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Bounded State Reconstruction Error for LPV Systems With Estimated Parameters

Gilles Millerioux, Lionel Rosier, Gerard Bloch, and Jamal Daafouz

***Abstract*—This note deals with the state reconstruction of a class of discrete-time systems with time-varying parameters. While usually, the parameters are assumed to be online available and exactly known, the special and realistic situation when the parameters are known with a finite accuracy is considered. The main objective of the note is to show that, despite of the resulting mismatch between the true system and the model, the state reconstruction error boundedness can be guaranteed and an explicit bound can be derived. The proof is based upon the concept of input-to-state stability.**

***Index Terms*—Estimated parameters, input-to-state stability (ISS), ISS Lyapunov function, polytopic observers.**

I. Introduction

In this note, the state reconstruction for linear parameter varying (LPV) discrete-time systems is considered. While for such a class of systems, the parameters are usually assumed to be exactly known, we consider here that the parameters are estimated in the sense that they are available with only some degree of accuracy. It may also correspond to the case where bounded disturbances on the dynamics and/or the measurements act on the system. The problem under study is the impact of the parameter estimation error on the state reconstruction error. In particular, we wonder whether there exists a guarantee of the boundedness. The answer is not trivial. Indeed, it can be shown that the effect of a bounded estimation error is similar to the effect of a bounded unknown exogenous input acting on the system. And yet, it is well known that a bounded disturbance may drive to infinity a nonlinear system [1]. In this note, it is proved that such a guarantee exists and an explicit bound is derived by using the concept of input-to-state stability (ISS), a notion introduced by Sontag in [2] (see also [3] and the references therein). For discrete-time nonlinear systems, the reader can refer to [4]–[6]. Some works with quite similar issues for the continuous case can be found in [7] with a neural-network-based approach or in [1] where bounded disturbances are considered. We mention that the problem under study differs from the one involving adaptive approaches where the goal is to simultaneously estimate the state and the parameters. The corresponding design often requires the use of a global state space diffeomorphism such that, in the new coordinates, the nonlinearities are restricted to be functions of available signals and the system becomes linear with respect to both state and parameters [8], [9]. Here, the parameters are known with a given accuracy and no transformation on the LPV system is carried out.

The layout of the note is the following. In Section II, the state reconstruction error equation is established from the consideration that the time-varying parameters are not exactly estimated. The motivation of using a polytopic observer as a state reconstructor and a special parameter dependent Lyapunov function, called polyquadratic Lyapunov function, is carried out. The main contribution of the note lies in Section III. The state reconstruction error is proved to be bounded despite of the estimation error through the concept of ISS. Finally, the results are illustrated through an example borrowed from the chaos synchronization problem. **Notation:** $\mathbb{R}^n$, the real n-vectors; M^T, the transpose of the matrix M; $\lambda_{\min}(M), \lambda_{\max}(M)$, the minimum and maximum eigenvalue of the real matrix $M = M^T$, $\|x_k\|$, the usual Euclidean norm $\sqrt{x_k^T x_k}$ of the vector x_k; $\|x\|_\infty$, the supremum norm $\sup_{k \geq 0} \|x_k\|$ of a discrete sequence x; $\|M\|$, the spectral norm $\sqrt{\lambda_{\max}(M^T M)}$ of the matrix M.

II. Problem Statement

We concentrate on the class of LPV discrete-time systems with state-space realization

$$\begin{cases} x_{k+1} = A(\rho_k)x_k + Bu_k \\ y_k = Cx_k + Du_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^m, u_k \in \mathbb{R}^r, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times r}$. Some usual assumptions and considerations for LPV systems are recalled. In particular, A is of class C^1 with respect to the entries of a L-dimensional time-varying parameter vector $\rho_k = (\rho_k^1, \ldots, \rho_k^L)^T$ which is bounded. For a general parameter dependence of the system and a general parameter dependent Lyapunov function, it is known [10] that the design of controllers or observers may lead to a convex but infinitely constrained problem. Thus, one usually must resort to "gridding" the range of all admissible values of the parameter in order to obtain a finite set of constraints. To overpass it, a solution consists in carrying out a polytopic decomposition. Indeed, since ρ_k is bounded, A lies in a compact set which can always be embedded in a polytope, that is

$$A(\rho_k) = \sum_{i=1}^{N} \xi_k^i(\rho_k) A_i \tag{2}$$

where the A_i's correspond to the vertices of the convex hull $\mathbf{Co}\{A_1, \ldots, A_N\}$. The ξ_k's belong to the compact set $\mathcal{S} = \{\mu_k \in \mathbb{R}^N, \mu_k = (\mu_k^1, \ldots, \mu_k^N)^T, \mu_k^i \geq 0 \forall i$ and $\sum_{i=1}^N \mu_k^i = 1\}$ and they can always be expressed as functions of class C^1 with respect to

Manuscript received December 13, 2002; revised November 3, 2003 and April 7, 2004. Recommended by Associate Editor W. Kang.

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Digital Object Identifier 10.1109/TAC.2004.832669

the ρ_k's. Such a decomposition turns the design problems into the resolution of a finite set of constraints involving only the vertices of the convex hull, as it will be seen later.

In this note, we focus on the situation where the true parameter ρ_k is only known with a given degree of accuracy and thus fulfills $\|\rho_k - \hat{\rho}_k\|_\infty < \Delta$. Obviously, it includes the case where $\rho_k = \rho^*$, a constant value. When this parameter depends on the output, the problem is well-posed for the admissible initial states and inputs for which the discrete trajectory $x(k, x_0)$ is bounded, that is $\|x\|_\infty < \infty$. Some similar problems related to nonlinear identification can be found in [7], [11] with the specificity that a learning machine approximates the whole dynamics and not just the parameters.

For the reconstruction of the state x_k, the following so-called polytopic observer is proposed:

$$\begin{cases} \hat{x}_{k+1} = A(\hat{\rho}_k)\hat{x}_k + Bu_k + L(\hat{\rho}_k)(y_k - \hat{y}_k) \\ \hat{y}_k = C\hat{x}_k + Du_k \end{cases} \tag{3}$$

with

$$A(\hat{\rho}_k) = \sum_{i=1}^{N} \hat{\xi}_k^i(\hat{\rho}_k)\hat{A}_i \tag{4}$$

and $\hat{\xi}_k \in \mathcal{S}$, $\hat{A}_i \in \mathbf{Co}\{\hat{A}_1, \ldots, \hat{A}_N\}$ meaning that $A(\hat{\rho}_k)$ must also evolve in a polytope. L is a time-varying gain defined by $L(\hat{\rho}_k) = \sum_{i=1}^{N} \hat{\xi}_k^i(\hat{\rho}_k)L_i$. The L_i's are some constant gains to be computed. The motivation of such an observer stems from the fact that, for the polytopic decomposition (2) and a perfect estimation corresponding to $\hat{\rho}_k = \rho_k$ and so $\mathbf{Co}\{\hat{A}_1, \ldots, \hat{A}_N\} = \mathbf{Co}\{A_1, \ldots, A_N\}$, a global convergence of the state reconstruction error is obtained. This has been shown in [12] and [13] from which the strict necessary background is recalled.

On one hand, from (1) and (3), it is easy to see that for $\rho_k = \hat{\rho}_k$, the state reconstruction error $\epsilon_k \triangleq x_k - \hat{x}_k$ is governed by the dynamics

$$\epsilon_{k+1} = \mathcal{A}(\rho_k)\epsilon_k \tag{5}$$

with $\mathcal{A}(\rho_k) = \sum_{i=1}^{N} \xi_k^i(\rho_k)(A_i - L_iC)$. Let mention that an additional extra-term E can be added in an affine way to the dynamics of x_k and so on $\hat{x}_k$ without modifying (5).

On the other hand, let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a function defined by $V(z_k, \xi_k) = z_k^T \mathcal{P}_k z_k$ with $\mathcal{P}_k = \sum_{i=1}^{N} \xi_k^i P_i$ and $\xi_k \in \mathcal{S}$. Following similar details as in [14], it can be shown that, if the following set of linear matrix inequalities is feasible:

$$\begin{bmatrix} P_i & A_i^T G_i^T - C^T F_i^T \\ G_i A_i - F_i C & G_i^T + G_i - P_j \end{bmatrix} > 0 \quad \forall (i,j) \in \{1, \ldots, N\} \times \{1, \ldots, N\} \tag{6}$$

where the positive–definite matrices P_i, the matrices F_i and G_i are unknown, then the time-varying gain $L(\rho_k) = \sum_{i=1}^{N} \xi_k^i L_i$ with $L_i = G_i^{-1}F_i$, ensures

$$V(\epsilon_{k+1}, \xi_{k+1}) - V(\epsilon_k, \xi_k) = \epsilon_k^T(\mathcal{A}^T \mathcal{P}_{k+1}\mathcal{A} - \mathcal{P}_k)\epsilon_k < 0 \tag{7}$$

where $\mathcal{P}_k = \sum_{i=1}^{N} \xi_k^i P_i$. As a result, V acts as a parameter dependent Lyapunov function, called poly-quadratic Lyapunov function, and (7) is sufficient for global asymptotic stability of (5). Note that the formulation (6) differs from the one encountered in [12] and [13], but is strictly equivalent. The reason is that (6) will be more suited to cope with the specificity of the problem considered here.

Now, the situation when $\|\rho_k - \hat{\rho}_k\|_\infty < \Delta$ with $\Delta \neq 0$ is considered. In this case, (5) does no longer hold and turns into

$$\epsilon_{k+1} = \mathcal{A}(\hat{\rho}_k)\epsilon_k + v_k \tag{8}$$

where it can easily be seen that $v_k = (A(\rho_k) - A(\hat{\rho}_k))x_k$.

The main objective of this note is to show that the boundedness (in the sense of the supremum norm) of the resulting state reconstruction error is guaranteed. Besides, the goal is to derive an explicit bound in terms of the estimation error bound Δ. The ISS concept is used. The definition is recalled in the discrete-time case.

Definition 1 [5]: System (8) is said to be ISS if there exist a $\mathcal{KL}$[1] function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a $\mathcal{K}$ function γ such that, for each input sequence v fulfilling $\|v\|_\infty < \infty$ and each $\epsilon_0 \in \mathbb{R}^n$, the discrete trajectory associated with the initial condition ϵ_0 and the input v fulfills

$$\|\epsilon_k\| \leq \beta(\|\epsilon_0\|, k) + \gamma(\|v\|_\infty) \qquad \forall k. \tag{9}$$

III. Error Bounding

The main result of this note is a direct consequence of next lemma.

Lemma 1: The Lyapunov function V ensuring the poly-quadratic stability of (8) when $v_k = 0$ is an ISS-Lyapunov function for (8); that is, there exist two positive quantities α_1 and α_2 such that

$$V(\epsilon_{k+1}, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) \leq -\alpha_1\|\epsilon_k\|^2 + \alpha_2\|v_k\|^2 \qquad \forall k \qquad \forall \hat{\xi}_k \in \hat{\mathcal{S}}. \tag{10}$$

Proof: For any $z_k \in \mathbb{R}^n$ and for all $j = 1, \ldots, N$, the following inequalities hold:

$$\min_{1\leq i\leq N} \lambda_{\min}(P_i)\|z_k\|^2 \leq z_k^T P_j z_k \leq \max_{1\leq i\leq N} \lambda_{\max}(P_i)\|z_k\|^2 \qquad \forall k.$$

For each $j = 1, \ldots, N$, multiply the aforementioned inequality by $\hat{\xi}_k^j$ and sum. This leads to

$$\min_{1\leq i\leq N} \lambda_{\min}(P_i)\|z_k\|^2 \leq z_k^T \mathcal{P}_k z_k \leq \max_{1\leq i\leq N} \lambda_{\max}(P_i)\|z_k\|^2 \qquad \forall \hat{\xi}_k \in \mathcal{S} \qquad \forall k \tag{11}$$

Thus, we have

$$c_1\|z_k\|^2 \leq V(z_k, \hat{\xi}_k) \leq c_2\|z_k\|^2 \qquad \forall z_k \in \mathbb{R}^n \qquad \forall \hat{\xi}_k \in \mathcal{S} \qquad \forall k \tag{12}$$

with $c_1 = \min_{1\leq i\leq N} \lambda_{\min}(P_i)$ and $c_2 = \max_{1\leq i\leq N} \lambda_{\max}(P_i)$ as best possible constants.

[1]A function $\gamma : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$.
A function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{KL}$ function if, for each $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$ function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

Besides, it can be shown (see details in the Appendix) that there exists a strictly positive quantity, namely

$$c_3 = \min_{1 \leq i \leq N, 1 \leq j \leq N} \lambda_{\min}(P_i - (\hat{A}_i - L_i C)^T P_j (\hat{A}_i - L_i C))$$

such that

$$V(\mathcal{A}\epsilon_k, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) \leq -c_3 \|\epsilon_k\|^2 \quad \forall \epsilon_k \in \mathbb{R}^n \quad \forall (\hat{\xi}_k, \hat{\xi}_{k+1}) \in \mathcal{S}^2 \quad (13)$$

where $\mathcal{A} = \mathcal{A}(\hat{\rho}_k)$. Since by virtue of (13), $c_3\|\epsilon_k\|^2 \leq V(\epsilon_k, \hat{\xi}_k)$, we obtain the inequality $0 < c_3 \leq c_1 \leq c_2$. From (8), the difference $V(\epsilon_{k+1}, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k)$ can be expressed as follows:

$$V(\epsilon_{k+1}, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) = V(\mathcal{A}\epsilon_k, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) + V(v_k, \hat{\xi}_{k+1}) + 2v_k^T \mathcal{P}_{k+1}\mathcal{A}\epsilon_k.$$

By using (12) and (13), we infer that

$$V(\epsilon_{k+1}, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) \leq -c_3\|\epsilon_k\|^2 + c_2\|v_k\|^2 + 2\|v_k\| \cdot \|\mathcal{P}_{k+1}\| \cdot \|\mathcal{A}\| \cdot \|\epsilon_k\| \quad (14)$$

with $\|\mathcal{A}\| = \|\sum_{i=1}^N \hat{\xi}_k^i (\hat{A}_i - L_i C)\|$. Furthermore, one has $\|\sum_{i=1}^N \hat{\xi}_k^i (\hat{A}_i - L_i C)\| \leq \sum_{i=1}^N \hat{\xi}_k^i \|\hat{A}_i - L_i C\| \leq \max_{1 \leq i \leq N} \|\hat{A}_i - L_i C\|$ and $\|\mathcal{P}_{k+1}\| = \|\sum_{i=1}^N \hat{\xi}_{k+1}^i P_i\| \leq \max_{1 \leq i \leq N} \|P_i\|$. Defining the strictly positive quantity $c_4 = (\max_{1 \leq i \leq N} \|\hat{A}_i - L_i C\|) \cdot (\max_{1 \leq i \leq N} \|P_i\|)$, from the previous inequalities, one obtains

$$2\|v_k\| \cdot \|\mathcal{P}_{k+1}\| \cdot \|\mathcal{A}\| \cdot \|\epsilon_k\| \leq 2c_4\|v_k\| \cdot \|\epsilon_k\|. \quad (15)$$

By invoking the well-known inequality $2ab \leq \delta a^2 + \delta^{-1} b^2 \forall (a,b) \in \mathbb{R}^2$ and $\forall \delta > 0$, (15) turns into

$$2\|v_k\| \cdot \|\mathcal{P}_{k+1}\| \cdot \|\mathcal{A}\| \cdot \|\epsilon_k\| \leq \delta\|\epsilon_k\|^2 + \delta^{-1} c_4^2 \|v_k\|^2. \quad (16)$$

Finally, from (16), (14) turns into

$$V(\epsilon_{k+1}, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) \leq -(c_3 - \delta)\|\epsilon_k\|^2 + \left(c_2 + \delta^{-1} c_4^2\right)\|v_k\|^2 \quad (17)$$

that is (10) with $\alpha_1 = c_3 - \delta, \alpha_2 = c_2 + \delta^{-1} c_4^2$ and the constraint $\delta \in]0, c_3[$. □

Theorem 1: (8) is ISS, that is there exist a $\mathcal{KL}$ function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a strictly positive quantity α_3 such that

$$\|\epsilon_k\| \leq \beta(\|\epsilon_0\|, k) + \alpha_3\|v\|_\infty \qquad \forall k. \quad (18)$$

Proof: First, we observe that the supremum norm of the sequence v can be expressed as the product of two bounded terms and so $\|v\|_\infty < \infty$. Indeed, $\|v\|_\infty \leq c_0\|\rho_k - \hat{\rho}_k\|_\infty \cdot \|x\|_\infty$ where c_0 denotes some Lipschitz constant for the function A of class C^1 on Θ and $\hat{\Theta}$.

Besides, from (12) and (17) of Lemma 1, one has

$$V(\epsilon_{k+1}, \hat{\xi}_{k+1}) \leq \left(1 - c_2^{-1}(c_3 - \delta)\right) V(\epsilon_k, \hat{\xi}_k) + \left(c_2 + \delta^{-1} c_4^2\right)\|v_k\|^2 \quad (19)$$

with the constraint $1 - c_2^{-1}(c_3 - \delta) < 1$. Letting $h = 1 - c_2^{-1}(c_3 - \delta)$ and applying the Gronwall lemma in the discrete-time case, we obtain

$$\begin{aligned} V(\epsilon_k, \hat{\xi}_k) &\leq h^k V(\epsilon_0, \hat{\xi}_0) + \left(c_2 + \delta^{-1} c_4^2\right) \sum_{l=0}^{k-1} h^{k-l-1} \|v_l\|^2 \\ &\leq h^k V(\epsilon_0, \hat{\xi}_0) + \left(c_2 + \delta^{-1} c_4^2\right) \frac{1}{1-h} \|v\|_\infty^2. \end{aligned}$$

Finally, by using again (12), substituting $1 - c_2^{-1}(c_3 - \delta)$ to h and taking the square root, the main inequality is obtained

$$\|\epsilon_k\| \leq \sqrt{\frac{c_2}{c_1}} \left(1 - \frac{c_3 - \delta}{c_2}\right)^{k/2} \|\epsilon_0\| + \sqrt{\frac{c_2 + \delta^{-1} c_4^2}{c_1} \cdot \frac{c_2}{c_3 - \delta}} \cdot \|v\|_\infty. \quad (20)$$

This inequality completes the proof according to the definition of ISS. The proof is constructive in the sense that it provides both the function β and the quantity α_3 which explicitly bounds the state reconstruction error in the steady state. □

Remark 1: It is worth noticing that the bound $\alpha_3\|v\|_\infty$ is not linear with respect to Δ. Indeed, in spite of the fact that the expression of $\|v\|_\infty$ is linear with respect to Δ, it is not the same for $\|\epsilon_k\|$ since α_3 depends on the quantities c_1, c_2, c_3, and c_4, which depend on $\hat{\rho}_k$ and so on Δ in a nonlinear way.

Remark 2: It is interesting to mention that ISS is preserved when additional bounded (in the sense of the supremum norm) disturbances w_k^d on the dynamics or w_k^m on the measurement of y_k are considered. Indeed, it can be easily seen that (8) holds with $v_k = (A(\rho_k) - A(\hat{\rho}_k))x_k + w_k^d - L(\hat{\rho}_k)w_k^m$ while $\|v\|_\infty < \infty$ is still true.

IV. Illustrative Example

Chaos synchronization of nonlinear systems is an interesting and open problem of the automatic control field [15]. A large number of papers is concerned with observer-based chaos synchronization approaches to deal in particular with a noisy context. References have voluntarily not been incorporated since detailing the topic is beyond the scope of the note. For our illustrative example, we consider a chaos synchronization problem involving the well-known two-dimensional chaotic Henon map. This map can be described in the form (1) with the state space matrices: $B = \mathbf{0}, D = \mathbf{0}$ since it is an autonomous map, $C = [1\ 0]$ which corresponds to a transmitted signal $y_k = x_k^{(1)}, E = [1\ 0]^T$ which corresponds to the constant part of the affine description, as mentioned at the beginning of this note, and

$$A(x_k) = \begin{bmatrix} -1.4x_k^{(1)} & 1 \\ q & 0 \end{bmatrix}.$$

For $q = 0.3$, the motion exhibited by this map is known to be chaotic and the corresponding attractor is depicted in Fig. 1(A). Our goal is to assess the impact of a bounded disturbance w_k^m acting on the signal y_k, coupling in a unidirectional way the chaotic system and a so-called "response" system. Actually, both systems should ideally synchronize each other from the scalar signal y_k. It is assumed that the disturbance is uniformly distributed in the range -0.0025 and 0.0025. As usual for synchronization problems, the "response" system is chosen to have an observer structure. Since the Henon map is viewed here as an LPV

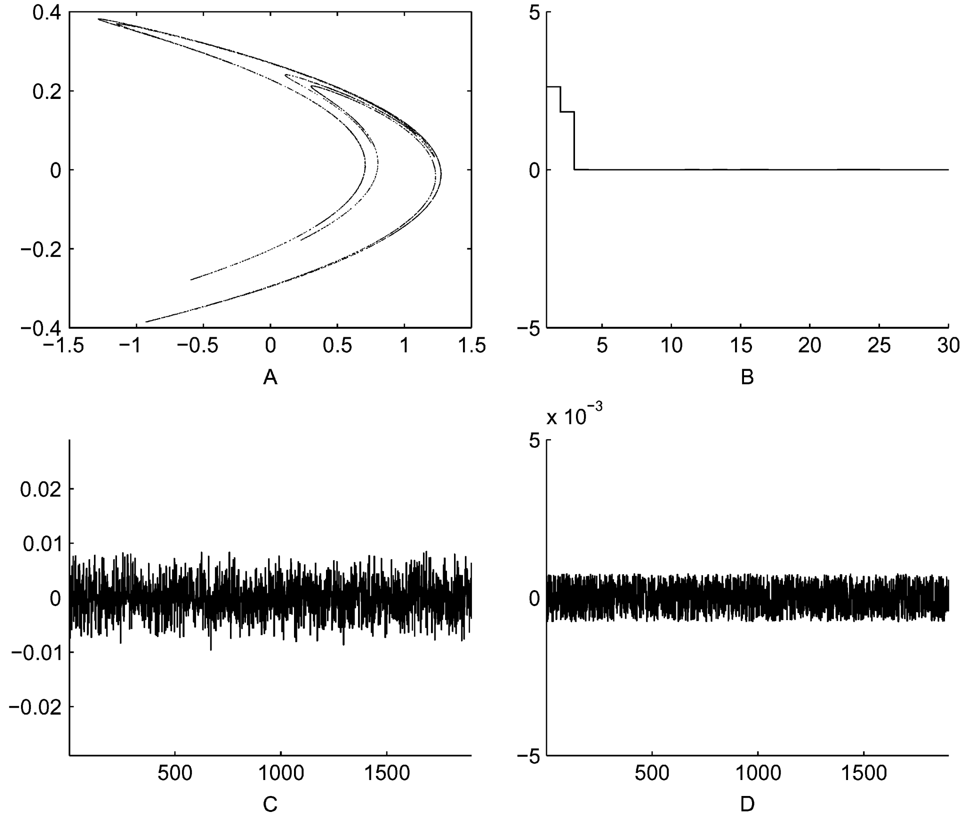

Fig. 1. (A) Chaotic attractor of the Henon map. (B) $\|\varepsilon_k\|$ with respect to k. (C) and (D) Each component of ϵ_k in the steady state.

system, the polytopic observer (3) is proposed to achieve the synchronization. Indeed, it can be stressed that this synchronization issue fulfills all the required assumptions for assessing the impact of the disturbance with the previous theoretical developments.

H1) Setting $\rho_k = -1.4x_k^{(1)}$ and $\hat{\rho}_k = -1.4y_k$, the problem is well-posed since the discrete trajectory $x(k, x_0)$, for any initial state x_0 lying in the chaotic attractor, is bounded, that is $\|x\|_\infty < \infty$. From a simple numerical study, it is inferred that $\|x\|_\infty = 1.3401$. Thus, $\|\rho\|_\infty$ and $\|\hat{\rho}\|_\infty$ are also bounded. Moreover, one has $\|\rho_k - \hat{\rho}_k\|_\infty < \Delta$ with $\Delta = 1.4\|w_k^m\|_\infty = 0.007$.

H2) According to Remark 2, this situation corresponds to $v_k = (A(\rho_k) - A(\hat{\rho}_k))x_k - L(\hat{\rho}_k)w_k^m$.

H3) Since $\hat{\rho}_k$ is bounded, it can be embedded in a polytope and, thus, $A(\hat{\rho}_k)$ can be described in a polytopic way with a corresponding convex hull $\mathbf{Co}\{\hat{A}_1, \hat{A}_2\}$. $A(\hat{\rho}_k)$ takes values between two vertices

$$\hat{A}_1 = \begin{bmatrix} -1.7850 & 1 \\ 0.3 & 0 \end{bmatrix} \quad \hat{A}_2 = \begin{bmatrix} 1.7995 & 1 \\ 0.3 & 0 \end{bmatrix}$$

The gains of the observer have been designed from the solution of the set of linear matrix inequalities (6)

$$L_1 = \begin{bmatrix} -1.7878 \\ 0.3 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1.7982 \\ 0.3 \end{bmatrix}.$$

The computation of the bound of the state reconstruction error involved in (20) gives $\alpha_3\|v\|_\infty = 0.1657$. The Euclidean norm of the reconstruction error ϵ_k during the transient is presented In Fig. 1(B). The numerical computation of $\|\epsilon_k\|$ in the steady state shows that the norm is always less than 0.01 and so, less than $\alpha_3\|v\|_\infty$. In Fig. 1(C) and (D), the steady state is depicted for each of the components of ϵ_k, showing that the reconstruction error is bounded. It is consistent with the theoretical results.

V. Concluding Remarks

The boundedness of the reconstruction error for LPV discrete-time systems involving parameters estimated with a bounded error has been investigated. It has been shown that the dynamics of the state reconstruction error is also bounded and an explicit bound has been derived from the concept of ISS. The proof is based on a special parameter dependent Lyapunov function called polyquadratic which plays the role of an ISS Lyapunov function. The result holds when bounded disturbances on both the dynamics and the measurements act on the system. In the near future, the issue of incorporating an adaptive estimation of the time-varying quantity, the minimization of the bound and a strict analysis of the sensitivity of this bound with respect to estimation error will be considered.

Appendix

If V is a Lyapunov function ensuring the polyquadratic stability of (8) when $v_k = 0$, then for all $\epsilon_k \in \mathbb{R}^n$, and for all $(\hat{\xi}_k, \hat{\xi}_{k+1}) \in \mathcal{S}^2$

$$V(\mathcal{A}\epsilon_k, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) = \epsilon_k^T(\mathcal{A}^T\mathcal{P}_{k+1}\mathcal{A} - \mathcal{P}_k)\epsilon_k < 0$$

with $V(z_k, \xi_k) = z_k^T\mathcal{P}_k z_k, \mathcal{A} = \sum_{i=1}^N \hat{\xi}_k^i(\hat{A}_i - L_iC)$ and $\mathcal{P}_k = \sum_{i=1}^N \hat{\xi}_k^i P_i$, the P_i's resulting from the solution of (6) after replacing A_i by $\hat{A}_i$.

It follows that $V(\mathcal{A}\epsilon_k, \hat{\xi}_{k+1}) - V(\epsilon_k, \hat{\xi}_k) \leq -c_3\|\epsilon_k\|^2$, where c_3 denotes the nonnegative constant $c_3 \triangleq \inf_{(\hat{\xi}_k, \hat{\xi}_{k+1}) \in \mathcal{S}^2} \lambda_{\min}(\mathcal{P}_k - \mathcal{A}^T\mathcal{P}_{k+1}\mathcal{A})$. Notice that $c_3 > 0$, as $\mathcal{S}^2$ is a compact set and the map

$(\hat{\xi}_k, \hat{\xi}_{k+1}) \mapsto \lambda_{\min}(\mathcal{P}_k - \mathcal{A}^T \mathcal{P}_{k+1} \mathcal{A})$ is continuous. The following equalities hold:

$$\begin{aligned} c_3 &= \inf_{\epsilon_k \in \mathbb{R}^n, \|\epsilon_k\|=1} \inf_{\hat{\xi}_{k+1} \in \mathcal{S}} \inf_{\hat{\xi}_k \in \mathcal{S}} \epsilon_k^T (\mathcal{P}_k - \mathcal{A}^T \mathcal{P}_{k+1} \mathcal{A}) \epsilon_k \\ &= \inf_{\epsilon_k \in \mathbb{R}^n, \|\epsilon_k\|=1} \inf_{\hat{\xi}_{k+1} \in \mathcal{S}} \inf_{\hat{\xi}_k \in \mathcal{S}} \left\{ \epsilon_k^T \left(\sum_{i=1}^N \hat{\xi}_k^i P_i \right) \epsilon_k \right. \\ &\quad - \epsilon_k^T \left(\left(\sum_{i=1}^N \hat{\xi}_k^i (\hat{A}_i - L_i C)^T \right) \cdot \left(\sum_{j=1}^N \hat{\xi}_{k+1}^j P_j \right) \right. \\ &\quad \left. \left. \cdot \left(\sum_{l=1}^N \hat{\xi}_k^l (\hat{A}_l - L_l C) \right) \right) \epsilon_k \right\}. \end{aligned}$$

Define the canonical basis of $\mathbb{R}^n : \mathcal{E} = \{(1,0,\ldots,0),\ldots,(0,0,\ldots,1)\}$. Furthermore, for a fixed $\epsilon_k \in \mathbb{R}^n$ and a fixed $\hat{\xi}_{k+1} \in \mathcal{S}$, define the function $f : \mathcal{S} \to \mathbb{R}$ by $f(\hat{\xi}_k) = f_1(\hat{\xi}_k) - f_2(\hat{\xi}_k)$ with $f_1(\hat{\xi}_k) = \sum_{i=1}^N \hat{\xi}_k^i P_i, f_2(\hat{\xi}_k) = (\sum_{i=1}^N \hat{\xi}_k^i (\hat{A}_i - L_i C)^T) \cdot (\sum_{j=1}^N \hat{\xi}_{k+1}^j P_j) \cdot (\sum_{l=1}^N \hat{\xi}_k^l (\hat{A}_l - L_l C))$. We claim that

$$\inf_{\hat{\xi}_k \in \mathcal{S}} f(\hat{\xi}_k) = \inf_{\hat{\xi}_k \in \mathcal{E}} f(\hat{\xi}_k). \tag{21}$$

Indeed, clearly, f_1 is linear and so concave. f_2 is a positive (hence convex) quadratic form. Hence, the function $-f_2$ is concave. As a consequence, f is concave as a sum of two concave functions and (21) holds. Consequently, the following equalities hold:

$$\begin{aligned} c_3 &= \inf_{\epsilon_k \in \mathbb{R}^n, \|\epsilon_k\|=1} \inf_{\hat{\xi}_{k+1} \in \mathcal{S}} \inf_{\hat{\xi}_k \in \mathcal{E}} \epsilon_k^T (\mathcal{P}_k - \mathcal{A}^T \mathcal{P}_{k+1} \mathcal{A}) \epsilon_k \\ &= \inf_{\epsilon_k \in \mathbb{R}^n, \|\epsilon_k\|=1} \inf_{\hat{\xi}_k \in \mathcal{E}} \inf_{\hat{\xi}_{k+1} \in \mathcal{S}} \epsilon_k^T (\mathcal{P}_k - \mathcal{A}^T \mathcal{P}_{k+1} \mathcal{A}) \epsilon_k \\ &= \inf_{\epsilon_k \in \mathbb{R}^n, \|\epsilon_k\|=1} \inf_{\hat{\xi}_k \in \mathcal{E}} \inf_{\hat{\xi}_{k+1} \in \mathcal{E}} \epsilon_k^T (\mathcal{P}_k - \mathcal{A}^T \mathcal{P}_{k+1} \mathcal{A}) \epsilon_k \\ &= \inf_{(\hat{\xi}_k, \hat{\xi}_{k+1}) \in \mathcal{E}^2} \lambda_{\min}(\mathcal{P}_k - \mathcal{A}^T \mathcal{P}_{k+1} \mathcal{A}). \end{aligned}$$

The first equation results from what has been claimed based on concavity property, the second equation stems from the fact that "inf" operator is commutative, the third equation is explained by the affine dependence on $\hat{\xi}_{k+1}$ and so concavity. Thus, $c_3 = \min_{1 \le i \le N, 1 \le j \le N} \lambda_{\min}(P_i - (\hat{A}_i - L_i C)^T P_j (\hat{A}_i - L_i C))$.

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Robust Hurwitz Stability Test for Linear Systems With Uncertain Commensurate Time Delays

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***Abstract*—This note deals with the robust Hurwitz stability of a class of linear systems with uncertain commensurate time delays, which is formulated as the robust Hurwitz stability of a nonpolytopic family of quasi-polynomials, a NP-hard problem from the viewpoint of computational complexity. On the basis of the "Edge Theorem" and Sylvester resultant elimination, a necessary and sufficient condition is developed for the robust stability of the whole family, and it yields an effective testing procedure. An illustrative example is given to show the effectiveness of the method.**

***Index Terms*—Nonpolytope, quasi-polynomial, resultant elimination, robust stability, time delay.**

I. INTRODUCTION

This note focuses on the robust Hurwitz stability of a class of linear dynamical systems with commensurate delays, described by

$$\dot{\mathbf{x}}(t) = \sum_{i=0}^{l} \mathbf{A}_i(\mathbf{q}) \mathbf{x}(t - k_i \tau), \qquad \mathbf{x} \in \mathcal{R}^n \tag{1}$$

where $0 = k_0 \tau < k_1 \tau \le k_2 \tau \le \cdots \le k_l \tau$ are the time delays, $\mathbf{A}_i(\mathbf{q})$ are constant matrices depending on a parametric vector $\mathbf{q} =$

Manuscript received June 6, 2003; revised December 14, 2004. Recommended by Associate Editor A. Datta. This work was supported by the National Natural Science Foundation of China under Grants 10372116 and 50135030 and in part by the Ministry of Education under Grant GG-130-10287-1593. This work has benefited from the Alexander von Humboldt Foundation, Germany, which supported Z. H. Wang's visit to the Institute of Mathematics, University of Cologne, Germany.

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Digital Object Identifier 10.1109/TAC.2004.832677