

NULL CONTROLLABILITY OF A PARABOLIC SYSTEM WITH A CUBIC COUPLING TERM*

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Abstract. We consider a system of two parabolic equations with a forcing term present in one equation and a cubic coupling term in the other one. We prove that the system is locally null controllable.

Key words. null controllability, parabolic system, nonlinear coupling, Carleman estimate, return method

AMS subject classifications. 35K40, 93B05

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1. Introduction. The control of coupled parabolic systems is a challenging issue, which has attracted the interest of the control community in the last decade. Let Ω be a nonempty connected bounded subset of \mathbb{R}^N of class C^2 . Let ω be a nonempty open subset of Ω . In [2, 1], the authors identified sharp conditions for the control of systems of the form

$$(1.1) \quad w_t = (D\Delta + A)w + Bh,$$

where $w = (w_1, \dots, w_N) : \Omega \rightarrow \mathbb{R}^N$ is the state to be controlled, $h = h(t, \cdot) : \Omega \rightarrow \mathbb{R}^M$ is the control input supported in ω , and $D : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$, and $B : \mathbb{R}^M \rightarrow \mathbb{R}^N$ are linear maps. In general, the rank of B is less than N , so that the controllability of the full system depends strongly on the (linear) coupling present in the system. See also [9, 11] for the Stokes system.

Here, we are concerned with the control of systems of semilinear parabolic systems in which the coupling occurs through *nonlinear* terms only. More precisely, we study the control properties of systems of the form

$$(1.2) \quad \begin{cases} u_t - \Delta u = g(u, v) + h1_\omega & \text{in } (0, T) \times \Omega, \\ v_t - \Delta v = u^3 + Rv & \text{in } (0, T) \times \Omega, \\ u = 0, \quad v = 0 & \text{in } (0, T) \times \partial\Omega, \end{cases}$$

where $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function of class C^∞ vanishing at $(0, 0) \in \mathbb{R} \times \mathbb{R}$, R is a given real number, and 1_ω is the characteristic function of ω . This is a control

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system where, at time $t \in [0, T]$, the state is $(u(t, \cdot), v(t, \cdot)) : \Omega \rightarrow \mathbb{R}^2$ and the control is $h(t, \cdot) : \Omega \rightarrow \mathbb{R}$.

The goal of this paper is to prove the local null controllability of system (1.2). Our main result is as follows.

THEOREM 1. *There exists $\delta > 0$ such that, for every $(u_0, v_0) \in L^\infty(\Omega)^2$ satisfying*

$$\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} < \delta,$$

there exists a control $h \in L^\infty((0, T) \times \Omega)$ such that the solution $(u, v) \in L^\infty([0, T] \times \Omega)^2$ of the Cauchy problem

$$(1.3) \quad \begin{cases} u_t - \Delta u = g(u, v) + h1_\omega & \text{in } (0, T) \times \Omega, \\ v_t - \Delta v = u^3 + Rv & \text{in } (0, T) \times \Omega, \\ u = 0, \quad v = 0 & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot), \quad v(0, \cdot) = v_0(\cdot) & \text{in } \Omega \end{cases}$$

satisfies

$$(1.4) \quad u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0 \quad \text{in } \Omega.$$

Let us give a system from chemistry to which our result applies. A reaction-diffusion system describing a reversible chemical reaction (see [4, 5, 10]) takes the form

$$(1.5) \quad u_t = \Delta u - ak(u^k - v^m) \quad \text{in } (0, T) \times \Omega,$$

$$(1.6) \quad v_t = \Delta v + bk(u^k - v^m) \quad \text{in } (0, T) \times \Omega,$$

together with homogeneous Neumann boundary conditions. In (1.5)–(1.6), a and b denote some positive numbers, and k and m are positive integers. The corresponding reversible chemical reaction reads $kA \rightleftharpoons mB$. Incorporating a forcing term $1_\omega h$ in (1.5), we obtain a system of the form (1.3) when $k = 3$ and $m = 1$, so that Theorem 1 may be applied. (In fact, Theorem 1 deals with Dirichlet homogeneous boundary conditions, but the proof we give here can easily be adapted to deal with homogeneous Neumann boundary conditions as well, and a scaling argument shows that one may assume without loss of generality that $b = 1/3$.)

Remark 2. In Theorem 1, it is not possible to replace u^3 by u^2 . Indeed, by the maximum principle, for every $(u_0, v_0) \in L^\infty(\Omega)^2$ and for every $h \in L^\infty((0, T) \times \Omega)$, the solution $(u, v) \in L^\infty([0, T] \times \Omega)^2$ of the Cauchy problem

$$\begin{cases} u_t - \Delta u = g(u, v) + h1_\omega & \text{in } (0, T) \times \Omega, \\ v_t - \Delta v = u^2 + Rv & \text{in } (0, T) \times \Omega, \\ u = 0, \quad v = 0 & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot), \quad v(0, \cdot) = v_0(\cdot) & \text{in } \Omega, \end{cases}$$

if it exists, satisfies

$$v(T, \cdot) \geq v^*(T, \cdot) \quad \text{in } \Omega,$$

where $v^* \in L^\infty([0, T] \times \Omega)$ is the solution of the linear Cauchy problem

$$\begin{cases} v_t^* - \Delta v^* = Rv^* & \text{in } (0, T) \times \Omega, \\ v^* = 0 & \text{in } (0, T) \times \partial\Omega, \\ v^*(0, \cdot) = v_0(\cdot) & \text{in } \Omega. \end{cases}$$

In particular, by the (strong) maximum principle, if $v_0 \geq 0$ and $v_0 \neq 0$, then $v(T, \cdot) > 0$ in Ω . L. Robbiano asked one of the authors whether the result in Theorem 1, with u^3 still replaced by u^2 , could be true if we consider *complex-valued* functions. The following result, whose proof is sketched in the appendix, shows that this is indeed the case.

THEOREM 3. *There exists $\delta > 0$ such that, for every $(u_0, v_0) \in L^\infty(\Omega; \mathbb{C})^2$ satisfying*

$$\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} < \delta,$$

there exists a control $h \in L^\infty((0, T) \times \Omega; \mathbb{C})$ such that the solution $(u, v) \in L^\infty([0, T] \times \Omega; \mathbb{C})^2$ of the Cauchy problem

$$(1.7) \quad \begin{cases} u_t - \Delta u = g(u, v) + h1_\omega & \text{in } (0, T) \times \Omega, \\ v_t - \Delta v = u^2 + Rv & \text{in } (0, T) \times \Omega, \\ u = 0, \quad v = 0 & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot), \quad v(0, \cdot) = v_0(\cdot) & \text{in } \Omega \end{cases}$$

satisfies

$$(1.8) \quad u(T, \cdot) = 0 \quad \text{and} \quad v(T, \cdot) = 0 \quad \text{in } \Omega.$$

When trying to prove a local null controllability result, the first thing to do is to look at the null controllability of the linearized control system around 0. Here, the linearized control system reads as

$$(1.9) \quad \begin{cases} u_t - \Delta u = \partial_u g(0, 0)u + \partial_v g(0, 0)v + h1_\omega & \text{in } (0, T) \times \Omega, \\ v_t - \Delta v = Rv & \text{in } (0, T) \times \Omega, \\ u = 0, \quad v = 0 & \text{in } (0, T) \times \partial\Omega. \end{cases}$$

Clearly, the control h has no influence on v , and if $v(0, \cdot) \neq 0$, then $v(T, \cdot) \neq 0$. Hence the linearized control system (1.9) is not null controllable and this strategy cannot be applied to prove Theorem 1.

Our proof of Theorem 1 relies on the return method, a method introduced in [6] for a stabilization problem and in [7] for the controllability of the Euler equations of incompressible fluids (see [8, Chapter 6] and the references therein for other applications of this method). Applied to the control system (1.2), it consists in looking for a trajectory $((\bar{u}, \bar{v}), \bar{h})$ of the control system (1.2) such that the following hold:

- (i) it goes from $(0, 0)$ to $(0, 0)$, i.e., $\bar{u}(0, \cdot) = \bar{v}(0, \cdot) = \bar{u}(T, \cdot) = \bar{v}(T, \cdot) = 0$;
- (ii) the linearized control system around that trajectory is null controllable.

With this trajectory and a suitable fixed point theory at hand, one can hope to get the null controllability stated in Theorem 1. We shall see that this is indeed the case.

This paper is organized as follows. Section 2 is devoted to the construction of the trajectory $((\bar{u}, \bar{v}), \bar{h})$. In section 3, using some Carleman inequality, we prove that the linearized control system around $((\bar{u}, \bar{v}), \bar{h})$ is null controllable (subsection 3.1). Next, we deduce the local null controllability around this trajectory by using the Kakutani fixed point theorem (subsection 3.2). The appendix contains a sketch of the proof of Theorem 3.

2. Construction of the trajectory $((\bar{u}, \bar{v}), \bar{h})$. Let us define $Q := (0, T) \times \Omega$. The goal of this section is to prove the existence of $\bar{u} \in C^\infty(Q)$, $\bar{v} \in C^\infty(Q)$, and $\bar{h} \in C^\infty(Q)$ such that

$$(2.1) \quad \text{the supports of } \bar{u}, \bar{v}, \text{ and } \bar{h} \text{ are compact and included in } (0, T) \times \omega,$$

$$(2.2) \quad \bar{u}_t - \Delta \bar{u} = g(\bar{u}, \bar{v}) + \bar{h} \quad \text{in } Q,$$

$$(2.3) \quad \bar{v}_t - \Delta \bar{v} = \bar{u}^3 + R\bar{v} \quad \text{in } Q,$$

$$(2.4) \quad \bar{u} \neq 0.$$

The existence of such $\bar{u} \in C^\infty(Q)$, $\bar{v} \in C^\infty(Q)$, and $\bar{h} \in C^\infty(Q)$ follows from the following theorem.

THEOREM 4. *Let $\rho > 0$ and R be two constants. Then there exist two functions $V : (t, x) \in \mathbb{R} \times \mathbb{R}^N \mapsto V(t, x) \in \mathbb{R}$ and $K : (t, x) \in \mathbb{R} \times \mathbb{R}^N \mapsto K(t, x) \in \mathbb{R}$ such that*

$$(2.5) \quad V \in C^\infty(\mathbb{R} \times \mathbb{R}^N), \text{ and } V(t, x) = 0 \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \text{ with } \max(|t|, |x|) \geq \rho,$$

$$(2.6) \quad K \in C^\infty(\mathbb{R} \times \mathbb{R}^N), \text{ and } K(t, x) = 0 \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \text{ with } \max(|t|, |x|) \geq \rho,$$

$$(2.7) \quad K \neq 0,$$

$$(2.8) \quad V_t = \Delta V + RV + K^3.$$

Indeed, let $x_0 \in \omega$, and let $\rho > 0$ be small enough so that

$$\max \left(\left| t - \frac{T}{2} \right|, |x - x_0| \right) \leq \rho \Rightarrow (t, x) \in (0, T) \times \omega.$$

Then, it suffices to define $\bar{u} \in C^\infty(Q)$, $\bar{v} \in C^\infty(Q)$, and $\bar{h} \in C^\infty(Q)$ by

$$\begin{aligned} \bar{u}(t, x) &:= K \left(t - \frac{T}{2}, x - x_0 \right), \quad \bar{v}(t, x) := V \left(t - \frac{T}{2}, x - x_0 \right) \quad \forall (t, x) \in Q, \\ \bar{h} &:= \bar{u}_t - \Delta \bar{u} - g(\bar{u}, \bar{v}) \quad \text{in } Q. \end{aligned}$$

Proof of Theorem 4. Note first that we may assume without loss of generality that $R = 0$. Indeed, setting

$$\tilde{V}(t, x) = e^{-Rt}V(t, x), \quad \tilde{K}(t, x) = e^{-Rt/3}K(t, x),$$

(2.8) is transformed into

$$\tilde{V}_t = \Delta \tilde{V} + \tilde{K}^3.$$

From now on, we assume that $R = 0$. We may also assume that $\rho = 1$. Indeed, if the construction has been done for $\rho = 1$ and $R = 0$, then for any $\rho > 0$ the functions

$$\tilde{V}(t, x) = V(\rho^{-2}t, \rho^{-1}x), \quad \tilde{K}(t, x) = \rho^{-\frac{2}{3}}K(\rho^{-2}t, \rho^{-1}x),$$

with support in $[-\rho^2, \rho^2] \times \{|x| \leq \rho\}$, satisfy the equation $\tilde{V}_t = \Delta \tilde{V} + \tilde{K}^3$. We assume from now on that $R = 0$ and that $\rho = 1$. Let $r = |x|$. We seek a radial function $V(t, x) = v(t, r)$ fulfilling the following properties:

$$(2.9) \quad v \in C^\infty(\mathbb{R} \times \mathbb{R}^+), \quad v(t, r) = 0 \quad \text{for } |t| \geq 1 \text{ or } r \geq 1,$$

$$(2.10) \quad k := \left(v_t - v_{rr} - \frac{N-1}{r} v_r \right)^{\frac{1}{3}} \in C^\infty(\mathbb{R} \times \mathbb{R}^+).$$

The smoothness of V and $K := (V_t - \Delta V)^{\frac{1}{3}}$ at the points $(t, 0)$, $t \in [-1, 1]$, will follow from additional properties of v (see below). As far as the construction of v is concerned, the idea is to have a precise knowledge of the place where k vanishes and a good “behavior” of v near the place where k vanishes to ensure that k is of class C^∞ . For the v we are going to construct, we shall have

$$\begin{aligned} \{(t, r); k(t, r) < 0\} &= \{(t, r); 0 < \lambda(t)/2 < r < \lambda(t)\}, \\ \{(t, r); k(t, r) > 0\} &= \{(t, r); 0 < r < \lambda(t)/2\}. \end{aligned}$$

See Figure 1.

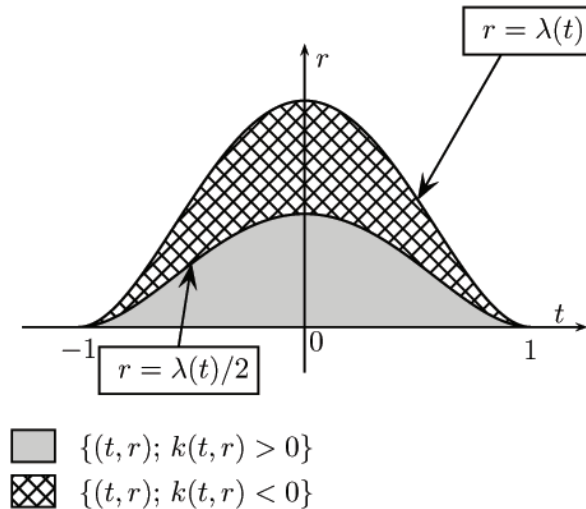


FIG. 1. $\{(t, r); k(t, r) > 0\}$ and $\{(t, r); k(t, r) < 0\}$.

Let us introduce some notation. Let

$$(2.11) \quad \lambda(t) = \varepsilon(1 - t^2)^2, \quad |t| < 1,$$

$$(2.12) \quad f_0(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1, \end{cases}$$

where $\varepsilon > 0$ is a (small) parameter chosen later. We search for v in the form

$$(2.13) \quad v(t, r) = \sum_{i=0}^3 f_i(t)g_i(z),$$

where $z := r/\lambda(t)$, g_0 is defined in Lemma 5 (see below), and the functions $f_i = f_i(t)$, $1 \leq i \leq 3$, and $g_i = g_i(z)$, $1 \leq i \leq 3$, defined during the proof, are in $C^\infty(\mathbb{R})$ and fulfill

$$(2.14) \quad \text{supp } f_i \subset [-1, 1],$$

$$(2.15) \quad \text{supp } g_i \subset \left[\frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} + \frac{\delta}{2} \right].$$

In (2.15), $\delta \in (0, 1/10)$ is a given number. Let us begin with the construction of g_0 .

LEMMA 5. *There exists a function $G \in C^\infty([0, +\infty))$ such that*

$$(2.16) \quad G(z) = \left(z - \frac{1}{2} \right)^3 \quad \text{for } \frac{1}{2} - \delta < z < \frac{1}{2} + \delta,$$

$$(2.17) \quad \left(z - \frac{1}{2} \right) G(z) > 0 \quad \text{for } 0 < z < 1, \quad z \neq \frac{1}{2},$$

and such that the solution g_0 to the Cauchy problem

$$(2.18) \quad g_0''(z) + \frac{N-1}{z} g_0'(z) = G(z), \quad z > 0,$$

$$(2.19) \quad g_0(1) = g_0'(1) = 0$$

satisfies

$$(2.20) \quad g_0(z) = 1 - z^2 \quad \text{if } 0 < z < \delta,$$

$$(2.21) \quad g_0(z) = e^{-\frac{1}{1-z^2}} \quad \text{if } 1 - \delta < z < 1,$$

$$(2.22) \quad g_0(z) = 0 \quad \text{if } z \geq 1.$$

Proof of Lemma 5. Note first that, by (2.18), (2.20)–(2.22),

$$(2.23) \quad G(z) = \begin{cases} -2N & \text{if } 0 < z < \delta, \\ [-2N(1-z^2)^{-2} - 8z^2(1-z^2)^{-3} + 4z^2(1-z^2)^{-4}] e^{-\frac{1}{1-z^2}} & \text{if } 1 - \delta < z < 1, \\ 0 & \text{if } z \geq 1. \end{cases}$$

Hence only the values of G on $[\delta, \frac{1}{2} - \delta]$ and on $[\frac{1}{2} + \delta, 1 - \delta]$ remain to be defined. Let $G \in C^\infty(0, +\infty)$ be any function satisfying (2.16) and (2.23), and denote by g_0 the solution of (2.18)–(2.19). Clearly, (2.21)–(2.22) are satisfied. Finally, it is clear that (2.20) holds if and only if $g_0(0^+) = 1$ and $g_0'(0^+) = 0$. Note that (2.18) may be written as

$$(2.24) \quad \frac{1}{z^{N-1}} (z^{N-1} g_0')' = G.$$

Using (2.19), this gives upon integration

$$(2.25) \quad -z^{N-1} g_0'(z) = \int_z^1 s^{N-1} G(s) ds.$$

This imposes the condition

$$(2.26) \quad \int_0^1 s^{N-1} G(s) ds = 0.$$

Note that if (2.26) holds, then, by (2.25),

$$(2.27) \quad g'_0(z) = \frac{1}{z^{N-1}} \int_0^z s^{N-1} G(s) ds,$$

which, combined with (2.23), yields

$$g'_0(z) = -2z, \quad 0 < z < \delta,$$

and $g'_0(0^+) = 0$. Integrating (2.27) on $[z, 1]$ and using (2.19), (2.23), (2.26), and an integration by parts, we obtain, for $0 < z < \delta$,

$$(2.28) \quad g_0(z) = - \int_z^1 \frac{1}{y^{N-1}} \left(\int_0^y s^{N-1} G(s) ds \right) dy$$

$$(2.29) \quad = \begin{cases} \frac{1}{2-N} \int_z^1 y G(y) dy - \frac{2}{2-N} z^2 & \text{if } N \neq 2, \\ \int_z^1 y(\ln y) G(y) dy - 2z^2 \ln z & \text{if } N = 2. \end{cases}$$

Then $g_0(0^+) = 1$, provided that

$$(2.30) \quad \begin{cases} \int_0^1 y G(y) dy = 2 - N & \text{if } N \neq 2, \\ \int_0^1 y(\ln y) G(y) dy = 1 & \text{if } N = 2. \end{cases}$$

It is then an easy exercise to extend G on $[\delta, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1 - \delta]$ in such a way that G is smooth and (2.17), (2.26), and (2.30) are satisfied. This concludes the proof of Lemma 5. \square

Let us turn now to the definition of the functions f_i and g_i for $1 \leq i \leq 3$. Let (t, z) range over $(-1, 1) \times (0, 1)$, so that (t, r) ranges over the domain

$$\mathcal{O} := \{(t, r); -1 < t < 1, 0 < r < \lambda(t)\}.$$

Differentiating in (2.13), we obtain

$$v_t = \sum_{i=0}^3 \left(\dot{f}_i g_i - \frac{\dot{\lambda}}{\lambda} f_i z g_i^{(1)} \right),$$

$$v_{rr} + \frac{N-1}{r} v_r = \sum_{i=0}^3 \lambda^{-2} f_i \left(g_i^{(2)} + \frac{N-1}{z} g_i^{(1)} \right),$$

where $\dot{f}_i := df_i/dt$ and $g_i^{(j)} := d^j g_i/dz^j$. Let us introduce the function $\mathcal{V} = \mathcal{V}(t, z)$ defined by

$$(2.31) \quad \mathcal{V} := \lambda^2 \left[v_{rr} + \frac{N-1}{r} v_r - v_t \right]$$

$$= \sum_{i=0}^3 \left[f_i \left(g_i^{(2)} + \frac{N-1}{z} g_i^{(1)} \right) + z \lambda \dot{f}_i g_i^{(1)} - \lambda^2 \dot{f}_i g_i \right].$$

We aim to define f_i and g_i so that

$$\begin{aligned} \mathcal{V} = \mathcal{V}_z = \mathcal{V}_{zz} = 0 & \quad \text{for } -1 < t < 1, \quad z = \frac{1}{2}, \\ \mathcal{V}_{zzz} \geq cf_0 & \quad \text{for } -1 < t < 1, \quad z = \frac{1}{2}, \end{aligned}$$

for some constant $c > 0$. By (2.16), (2.18), and (2.31),

$$\begin{aligned} \mathcal{V}\left(\cdot, \frac{1}{2}\right) &= \frac{1}{2}\lambda\dot{\lambda}f_0g_0^{(1)}\left(\frac{1}{2}\right) - \lambda^2\dot{f}_0g_0\left(\frac{1}{2}\right) \\ &+ \sum_{i=1}^3 \left[f_i\left(g_i^{(2)}\left(\frac{1}{2}\right) + 2(N-1)g_i^{(1)}\left(\frac{1}{2}\right)\right) + \frac{1}{2}\lambda\dot{\lambda}f_i g_i^{(1)}\left(\frac{1}{2}\right) \right. \\ &\quad \left. - \lambda^2\dot{f}_i g_i\left(\frac{1}{2}\right) \right]. \end{aligned}$$

We impose the condition

$$(2.32) \quad g_i^{(j)}\left(\frac{1}{2}\right) = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2, \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 1 \leq i \leq 3, \quad 0 \leq j \leq 2.$$

It follows that

$$\mathcal{V}\left(\cdot, \frac{1}{2}\right) = \frac{1}{2}\lambda\dot{\lambda}f_0g_0^{(1)}\left(\frac{1}{2}\right) - \lambda^2\dot{f}_0g_0\left(\frac{1}{2}\right) + f_1.$$

The function f_1 is then defined by

$$(2.33) \quad f_1 := -\frac{1}{2}\lambda\dot{\lambda}f_0g_0^{(1)}\left(\frac{1}{2}\right) + \lambda^2\dot{f}_0g_0\left(\frac{1}{2}\right),$$

so that

$$(2.34) \quad \mathcal{V}\left(\cdot, \frac{1}{2}\right) = 0 \quad \text{on } (-1, 1).$$

Differentiating with respect to z in (2.31) yields, once more using (2.18),

$$(2.35) \quad \begin{aligned} \mathcal{V}_z &= f_0G^{(1)} + z\lambda\dot{\lambda}f_0g_0^{(2)} + (\lambda\dot{\lambda}f_0 - \lambda^2\dot{f}_0)g_0^{(1)} \\ &+ \sum_{i=1}^3 \left[f_i\left(g_i^{(3)} + \frac{N-1}{z}g_i^{(2)} - \frac{N-1}{z^2}g_i^{(1)}\right) + z\lambda\dot{\lambda}f_i g_i^{(2)} + (\lambda\dot{\lambda}f_i - \lambda^2\dot{f}_i)g_i^{(1)} \right]. \end{aligned}$$

We infer from (2.16), (2.32), and (2.35) that

$$\begin{aligned} \mathcal{V}_z\left(\cdot, \frac{1}{2}\right) &= \frac{1}{2}\lambda\dot{\lambda}f_0g_0^{(2)}\left(\frac{1}{2}\right) + (\lambda\dot{\lambda}f_0 - \lambda^2\dot{f}_0)g_0^{(1)}\left(\frac{1}{2}\right) \\ &+ \left(\sum_{i=1}^3 f_i g_i^{(3)}\left(\frac{1}{2}\right)\right) + f_1\left(2(N-1) + \frac{1}{2}\lambda\dot{\lambda}\right). \end{aligned}$$

We impose the condition

$$(2.36) \quad g_i^{(3)}\left(\frac{1}{2}\right) = \begin{cases} 1 & \text{if } i = 2, \\ 0 & \text{if } i \in \{1, 3\} \end{cases}$$

and define f_2 as

$$(2.37) \quad f_2 := - \left[f_1 \left(2(N-1) + \frac{1}{2} \lambda \dot{\lambda} \right) + \frac{1}{2} \lambda \dot{\lambda} f_0 g_0^{(2)} \left(\frac{1}{2} \right) + (\lambda \dot{\lambda} f_0 - \lambda^2 \dot{f}_0) g_0^{(1)} \left(\frac{1}{2} \right) \right].$$

It follows that

$$(2.38) \quad \mathcal{V}_z \left(\cdot, \frac{1}{2} \right) = 0 \quad \text{on } (-1, 1).$$

Differentiating (2.35) with respect to z , we get

$$(2.39) \quad \begin{aligned} \mathcal{V}_{zz} = & f_0 G^{(2)} + z \lambda \dot{\lambda} f_0 g_0^{(3)} + (2 \lambda \dot{\lambda} f_0 - \lambda^2 \dot{f}_0) g_0^{(2)} \\ & + \sum_{i=1}^3 \left[f_i \left(g_i^{(4)} + \frac{N-1}{z} g_i^{(3)} - 2 \frac{N-1}{z^2} g_i^{(2)} + 2 \frac{N-1}{z^3} g_i^{(1)} \right) + z \lambda \dot{\lambda} f_i g_i^{(3)} \right. \\ & \left. + (2 \lambda \dot{\lambda} f_i - \lambda^2 \dot{f}_i) g_i^{(2)} \right], \end{aligned}$$

which, together with (2.16), (2.32), and (2.36), leads to

$$(2.40) \quad \begin{aligned} \mathcal{V}_{zz} \left(\cdot, \frac{1}{2} \right) = & \frac{1}{2} \lambda \dot{\lambda} f_0 g_0^{(3)} \left(\frac{1}{2} \right) + (2 \lambda \dot{\lambda} f_0 - \lambda^2 \dot{f}_0) g_0^{(2)} \left(\frac{1}{2} \right) \\ & + \left(\sum_{i=1}^3 f_i g_i^{(4)} \left(\frac{1}{2} \right) \right) + 2(N-1) f_2 - 8(N-1) f_1 + \frac{1}{2} \lambda \dot{\lambda} f_2 + (2 \lambda \dot{\lambda} f_1 - \lambda^2 \dot{f}_1). \end{aligned}$$

We impose the condition

$$(2.41) \quad g_i^{(4)} \left(\frac{1}{2} \right) = \begin{cases} 1 & \text{if } i = 3, \\ 0 & \text{if } i \in \{1, 2\} \end{cases}$$

and define f_3 as

$$(2.42) \quad \begin{aligned} f_3 := & - \left[\left(2(N-1) + \frac{1}{2} \lambda \dot{\lambda} \right) f_2 + (2 \lambda \dot{\lambda} - 8(N-1)) f_1 - \lambda^2 \dot{f}_1 \right. \\ & \left. + \frac{1}{2} \lambda \dot{\lambda} f_0 g_0^{(3)} \left(\frac{1}{2} \right) + (2 \lambda \dot{\lambda} f_0 - \lambda^2 \dot{f}_0) g_0^{(2)} \left(\frac{1}{2} \right) \right]. \end{aligned}$$

This gives

$$(2.43) \quad \mathcal{V}_{zz} \left(\cdot, \frac{1}{2} \right) = 0 \quad \text{on } (-1, 1).$$

By (2.16) and (2.39), we have, for $(t, z) \in (-1, 1) \times (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$,

$$\mathcal{V}_{zzz} = 6f_0 + \mathcal{R},$$

where

$$(2.44) \quad \begin{aligned} \mathcal{R} := & z \lambda \dot{\lambda} f_0 g_0^{(4)} + (3 \lambda \dot{\lambda} f_0 - \lambda^2 \dot{f}_0) g_0^{(3)} \\ & + \sum_{i=1}^3 \left[f_i \left(g_i^{(5)} + \frac{N-1}{z} g_i^{(4)} - 3 \frac{N-1}{z^2} g_i^{(3)} + 6 \frac{N-1}{z^3} g_i^{(2)} - 6 \frac{N-1}{z^4} g_i^{(1)} \right) \right. \\ & \left. + z \lambda \dot{\lambda} f_i g_i^{(4)} + (3 \lambda \dot{\lambda} f_i - \lambda^2 \dot{f}_i) g_i^{(3)} \right]. \end{aligned}$$

Let C denote various constants independent of ε , t , and z , which may vary from line to line. We claim that

$$(2.45) \quad |\mathcal{R}| \leq C\varepsilon^2 f_0 \quad \text{for } (t, z) \in (-1, 1) \times \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right).$$

First, we have that

$$|\mathcal{R}| \leq C \left(|\lambda\dot{\lambda}f_0| + \lambda^2|\dot{f}_0| + \sum_{i=1}^3 (|f_i| + |\lambda\dot{\lambda}f_i| + \lambda^2|\dot{f}_i|) \right).$$

Since

$$(2.46) \quad \lambda\dot{\lambda}f_0 = -4\varepsilon^2 t(1-t^2)^3 f_0, \quad \lambda^2\dot{f}_0 = -2\varepsilon^2 t(1-t^2)^2 f_0,$$

one may write for each $i \in \{1, \dots, 3\}$

$$f_i(t) = \varepsilon^2 p_i(t, \varepsilon) f_0(t),$$

where $p_i \in \mathbb{R}[t, \varepsilon]$. Therefore, there exists some constant $C > 0$ such that

$$|\lambda\dot{\lambda}f_0| + \lambda^2|\dot{f}_0| + \sum_{i=1}^3 (|f_i| + \lambda^2|\dot{f}_i|) \leq C\varepsilon^2 f_0,$$

and (2.45) follows. We infer that for ε small enough

$$\mathcal{V}_{zzz} \geq (6 - C\varepsilon^2)f_0 \geq f_0 \quad \text{for } (t, z) \in [-1, 1] \times \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right).$$

In view of the definitions of \mathcal{V} and of f_i for $1 \leq i \leq 3$, we may write

$$(2.47) \quad \mathcal{V}(t, z) = f_0(t) \sum_{j=1}^p P_j(t) k_j(z) =: f_0(t) A(t, z),$$

where $p \geq 1$, $P_j \in \mathbb{R}[t]$, and $k_j \in C^\infty([0, +\infty))$. Since

$$A\left(\cdot, \frac{1}{2}\right) = A_z\left(\cdot, \frac{1}{2}\right) = A_{zz}\left(\cdot, \frac{1}{2}\right) = 0$$

while

$$A_{zzz}(t, z) \geq 1 \quad \text{for } (t, z) \in [-1, 1] \times \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right),$$

we conclude that we can write

$$(2.48) \quad A(t, z) = \left(z - \frac{1}{2}\right)^3 \varphi(t, z) \quad \text{for } t \in [-1, 1], z \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right),$$

where $\varphi \in C^\infty([-1, 1]_t \times (\frac{1}{2} - \delta, \frac{1}{2} + \delta)_z)$ and

$$\varphi(t, z) > 0 \quad \text{for } t \in [-1, 1], z \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right).$$

From (2.15), (2.18), (2.31), and (2.47) we have that, for $t \in [-1, 1]$ and $|z - \frac{1}{2}| \geq \frac{\delta}{2}$,

$$A(t, z) = G + z\lambda\dot{\lambda}g_0^{(1)} - \lambda^2 \frac{f_0}{f_0} g_0,$$

which, combined with (2.46) and (2.17), yields

$$|A(t, z) - G| \leq C\varepsilon^2 |G(z)|.$$

It follows that for $\varepsilon > 0$ small enough

$$(2.49) \quad |A(t, z)| > \frac{1}{2}|G(z)| > 0 \quad \text{for } t \in [-1, 1], z \in \left[0, \frac{1}{2} - \frac{\delta}{2}\right] \cup \left[\frac{1}{2} + \frac{\delta}{2}, 1\right).$$

Gathering (2.48)–(2.49) we obtain that

$$\lambda^{-2}\mathcal{V} = \lambda^{-2}f_0(t)A(t, z) = B(t, z)^3, \quad t \in (-1, 1), \quad z \in [0, 1),$$

for some $B \in C^\infty((-1, 1) \times [0, 1))$. Now define $V(t, x)$ by

$$V(t, x) := v(t, |x|) = \sum_{i=0}^3 f_i(t)g_i\left(\frac{|x|}{\lambda(t)}\right).$$

From (2.15) and (2.20) we have that

$$V(t, x) = f_0(t) \left(1 - \frac{|x|^2}{\lambda^2(t)}\right) \quad \text{for } |x| < \delta|\lambda(t)|.$$

Combined with (2.11), (2.12), and (2.21), this yields

$$(2.50) \quad V \in C^\infty(\mathbb{R} \times \mathbb{R}^N).$$

On the other hand, it follows from (2.15), (2.23), (2.31), (2.47), (2.48), and (2.49) that

$$\lambda^{-2}\mathcal{V} = \lambda^{-2}f_0 \left(z - \frac{1}{2}\right)^3 (1 - z^2)^{-4} e^{-\frac{1}{1-z^2}} \psi(t, z) \quad \text{for } (t, z) \in (-1, 1) \times [0, 1)$$

for some function $\psi \in C^\infty([-1, 1] \times [0, 1])$ with $|\psi(t, z)| \geq \eta > 0$ on $[-1, 1] \times [0, 1]$. We observe that the function

$$(2.51) \quad (\lambda^{-2}\mathcal{V})^{\frac{1}{3}} = (\lambda^{-2}f_0)^{\frac{1}{3}} \left(\frac{r}{\lambda} - \frac{1}{2}\right) \left(1 - \frac{r^2}{\lambda^2}\right)^{-\frac{4}{3}} e^{-\frac{1}{3} \frac{1}{1-\frac{r^2}{\lambda^2}}} \psi^{\frac{1}{3}}\left(t, \frac{r}{\lambda}\right)$$

when extended by 0 for $|t| \geq 1$ or $r \geq \lambda(t)$ is of class C^∞ on $\mathbb{R}_t \times [0, +\infty)_r$. Therefore

$$(\Delta V - V_t)^{\frac{1}{3}}(t, x) = (\lambda^{-2}\mathcal{V})^{\frac{1}{3}}\left(t, \frac{|x|}{\lambda(t)}\right)$$

is of class C^∞ on $\mathbb{R} \times \mathbb{R}^N \setminus ([-1, 1] \times \{0\})$ and on a neighborhood of $(-1, 1) \times \{0\}$ by (2.50) and the fact that

$$(\Delta V - V_t)(t, 0) = -2N\varepsilon^{-2}(1 - t^2)^{-4}f_0(t) + 2t(1 - t^2)^{-2}f_0(t) < 0$$

for $\varepsilon > 0$ small enough. The smoothness of $(v_t - \Delta v)^{1/3}$ near $(\pm 1, 0)$ follows from (2.12) and (2.51). The proof of Theorem 4 is complete. \square

3. Local null controllability around the trajectory $((\bar{u}, \bar{v}), \bar{h})$. We consider the trajectory $((\bar{u}, \bar{v}), \bar{h})$ of the control system (1.2) constructed in section 2. Let $((u, v), h) : Q \rightarrow \mathbb{R}^2 \times \mathbb{R}$, and let $((\zeta_1, \zeta_2), \tilde{h}) := ((u - \bar{u}, v - \bar{v}), h - \bar{h})$. Then $((u, v), h)$ is a trajectory of (1.2) if and only if $((\zeta_1, \zeta_2), \tilde{h})$ fulfills

$$(3.1) \quad \begin{cases} \zeta_{1,t} - \Delta \zeta_1 = G_{11}(\zeta_1, \zeta_2)\zeta_1 + G_{12}(\zeta_1, \zeta_2)\zeta_2 + \tilde{h}1_\omega & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t} - \Delta \zeta_2 = G_{21}(\zeta_1, \zeta_2)\zeta_1 + G_{22}(\zeta_1, \zeta_2)\zeta_2 & \text{in } (0, T) \times \Omega, \\ \zeta_1 = 0, \quad \zeta_2 = 0 & \text{in } (0, T) \times \partial\Omega, \end{cases}$$

where

$$\begin{aligned} G_{11}(\zeta_1, \zeta_2)(t, x) &:= \int_0^1 \frac{\partial g}{\partial u}(\lambda \zeta_1(t, x) + \bar{u}(t, x), \zeta_2(t, x) + \bar{v}(t, x)) d\lambda \\ &= \begin{cases} \frac{g(\zeta_1 + \bar{u}, \zeta_2 + \bar{v}) - g_j(\bar{u}, \zeta_2 + \bar{v})}{\zeta_1}(t, x) & \text{if } \zeta_1(t, x) \neq 0, \\ \partial_u g(\bar{u}(t, x), \zeta_2(t, x) + \bar{v}(t, x)) & \text{if } \zeta_1(t, x) = 0, \end{cases} \end{aligned}$$

$$\begin{aligned} G_{12}(\zeta_1, \zeta_2)(t, x) &:= \int_0^1 \frac{\partial g}{\partial v}(\bar{u}(t, x), \lambda \zeta_2(t, x) + \bar{v}(t, x)) d\lambda \\ &= \begin{cases} \frac{g(\bar{u}, \zeta_2 + \bar{v}) - g(\bar{u}, \bar{v})}{\zeta_2}(t, x) & \text{if } \zeta_2(t, x) \neq 0, \\ \partial_v g(\bar{u}(t, x), \bar{v}(t, x)) & \text{if } \zeta_2(t, x) = 0, \end{cases} \end{aligned}$$

$$\begin{aligned} G_{21}(\zeta_1, \zeta_2) &:= (3\bar{u}^2 + 3\bar{u}\zeta_1 + \zeta_1^2), \\ \text{and } G_{22}(\zeta_1, \zeta_2) &:= R. \end{aligned}$$

Note that

$$(3.2) \quad G_{21}(0, 0) = 3\bar{u}^2.$$

By (2.4) and (3.2), there exist $t_1 \in (0, T)$, $t_2 \in (0, T)$, a nonempty open subset ω_0 of Ω , and $\bar{M} > 0$ such that

$$(3.3) \quad \bar{\omega}_0 \subset \omega, \quad t_1 < t_2,$$

$$(3.4) \quad G_{21}(0, 0)(t, x) \geq \frac{2}{\bar{M}} \quad \forall (t, x) \in (t_1, t_2) \times \omega_0.$$

Increasing $\bar{M} > 0$ if necessary, we may also assume that

$$(3.5) \quad \|G_{ij}(0, 0)\|_{L^\infty(Q)} \leq \frac{\bar{M}}{2} \quad \forall (i, j) \in \{1, 2\}^2.$$

It is therefore natural to study the null controllability of the linear systems

$$(3.6) \quad \begin{cases} \zeta_{1,t} - \Delta \zeta_1 = a_{11}(t, x)\zeta_1 + a_{12}(t, x)\zeta_2 + h1_\omega & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t} - \Delta \zeta_2 = a_{21}(t, x)\zeta_1 + a_{22}(t, x)\zeta_2 & \text{in } (0, T) \times \Omega, \\ \zeta_1 = 0, \quad \zeta_2 = 0 & \text{in } (0, T) \times \partial\Omega \end{cases}$$

under the following assumptions:

$$(3.7) \quad a_{21}(t, x) \geq \frac{1}{\overline{M}} \quad \forall (t, x) \in (t_1, t_2) \times \omega_0,$$

$$(3.8) \quad \|a_{ij}\|_{L^\infty(Q)} \leq \overline{M} \quad \forall (i, j) \in \{1, 2\}^2.$$

We will do this study in subsection 3.1. In subsection 3.2, we deduce from this study the local null controllability around the trajectory $((\bar{u}, \bar{v}), \bar{h})$ and therefore get Theorem 1.

3.1. Null controllability of a family of linear control systems. Let \mathcal{E} be the set of $(a_{11}, a_{12}, a_{21}, a_{22}) \in L^\infty(Q)^4$ such that (3.7) and (3.8) hold. The goal of this subsection is to prove the following lemma.

LEMMA 6. *There exists $C > 0$ such that, for every $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathcal{E}$ and for every $(\alpha_1, \alpha_2) \in L^2(\Omega)^2$, there exists a control $h \in L^\infty(Q)$ satisfying*

$$(3.9) \quad \|h\|_{L^\infty(Q)} \leq C(\|\alpha_1\|_{L^2(\Omega)} + \|\alpha_2\|_{L^2(\Omega)})$$

such that the solution to the Cauchy problem

$$(3.10) \quad \begin{cases} \zeta_{1,t} - \Delta \zeta_1 = a_{11}(t, x)\zeta_1 + a_{12}(t, x)\zeta_2 + h1_\omega & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t} - \Delta \zeta_2 = a_{21}(t, x)\zeta_1 + a_{22}(t, x)\zeta_2 & \text{in } (0, T) \times \Omega, \\ \zeta_1 = 0, \quad \zeta_2 = 0 & \text{in } (0, T) \times \partial\Omega, \\ \zeta_1(0, \cdot) = \alpha_1, \quad \zeta_2(0, \cdot) = \alpha_2 & \text{in } \Omega \end{cases}$$

satisfies

$$(3.11) \quad \zeta_1(T, \cdot) = 0 \quad \text{and} \quad \zeta_2(T, \cdot) = 0.$$

Note that the coefficients a_{jk} are in $L^\infty(Q)$, so that we have existence and uniqueness of solutions of (3.10) in $C^0([0, T]; L^2(\Omega)^2) \cap L^2(0, T; H^1(\Omega)^2)$. In order to prove Lemma 6, we take $h(t, \cdot) = 0$ for every $t \in (0, t_1)$. Note that there then exists $C > 0$ such that, for every $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathcal{E}$ and for every $(\alpha_1, \alpha_2) \in L^2(\Omega)^2$, the solution to the Cauchy problem (3.10) satisfies

$$\|(\alpha_1^*, \alpha_2^*)\|_{L^2(\Omega)^2} \leq C\|(\alpha_1, \alpha_2)\|_{L^2(\Omega)^2},$$

with

$$(\alpha_1^*, \alpha_2^*) := (\zeta_1(t_1, \cdot), \zeta_2(t_1, \cdot)).$$

Then, our goal will be to find $h : (t_1, t_2) \times \Omega \rightarrow \mathbb{R}$ satisfying

$$(3.12) \quad \|h\|_{L^\infty((t_1, t_2) \times \Omega)} \leq C(\|\zeta_1(t_1, \cdot)\|_{L^2(\Omega)} + \|\zeta_2(t_1, \cdot)\|_{L^2(\Omega)})$$

such that the solution (ζ_1, ζ_2) of the Cauchy problem

$$(3.13) \quad \begin{cases} \zeta_{1,t} - \Delta \zeta_1 = a_{11}(t, x)\zeta_1 + a_{12}(t, x)\zeta_2 + h1_\omega & \text{in } (t_1, t_2) \times \Omega, \\ \zeta_{2,t} - \Delta \zeta_2 = a_{21}(t, x)\zeta_1 + a_{22}(t, x)\zeta_2 & \text{in } (t_1, t_2) \times \Omega, \\ \zeta_1 = 0, \quad \zeta_2 = 0 & \text{in } (t_1, t_2) \times \partial\Omega, \\ \zeta_1(t_1, \cdot) = \alpha_1^*, \quad \zeta_2(t_1, \cdot) = \alpha_2^* & \text{in } \Omega \end{cases}$$

satisfies

$$(3.14) \quad \zeta_1(t_2, \cdot) = 0 \quad \text{and} \quad \zeta_2(t_2, \cdot) = 0.$$

Finally, we take $h(t, \cdot) = 0$ for every $t \in (t_2, T)$. Of course, this construction provides a control $h \in L^\infty((0, T) \times \Omega)$ driving the solution of (3.10) to $(0, 0)$ at time $t = T$. In subsection 3.1.1 we prove the existence of $h : (t_1, t_2) \times \Omega \rightarrow \mathbb{R}$ satisfying the required property but with an L^2 -bound instead of (3.12). In subsection 3.1.2 we deal with the condition (3.12). For the sake of simplicity, in these two subsections, we write $(0, T)$ instead of (t_1, t_2) and Q instead of $(t_1, t_2) \times \Omega$.

3.1.1. Controls in L^2 . The goal of this subsection is to prove a null controllability result for the linear control systems (3.6) with L^2 controls.

LEMMA 7. *There exists $C > 0$ such that, for every $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathcal{E}$ and for every $(\alpha_1, \alpha_2) \in L^2(\Omega)^2$, there exists a control $h \in L^2(Q)$ satisfying*

$$\|h\|_{L^2(Q)} \leq C(\|\alpha_1\|_{L^2(\Omega)} + \|\alpha_2\|_{L^2(\Omega)})$$

such that the solution to the Cauchy problem (3.10) satisfies (3.11).

In order to prove Lemma 7, we consider the associated adjoint system

$$(3.15) \quad \begin{cases} -\varphi_{1,t} - \Delta\varphi_1 = a_{11}(t, x)\varphi_1 + a_{21}(t, x)\varphi_2 & \text{in } (0, T) \times \Omega, \\ -\varphi_{2,t} - \Delta\varphi_2 = a_{12}(t, x)\varphi_1 + a_{22}(t, x)\varphi_2 & \text{in } (0, T) \times \Omega, \\ \varphi_1 = 0, \quad \varphi_2 = 0 & \text{in } (0, T) \times \partial\Omega, \\ \varphi_1(T, \cdot) = \varphi_{1,T}(\cdot), \quad \varphi_2(T, \cdot) = \varphi_{2,T}(\cdot) & \text{in } \Omega \end{cases}$$

and set $\varphi := (\varphi_1, \varphi_2)$. For this system, we intend to prove the following observability inequality:

$$(3.16) \quad \int_{\Omega} |\varphi(0, x)|^2 dx \leq C \iint_{(0,T) \times \omega_0} |\varphi_1|^2 dx dt.$$

From estimate (3.16), with $C > 0$ independent of $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathcal{E}$, it is classical to deduce Lemma 7.

Let us recall the following Carleman inequality, proved in [12, Chapter 1], for the heat equation with Dirichlet boundary conditions.

LEMMA 8. *Let $\eta(t) = t^{-1}(T - t)^{-1}$. Let ω_1 be a nonempty open set included in Ω . There exist a constant $C > 0$ and a function $\rho \in C^2(\overline{\Omega}; (0, +\infty))$ such that, for every $z \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and for every $s \geq C$,*

$$(3.17) \quad \begin{aligned} & \iint_{(0,T) \times \Omega} e^{-s\rho(x)\eta(t)} ((s\eta)^3 |z|^2 + s\eta |\nabla z|^2 + (s\eta)^{-1} (|\Delta z|^2 + |z_t|^2)) dx dt \\ & \leq C \left(\iint_{(0,T) \times \Omega} e^{-s\rho(x)\eta(t)} |z_t + \Delta z|^2 dx dt + \iint_{(0,T) \times \omega_1} e^{-s\rho(x)\eta(t)} (s\eta)^3 |z|^2 dx dt \right). \end{aligned}$$

For ω_1 we take a nonempty open set of \mathbb{R}^N whose closure (in \mathbb{R}^N) is included in ω_0 . Unless otherwise specified, we denote by C various positive constants varying from line to line which may depend on $\Omega, \omega_0, \omega_1, T, \rho, \overline{M}$, and other variables which will be specified later. However, they are independent of $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathcal{E}$, $(\varphi_{1,T}, \varphi_{2,T})$, and of other variables which will be specified later.

We start by applying (3.17) to φ_1 and φ_2 (the solution of (3.15)):

$$\begin{aligned} & \iint_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}((s\eta)^3|\varphi|^2 + s\eta|\nabla\varphi|^2 + (s\eta)^{-1}(|\Delta\varphi|^2 + |\varphi_t|^2)) dx dt \\ & \leq C \left(\iint_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}(|a_{11}(t,x)\varphi_1 + a_{21}(t,x)\varphi_2|^2 + |a_{12}(t,x)\varphi_1 \right. \\ & \quad \left. + a_{22}(t,x)\varphi_2|^2) dx dt + \iint_{(0,T)\times\omega_1} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi|^2 dx dt \right) \end{aligned}$$

for every $s \geq C$. Using (3.8) and taking s large enough, we have

$$\begin{aligned} (3.18) \quad & \iint_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}((s\eta)^3|\varphi|^2 + s\eta|\nabla\varphi|^2 + (s\eta)^{-1}(|\Delta\varphi|^2 + |\varphi_t|^2)) dx dt \\ & \leq C \iint_{(0,T)\times\omega_1} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi|^2 dx dt \end{aligned}$$

for every $s \geq C$. Finally, we estimate the local integral of φ_2 . We multiply the first equation in (3.15) by $\chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3\varphi_2$, where

$$(3.19) \quad \chi \in C^2(\bar{\Omega}; [0, +\infty)), \quad \text{the support of } \chi \text{ is included in } \omega_0 \text{ and } \chi = 1 \text{ in } \omega_1.$$

Integrating in $(0, T) \times \omega_0$, this gives

$$\begin{aligned} (3.20) \quad & \iint_{(0,T)\times\omega_0} a_{21}(t,x)\chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi_2|^2 dx dt \\ & = \iint_{(0,T)\times\omega_0} \chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3\varphi_2(-\varphi_{1,t} - \Delta\varphi_1 - a_{11}(t,x)\varphi_1) dx dt. \end{aligned}$$

Thanks to (3.7) and (3.19), the integral in the left-hand side of (3.20) is bounded from below by

$$C^{-1} \iint_{(0,T)\times\omega_1} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi_2|^2 dx dt.$$

Let us now estimate the integral in the right-hand side of (3.20). Let $\varepsilon \in (0, 1)$. From now on the constant $C > 0$ may depend on $\varepsilon \in (0, 1)$. Using (3.8) (for $(i, j) = (1, 1)$), we have that

$$\begin{aligned} (3.21) \quad & \left| \iint_{(0,T)\times\omega_0} \chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3\varphi_2a_{11}(t,x)\varphi_1 dx dt \right| \\ & \leq \varepsilon \iint_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi_2|^2 dx dt + C \iint_{(0,T)\times\omega_0} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi_1|^2 dx dt. \end{aligned}$$

Next, for the time derivative term, we integrate by parts with respect to t . We get

$$\begin{aligned} (3.22) \quad & - \iint_{(0,T)\times\omega_0} \chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3\varphi_2\varphi_{1,t} dx dt \\ & = \iint_{(0,T)\times\omega_0} \chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3\varphi_{2,t}\varphi_1 dx dt \\ & \quad + \iint_{(0,T)\times\omega_0} \chi(x)(e^{-s\rho(x)\eta(t)}(s\eta)^3)_t\varphi_2\varphi_1 dx dt. \end{aligned}$$

Using that $|(e^{-s\rho(x)\eta(t)}(s\eta)^3)_t| \leq C s^4 e^{-s\rho(x)\eta(t)} \eta(t)^5$ and Cauchy–Schwarz’s inequality, we can estimate this term in the following way:

$$\begin{aligned}
 & \left| \iint_{(0,T) \times \omega_0} \chi(x) e^{-s\rho(x)\eta(t)} (s\eta)^3 \varphi_2 \varphi_{1,t} \, dx \, dt \right| \\
 (3.23) \quad & \leq \epsilon \iint_{(0,T) \times \Omega} e^{-s\rho(x)\eta(t)} (s\eta)^{-1} (|\varphi_{2,t}|^2 + (s\eta)^4 |\varphi_2|^2) \, dx \, dt \\
 & \quad + C \iint_{(0,T) \times \omega_0} e^{-s\rho(x)\eta(t)} (s\eta)^7 |\varphi_1|^2 \, dx \, dt
 \end{aligned}$$

for $s \geq C$. Finally, for the integral term with $\Delta\varphi_1$, we integrate by parts twice with respect to x to get

$$\begin{aligned}
 & - \iint_{(0,T) \times \omega_0} \chi(x) e^{-s\rho(x)\eta(t)} (s\eta)^3 \varphi_2 \Delta\varphi_1 \, dx \, dt \\
 & = - \iint_{(0,T) \times \omega_0} \Delta(\chi(x) e^{-s\rho(x)\eta(t)} (s\eta)^3 \varphi_2) \varphi_1 \, dx \, dt.
 \end{aligned}$$

Using that

$$\begin{aligned}
 & |\Delta(\chi(x) e^{-s\rho(x)\eta(t)} (s\eta)^3 \varphi_2)| \\
 & \leq C e^{-s\rho(x)\eta(t)} (s\eta)^3 (|\Delta\varphi_2| + s\eta |\nabla\varphi_2| + (s\eta)^2 |\varphi_2|), \quad (t, x) \in (0, T) \times \omega_0
 \end{aligned}$$

in the previous identity, together with Cauchy–Schwarz’s inequality, we deduce that

$$\begin{aligned}
 & - \iint_{(0,T) \times \omega_0} \chi(x) e^{-s\rho(x)\eta(t)} (s\eta)^3 \varphi_2 \Delta\varphi_1 \, dx \, dt \\
 (3.24) \quad & \leq \epsilon \iint_{(0,T) \times \Omega} e^{-s\rho(x)\eta(t)} (s\eta)^{-1} (|\Delta\varphi_2|^2 + (s\eta)^2 |\nabla\varphi_2|^2 + (s\eta)^4 |\varphi_2|^2) \, dx \, dt \\
 & \quad + C \iint_{(0,T) \times \omega_0} e^{-s\rho(x)\eta(t)} (s\eta)^7 |\varphi_1|^2 \, dx \, dt.
 \end{aligned}$$

Combining inequalities (3.21), (3.23), and (3.24) with (3.18) and (3.20) and taking $\epsilon \in (0, 1)$ small enough, we obtain

$$\begin{aligned}
 & \iint_{(0,T) \times \Omega} e^{-s\rho(x)\eta(t)} ((s\eta)^3 |\varphi|^2 + s\eta |\nabla\varphi|^2 + (s\eta)^{-1} (|\Delta\varphi|^2 + |\varphi_t|^2)) \, dx \, dt \\
 (3.25) \quad & \leq C \iint_{(0,T) \times \omega_0} e^{-s\rho(x)\eta(t)} (s\eta)^7 |\varphi_1|^2 \, dx \, dt
 \end{aligned}$$

for every $s \geq C$. From this estimate and taking into account the dissipation (in time) of the heat system (3.15), one gets for s large enough

$$(3.26) \quad \|\varphi(0, \cdot)\|_{L^2(\Omega)^2}^2 \leq C \iint_{(0,T) \times \omega_0} e^{-s\rho(x)\eta(t)} (s\eta)^7 |\varphi_1|^2 \, dx \, dt,$$

which gives (3.16). Note that the constant C in (3.26) may depend on s at this time. The proof of Lemma 7 is finished. \square

3.1.2. Controls in L^∞ . Let us remark that the proof in this subsection follows ideas of [3]. Let $\varepsilon \in (0, 1)$. In this subsection the constants $C > 0$ do not depend on $\varepsilon \in (0, 1)$. We choose $s > 0$ large enough so that (3.25) (and therefore also (3.26)) holds. Let $(\alpha_1, \alpha_2) \in L^2(\Omega)^2$. Let us consider, for each $\varepsilon > 0$, the extremal problem

$$(3.27) \quad \inf_{h \in L^2((0,T) \times \omega_0)} \frac{1}{2} \iint_{(0,T) \times \omega_0} e^{s\rho(x)\eta(t)} (s\eta)^{-7} |h|^2 dx dt + \frac{1}{2\varepsilon} \|\zeta(T, \cdot)\|_{L^2(\Omega)^2}^2,$$

where $\zeta := (\zeta_1, \zeta_2)$ is the solution of (3.6) satisfying the initial condition

$$(3.28) \quad \zeta_1(0, \cdot) = \alpha_1, \quad \zeta_2(0, \cdot) = \alpha_2.$$

Clearly, there exists a (unique) solution of (3.27) h^ε with $(e^{s\rho(x)\eta(t)}(s\eta)^{-7})^{1/2} h^\varepsilon$ belonging to $L^2((0, T) \times \omega_0)$. We extend h^ε to all of Q by letting $h^\varepsilon := 0$ on $Q \setminus (0, T) \times \omega_0$. Let us call $\zeta^\varepsilon := (\zeta_1^\varepsilon, \zeta_2^\varepsilon)$ the solution of (3.6) associated with h^ε with, again, the initial condition (3.28). The necessary condition of the minimum yields

$$(3.29) \quad \iint_{(0,T) \times \omega_0} e^{s\rho(x)\eta(t)} (s\eta)^{-7} h^\varepsilon h dx dt + \frac{1}{\varepsilon} \int_{\Omega} \zeta^\varepsilon(T) \cdot \zeta(T) dx = 0 \quad \forall h \in L^2((0, T) \times \omega_0),$$

where $\zeta := (\zeta_1, \zeta_2)$ is the solution of

$$(3.30) \quad \begin{cases} \zeta_{1,t} - \Delta \zeta_1 = a_{11}(t, x)\zeta_1 + a_{12}(t, x)\zeta_2 + h1_{\omega_0} & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t} - \Delta \zeta_2 = a_{21}(t, x)\zeta_1 + a_{22}(t, x)\zeta_2 & \text{in } (0, T) \times \Omega, \\ \zeta_1 = 0, \quad \zeta_2 = 0 & \text{on } (0, T) \times \partial\Omega, \\ \zeta_1(0, \cdot) = \zeta_2(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

Let us now introduce $(\varphi_1^\varepsilon, \varphi_2^\varepsilon)$ as the solution of the following homogeneous adjoint system:

$$(3.31) \quad \begin{cases} -\varphi_{1,t}^\varepsilon - \Delta \varphi_1^\varepsilon = a_{11}(t, x)\varphi_1^\varepsilon + a_{21}(t, x)\varphi_2^\varepsilon & \text{in } (0, T) \times \Omega, \\ -\varphi_{2,t}^\varepsilon - \Delta \varphi_2^\varepsilon = a_{12}(t, x)\varphi_1^\varepsilon + a_{22}(t, x)\varphi_2^\varepsilon & \text{in } (0, T) \times \Omega, \\ \varphi_1^\varepsilon = \varphi_2^\varepsilon = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi^\varepsilon(T, \cdot) = -\frac{1}{\varepsilon} \zeta^\varepsilon(T, \cdot) & \text{in } \Omega. \end{cases}$$

Then, the duality properties between φ^ε and ζ provide

$$-\frac{1}{\varepsilon} \int_{\Omega} \zeta^\varepsilon(T) \cdot \zeta(T) dx = \iint_{(0,T) \times \omega_0} h \varphi_1^\varepsilon dx dt,$$

which, combined with (3.29), yields

$$\iint_{(0,T) \times \omega_0} h \varphi_1^\varepsilon dx dt = \iint_{(0,T) \times \omega_0} e^{s\rho(x)\eta(t)} (s\eta)^{-7} h^\varepsilon h dx dt \quad \forall h \in L^2((0, T) \times \omega_0).$$

Consequently, we can identify h^ε :

$$(3.32) \quad h^\varepsilon = e^{-s\rho(x)\eta(t)} (s\eta)^7 \varphi_1^\varepsilon 1_{\omega_0}.$$

From the systems fulfilled by ζ^ε and φ^ε , we find, using (3.32),

$$-\frac{1}{\varepsilon} \|\zeta^\varepsilon(T, \cdot)\|_{L^2(\Omega)^2}^2 = \iint_{(0,T) \times \omega_0} e^{-s\rho(x)\eta(t)} (s\eta)^7 |\varphi_1^\varepsilon|^2 dx dt + \int_\Omega \varphi^\varepsilon(0, \cdot) \cdot \alpha dx,$$

with $\alpha := (\alpha_1, \alpha_2)$. Inequality (3.26) used for φ^ε tells us that

$$\|\varphi^\varepsilon(0, \cdot)\|_{L^2(\Omega)^2}^2 \leq C \iint_{(0,T) \times \omega_0} e^{-s\rho(x)\eta(t)} (s\eta)^7 |\varphi_1^\varepsilon|^2 dx dt,$$

so, once more using (3.32),

$$(3.33) \quad \frac{1}{\varepsilon} \|\zeta^\varepsilon(T, \cdot)\|_{L^2(\Omega)^2}^2 + \|(e^{s\rho(x)\eta(t)}(s\eta)^{-7})^{1/2} h^\varepsilon\|_{L^2((0,T) \times \omega_0)}^2 \leq C \|\alpha\|_{L^2(\Omega)^2}^2.$$

Consequently, we deduce the existence of a control h such that $(e^{s\rho(x)\eta(t)}(s\eta)^{-7})^{1/2} h \in L^2((0, T) \times \omega_0)$ (whose corresponding solution we denote by ζ) such that $\zeta(T, \cdot) = 0$ and

$$(3.34) \quad \|(e^{s\rho(x)\eta(t)}(s\eta)^{-7})^{1/2} h\|_{L^2((0,T) \times \omega_0)} \leq C \|\alpha\|_{L^2(\Omega)^2}.$$

Let us finally bound the L^∞ -norm of the control h^ε . For this, we now develop a bootstrap argument.

• Let

$$\begin{aligned} \psi^{\varepsilon,0} &:= e^{-s\rho(x)\eta(t)/2} (s\eta)^{-1/2} \varphi^\varepsilon, \\ \psi^{\varepsilon,1} &:= e^{-s\rho(x)\eta(t)/2} (s\eta)^{-5/2} \varphi^\varepsilon = \frac{1}{(s\eta)^2} \psi^{\varepsilon,0}. \end{aligned}$$

Let $\mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)$ be the vector space of linear maps from \mathbb{R}^2 into itself. Using (3.31), one easily checks that $\psi^{\varepsilon,1}$ fulfills a backward heat system with a homogeneous Dirichlet boundary condition and a final null condition of the following form:

$$(3.35) \quad \begin{cases} -\psi_t^{\varepsilon,1} - \Delta \psi^{\varepsilon,1} = d_1 & \text{in } (0, T) \times \Omega, \\ \psi^{\varepsilon,1} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi^{\varepsilon,1}(T, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

with

$$(3.36) \quad d_1(t, x) = A_1(t, x) \psi^{\varepsilon,0} + (s\eta)^{-1} \nabla \psi^{\varepsilon,0} \cdot \nabla \rho,$$

where $A_1 \in L^\infty(Q; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2))$ satisfies (see, in particular, (3.8))

$$(3.37) \quad \|A_1\|_{L^\infty(Q; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2))} \leq C.$$

In (3.36) and in the following, we use the notation

$$(\nabla \theta \cdot \nabla \rho)(t, x) := (\nabla \theta_1(t, x) \cdot \nabla \rho(x), \nabla \theta_2(t, x) \cdot \nabla \rho(x)) \quad \text{for } \theta = (\theta_1, \theta_2) : Q \rightarrow \mathbb{R}^2.$$

Thanks to (3.25), (3.32), (3.33), (3.36), and (3.37),

$$(3.38) \quad d_1 \in L^2(Q)^2 \quad \text{and} \quad \|d_1\|_{L^2(Q)^2} \leq C \|\alpha\|_{L^2(\Omega)^2}.$$

For $r \in [1, +\infty]$, let $X_r := L^r(0, T; W^{2,r}(\Omega)^2) \cap W^{1,r}(0, T; L^r(\Omega)^2)$. We denote by $\|\cdot\|_{X_r}$ its usual norm. Let

$$(3.39) \quad p_1 := 2.$$

From (3.35), (3.38), (3.39), and a standard parabolic regularity theorem, we have

$$(3.40) \quad \psi^{\varepsilon,1} \in X_{p_1}, \quad \|\psi^{\varepsilon,1}\|_{X_{p_1}} \leq C\|\alpha\|_{L^2(\Omega)^2}.$$

- For $k \in \mathbb{N} \setminus \{0\}$, let

$$\psi^{\varepsilon,k} := e^{-s\rho(x)\eta(t)/2}(s\eta)^{-1/2-2k}\varphi^\varepsilon = \frac{1}{(s\eta)^{2k}}\psi^{\varepsilon,0}.$$

Let us define, by induction on k , a sequence $(p_k)_{k \in \mathbb{N} \setminus \{0\}}$ of elements of $[2, +\infty]$ by

$$p_k := \begin{cases} \frac{(N+2)p_{k-1}}{N+2-p_{k-1}} & \text{if } p_{k-1} < N+2, \\ 2p_{k-1} & \text{if } p_{k-1} = N+2, \\ +\infty & \text{if } p_{k-1} > N+2. \end{cases}$$

One easily checks the following:

- If $N = 2l$, with $l \in \mathbb{N} \setminus \{0\}$, one has

$$\begin{aligned} p_k &= 2\frac{N+2}{N-2(k-2)} & \text{if } k < l+2, \\ p_{l+2} &= 2(N+2), \\ p_k &= +\infty & \forall k \geq l+3. \end{aligned}$$

- If $N = 2l + 1$, with $l \in \mathbb{N}$, one has

$$\begin{aligned} p_k &= 2\frac{N+2}{N-2(k-2)} & \text{if } k \leq l+2, \\ p_k &= +\infty & \forall k > l+2. \end{aligned}$$

In particular,

$$(3.41) \quad p_k = +\infty \quad \forall k \geq \frac{N}{2} + 3.$$

We now use an induction argument on k . We assume that

$$(3.42) \quad \psi^{\varepsilon,k-1} \in X_{p_{k-1}} \quad \text{and} \quad \|\psi^{\varepsilon,k-1}\|_{X_{p_{k-1}}} \leq C\|\alpha\|_{L^2(\Omega)^2}$$

(now C is allowed to depend on k) and that $\psi^{\varepsilon,k-1}$ fulfills a heat system of the following form:

$$(3.43) \quad \begin{cases} -\psi_t^{\varepsilon,k-1} - \Delta\psi^{\varepsilon,k-1} = d_{k-1} & \text{in } (0, T) \times \Omega, \\ \psi^{\varepsilon,k-1} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi^{\varepsilon,k-1}(T, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

with

$$(3.44) \quad d_{k-1}(t, x) = A_{k-1}(t, x)\psi^{\varepsilon,k-2} + (s\eta)^{-1}\nabla\psi^{\varepsilon,k-2} \cdot \nabla\rho,$$

where $A_{k-1} \in L^\infty(Q; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2))$ satisfies

$$(3.45) \quad \|A_{k-1}\|_{L^\infty(Q; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2))} \leq C.$$

Note that we have just proved above that this induction assumption holds for $k = 2$. Using (3.43) and (3.44), one gets that $\psi^{\varepsilon, k}$ fulfills the following backward heat system with a homogeneous Dirichlet boundary condition and a final null condition:

$$(3.46) \quad \begin{cases} -\psi_t^{\varepsilon, k} - \Delta \psi^{\varepsilon, k} = d_k & \text{in } (0, T) \times \Omega, \\ \psi^{\varepsilon, k} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi^{\varepsilon, k}(T, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

with

$$(3.47) \quad d_k(t, x) = A_k(t, x)\psi^{\varepsilon, k-1} + (s\eta)^{-1} \nabla \psi^{\varepsilon, k-1} \cdot \nabla \rho,$$

where $A_k : Q \rightarrow \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)$ is defined by

$$(3.48) \quad A_k := A_{k-1} + 2 \frac{\eta t}{s^2 \eta^3} \text{Id},$$

Id denoting the identity map of \mathbb{R}^2 . From (3.45) and (3.48), one gets that

$$(3.49) \quad A_k \in L^\infty(Q; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)) \quad \text{and} \quad \|A_k\|_{L^\infty(Q; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2))} \leq C.$$

Let us recall the following embeddings between Sobolev spaces (see, e.g., [13, Lemma 3.3, page 80]).

LEMMA 9. *Let $p \in (1, +\infty)$.*

(i) *If $p < N + 2$, let*

$$r := \frac{(N + 2)p}{N + 2 - p}.$$

Then X_p is continuously embedded in $L^r(0, T; W^{1, r}(\Omega)^2)$.

(ii) *If $p = N + 2$, for every $r \in [1, +\infty)$, X_p is continuously embedded in $L^r(0, T; W^{1, r}(\Omega)^2)$.*

(iii) *If $p > N + 2$, X_p is continuously embedded in $L^\infty(0, T; W^{1, \infty}(\Omega)^2)$.*

Applying Lemma 9 with $p = p_{k-1}$ and using (3.42), we get that

$$(3.50) \quad \psi^{\varepsilon, k-1} \in L^{p_k}(0, T; W^{1, p_k}(\Omega)^2) \quad \text{and} \quad \|\psi^{\varepsilon, k-1}\|_{L^{p_k}(0, T; W^{1, p_k}(\Omega)^2)} \leq C \|\alpha\|_{L^2(\Omega)^2}.$$

From (3.47), (3.49), and (3.50), we have

$$(3.51) \quad d_k \in L^{p_k}(Q)^2 \quad \text{and} \quad \|d_k\|_{L^{p_k}(Q)^2} \leq C \|\alpha\|_{L^2(\Omega)^2}.$$

Using (3.46), (3.51), and a classical parabolic regularity theorem (see, e.g., [13, Theorem 9.1, pages 341–342]), we have

$$(3.52) \quad \psi^{\varepsilon, k} \in X_{p_k} \quad \text{and} \quad \|\psi^{\varepsilon, k}\|_{X_{p_k}} \leq C \|\alpha\|_{L^2(\Omega)^2}.$$

Hence (3.52) holds for every positive integer k . Let us choose an integer k such that $k \geq (N/2) + 3$. Then using (3.41) and (3.52) we get that

$$(3.53) \quad \psi^{\varepsilon, k} \in L^\infty(Q)^2 \quad \text{and} \quad \|\psi^{\varepsilon, k}\|_{L^\infty(Q)^2} \leq C \|\alpha\|_{L^2(\Omega)^2}.$$

This shows that the control defined in (3.32) is bounded in $L^\infty(Q)$ independently of ε by $C \|\alpha\|_{L^2(\Omega)^2}$. This concludes the proof of Lemma 6. \square

3.2. Local null controllability around the trajectory $((\bar{u}, \bar{v}), \bar{h})$. Let $\nu > 0$ be small enough so that, for every $z = (z_1, z_2) \in L^\infty(Q)^2$,

$$(3.54) \quad (\|z\|_{L^\infty(Q)^2} \leq \nu) \Rightarrow ((G_{11}(z_1, z_2), G_{12}(z_1, z_2), G_{21}(z_1, z_2), G_{22}(z_1, z_2)) \in \mathcal{E}).$$

(The existence of such a $\nu > 0$ follows from (3.4) and (3.5).) Let \mathcal{Z} be the set of $z = (z_1, z_2) \in L^\infty(Q)^2$ such that $\|z\|_{L^\infty(Q)^2} \leq \nu$. By (3.54) and Lemma 6, there exists $C_0 > 0$ such that, for every $z = (z_1, z_2) \in \mathcal{Z}$ and for every $(\alpha_1, \alpha_2) \in L^\infty(\Omega)^2$, there exists a control $h \in L^\infty(Q)$ satisfying

$$(3.55) \quad \|h\|_{L^\infty(Q)} \leq C_0(\|\alpha_1\|_{L^2(\Omega)} + \|\alpha_2\|_{L^2(\Omega)})$$

such that the solution (ζ_1, ζ_2) to the Cauchy problem

$$(3.56) \quad \begin{cases} \zeta_{1,t} - \Delta \zeta_1 = G_{11}(z_1, z_2)\zeta_1 + G_{12}(z_1, z_2)\zeta_2 + h1_\omega & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t} - \Delta \zeta_2 = G_{21}(z_1, z_2)\zeta_1 + G_{22}(z_1, z_2)\zeta_2 & \text{in } (0, T) \times \Omega, \\ \zeta_1 = 0, \quad \zeta_2 = 0 & \text{on } (0, T) \times \partial\Omega, \\ \zeta_1(0, \cdot) = \alpha_1, \quad \zeta_2(0, \cdot) = \alpha_2 & \text{in } \Omega \end{cases}$$

satisfies (3.11). We now define a set-valued mapping $B : \mathcal{Z} \rightarrow L^\infty(Q)^2$ in the following way. Fix first any $(\alpha_1, \alpha_2) \in L^\infty(\Omega)^2$. For any $z = (z_1, z_2) \in \mathcal{Z}$, $B(z)$ is the set of $(\zeta_1, \zeta_2) \in L^\infty(Q)^2$ such that, for some $h \in L^\infty(Q)$ fulfilling (3.55), (ζ_1, ζ_2) is the solution of (3.56) and this solution satisfies (3.11). As we have just pointed out, $B(z)$ is never empty. Theorem 1 will be proved if one can check that the set-valued mapping $z \mapsto B(z)$ has a fixed point (i.e., a point z such that $z \in B(z)$) with the additional hypothesis $\|(\alpha_1, \alpha_2)\|_{L^\infty(\Omega)^2}$ being small enough. To get the existence of this fixed point, we apply Kakutani’s fixed point theorem (see, e.g., [14, Theorem 9.B, page 452]):

- (i) for every $z \in \mathcal{Z}$, $B(z)$ is a nonempty closed convex subset of $L^\infty(Q)^2$;
- (ii) there exists a convex compact set $K \subset \mathcal{Z}$ such that

$$(3.57) \quad B(z) \subset K \quad \forall z \in \mathcal{Z};$$

- (iii) B is upper semicontinuous in $L^\infty(Q)^2$; i.e., for every closed subset \mathcal{A} of \mathcal{Z} , $B^{-1}(\mathcal{A}) := \{z \in \mathcal{Z}; B(z) \cap \mathcal{A} \neq \emptyset\}$ is closed (see, e.g., [14, Definition 9.3, page 450]).

If (i)–(iii) hold, then there exists $z \in \mathcal{Z}$ such that $z \in B(z)$.

Clearly (i) holds. Let us prove that (ii) holds. By standard estimates, there exists $C_1 > 0$ such that

$$(3.58) \quad \|\zeta\|_{L^\infty(Q)^2} \leq C_1(\|\alpha_1\|_{L^\infty(\Omega)} + \|\alpha_2\|_{L^\infty(\Omega)}) \quad \forall z \in \mathcal{Z}, \forall \zeta \in B(z).$$

From now on we assume that $(\alpha_1, \alpha_2) \in L^\infty(\Omega)^2$ satisfies

$$(3.59) \quad \|\alpha_1\|_{L^\infty(\Omega)} + \|\alpha_2\|_{L^\infty(\Omega)} \leq \frac{\nu}{C_1}.$$

From (3.58) and (3.59), one has

$$(3.60) \quad B(z) \subset \mathcal{Z} \quad \forall z \in \mathcal{Z}.$$

Let $(\lambda_1, \lambda_2) \in L^\infty(Q)^2$ be the solution to the following Cauchy problem:

$$(3.61) \quad \begin{cases} \lambda_{1,t} - \Delta \lambda_1 = 0 & \text{in } (0, T) \times \Omega, \\ \lambda_{2,t} - \Delta \lambda_2 = 0 & \text{in } (0, T) \times \Omega, \\ \lambda_1 = 0, \quad \lambda_2 = 0 & \text{on } (0, T) \times \partial\Omega, \\ \lambda_1(0, \cdot) = \alpha_1, \quad \lambda_2(0, \cdot) = \alpha_2 & \text{in } \Omega. \end{cases}$$

Let $\zeta_1^* := \zeta_1 - \lambda_1$ and $\zeta_2^* := \zeta_2 - \lambda_2$. Then (ζ_1^*, ζ_2^*) is the solution to the following Cauchy problem:

$$(3.62) \quad \begin{cases} \zeta_{1,t}^* - \Delta \zeta_1^* = D_1 & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t}^* - \Delta \zeta_2^* = D_2 & \text{in } (0, T) \times \Omega, \\ \zeta_1^* = 0, \quad \zeta_2^* = 0 & \text{on } (0, T) \times \partial\Omega, \\ \zeta_1^*(0, \cdot) = 0, \quad \zeta_2^*(0, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

with

$$(3.63) \quad D_1 := G_{11}(z_1, z_2)\zeta_1 + G_{12}(z_1, z_2)\zeta_2 + h1_\omega,$$

$$(3.64) \quad D_2 := G_{21}(z_1, z_2)\zeta_1 + G_{22}(z_1, z_2)\zeta_2.$$

Note that there exists $C_2 > 0$ such that

$$(3.65) \quad \|D_1\|_{L^\infty(Q)} + \|D_2\|_{L^\infty(Q)} \leq C_2 \quad \forall z \in \mathcal{Z}, \forall \zeta \in B(z).$$

From (3.62), (3.65), and a classical parabolic regularity theorem (see, e.g., [13, Lemma 3.3, page 80 and Theorem 9.1, pages 341–342]), $\zeta^* := (\zeta_1^*, \zeta_2^*) \in C^0(\overline{Q})^2$ and there exists $C_3 > 0$ such that, for every $z \in \mathcal{Z}$ and for every $\zeta \in B(z)$,

$$(3.66) \quad |\zeta^*(t, x) - \zeta^*(t', x')| \leq C_3(|t - t'|^{1/2} + |x - x'|) \quad \forall (t, x) \in \overline{Q}, \forall (t', x') \in \overline{Q}.$$

Let K^* be the set of $\zeta^* = (\zeta_1^*, \zeta_2^*) \in C^0(\overline{Q})^2$ such that (3.66) holds. Then $(\lambda_1, \lambda_2) + K^*$ is a compact convex subset of $L^\infty(Q)^2$ and

$$(3.67) \quad B(z) \subset (\lambda_1, \lambda_2) + K^* \quad \forall z \in \mathcal{Z}.$$

Then $K := ((\lambda_1, \lambda_2) + K^*) \cap \mathcal{Z}$ is a convex compact subset of \mathcal{Z} such that (3.57) holds.

Let us finally prove the upper semicontinuity of B . Let \mathcal{A} be a closed subset of \mathcal{Z} . Let $(z^k)_{k \in \mathbb{N}}$ be a sequence of elements in \mathcal{Z} , let $(\zeta^k)_{k \in \mathbb{N}}$ be a sequence of elements in $L^\infty(Q)^2$, and let $z \in \mathcal{Z}$ be such that

$$(3.68) \quad z^k \rightarrow z \quad \text{in } L^\infty(Q) \text{ as } k \rightarrow +\infty,$$

$$(3.69) \quad \zeta^k \in \mathcal{A} \quad \forall k \in \mathbb{N},$$

$$(3.70) \quad \zeta^k \in B(z^k) \quad \forall k \in \mathbb{N}.$$

By (3.70), for every $k \in \mathbb{N}$ there exists a control $h^k \in L^\infty(Q)$ satisfying

$$(3.71) \quad \|h^k\|_{L^\infty(Q)} \leq C_0(\|\alpha_1\|_{L^2(\Omega)} + \|\alpha_2\|_{L^2(\Omega)})$$

such that $\zeta^k = (\zeta_1^k, \zeta_2^k)$ is the solution of the Cauchy problem

$$(3.72) \quad \begin{cases} \zeta_{1,t}^k - \Delta \zeta_1^k = G_{11}(z_1^k, z_2^k)\zeta_1^k + G_{12}(z_1^k, z_2^k)\zeta_2^k + h^k 1_\omega & \text{in } (0, T) \times \Omega, \\ \zeta_{2,t}^k - \Delta \zeta_2^k = G_{21}(z_1^k, z_2^k)\zeta_1^k + G_{22}(z_1^k, z_2^k)\zeta_2^k & \text{in } (0, T) \times \Omega, \\ \zeta_1^k = 0, \quad \zeta_2^k = 0 & \text{on } (0, T) \times \partial\Omega, \\ \zeta_1^k(0, \cdot) = \alpha_1, \quad \zeta_2^k(0, \cdot) = \alpha_2 & \text{in } \Omega, \end{cases}$$

and this solution satisfies

$$(3.73) \quad \zeta_1^k(T, \cdot) = 0, \quad \zeta_2^k(T, \cdot) = 0.$$

From (ii) and (3.71), there exist a strictly increasing sequence $(k_l)_{l \in \mathbb{N}}$ of integers, $h \in L^\infty(Q)$, and $\zeta \in \mathcal{Z}$ such that

$$(3.74) \quad h^{k_l} \rightharpoonup h \quad \text{for the weak-* topology on } L^\infty(Q) \text{ as } l \rightarrow +\infty,$$

$$(3.75) \quad \zeta^{k_l} \rightarrow \zeta \quad \text{in } L^\infty(Q)^2 \text{ as } l \rightarrow +\infty.$$

Note that, since \mathcal{A} is closed, (3.69) and (3.75) imply that $\zeta \in \mathcal{A}$. Hence, in order to prove (iii), it suffices to check that

$$(3.76) \quad \zeta \in B(z).$$

Letting $l \rightarrow +\infty$ in (3.72) and (3.73) and using (3.68), (3.74), and (3.75), we get (3.11) and (3.56). (The two equalities in (3.11) and the last two equalities of (3.56) have to be understood as equalities in $L^2(\Omega)$ for $\zeta^{k_l} \rightharpoonup \zeta$ in X_2 .) Letting $l \rightarrow +\infty$ in (3.71) and using (3.74), we get (3.55). Hence (3.76) holds. This concludes the proof of (iii) and of Theorem 1. \square

4. Appendix: Sketch of the proof of Theorem 3. As the proof is very similar to that of Theorem 1, we limit ourselves to pointing out only the differences. First, Lemma 5 should be replaced by the following lemma.

LEMMA 10. *There exists a function $G \in C^\infty([0, +\infty); \mathbb{C})$ such that*

$$(4.1) \quad G(z) = \left(z - \frac{1}{2}\right)^2 \quad \text{for } \frac{1}{2} - \delta < z < \frac{1}{2} + \delta,$$

$$(4.2) \quad \text{Im } G(z) < 0 \quad \text{for } 0 < z < \frac{1}{2} - \delta,$$

$$(4.3) \quad \text{Re } G(z) > \text{Im } G(z) > 0 \quad \text{for } \frac{1}{2} + \delta < z < 1 - \delta$$

and such that the solution g_0 to the Cauchy problem

$$(4.4) \quad g_0''(z) + \frac{N-1}{z} g_0'(z) = G(z), \quad z > 0,$$

$$(4.5) \quad g_0(1) = g_0'(1) = 0$$

satisfies

$$(4.6) \quad g_0(z) = 1 - z^2 \quad \text{if } 0 < z < \delta,$$

$$(4.7) \quad g_0(z) = e^{-\frac{1}{1-z^2}} \quad \text{if } 1 - \delta < z < 1,$$

$$(4.8) \quad g_0(z) = 0 \quad \text{if } z \geq 1.$$

The proof is carried out in the same way as for Lemma 5. Note that the conditions (2.26) and (2.30) are easily satisfied thanks to the change of sign of $\operatorname{Re} G$ and $\operatorname{Im} G$. Theorem 4 is still true for some functions $V : (t, x) \in \mathbb{R} \times \mathbb{R}^N \mapsto V(t, x) \in \mathbb{C}$ and $K : (t, x) \in \mathbb{R} \times \mathbb{R}^N \mapsto K(t, x) \in \mathbb{C}$ when (2.8) is replaced by

$$V_t = \Delta V + RV + V^2.$$

In the proof, we consider the same functions $\lambda(t)$ and $f_0(t)$ and search for $v(t, r)$ in the form

$$v(t, r) = \sum_{i=0}^2 f_i(t)g_i(z).$$

f_1, f_2, g_1 , and g_2 are defined in the same way as for Theorem 4, so that

$$\begin{aligned} \mathcal{V} = \mathcal{V}_x = 0 & \quad \text{for } -1 < t < 1, \quad z = \frac{1}{2}, \\ \mathcal{V}_{xx} = 2f_0 + \mathcal{R} & \quad \text{for } -1 < t < 1, \quad \left| z - \frac{1}{2} \right| < \delta, \end{aligned}$$

with $|\mathcal{R}| \leq C\varepsilon^2 f_0$. Letting $A(t, z) = f_0(t)^{-1}\mathcal{V}(t, z)$, we have that

$$A(t, z) = \left(z - \frac{1}{2} \right)^2 \varphi(t, z) \quad \text{for } t \in [-1, 1], \quad z \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta \right),$$

where $\varphi \in C^\infty([-1, 1]_t \times (\frac{1}{2} - \delta, \frac{1}{2} + \delta))$ and

$$\operatorname{Re} \varphi(t, x) > f_0(t) \quad \text{for } t \in [-1, 1], \quad z \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta \right).$$

On the other hand,

$$|A(t, z) - G| \leq C\varepsilon^2 |G(z)|$$

for $-1 \leq t \leq 1$ and $|z - \frac{1}{2}| \geq \delta/2$. Using (4.2), (4.3), it is then clear that for ε small enough we have $A(t, z) \notin i\mathbb{R}^+$ for $-1 \leq t \leq 1, z \in (0, 1) \setminus \{\frac{1}{2}\}$. Defining the square root as an analytic function on the complement of $i\mathbb{R}^+$, we see that $\lambda^{-2}\mathcal{V} = \lambda^{-2}f_0(t)A(t, z) = B(t, z)^2$ for some $B \in C^\infty((-1, 1) \times [0, 1])$. The end of the construction of (V, K) is as in the proof of Theorem 4. In the study of the local null controllability around the trajectory $((\bar{u}, \bar{v}), \bar{h})$, the functions G_{ij} are defined in the same way, except

$$G_{21}(\zeta_1, \zeta_2) = 2\bar{u} + \zeta_1.$$

Therefore, (3.4) and (3.7) have to be changed to

$$\begin{aligned} |\operatorname{Im} G_{21}(0, 0)(t, x)| &\geq \frac{2}{M} \quad \forall (t, x) \in (t_1, t_2) \times \omega_0, \\ |\operatorname{Im} a_{21}(t, x)| &\geq \frac{1}{M} \quad \forall (t, x) \in (t_1, t_2) \times \omega_0, \end{aligned}$$

respectively. Note that the functions in the control systems are complex-valued, so that we have to conjugate the coefficients in the right-hand side of the adjoint system (3.15). To estimate the local integral of φ_2 , we multiply the first equation in (3.15) by $\chi(x)e^{-s\rho(x)\eta(t)}(s\eta)^3\overline{\varphi_2}$ (where $\overline{\varphi_2}$ stands for the conjugate of φ_2) and take the absolute value of the imaginary part of the left-hand side of (3.20). The remaining part of the proof is the same as for Theorem 1.

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