+|q|^2\rangle^{2b}|\widehat{\eta u_q}(\tau)|^2 \le c\sum_p \langle p\rangle^{2s+4}|v_p|^2.$$

Using (2.34), we may write

$$I \le c(I_1 + I_2 + I_3),$$

where

$$I_1 = \sum_q \int d\tau \langle q\rangle^{2s}\langle\tau+|q|^2\rangle^{2b}\left(\sum_{p;|p|=|q|} |v_p\widehat{t\eta(t)}(\tau+|q|^2)|p_nq_n\prod_{j=1}^{n-1}\langle p_j-q_j\rangle^{-k}\right)^2,$$

$$I_2 = \sum_q \int d\tau \langle q\rangle^{2s}\langle\tau+|q|^2\rangle^{2b}\left(\sum_{p;|p|\neq|q|} \left|v_p\frac{\hat{\eta}(\tau+|q|^2)}{|q|^2-|p|^2}\right| p_nq_n\prod_{j=1}^{n-1}\langle p_j-q_j\rangle^{-k}\right)^2,$$

$$I_3 = \sum_q \int d\tau \langle q\rangle^{2s}\langle\tau+|q|^2\rangle^{2b}\left(\sum_{p;|p|\neq|q|} \left|v_p\frac{\hat{\eta}(\tau+|p|^2)}{|q|^2-|p|^2}\right| p_nq_n\prod_{j=1}^{n-1}\langle p_j-q_j\rangle^{-k}\right)^2.$$

We bound separately I_1, I_2 and I_3.

(a)

$$I_1 \leq C\left(\int d\sigma \langle\sigma\rangle^{2b} |\widehat{t\eta(t)}(\sigma)|^2\right) \sum_q \langle q\rangle^{2s} q_n^2 \left(\sum_{p;|p|=|q|} |v_p| p_n \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right)^2$$

$$\leq C \sum_q \langle q\rangle^{2s} q_n^2 \left(\sum_{p;|p|=|q|} |v_p|^2 p_n^2 \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right)\left(\sum_{p;|p|=|q|} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right),$$

where we used successively a change of variables in the integral term, the fact that $\eta \in \mathcal{S}(\mathbb{R})$ and Cauchy–Schwarz inequality. From

$$\sum_{p;|p|=|q|} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \leq \sum_{p_1,\ldots,p_{n-1}} \left(\prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \sum_{p_n;|p|=|q|} 1\right)$$
$$\leq \prod_{j=1}^{n-1} \sum_{p_j\in\mathbb{Z}} \langle p_j\rangle^{-k} < \infty$$

we deduce that

$$I_1 \leq C \sum_p |v_p|^2 |p|^2 \sum_{q;|q|=|p|} \langle q\rangle^{2s+2} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}$$
$$\leq C \sum_p |v_p|^2 |p|^{2s+4}.$$

(b)

$$I_2 = C\left(\int d\sigma \langle\sigma\rangle^{2b} |\hat\eta(\sigma)|^2\right) \sum_q \langle q\rangle^{2s} q_n^2 \left(\sum_{p;|p|\neq|q|} \left|\frac{v_p}{|q|^2 - |p|^2}\right| p_n \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right)^2$$

$$\leq c \sum_q \langle q\rangle^{2s} q_n^2 \left(\sum_{p;|p|\neq|q|} \frac{|v_p|^2 p_n^2}{||q|^2 - |p|^2|^{2(1-\delta)}} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right)$$
$$\times \left(\sum_{p;|p|\neq|q|} ||q|^2 - |p|^2|^{-2\delta} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}\right),$$

where we used Cauchy–Schwarz inequality, and $\delta > 1/4$ was chosen so that $s + 2\delta < 1/2$. From Claim 3, we obtain that

$$\sum_{p;|p|\neq|q|} ||q|^2 - |p|^2|^{-2\delta} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k}$$
$$\leq C \sum_{p'} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \sum_{p_n;|p|\neq|q|} \langle |q|^2 - |p|^2\rangle^{-2\delta} < \text{const}.$$

Therefore, since $s < 0$, we see that

$$I_2 \leq C \sum_q q_n^{2s+2} \sum_{p;|p|\neq|q|} \frac{|v_p|^2 p_n^2}{||q|^2 - |p|^2|^{2(1-\delta)}} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k}$$

and from Claim 4

$$I_2 \leq C \sum_p |v_p|^2 |p|^{2s+4}.$$

(c) From the elementary estimate

$$\langle \tau + |q|^2 \rangle \leq c \langle \tau + |p|^2 \rangle \langle |q|^2 - |p|^2 \rangle$$

we infer that

$$I_3 \leq C \sum_q \int d\tau \langle q \rangle^{2s} |q_n|^2 \left(\sum_{p;|p|\neq|q|} |v_p| \frac{|\hat{\eta}(\tau + |p|^2)| \langle \tau + |p|^2 \rangle^b}{||q|^2 - |p|^2|^{1-b}} p_n \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right)^2. \tag{3.15}$$

For any fixed $\gamma > 1$, we have that for some constant $c > 0$

$$\langle \sigma \rangle^b |\hat{\eta}(\sigma)| \leq c \langle \sigma \rangle^{-\gamma} \quad \forall\, \sigma \in \mathbb{R}.$$

Expanding the squared term in (3.15) results in

$$\begin{aligned} I_3 &\leq C \sum_q \langle q \rangle^{2s} |q_n|^2 \sum_{p;|p|\neq|q|} \sum_{\tilde{p};|\tilde{p}|\neq|q|} \frac{|v_p|\, |v_{\tilde{p}}| p_n \tilde{p}_n}{||q|^2 - |p|^2|^{1-b} ||q|^2 - |\tilde{p}|^2|^{1-b}} \\ &\quad \times \left(\prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \langle \tilde{p}_j - q_j \rangle^{-k} \right) \int d\tau \langle \tau + |p|^2 \rangle^{-\gamma} \langle \tau + |\tilde{p}|^2 \rangle^{-\gamma} \\ &\leq C \sum_q \langle q \rangle^{2s} |q_n|^2 \sum_{p;|p|\neq|q|} \sum_{\tilde{p};|\tilde{p}|\neq|q|} \frac{|v_p|\, |v_{\tilde{p}}| p_n \tilde{p}_n}{||q|^2 - |p|^2|^{1-b} ||q|^2 - |\tilde{p}|^2|^{1-b}} \\ &\quad \times \left(\prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \langle \tilde{p}_j - q_j \rangle^{-k} \right) \langle |p|^2 - |\tilde{p}|^2 \rangle^{-\gamma}, \end{aligned}$$

where we used the following estimate valid for $\gamma > 1$ (see e.g. Lemma 7.34 in Ref. 33)

$$\int d\tau \langle \tau + \tau_1 \rangle^{-\gamma} \langle \tau + \tau_2 \rangle^{-\gamma} \leq c \langle \tau_1 - \tau_2 \rangle^{-\gamma}.$$

Thus

$$\begin{aligned} I_3 &\leq C \sum_q \langle q \rangle^{2s} q_n^2 \sum_{p;|p|\neq|q|} \frac{|v_p|^2 p_n^2}{||q|^2 - |p|^2|^{2(1-b)}} \left(\prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right) \\ &\quad \times \sum_{\tilde{p};|\tilde{p}|\neq|q|} \prod_{j=1}^{n-1} \langle \tilde{p}_j - q_j \rangle^{-k} \langle |p|^2 - |\tilde{p}|^2 \rangle^{-\gamma}. \end{aligned}$$

Since $\gamma > 1/2$, it follows from Claim 3 that

$$\sum_{\tilde{p}} \prod_{j=1}^{n-1} \langle \tilde{p}_j - q_j \rangle^{-k} \langle |p|^2 - |\tilde{p}| \rangle^{-\gamma}$$
$$\leq \sum_{\tilde{p}_1,\dots,\tilde{p}_{n-1}} \prod_{j=1}^{n-1} \langle \tilde{p}_j - q_j \rangle^{-k} \sum_{\tilde{p}_n} \langle \tilde{p}_n^2 + |\tilde{p}'|^2 - |p|^2 \rangle^{-\gamma} < \text{const.}$$

Thus

$$I_3 \leq C \sum_q \langle q \rangle^{2s} q_n^2 \sum_{p;|p|\neq|q|} \frac{|v_p|^2 p_n^2}{||q|^2 - |p|^2|^{2(1-b)}} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k}.$$

Using Claim 4 and the fact that $s \in [-1, 0)$, we have that

$$I_3 \leq C \sum_p |v_p|^2 |p|^2 \sum_{q;|q|\neq|p|} \frac{q_n^{2s+2}}{||q|^2 - |p|^2|^{2(1-b)}} \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \leq \sum_p |v_p|^2 |p|^{2s+4}.$$

Step 3. Assume that $s + 2b < 1/2$ with $s \in [0, 1/2)$. Let u_T, v_0, u and η be as in Step 2. Then

$$\begin{aligned} \|\eta(t)\Gamma v_0\|_{X_{s,b}} &\leq C\|\eta D^{s+1}\Gamma v_0\|_{X_{-1,b}} \\ &\leq C(\|\eta(t)\Gamma(D^{s+1}v_0)\|_{X_{-1,b}} + \|\eta(t)[\Gamma, D^{s+1}]v_0\|_{X_{-1,b}}). \end{aligned} \tag{3.16}$$

According to Step 2, the first term on the R.H.S. of (3.16) is less than $C\|D^{s+1}v_0\|_1 \leq C\|v_0\|_{s+2}$, for $-1 + 2b < 1/2$. The contribution due to $F_0 = \{x \in \partial\Omega;\ x_n = 0\}$ in $\|\eta(t)[\Gamma, D^{s+1}]v_0\|_{-1,b}^2$ is estimated by

$$\begin{aligned} C_{F_0} &\leq \sum_q \int d\tau \langle q \rangle^{-2} \langle \tau + |q|^2 \rangle^{2b} \\ &\quad \times \left| \sum_{p;|p|\neq|q|} v_p \frac{|p|^{s+1} - |q|^{s+1}}{|q|^2 - |p|^2} (\hat{\eta}(\tau + |p|^2) - \hat{\eta}(\tau + |q|^2)) I_{F_0} \right|^2 \\ &\leq C(I_2' + I_3'), \end{aligned}$$

where

$$\begin{aligned} I_2' &= \sum_q \int d\tau \langle q \rangle^{-2} \langle \tau + |q|^2 \rangle^{2b} \\ &\quad \times \left(\sum_{p;|p|\neq|q|} |v_p \hat{\eta}(\tau + |q|^2)| \frac{|p|^s + |q|^s}{|p| + |q|} p_n q_n \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right)^2, \\ I_3' &= \sum_q \int d\tau \langle q \rangle^{-2} \langle \tau + |q|^2 \rangle^{2b} \\ &\quad \times \left(\sum_{p;|p|\neq|q|} |v_p \hat{\eta}(\tau + |p|^2)| \frac{|p|^s + |q|^s}{|p| + |q|} p_n q_n \prod_{j=1}^{n-1} \langle p_j - q_j \rangle^{-k} \right)^2. \end{aligned}$$

We bound separately I_2' and I_3'.

(a) We have that

$$
\begin{aligned}
I_2' &\le C\left(\int d\sigma \langle\sigma\rangle^{2b} |\hat{\eta}(\sigma)|^2\right) \\
&\quad \times \sum_q \langle q\rangle^{-2} |q_n|^2 \left| \sum_{p;|p|\neq|q|} |v_p| \frac{|p|^s + |q|^s}{|p| + |q|} p_n \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \right|^2 \\
&\le C \sum_q \left| \sum_p |v_p| \frac{|p|^s + |q|^s}{|p| + |q|} p_n \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \right|^2 \\
&\le C \sum_p |p|^{3+2s} |v_p|^2 \\
&\le C \|v_0\|^2_{s+\frac{3}{2}},
\end{aligned}
$$

where we used (2.36) and (2.37).

(b) Doing computations similar to those performed in Step 2, we obtain that

$$
\begin{aligned}
I_3' &\le C \sum_q \langle q\rangle^{-2} q_n^2 \sum_{p;|p|\neq|q|} |v_p|^2 p_n^2 \frac{|p|^{2s} + |q|^{2s}}{(|p| + |q|)^2} ||q|^2 - |p|^2|^{2b} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \\
&\le C \sum_p |v_p|^2 |p|^2 \sum_{q;|q|\neq|p|} (|p| + |q|)^{2s+4b-2} \prod_{j=1}^{n-1} \langle p_j - q_j\rangle^{-k} \\
&\le C \|v_0\|_1^2,
\end{aligned}
$$

where we used the fact that $s + 2b < 1/2$. Since $s + 2 \ge 1$, we finally have that

$$
C_{F_0} \le C \|v_0\|^2_{H_D^{s+2}(\Omega)}.
$$

This completes the proof of Proposition 3.2.

We can now complete the proof of Theorem 3.2. Let s, b, u_0 and u_T be as in the statement of the theorem. Using Proposition 3.2 and proceeding as in the proof of Theorem 3.1, one can show that the map

$$
\begin{aligned}
\Gamma(v) = W_D(t)u_0 + i \int_0^t W_D(t-\tau) N(v)(\tau) d\tau \\
+ \Lambda(u_T - W_D(T)u_0 - \omega(v,T))
\end{aligned}
\tag{3.17}
$$

has a fixed-point $\Gamma(v) = v$ in some closed ball $B_M \subset X^T_{s,b}(\Omega)$ provided that $\|u_0\|_{H_D^s(\Omega)} + \|u_T\|_{H_D^s(\Omega)}$ is small enough. Such a trajectory v fulfills all the requirements of Theorem 3.2. In particular, $v \in X^T_{s,b}(\Omega) \cap C([0,T]; H_D^s(\Omega))$. The smoothness of the last term in (3.17) follows from Proposition 3.2. In (3.17), we used the notation

$$
\omega(v,T) = i \int_0^T W_D(T-\tau) N(v)(\tau) d\tau.
$$

Note that $\int_0^t W_D(t-\tau)N(v)(\tau)d\tau \in X^T_{s,b'+1}(\Omega) \subset C([0,T]; H^s_D(\Omega))$, by Lemma 3.2, (3.14), and the fact that $b' > -1/2$. In particular, $\omega(v,T) \in H^s_D(\Omega)$. The proof of Theorem 3.2 is achieved. □

Remark 3.2. (a) Using ideas from Ref. 8, it is likely that Theorem 3.2 may be applied when $n \geq 2$, Γ_0 is a neighborhood of a vertex, and $N(u) = \lambda|u|^\alpha u$ with $\alpha > 0$ small enough.

(b) The condition $s + 2b < 1/2$ in Proposition 3.2 is actually sharp. Indeed, let us take $n = 1$ and pick any $p \in \mathbb{N}^*$ and any $\eta \in \mathcal{S}(\mathbb{R})$ with $|\hat{\eta}(\tau)| > 1$ for $-1 \leq \tau \leq 1$. Set $v_0(x) = \sin(px)$ for $x \in \Omega = (0,\pi)$. With $\Gamma_0 = \{0\}$, we have that $I_{F_0} = pq$ with

$$\widehat{\eta u_q}(\tau) = \begin{cases} -\dfrac{2i}{\pi}\widehat{t\eta(t)}(\tau+p^2)p^2 & \text{if } q = p; \\ -\dfrac{2}{\pi}\dfrac{\hat{\eta}(\tau+p^2)-\hat{\eta}(\tau+q^2)}{q^2-p^2}pq & \text{if } q \neq p. \end{cases}$$

Therefore

$$\begin{aligned} \frac{\pi^2}{4}\|\eta u\|^2_{X_{s,b}(\Omega)} &= \int d\tau \sum_{q;q\neq p} \langle q\rangle^{2s}\langle \tau+q^2\rangle^{2b}\left|\frac{\hat{\eta}(\tau+p^2)-\hat{\eta}(\tau+q^2)}{q^2-p^2}\right|^2 p^2q^2 \\ &\quad + \left(\int d\tau \langle\tau+p^2\rangle^{2b}|\widehat{t\eta(t)}(\tau+p^2)|^2\right)\langle p\rangle^{2s}p^4 \\ &= \int d\tau \sum_{q;q\neq p}\langle q\rangle^{2s}\langle\tau+q^2\rangle^{2b}\frac{|\hat{\eta}(\tau+p^2)|^2}{|q^2-p^2|^2}p^2q^2 + J(p), \end{aligned}$$

where $|J(p)| \leq Cp^{2s+4} \leq C\|v_0\|^2_{s+2}$, according to the estimations of I_1, I_2 and the fact that

$$\int d\tau\langle\tau+q^2\rangle^{2b}|\hat{\eta}(\tau+p^2)\hat{\eta}(\tau+q^2)|d\tau \leq \text{const.} < \infty.$$

Since for $q \neq p$

$$\int d\tau\langle\tau+q^2\rangle^{2b}|\hat{\eta}(\tau+p^2)|^2 \geq \int_{-p^2-1}^{-p^2+1} d\tau\langle\tau+q^2\rangle^{2b} \geq C|q^2-p^2|^{2b}$$

we have that for $s + 2b \geq 1/2$,

$$\int d\tau \sum_{q;q\neq p}\langle q\rangle^{2s}\langle\tau+q^2\rangle^{2b}\frac{|\hat{\eta}(\tau+p^2)|^2}{|q^2-p^2|^2}p^2q^2 \geq Cp^2\sum_{q;q>p}|q^2-p^2|^{2b-2}\langle q\rangle^{2s}q^2 = \infty,$$

therefore $\eta u \notin X_{s,b}(\Omega)$. The condition $s + 2b < 1/2$ seems related to the fact that any smooth function on $\mathbb{T}^n$ with non-null boundary values belongs to the space $H^s_D(\Omega)$ for $s < 1/2$ only. Better results would probably require to consider Bourgain spaces other than $X_{s,b}(\Omega)$.

Corollary 3.4. *Let $n = 1$, $\Omega = (0, \pi)$, $\Gamma_0 = \{0\}$, and let the nonlinear term $N(u)$ satisfy*

$$|N(u) - N(v)| \leq C(|u|^\alpha + |v|^\alpha)|u - v|, \quad \forall\, u, v \in \mathbb{R}$$

for some $\alpha \in [0, 5/4)$. Let $p = \frac{4}{3}(\alpha + 1) < 3$. Then there exists a number $\delta > 0$ such that for any $u_0, u_T \in L^2(\Omega)$ satisfying

$$\|u_0\|_{L^2(\Omega)} < \delta, \quad \|u_T\|_{L^2(\Omega)} < \delta$$

one may find a function $h \in H^{\frac{1}{2}}(0, T)$ and a solution $u \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; L^p(\Omega))$ of (3.10) and (3.11) such that $u(0) = u_0$ and $u(T) = u_T$.

For instance, $N_1(u) = \lambda|u|^\alpha u$ with $0 \leq \alpha < 5/4$, and $N_2(u)$ of the form (3.2) with $\alpha = 1$ are concerned.

Proof. From the classical Strichartz estimate (see e.g. Ref. 50)

$$\|u\|_{L^4(\mathbb{R}; L^4(\mathbb{T}))} \leq C\|u\|_{X_{0,\frac{3}{8}}}$$

we obtain at once the following estimates involving the spaces $X^T_{s,b}(\Omega)$

$$\|u\|_{L^4(0,T;L^4(\Omega))} \leq C\|u\|_{X^T_{0,\frac{3}{8}}(\Omega)},$$
$$\|u\|_{X^T_{0,-\frac{3}{8}}(\Omega)} \leq C\|u\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))}.$$

Notice that for $v \in L^p(0, T; L^p(\Omega))$, we have that

$$\int_0^t W_D(t - \tau)N(v)(\tau)d\tau \in X^T_{0,\frac{5}{8}}(\Omega) \subset C([0, T]; L^2(\Omega)) \cap L^p(0, T; L^p(\Omega)).$$

Indeed,

$$\begin{aligned}\left\|\int_0^t W_D(t - \tau)N(v)(\tau)d\tau\right\|_{X^T_{0,\frac{5}{8}}(\Omega)} &\leq C\|N(v)\|_{X^T_{0,-\frac{3}{8}}(\Omega)} \\ &\leq C\|N(v)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))} \\ &\leq C\|v\|^{\alpha+1}_{L^p(0,T;L^p(\Omega))} < \infty.\end{aligned}$$

In particular, $\omega(v, T) = i\int_0^T W_D(T - \tau)N(v)(\tau)d\tau \in L^2(\Omega)$. On the other hand, by Proposition 3.2, Λ maps continuously $L^2(\Omega)$ into $C([0, T]; L^2(\Omega)) \cap X^T_{0,b}(\Omega)$ for any $b < 1/4$. Interpolating between

$$X_{0,\frac{3}{8}} \subset L^4(\mathbb{R}; L^4(\mathbb{T})) \quad \text{and} \quad X_{0,0} = L^2(\mathbb{R}; L^2(\mathbb{T}))$$

we obtain that

$$X_{0,b} \subset L^p(\mathbb{R}; L^p(\mathbb{T})) \quad \text{for } b = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{4}.$$

Therefore

$$\Lambda(L^2(\Omega)) \subset C([0, T]; L^2(\Omega)) \cap L^p(0, T; L^p(\Omega)).$$

It follows that the map

$$\Gamma(v) = W_D(t)u_0 + i\int_0^t W_D(t-\tau)N(v)(\tau)d\tau + \Lambda(u_T - W_D(T)u_0 - \omega(v,T))$$

is well-defined from $L^p(0,T;L^p(\Omega))$ into $C([0,T];L^2(\Omega)) \cap L^p(0,T;L^p(\Omega))$. Using the computations above, one readily sees that Γ contracts in some ball $B_M \subset L^p(0,T;L^p(\Omega))$, provided that $\|u_0\|_{L^2(\Omega)} + \|u_T\|_{L^2(\Omega)}$ is small enough. □

Corollary 3.5. *Theorem 3.2 may be applied when* $n=2$, $\Omega=(0,\pi)^2$, *g is a smooth Dirichlet controller,* $N(u)=\bar{u}^2$, $s\in(-\frac{3}{8},-\frac{1}{3})$, $b\in(\frac{3}{8},\frac{1}{2})$ *with* $s+2b<\frac{1}{2}$, *and* $b'>-\frac{1}{2}$ *is sufficiently close to* $-\frac{1}{2}$.

Corollary 3.5 is a direct consequence of Theorem 3.2 and of the following result, whose proof will be given in the Appendix.

Proposition 3.3. *Let* $s\in(-\frac{3}{8},-\frac{1}{3})$ *and* $b\in(\frac{3}{8},\frac{1}{2})$. *Then there exists* $b'\in(-\frac{1}{2},-\frac{5}{12})$ *and* $C>0$ *such that*

$$\|\bar{v}_1\bar{v}_2\|_{X_{s,b'}(\mathbb{T}^2)} \le C\|v_1\|_{X_{s,b}(\mathbb{T}^2)}\|v_2\|_{X_{s,b}(\mathbb{T}^2)}, \quad \forall\, v_1,v_2\in X_{s,b}(\mathbb{T}^2), \tag{3.18}$$

$$\|\bar{u}_1\bar{u}_2\|_{X_{s,b'}(\Omega)} \le C\|u_1\|_{X_{s,b}(\Omega)}\|u_2\|_{X_{s,b}(\Omega)}, \quad \forall\, u_1,u_2\in X_{s,b}(\Omega). \tag{3.19}$$

Notice that if we increase the value of s, the state space in which the controllability result holds has to take into account the fact that the value (or the normal derivative) of the function vanishes on $\partial\Omega\backslash\Gamma_0$. To state a result of this kind, we limit ourselves to the situation when Γ_0 is a side, e.g.

$$\Gamma_0 = \{0\}\times(0,l_2)\times\cdots\times(0,l_n).$$

Introduce the domain $\tilde{\Omega}=(-1,l_1)\times(0,l_2)\times\cdots\times(0,l_n)$, a function $a\in C_0^\infty(\tilde{\Omega}\backslash\overline{\Omega})$, and consider the internal control problem

$$iu_t + \Delta u + N(u) = ia(x)h(x,t), \quad x\in\tilde{\Omega}, \quad t\in(0,T). \tag{3.20}$$

Taking the restriction to $\Omega\times(0,T)$ of solutions of (3.20), we obtain as a corollary of Theorem 3.1 that both systems (3.10)−(3.11) and (3.10)−(3.12) are locally exactly controllable in some subspace of $H^s(\Omega)$ for any $s>s_{\alpha,n}$.

Corollary 3.6. *For given* $\alpha\ge1$, $n\ge2$, $\lambda\in\mathbb{R}$, $s>s_{\alpha,n}$ *and* $T>0$, *there exists a constant* $\delta>0$ *such that for any* $u_0,u_1\in H^s(\Omega)$ *satisfying*

$$\|u_i\|_{H^s(\Omega)}\le\delta, \quad i=0,1$$

and

$$u_i=\Delta u_i=\cdots=\Delta^p u_i=0 \quad x\in\partial\Omega\backslash\Gamma_0, \quad p\le\left[\frac{2s-1}{4}\right], \quad i=0,1$$

$$\left(\textit{resp. } \frac{\partial u_i}{\partial\nu}=\frac{\partial\Delta u_i}{\partial\nu}=\cdots=\frac{\partial\Delta^p u_i}{\partial\nu}=0,\ x\in\partial\Omega\backslash\Gamma_0,\ p\le\left[\frac{2s-3}{4}\right],\ i=0,1\right),$$

then one can choose a control input h such that system (3.10)–(3.11) (*resp. system* (3.10)–(3.12)) *admits a solution $u \in C([0,T]; H^s(\Omega))$ with*

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x).$$

Remark 3.3. By using the same extension and restriction argument, one can derive a local controllability result in the space $H^s(\Omega)$ when $s > s_{\alpha,n}$ and for any given bounded smooth set Ω, provided that the control is applied on the whole boundary (i.e. $\Gamma_0 = \partial\Omega$). A result of this kind for which the critical Sobolev exponent $s = s_c = s_{2,2} = 0$ is reached, is given in Ref. 47.

4. Stabilization

In this section we focus on the internal stabilization of the semilinear Schrödinger equation on the torus $\mathbb{T}^n$

$$iu_t + \Delta u + N(u) = -ia^2(x)u, \quad x \in \mathbb{T}^n, \tag{4.1}$$

where a is any smooth real function with $a \not\equiv 0$.

We have the following local exponential stability result which does not require the Geometric Control Condition.

Theorem 4.1. *Let $a \in C_0^\infty(\mathbb{T}^n)$, $a \not\equiv 0$ and let $s > s_{\alpha,N}$. Then there exist some constants ν, C such that every solution u of* (4.1) *issued from the initial state $u_0 \in H^s(\mathbb{T}^n)$ satisfies*

$$\|u(t)\|_s \le Ce^{-\nu t}\|u_0\|_s \quad \forall\, t \ge 0. \tag{4.2}$$

Proof. We proceed as in Ref. 46. The operator $A_a = i\Delta - a^2$ with domain $\mathcal{D}(A_a) = H^{s+2}(\mathbb{T}^n)$ generates a continuous group $(W_a(t))_{t\in\mathbb{R}}$ of operators on $H^s(\mathbb{T}^n)$. The first step is to check that the semigroup $(W_a(t))_{t\in\mathbb{R}^+}$ is exponentially stable in $H^s(\mathbb{T}^n)$. This is done in the following. □

Proposition 4.1. *There exist positive constants $C > 0$ and $\nu > 0$ such that*

$$\|W_a(t)u_0\|_s \le Ce^{-\nu t}\|u_0\|_s \quad \forall\, t \ge 0. \tag{4.3}$$

Proof of Proposition 4.1. When $s = 0$, the exponential stability of $(W_a(t))_{t\in\mathbb{R}^+}$ is a direct consequence of Theorem 2.1, according to Ref. 36. To prove (4.3) when $s = 2$, we pick any $u_0 \in H^2(\mathbb{T}^n)$ and set $v := u_t$. Then v solves the system

$$\begin{cases} v_t = i\Delta v - a^2(x)v, & x \in \mathbb{T}^n, \\ v(x,0) = v_0(x) := i\Delta u_0(x) - a^2(x)u_0(x). \end{cases} \tag{4.4}$$

By the property (4.3) established when $s = 0$, we have

$$\|u(t)\|_0 \le Ce^{-\nu t}\|u_0\|_0, \quad \|v(t)\|_0 \le Ce^{-\nu t}\|v_0\|_0.$$

Since $i\Delta u = v + a^2 u$, we conclude that

$$\|u(t)\|_2 \le Ce^{-\nu t}\|u_0\|_2 \quad \forall\, t \ge 0.$$

An easy induction yields (4.3) for any $s \in 2\mathbb{N}$. The proposition then follows by a classical interpolation argument.

Let us now turn our attention to the stability properties of the nonlinear system

$$u_t = A_a u + iN(u), \quad u(\cdot, 0) = u_0$$

that we shall write in its integral form

$$u(t) = W_a(t)u_0 + i\int_0^t W_a(t-\tau)N(u)(\tau)d\tau. \tag{4.5}$$

At this point, we need to establish linear estimates when W_a is substituted to W.

Lemma 4.1. *Let $T > 0$, $s \geq 0$ and $0 \leq b \leq 1$ be given. Then there exists a constant $C > 0$ depending only on T, s and b such that*

$$\|W_a(t)\phi\|_{X^T_{s,b}} \leq C\|\phi\|_s$$

for any $\phi \in H^s(\mathbb{T}^n)$

Proof of Lemma 4.1. An application of Duhamel formula gives

$$W_a(t)\phi = W(t)\phi - \int_0^t W(t-\tau)(a^2 W_a(\tau)\phi)d\tau. \tag{4.6}$$

It follows that

$$\begin{aligned}
\|W_a(t)\phi\|_{X^T_{s,b}} &\leq \|W(t)\phi\|_{X^T_{s,b}} + \left\| \int_0^t W(t-\tau)(a^2 W_a(\tau)\phi)d\tau \right\|_{X^T_{s,b}} \\
&\leq C\|\phi\|_s + C\|a^2 W_a(t)\phi\|_{X^T_{s,b-1}} \\
&\leq C\|\phi\|_s + C\|W_a(t)\phi\|_{L^2(0,T;H^s(\mathbb{T}^n))} \quad (\text{as } b-1 \leq 0) \\
&\leq C\|\phi\|_s,
\end{aligned}$$

as desired. □

Lemma 4.2. *Let $T > 0$, $s \geq 0$ and $b \in (\frac{1}{2}, 1)$ be given. Then there exists a constant $C > 0$ depending only on T, s and b such that*

$$\left\| \int_0^t W_a(t-\tau)f(\tau)d\tau \right\|_{X^T_{s,b}} \leq C\|f\|_{X^T_{s,b-1}}$$

for any $f \in X^T_{s,b-1}$.

Proof of Lemma 4.2. It follows from (4.6) that

$$\begin{aligned}
&\int_0^t W_a(t-\tau)f(\tau)d\tau \\
&\quad = \int_0^t W(t-\tau)f(\tau)d\tau - \int_0^t W(t-\tau)a^2\left(\int_0^\tau W_a(\tau-s)f(s)ds\right)d\tau,
\end{aligned}$$

hence

$$\begin{aligned}\left\|\int_0^t W_a(t-\tau)f(\tau)d\tau\right\|_{X^T_{s,b}} &\le C\|f\|_{X^T_{s,b-1}} + C\left\|a^2\int_0^t W_a(t-s)f(s)ds\right\|_{X^T_{s,b-1}}\\ &\le C\|f\|_{X^T_{s,b-1}} + C\left\|\int_0^t W_a(t-s)f(s)ds\right\|_{X^T_{s,0}}\\ &\le C\|f\|_{X^T_{s,b-1}} + CT^\alpha\left\|\int_0^t W_a(t-s)f(s)ds\right\|_{X^T_{s,b}}\end{aligned}$$

for some constant $\alpha > 0$, by virtue of Lemma 3.2 and of Lemma 2.11 in Ref. 50. The result follows at once if T is small enough, say $T < T_0$. For $T \ge T_0$, the result follows from Lemma 4.1 and an easy induction. □

Let us now proceed to the proof of the exponential stability of the system (4.1). Pick a number $s \ge 0$. According to Proposition 4.1, there exist positive constants C, ν such that

$$\|W_a(t)u_0\|_s \le Ce^{-\nu t}\|u_0\|_s \quad \forall\, t \ge 0.$$

Pick a time $T > 0$ such that

$$Ce^{-\nu T} < \frac{1}{4}$$

and fix a number $b \in (\frac{1}{2}, 1)$. We seek a solution u of the integral equation (4.5) in the form of a fixed point of the map

$$\Gamma(u) = W_a(t)u_0 + i\int_0^t W_a(t-\tau)N(u)(\tau)d\tau$$

in some ball B_M of the space $X^T_{s,b}$. This will be done provided that $\|u_0\|_s \le \delta$, where δ is a small number to be determined. Furthermore, to ensure the exponential stability, δ and M will be chosen in such a way that $\|u(T)\|_s \le \|u_0\|_s/2$. Pick for the moment any $\delta > 0$ and $M > 0$, and let $u_0 \in H^s(\mathbb{T}^n)$ be such that $\|u_0\|_s \le \delta$. By computations similar to those displayed in the proof of Theorem 3.1 with $W_a(t)$ substituted to $W(t)$, we arrive at

$$\|\Gamma(u)\|_{X^T_{s,b}} \le c\|u_0\|_s + cM^{\alpha+1} \quad \forall\, u \in B_M$$

and

$$\|\Gamma(u) - \Gamma(v)\|_{X^T_{s,b}} \le cM^\alpha\|u - v\|_{X^T_{s,b}} \quad \forall\, u, v \in B_M$$

for some constant $c > 0$ independent of δ, M, and u_0. On the other hand, using the estimate of $\|\omega(T,u)\|_s$ in the proof of Theorem 3.1, we obtain

$$\begin{aligned}\|\Gamma(u)(T)\|_s &\le \|W_a(T)u_0\|_s + \left\|\int_0^T W_a(T-t)N(u)(t)dt\right\|_s\\ &\le \frac{1}{4}\|u_0\|_s + cM^{\alpha+1}.\end{aligned}$$

Pick $\delta = 4cM^{\alpha+1}$ where $M > 0$ is chosen so that

$$(4c^2 + c)M^{\alpha+1} \le M \quad \text{and} \quad cM^\alpha \le \frac{1}{2}.$$

Then we have

$$\|\Gamma(u)\|_{X^T_{s,b}} \le M \quad \forall\, u \in B_M$$
$$\|\Gamma(u) - \Gamma(v)\|_{X^T_{s,b}} \le \frac{1}{2}\|u - v\|_{X^T_{s,b}} \quad \forall\, u, v \in B_M.$$

Thus the map Γ, which is a contraction in B_M, has a fixed point $u \in B_M$. By construction, u fulfills

$$\|u(T)\|_s = \|\Gamma(u)(T)\|_s \le \frac{\delta}{2}.$$

Assume now that $0 < \|u_0\|_s < \delta$. Changing δ into $\delta' := \|u_0\|_s$ and M into $M' := (\delta'/\delta)^{\frac{1}{\alpha+1}}M$, we obtain that $\|u(T)\|_s \le \|u_0\|_s/2$, and an obvious induction yields $\|u(kT)\|_s \le 2^{-k}\|u_0\|_s$ for any $k \ge 0$. As $X^T_{s,b} \subset C([0,T]; H^s(\mathbb{T}^n))$ for $b > 1/2$, and $\|u\|_{X^T_{s,b}} \le M = (\delta/(4c))^{\frac{1}{\alpha+1}}$, we infer by the semigroup property that there exist some constants $C' > 0, \nu' > 0$ such that

$$\|u(t)\|_s \le C'e^{-\nu' t}\|u_0\|_s.$$

The proof is complete. □

Appendix A

A.1. *Proof of Proposition* 3.1

We proceed as in Ref. 9, pp. 115–118. We first introduce some notations. Let $|x|_\infty := \sup_{1\le i\le n} |x_i|$ for $x = (x_i)_{1\le i\le n} \in \mathbb{R}^n$. We introduce a dyadic partition of $\mathbb{Z}^n$

$$\mathbb{Z}^n = \bigcup_{j\ge 0} D_j,$$

where $D_0 = \{0\}$ and $D_j = \{k \in \mathbb{Z}^n;\ 2^{j-1} \le |k|_\infty < 2^j\}$ for $j \ge 1$. For any Hölder exponent $p, q \in [1, +\infty]$, we write $L^p_t L^q_x$ for $L^p(\mathbb{R}_t, L^q(\mathbb{T}^n_x))$. The (discrete) cube of center $x_0 \in \mathbb{R}^n$ and sidelength $2R > 0$ is

$$Q(x_0, R) = \{k \in \mathbb{Z}^n;\ |k - x_0|_\infty \le R\}.$$

The Strichartz estimate (see Refs. 8 and 16)

$$\|u\|_{L^4_t L^4_x} \le c\|u\|_{X_{s,b}}, \quad s > \frac{n}{2} - \frac{n+2}{4}, \quad b > \frac{1}{2},$$

when combined with the standard estimates

$$\|u\|_{L^\infty_t L^2_x} \le c\|u\|_{X_{0,b}}, \quad b > \frac{1}{2}$$
$$\|u\|_{L^2_t L^2_x} = \|u\|_{X_{0,0}}$$

and Sobolev embedding theorem, gives by interpolation the following result.

Lemma A.1. (Corollary 2.2 in Ref. 16) *Let* $n \geq 2$.

(i) *For all* p, q, s *satisfying*

$$0 < \frac{1}{p} \leq \frac{1}{4}, \quad 0 < \frac{1}{q} \leq \frac{1}{2} - \frac{1}{p}, \quad s > \frac{n}{2} - \frac{2}{p} - \frac{n}{q}, \tag{A.1}$$

there exists a number $b \in (0, \frac{1}{2})$ *such that for all* $u \in X_{s,b}$, *the following holds*

$$\|u\|_{L_t^p L_x^q} \leq c\|u\|_{X_{s,b}}. \tag{A.2}$$

(ii) *For all* p, q, s, b *satisfying*

$$\begin{gathered} 0 \leq \frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{2} \leq \frac{1}{p} + \frac{1}{q} \leq 1, \quad s > (n-2)\left(\frac{1}{2} - \frac{1}{q}\right), \quad \text{and} \\ b > 1 - \frac{1}{p} - \frac{1}{q} \end{gathered} \tag{A.3}$$

then for all $u \in X_{s,b}$, (A.2) *holds.*

Let $\mathcal{F}_x$ denote the Fourier transform in x, and let 1_Q denote the characteristic function of the cube Q. The following result, inspired by an observation made in Ref. 8, indicates that for a function spatially supported in a cube, only the sidelength of the cube (not its center) comes into play in (A.2).

Lemma A.2. (Lemma 2.4 in Ref. 16) *Assume that for* p, q, s, b *the estimate* (A.2) *is valid. Then there exists a constant* $c > 0$ *such that for any cube* Q *of center* $x_0 \in \mathbb{R}^n$ *and sidelength* $R > 0$, *the following holds*

$$\|(\mathcal{F}_x^{-1} 1_Q \mathcal{F}_x) u\|_{L_t^p L_x^q} \leq c R^s \|u\|_{X_{0,b}}. \tag{A.4}$$

It follows that if (A.1) (or (A.3)) holds and if $u = u(x,t)$ is a function decomposed as

$$u(x,t) = \sum_{|k-x_0|_\infty \leq R} \int_{\mathbb{R}} \hat{u}(k,\tau) e^{i(k\cdot x + \tau t)} d\tau,$$

then

$$\|u\|_{L_t^p L_x^q} \leq c R^s \|u\|_{X_{0,b}} = c R^s \left(\sum_{|k-x_0|_\infty \leq R} \int_{\mathbb{R}} \langle \tau + |k|^2 \rangle^{2b} |\hat{u}(k,\tau)|^2 d\tau \right)^{\frac{1}{2}}. \tag{A.5}$$

Let the functions $u_1, \ldots, u_{\alpha+1} \in X_{s,b}$ be given, where s and b denote some positive numbers, and let us set

$$u = \tilde{u}_1 \tilde{u}_2 \cdots \tilde{u}_{\alpha+1},$$

where $\tilde{u}_i$ is u_i or $\overline{u_i}$. To estimate $\|u\|_{X_{s,-b}}$ we proceed by duality, estimating the integral $\int_{\mathbb{R}}\int_{\mathbb{T}^n} u\bar{v}dxdt$ for any $v \in X_{-s,b}$ with $\|v\|_{X_{-s,b}} \le 1$. By Plancherel theorem

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^n} u\bar{v}dxdt &= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}} \hat{u}(k,\tau)\bar{\hat{v}}(k,\tau)d\tau \\ &= \sum_{k_1\cdots k_{\alpha+1}}\int_{\tau_1\cdots\tau_{\alpha+1}} \langle k\rangle^s\left(\prod_{i=1}^{\alpha+1}\hat{\tilde{u}}_i(k_i,\tau_i)\right)\langle k\rangle^{-s}\bar{\hat{v}}(k,\tau),\end{aligned}$$

where $k = k_1 + \cdots + k_{\alpha+1}$ and $\tau = \tau_1 + \cdots + \tau_{\alpha+1}$. Notice that $\hat{\tilde{u}}(k_i,\tau_i) = \overline{\hat{u}_i(-k_i,-\tau_i)}$. Writing $k_i \in D_{j_i}$, $j_i \ge 0$, we obtain

$$\left|\int_{\mathbb{R}}\int_{\mathbb{T}^n} u\bar{v}dxdt\right| \le \sum_{j_1\cdots j_{\alpha+1}}\sum_{k_i\in D_{j_i}}\int_{\tau_1\cdots\tau_{\alpha+1}} \langle k\rangle^s\left(\prod_{i=1}^{\alpha+1}|\hat{u}_i(k_i,\tau_i)|\right)\langle k\rangle^{-s}|\hat{v}(k,\tau)|,$$

where now $k = \pm k_1 \cdots \pm k_{\alpha+1}$, $\tau = \pm\tau_1\cdots\pm\tau_{\alpha+1}$ ($+k_i$ if $\tilde{u}_i = u_i$, $-k_i$ if $\tilde{u}_i = \overline{u_i}$, and the same for $\pm\tau_i$). We shall focus on the sum $\Sigma = \sum_{j_1\ge j_2\ge\cdots\ge j_{\alpha+1}}$, the other contributions leading to similar bounds. As $|k_i|_\infty \le 2|k_1|_\infty$ for $i \ge 2$, we have that

$$\Sigma \le c\sum_{j_1\ge\cdots\ge j_{\alpha+1}} 2^{j_1 s}\sum_{k_i\in D_{j_i}}\int_{\tau_1\cdots\tau_{\alpha+1}}\left(\prod_{i=1}^{\alpha+1}|\hat{u}_i(k_i,\tau_i)|\right)\langle k\rangle^{-s}|\hat{v}(k,\tau)|.$$

Pick $\gamma \in \mathbb{N}^*$ with

$$\alpha \le 2^{\gamma-2}$$

and split Σ into $\Sigma_1 + \Sigma_2$ where Σ_1 corresponds to the $j_1,\dots,j_{\alpha+1}$ for which

$$j_1 \ge j_2 + \gamma + 2 \ge j_2 \ge j_3 \ge \cdots \ge j_{\alpha+1}.$$

Consider a "partition" of D_{j_1} into a collection of cubes Q_l of sidelength 2^{j_2}

$$D_{j_1} = \bigcup_l Q_l.$$

Note that each $k \in D_{j_1}$ belongs to at most 2^n cubes Q_l. For any l, we denote by $\tilde{Q}_l$ the cube of sidelength $2^{j_2+\gamma}$ with the same center as Q_l if $k = k_1 \pm k_2,\dots$, and with center the opposite of that of Q_l if $k = -k_1 \pm k_2 \dots$. We claim that $k \in \tilde{Q}_l$ when $k_1 \in Q_l$ and $k_i \in D_{j_i}$ for $i \ge 2$. Indeed

$$|k_2|_\infty + \cdots + |k_{\alpha+1}|_\infty \le \alpha 2^{j_2} \le 2^{j_2+\gamma-2}, \tag{A.6}$$

hence if $Q_l = Q(x_0, 2^{j_2-1})$

$$\begin{aligned}|\pm x_0 - k|_\infty &\le |\pm x_0 - \pm k_1|_\infty + |k_2|_\infty + \cdots + |k_{\alpha+1}|_\infty \\ &\le 2^{j_2-1} + 2^{j_2+\gamma-2} \le 2^{j_2+\gamma-1}.\end{aligned}$$

Notice also that $\tilde{Q}_l \subset D_{j_1-1} \cup D_{j_1} \cup D_{j_1+1}$ since the sidelength of $\tilde{Q}_l$ is at most 2^{j_1-2} and $Q_l \subset D_{j_1}$. It follows that

$$\Sigma_1 \leq c \sum_{\substack{j_1 \geq j_2+\gamma+2 \\ j_2 \geq j_3 \geq \cdots \geq j_{\alpha+1}}} 2^{j_1 s} \sum_l \sum_{k_1 \in Q_l} \sum_{\substack{k_2 \in D_{j_2}, \\ k_{\alpha+1} \in D_{j_{\alpha+1}}}} \int_{\tau_1 \cdots \tau_{\alpha+1}} \times \left(\prod_{i=1}^{\alpha+1} |\hat{u}_i(k_i, \tau_i)| \right) 1_{\tilde{Q}_l}(k) \langle k \rangle^{-s} |\hat{v}(k,\tau)|.$$

Let us introduce the functions

$$f_l(x,t) = \sum_{k \in Q_l} \int_{\mathbb{R}} |\hat{u}_1(k,\tau)| e^{i(k\cdot x+\tau t)} d\tau,$$
$$g_l(x,t) = \sum_{k \in \tilde{Q}_l} \int_{\mathbb{R}} \langle k \rangle^{-s} |\hat{v}(k,\tau)| e^{i(k\cdot x+\tau t)} d\tau$$

and

$$h_i(x,t) = \sum_{k \in D_{j_i}} \int_{\mathbb{R}} |\hat{u}_i(k,\tau)| e^{i(k\cdot x+\tau t)} d\tau \quad \text{for } i = 2, \ldots, \alpha+1.$$

By Plancherel theorem

$$\Sigma_1 \leq c \sum_{\substack{j_1 \geq j_2+\gamma+2 \\ j_2 \geq j_3 \geq \cdots \geq j_{\alpha+1}}} 2^{j_1 s} \sum_l \int_{\mathbb{R}} \int_{\mathbb{T}^n} |f_l h_2 \cdots h_{\alpha+1} g_l| \, dx dt.$$

Pick Hölder exponents $p_1, q_1, p_2, q_2 \in [1,\infty)$ such that

$$\frac{3}{p_1} + \frac{\alpha-1}{p_2} = 1, \tag{A.7}$$

$$\frac{3}{q_1} + \frac{\alpha-1}{q_2} = 1. \tag{A.8}$$

We have that

$$\int_{\mathbb{R}} \int_{\mathbb{T}^n} |f_l h_2 \cdots h_{\alpha+1} g_l| dx dt \leq \|f_l\|_{L_t^{p_1} L_x^{q_1}} \|g_l\|_{L_t^{p_1} L_x^{q_1}} \|h_2\|_{L_t^{p_1} L_x^{q_1}} \prod_{i=3}^{\alpha+1} \|h_i\|_{L_t^{p_2} L_x^{q_2}}.$$

Assume that for some exponents s_1, b_1, s_2, b_2, the following estimates hold

$$\|u\|_{L_t^{p_1} L_x^{q_1}} \leq c \|u\|_{X_{s_1,b_1}}, \tag{A.9}$$

$$\|u\|_{L_t^{p_2} L_x^{q_2}} \leq c \|u\|_{X_{s_2,b_2}}. \tag{A.10}$$

Then, by (A.5) and the fact that the sidelength of Q_l (resp. $\tilde{Q}_l$) is 2^{j_2} (resp. $2^{j_2+\gamma}$), we have

$$\|f_l\|_{L_t^{p_1} L_x^{q_1}} \leq c 2^{j_2 s_1} \left(\sum_{k \in Q_l} \int_\tau \langle \tau + |k|^2 \rangle^{2b_1} |\hat{u}_1|^2 \right)^{\frac{1}{2}}, \tag{A.11}$$

$$\|g_l\|_{L_t^{p_1}L_x^{q_1}} \leq c2^{j_2 s_1}\left(\sum_{k\in\tilde{Q}_l}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}\langle k\rangle^{-2s}|\hat{v}|^2\right)^{\frac{1}{2}}, \tag{A.12}$$

$$\|h_2\|_{L_t^{p_1}L_x^{q_1}} \leq c2^{j_2 s_1}\left(\sum_{k\in D_{j_2}}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}|\hat{u}_2|^2\right)^{\frac{1}{2}} \tag{A.13}$$

and for $i=3,\ldots,\alpha+1$

$$\begin{aligned}\|h_i\|_{L_t^{p_2}L_x^{q_2}} &\leq c2^{j_i s_2}\left(\sum_{k\in D_{j_i}}\int_\tau \langle\tau+|k|^2\rangle^{2b_2}|\hat{u}_i|^2\right)^{\frac{1}{2}}\\ &\leq c\left(\sum_{k\in D_{j_i}}\int_\tau \langle\tau+|k|^2\rangle^{2b_2}\langle k\rangle^{2s_2}|\hat{u}_i|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{A.14}$$

Using Cauchy–Schwarz in $\sum_l$, we obtain

$$\begin{aligned}\Sigma_1 \leq c &\sum_{\substack{j_1\geq j_2+\gamma+2\\ j_2\geq j_3\geq\cdots\geq j_{\alpha+1}}} 2^{j_1 s+3j_2 s_1}\left(\sum_l\sum_{k\in Q_l}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}|\hat{u}_1|^2\right)^{\frac{1}{2}}\\ &\times\left(\sum_l\sum_{k\in\tilde{Q}_l}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}\langle k\rangle^{-2s}|\hat{v}|^2\right)^{\frac{1}{2}}\\ &\times\left(\sum_{k\in D_{j_2}}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}|\hat{u}_2|^2\right)^{\frac{1}{2}}\prod_{i=3}^{\alpha+1}\left(\sum_{k\in D_{j_i}}\int_\tau \langle\tau+|k|^2\rangle^{2b_2}\langle k\rangle^{2s_2}|\hat{u}_i|^2\right)^{\frac{1}{2}}\\ &\leq c\sum_{\substack{j_1\geq j_2+\gamma+2\\ j_2\geq j_3\geq\cdots\geq j_{\alpha+1}}}\left(\sum_{k\in D_{j_1}}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}\langle k\rangle^{2s}|\hat{u}_1|^2\right)^{\frac{1}{2}}\\ &\times\left(\sum_{k\in D_{j_1-1}\cup D_{j_1}\cup D_{j_1+1}}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}\langle k\rangle^{-2s}|\hat{v}|^2\right)^{\frac{1}{2}}\\ &\times\left(\sum_{k\in D_{j_2}}\int_\tau \langle\tau+|k|^2\rangle^{2b_1}\langle k\rangle^{6s_1}|\hat{u}_2|^2\right)^{\frac{1}{2}}\prod_{i=3}^{\alpha+1}\left(\sum_{k\in D_{j_i}}\int_\tau \langle\tau+|k|^2\rangle^{2b_2}\langle k\rangle^{2s_2}|\hat{u}_i|^2\right)^{\frac{1}{2}}.\end{aligned}$$

We used the fact that a point $k\in D_{j_1-1}\cup D_{j_1}\cup D_{j_1+1}$ belongs to (at most) a finite number of cubes $\tilde{Q}_l$, bounded by $(2^{\gamma+2}+1)^n$. A sum $\sum_{j_i\geq 0}(\sum_{k\in D_{j_i}}\int_\tau\langle\tau+$

$|k|^2\rangle^{2b_2}\langle k\rangle^{2s_2}|\hat{u}_i|^2)^{\frac{1}{2}}$ can be estimated by $c\|u_i\|_{X_{s_2+\varepsilon,b_2}}$ for any $\varepsilon > 0$ thanks to Cauchy–Schwarz. Summing successively in $k_{\alpha+1}, \ldots, k_1$, we arrive at

$$\Sigma_1 \le c\|u_1\|_{X_{s,b_1}}\|v\|_{X_{-s,b_1}}\|u_2\|_{X_{3s_1+\varepsilon,b_1}} \prod_{i=3}^{\alpha+1} \|u_i\|_{X_{s_2+\varepsilon,b_2}}.$$

The same bound for Σ_2 can be obtained by a more simple analysis. Indeed, as $j_1 \le j_2+\gamma+1$ in the sum over $j_1, \ldots, j_{\alpha+1}$, we obtain

$$\Sigma_2 \le c \sum_{\substack{j_1 \le j_2+\gamma+1 \\ j_2 \ge j_3 \ge \cdots \ge j_{\alpha+1}}} 2^{j_1 s} \int_{\mathbb{R}} \int_{\mathbb{T}^n} |f h_2 \cdots h_{\alpha+1} g| dx dt,$$

where

$$f(x,t) = \sum_{k \in D_{j_1}} \int_{\mathbb{R}} |\hat{u}_1(k,\tau)| e^{i(k\cdot x+\tau t)} d\tau$$

$$g(x,t) = \sum_{|k| \le (2^{\gamma+1}+\alpha)2^{j_2}} \int_{\mathbb{R}} \langle k\rangle^{-s} |\hat{v}(k,\tau)| e^{i(k\cdot x+\tau t)} d\tau$$

and $h_2, \ldots, h_{\alpha+1}$ as above. Since $2^{j_1 s_1} \le c 2^{j_2 s_1}$, we still have

$$\|f\|_{L_t^{p_1} L_x^{q_1}} \le c 2^{j_2 s_1} \left(\sum_{k \in D_{j_1}} \int_\tau \langle \tau + |k|^2\rangle^{2b_1} |\hat{u}_1|^2 \right)^{\frac{1}{2}},$$

$$\|g\|_{L_t^{p_1} L_x^{q_1}} \le c 2^{j_2 s_1} \left(\sum_{k \in \mathbb{Z}^n} \int_\tau \langle \tau + |k|^2\rangle^{2b_1} \langle k\rangle^{-2s} |\hat{v}|^2 \right)^{\frac{1}{2}}.$$

Next, Σ_2 is estimated as Σ_1 (see above). At this stage, we have proved that

$$\Sigma \le c\|u_1\|_{X_{s,b_1}}\|v\|_{X_{-s,b_1}}\|u_2\|_{X_{3s_1+\varepsilon,b_1}} \prod_{i=3}^{\alpha+1} \|u_i\|_{X_{s_2+\varepsilon,b_2}}, \tag{A.15}$$

where $\varepsilon > 0$ is arbitrarily small, the exponents s_1, b_1, s_2, b_2 are taken so that (A.9) and (A.10) are satisfied, with the Hölder exponents p_1, q_1, p_2, q_2 satisfying (A.7) and (A.8). The proof will be complete if, in addition, we have

$$s \ge \sup\{3s_1+\varepsilon, s_2+\varepsilon\}, \quad b_1 < \frac{1}{2}, \quad b_2 < \frac{1}{2}.$$

We distinguish three cases: (i) $\alpha \ge 3$; (ii) $\alpha = 2$; (iii) $\alpha = 1$.

(i) $\alpha \ge 3$

We aim to reach any value $s > s_c$. To find the sets of exponents (p_1, q_1, s_1, b_1), (p_2, q_2, s_2, b_2) satisfying (A.1), (A.7) and (A.8), and leading to the "smallest" value of s, we are led to minimize the functional $\sup\{3\sigma_1, \sigma_2\}$, where

$$\sigma_1 = \frac{n}{2} - \left(\frac{2}{p_1} + \frac{n}{q_1}\right), \tag{A.16}$$

$$\sigma_2 = \frac{n}{2} - \left(\frac{2}{p_2} + \frac{n}{q_2}\right) \tag{A.17}$$

under the constraints

$$4 \leq p_1 < \infty, \tag{A.18}$$

$$0 < \frac{1}{q_1} \leq \frac{1}{2} - \frac{1}{p_1}, \tag{A.19}$$

$$4 \leq p_2 < \infty, \tag{A.20}$$

$$0 < \frac{1}{q_2} \leq \frac{1}{2} - \frac{1}{p_2}, \tag{A.21}$$

$$\frac{3}{p_1} + \frac{\alpha - 1}{p_2} = 1, \tag{A.22}$$

$$\frac{3}{q_1} + \frac{\alpha - 1}{q_2} = 1. \tag{A.23}$$

At this point, it is convenient to introduce the numbers r_1, r_2 with

$$\frac{1}{r_1} = \frac{2}{p_1} + \frac{n}{q_1}, \tag{A.24}$$

$$\frac{1}{r_2} = \frac{2}{p_2} + \frac{n}{q_2}\cdot \tag{A.25}$$

Note that, by (A.22) and (A.23),

$$\frac{3}{r_1} + \frac{\alpha - 1}{r_2} = n + 2. \tag{A.26}$$

Therefore, $3\sigma_1 = \frac{n}{2} - 2 + \frac{\alpha-1}{r_2}$ (resp. $\sigma_2 = \frac{n}{2} - \frac{1}{r_2}$) is a non-increasing function (resp. a non-decreasing function) of r_2. Thus the least value of $\sup\{3\sigma_1, \sigma_2\}$ is achieved when $3\sigma_1 = \sigma_2$, which yields

$$r_2 = \frac{\alpha}{2}, \quad r_1 = 3\left(n + \frac{2}{\alpha}\right)^{-1}, \quad 3\sigma_1 = \sigma_2 = \frac{n}{2} - \frac{2}{\alpha}\cdot \tag{A.27}$$

It remains to find p_1, q_1, p_2, q_2 satisfying (A.18)–(A.25). Note first that (A.23) is satisfied whenever (A.22) is by (A.26). Taking p_1 as variable, we infer from (A.22), (A.24) and (A.25) that

$$\frac{1}{p_2} = \frac{1}{\alpha - 1}\left(1 - \frac{3}{p_1}\right), \quad \frac{1}{q_1} = \frac{1}{3}\left(1 + \frac{2}{n\alpha}\right) - \frac{2}{np_1},$$

$$\frac{1}{q_2} = \frac{2}{n(\alpha - 1)}\left(\frac{3}{p_1} - \frac{1}{\alpha}\right).$$

The constraints (A.20), (A.19) and (A.21) are found to be respectively equivalent to

$$p_1 \le 3\left(1 - \frac{\alpha - 1}{4}\right)^{-1} \quad (\text{for } \alpha \le 4),$$
$$p_1 \ge \sup\left\{6\left(n + \frac{2}{\alpha}\right)^{-1},\ 6\left(1 - \frac{2}{n}\right)\left(1 - \frac{4}{n\alpha}\right)^{-1}\right\}, \quad p_1 < 3\alpha. \tag{A.28}$$

The value $p_1 = 6$ fulfills all the requirements in (A.28). Now let $s > \frac{n}{2} - \frac{2}{\alpha}$ be given. Choose $\varepsilon > 0$ such that $4\varepsilon < s - (\frac{n}{2} - \frac{2}{\alpha})$, and pick $s_1 \in (\sigma_1, \sigma_1 + \varepsilon)$, and $s_2 \in (\sigma_2, \sigma_2 + \varepsilon)$. Then (A.9) and (A.10) hold for some numbers $b_1 < \frac{1}{2}$, $b_2 < \frac{1}{2}$, according to Lemma A.1. Finally set $b = \sup\{b_1, b_2\}$. Then we have

$$\Sigma \le c\left(\prod_{i=1}^{\alpha+1} \|u_i\|_{X_{s,b}}\right)\|v\|_{X_{-s,b}}$$

which gives (3.5).

(ii) $\alpha = 2$

Observe first that the approach followed in (i) does not work for $n > 2$. Indeed, the constraints (A.18)–(A.27) impose $p_1 = p_2 = q_1 = q_2 = 4$, and the equation $3\sigma_1 = \sigma_2$ is then satisfied only for $n = 2$. Assume $n \ge 3$. We now search a couple (p_1, q_1) satisfying

$$0 < \frac{1}{p_1} \le \frac{1}{q_1} \le \frac{1}{2} \le \frac{1}{p_1} + \frac{1}{q_1} \le 1, \quad s_1 > (n-2)\left(\frac{1}{2} - \frac{1}{q_1}\right),$$
$$b_1 > 1 - \frac{1}{p_1} - \frac{1}{q_1}, \tag{A.29}$$

while (p_2, q_2) still satisfies

$$0 < \frac{1}{p_2} \le \frac{1}{4}, \quad 0 \le \frac{1}{q_2} \le \frac{1}{2} - \frac{1}{p_2}, \quad s_2 > \frac{n}{2} - \frac{2}{p_2} - \frac{n}{q_2}. \tag{A.30}$$

The Hölder exponents (p_1, q_1) and (p_2, q_2) have to satisfy the relations

$$\frac{3}{p_1} + \frac{1}{p_2} = 1, \tag{A.31}$$

$$\frac{3}{q_1} + \frac{1}{q_2} = 1. \tag{A.32}$$

We still minimize the functional $\sup\{3\sigma_1, \sigma_2\}$, where

$$\sigma_1 = (n-2)\left(\frac{1}{2} - \frac{1}{q_1}\right), \quad \sigma_2 = \frac{n}{2} - \frac{2}{p_2} - \frac{n}{q_2} = \frac{n}{2} - \frac{2}{p_2} - n\left(1 - \frac{3}{q_1}\right)$$

by solving in q_1 the equation $3\sigma_1 = \sigma_2$. Taking $p_2 = 4$ to produce the least value of σ_2, we find as solution $q_1 = 3(1 + \frac{1}{4n-5}) \in (3, 4)$, which yields $p_1 = 4$ and $q_2 = 4(n-1)$ by

(A.31) and (A.32), and

$$3\sigma_1 = \sigma_2 = \frac{n}{2} - \frac{3}{4} - \frac{1}{4(n-1)}\cdot$$

The constraints on p_1, q_1, p_2, q_2 in (A.29)−(A.30) are clearly fulfilled, for $n > 2$. Pick now any $s > \frac{n}{2} - \frac{3}{4} - \frac{1}{4(n-1)}$ and $\varepsilon > 0$ such that $4\varepsilon < s - (\frac{n}{2} - \frac{3}{4} - \frac{1}{4(n-1)})$. We next pick $s_1 \in (\sigma_1, \sigma_1 + \varepsilon)$, $s_2 \in (\sigma_2, \sigma_2 + \varepsilon)$, $b_1 \in (1 - \frac{1}{p_1} - \frac{1}{q_1}, \frac{1}{2})$, and $b_2 < \frac{1}{2}$ so that (A.2) holds. Then (3.5) follows with $b = \sup\{b_1, b_2\}$.

(iii) $\alpha = 1$

In this case, we have with $p_1 = q_1 = 3$

$$\Sigma \leq c\|u_1\|_{X_{s,b_1}}\|u_2\|_{X_{3s_1+\varepsilon,b_1}}\|v\|_{X_{-s,b_1}}$$

provided that (A.29) is satisfied, i.e.

$$s_1 > \sigma_1 = \frac{n-2}{6}, \quad b_1 > \frac{1}{3}\cdot$$

Therefore, if $s > \frac{n}{2} - 1$, taking $\varepsilon > 0$ such that $4\varepsilon < s - (\frac{n}{2} - 1)$, $s_1 \in (\sigma_1, \sigma_1 + \varepsilon)$, and $b = b_1 \in (\frac{1}{3}, \frac{1}{2})$, we conclude that

$$\Sigma \leq c\|u_1\|_{X_{s,b}}\|u_2\|_{X_{s,b}}\|v\|_{X_{-s,b}}$$

and (3.5) follows. □

A.2. *Proof of Proposition* 3.3

We begin with the proof of (3.18) by following closely Ref. 17. Note, however, that the main concern here is to have the condition $s + 2b < 1/2$ fulfilled. Let s, b be as in the statement of Proposition 3.3, and let $v_1, v_2 \in X_{s,b}$ be decomposed as

$$v_i(x,t) = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^2} \mathcal{F}v_i(k,\tau)e^{i(k\cdot x+\tau t)}d\tau \quad i = 1, 2.$$

(Here, we use the symbol $\mathcal{F}$ instead of $\hat{}$ to denote Fourier transform in space and time.) Let

$$f_i(k,\tau) = \langle k\rangle^s\langle\tau - |k|^2\rangle^b\mathcal{F}\bar{v}_i(k,\tau), \qquad i = 1, 2.$$

Then

$$\|\bar{v}_1\bar{v}_2\|_{X_{s,b'}} = \|\langle k\rangle^s\langle\tau + |k|^2\rangle^{b'}\int_{\tau_1+\tau_2=\tau}\sum_{k_1+k_2=k}\prod_{i=1}^{2}\langle k_i\rangle^{-s}\langle\tau_i - |k_i|^2\rangle^{-b}f_i\|_{L^2_{k,\tau}}, \tag{A.33}$$

where $\int_{\tau_1+\tau_2=\tau}\sum_{k_1+k_2=k}$ stands for $\int_{\mathbb{R}} d\tau_1 \sum_{k_1\in\mathbb{Z}^2}$ with the relations $\tau_1 + \tau_2 = \tau$ and $k_1 + k_2 = k$ satisfied. Let A_0 (resp. A_i, $i = 1, 2$) denote the region where the largest number among $\langle\tau + |k|^2\rangle$, $\langle\tau_1 - |k_1|^2\rangle$ and $\langle\tau_2 - |k_2|^2\rangle$ is $\langle\tau + |k|^2\rangle$ (resp. $\langle\tau_i - |k_i|^2\rangle$,

$i = 1, 2$). We infer from the relation

$$\tau + |k|^2 - \sum_{i=1}^{2}(\tau_i - |k_i|^2) = |k|^2 + \sum_{i=1}^{2}|k_i|^2$$

that

$$\langle k\rangle^2 + \sum_{i=1}^{2}\langle k_i\rangle^2 \le C\left(\langle \tau + |k|^2\rangle + \sum_{i=1}^{2}\langle \tau_i - |k_i|^2\rangle\right). \tag{A.34}$$

Let us begin with the region A_0. (A.34) gives, with $0 < \varepsilon < \inf\{\frac{1}{2}(\frac{1}{2} - |s|), 2(b - |s|)\}$ and $-b' := \frac{1}{2}(\frac{1}{2} - s) + \varepsilon < \frac{1}{2}$

$$\langle k\rangle^{\frac{1}{2}+s} \prod_{i=1}^{2} \langle k_i\rangle^{-s+\varepsilon} \le C\langle \tau + |k|^2\rangle^{-b'}.$$

The contribution in (A.33) due to A_0 is therefore bounded by

$$\begin{aligned}
&C\left\| \langle k\rangle^{-\frac{1}{2}} \int_{\tau_1+\tau_2=\tau} \sum_{k_1+k_2=k} \langle k_i\rangle^{-\varepsilon}\langle \tau_i - |k_i|^2\rangle^{-b}|f_i| \right\|_{L^2_{k,\tau}}\\
&\quad = C\left\| \langle k\rangle^{-\frac{1}{2}} \int_{\tau_1+\tau_2=\tau} \sum_{k_1+k_2=k} \langle k_i\rangle^{s-\varepsilon}|\mathcal{F}\bar{v}_i| \right\|_{L^2_{k,\tau}}\\
&\quad = C\left\| \prod_{i=1}^{2} J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar{v}_i| \right\|_{L^2_t H^{-\frac{1}{2}}_x}\\
&\quad \le C\left\| \prod_{i=1}^{2} J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\,\bar{v}_i| \right\|_{L^2_t L^q_x}, \quad q > \frac{4}{3}\\
&\quad \le C\prod_{i=1}^{2} \|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\,\bar{v}_i|\|_{L^4_t L^{2q}_x}, \quad q > \frac{4}{3}\\
&\quad \le C\prod_{i=1}^{2} \|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar{v}_i|\|_{X^-_{\varepsilon,b}}\\
&\quad \le C\prod_{i=1}^{2} \|v_i\|_{X_{s,b}},
\end{aligned}$$

where we used the fact that $L^q(\mathbb{T}^2) \subset H^{-\frac{1}{2}}(\mathbb{T}^2)$ for $q > 4/3$ (by dualizing the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{T}^2) \subset L^p(\mathbb{T}^2)$ for $p < 4$), Hölder inequality, and (A.2)–(A.3). We also used the notation

$$\|u\|_{X^-_{s,b}} = \left(\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}^2} \langle k\rangle^{2s}\langle \tau - |k|^2\rangle^{2b}|\mathcal{F}u(k,\tau)|^2 d\tau\right)^{\frac{1}{2}} = \|\bar{u}\|_{X_{s,b}}$$

borrowed from Ref. 16. It remains to estimate the contributions in (A.33) due to the regions A_1 and A_2. By symmetry, we can consider only the region A_1. In A_1, since $-s+\frac{\varepsilon}{2}<b$, we have that

$$\langle k_2\rangle^{-s+\varepsilon}\langle k_1\rangle^{-s}\le C\langle \tau_1-|k_1|^2\rangle^{-s+\frac{\varepsilon}{2}}\le C\langle \tau_1-|k_1|^2\rangle^{b}$$

and therefore the contribution in (A.33) is bounded by

$$\left\|\langle k\rangle^s\langle \tau+|k|^2\rangle^{b'}\int_{\tau_1+\tau_2=\tau}\sum_{k_1+k_2=k}|f_1|\langle k_2\rangle^{-\varepsilon}\langle \tau_2-|k_2|^2\rangle^{-b}|f_2|\right\|_{L^2_{k,\tau}}$$

$$=C\|\mathcal{F}^{-1}|f_1|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar v_2|\|_{X_{s,b'}}.$$

By (A.1)–(A.2) with $-s>1/3$ and $-b'$ chosen sufficiently close to $\frac12$, we have that

$$X_{-s,-b'}\subset L^6(\mathbb{R};L^6(\mathbb{T}^2)),\quad \text{hence}\quad L^{\frac65}(\mathbb{R};L^{\frac65}(\mathbb{T}^2))\subset X_{s,b'}.$$

It follows that

$$\begin{aligned}\|\mathcal{F}^{-1}|f_1|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar v_2|\|_{X_{s,b'}} &\le C\|\mathcal{F}^{-1}|f_1|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar v_2|\|_{L_t^{\frac65}L_x^{\frac65}}\\ &\le C\|\mathcal{F}^{-1}|f_1|\|_{L_t^{2}L_x^{2}}\|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar v_2|\|_{L_t^{3}L_x^{3}}\\ &\le C\|\bar v_1\|_{X^-_{s,b}}\|J^{s-\varepsilon}\mathcal{F}^{-1}|\mathcal{F}\bar v_2|\|_{X^-_{\varepsilon,b}}\\ &\le C\|v_1\|_{X_{s,b}}\|v_2\|_{X_{s,b}},\end{aligned}$$

where we used Hölder inequality and (A.2)–(A.3) with $p=q=3$. This completes the proof of (3.18).

To derive (3.19) from (3.18), we consider two functions u_1,u_2 in $X_{0,b}(\Omega)\subset X_{s,b}(\Omega)$, and consider their odd extensions v_1,v_2 to $(-\pi,\pi)^2$; i.e. $v_i(\epsilon_1x_1,\epsilon_2x_2)=\epsilon_1\epsilon_2u_i(x_1,x_2)$ for $x=(x_1,x_2)\in\Omega$ and $\epsilon_i=\pm1$. Note that $v_1,v_2\in X_{0,b}$ and that $\bar u_1\bar u_2=(\bar v_1\bar v_2)_{|\Omega}$. For any function $w=\sum_{k\in\mathbb{N}^2}\int_{\mathbb{R}}\mathcal{F}w(k,\tau)e^{i\tau t}\cos(k_1x_1)\cos(k_2x_2)d\tau$, we set

$$\|w\|^2_{X_{s,b}(\Omega)_N}=\sum_{k\in\mathbb{N}^2}\int_{\mathbb{R}}\langle \tau+|k|^2\rangle^{2b}\langle k\rangle^{2s}|\mathcal{F}w(k,\tau)|^2d\tau.$$

The Bourgain space $X_{s,b}(\Omega)_N$ (with Neumann boundary conditions) is defined as the space of w's for which the norm $\|w\|_{X_{s,b}(\Omega)_N}$ is finite. Since the function $\bar v_1\bar v_2$ is even with respect to both x_1 and x_2, we have that

$$\|\bar u_1\bar u_2\|_{X_{s,b'}(\Omega)_N}\sim C\|\bar v_1\bar v_2\|_{X_{s,b'}}\le C\|v_1\|_{X_{s,b}}\|v_2\|_{X_{s,b}}\le C\|u_1\|_{X_{s,b}(\Omega)}\|u_2\|_{X_{s,b}(\Omega)}.$$

We claim that $X_{s,b}(\Omega)=X_{s,b}(\Omega)_N$ for $|s|<1/2$ and $|b|\le1$. Note first that this is true for $|s|<\frac12$ and $b=0$, since

$$X_{s,0}(\Omega)=L^2(\mathbb{R};H^s(\Omega))=X_{s,0}(\Omega)_N.$$

The claim is also true for $|s|<1/2$ and $b=1$, since

$$u\in X_{s,1}(\Omega)\Longleftrightarrow u\in X_{s,0}(\Omega)\quad\text{and}\quad iu_t+\Delta u\in X_{s,0}(\Omega)$$

and since a similar criterion may be written for $X_{s,1}(\Omega)_N$. The claim is therefore true for $|s| < 1/2$ and $0 \leq b \leq 1$ by interpolation, and for $|s| < 1/2$ and $|b| \leq 1$ by duality. (3.19) follows for $u_1, u_2 \in X_{0,b}(\Omega)$, and also for $u_1, u_2 \in X_{s,b}(\Omega)$ by density. This completes the proof of Proposition 3.3. □

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