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Regularity of Liapunov functions for stable systems

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Abstract

We consider the problem of characterizing those systems which admit (weak) Liapunov functions with nice analytic properties. Our investigation gives a rather complete picture of the situation for the one-dimensional case. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider a time-invariant, finite-dimensional system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where f is at least continuous, and $f(0) = 0$. It is proved in [2] that if the origin is stable and f fulfills same additional mild assumptions, then it is possible to construct a semi-continuous weak Liapunov function for (1). However, as pointed out independently by Ura and Krasovskii in 1959, an everywhere continuous weak Liapunov function need not exist, not even when $f \in C^\infty$. A celebrated two-dimensional example is reported for instance in [1], but there are also one-dimensional examples of this fact. In [1] we can find also a necessary and sufficient condition for the existence of a continuous Liapunov function.

The problem of finding Liapunov functions with nice analytic properties for a stable (but not asymptotically stable) system, is important for applications to automatic control theory: we limit ourselves to recall the classical Jurdjevic–Quinn method, where a weak Liapunov function is used in order to construct an asymptotically stabilizing feedback, provided that it is of class C^∞. However, apart from what is reported above, no further results seem to be available in the literature. This paper is a contribution to the investigation of this problem. Our main conclusions can be summarized as follows:

- the existence of a continuous Liapunov function does not imply the existence of a locally Lipschitz continuous Liapunov function;
- the existence of a Lipschitz continuous Liapunov function does not imply the existence of a Liapunov function of class C^1;
- with $n = 1$, the existence of a Liapunov function of class C^1 implies the existence of a Liapunov function of class C^∞, but with $n > 1$, the implication is no more valid; more precisely when $n > 1$, for each integer $p \geqslant 1$ the existence of a Liapunov function of class C^p does not imply the existence of a Liapunov function of class C^{p+1}.

Those and other results are proved in this paper. For reader's convenience, we recall that (1) is *stable*

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at the origin if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each initial state x_0 with $|x_0| < \delta$ and all the solutions $\varphi(t)$ issuing from x_0, it follows that φ is right continuable and $|\varphi(t)| < \varepsilon$ for all $t \geqslant 0$ (we stress that under our assumptions, uniqueness of solutions is not guaranteed). Moreover, we say that the function $V(x)$ is a *weak Liapunov function* for (1) if $V(x)$ is positive definite in a neighborhood X of the origin and for each solution $\varphi(t)$ of (1) defined on an interval I and lying in X for each $t \in I$, the composite map $V(\varphi(t))$ is non-increasing on I. Being only concerned with the stability of (1), we do not assume that V is radially unbounded when $X = \mathbb{R}$. Since we do not use other types of Liapunov functions, the qualifier *weak* will be omitted.

To conclude this introduction, we want to emphasize that time-invariance is the main concern of our investigation. If time-dependent Liapunov functions are allowed, the situation is much better understood (see [3,8]).

2. The one-dimensional case

As recalled in the Introduction, the existence of continuous Liapunov function has been characterized in [1]. It is natural to ask the question whether in general, a continuous Liapunov function can be further smoothed out. The answer is negative.

Theorem 1. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ of class C^∞ with the following properties*:
(i) *there exists a continuous Liapunov function for* (1),
(ii) *there exists no locally Lipschitz continuous Liapunov function for* (1).

Proof. We construct the function f with the required properties on $[0, +\infty)$. The construction on $(-\infty, 0]$ is completely analogous.

Let C be the Cantor set contained in the interval $[0, 1]$. Recall that C is closed, it is nowhere dense and its Lebesgue measure is zero. In particular, since C is closed, there exists a function f of class C^∞ such that $f(x) = 0$ for $x \in C \cup [1, +\infty)$ and $f(x) > 0$ for $x \in [0, 1] \setminus C$ (see [5, p. 17]).

Let $V(x)$ be any continuous, nondecreasing real function defined for $x \geqslant 0$ and coincident with the Cantor function on the interval [0,1] (see [7, p. 48]). We claim that $V(x)$ is a Liapunov function for (1).

Indeed, it is obvious that $V(0)=0$ and that $V(x) > 0$ for $x > 0$. As far as the monotonicity property is concerned, we shall prove that actually $V(\varphi(t))=$constant for each solution of (1). To this purpose, let us remark that since f is C^∞, (1) exhibits uniqueness of solution for each initial state x_0. Hence, the notation $\varphi(t, x_0)$ is meaningful.

If $x_0 \in C \cup [1, +\infty)$, then $f(x_0)=0$ and $\varphi(t, x_0)=x_0$ for each $t \in \mathbb{R}$. Hence, $V(\varphi(t, x_0)) =$ constant.

On the other hand, if $x \in [0, 1] \setminus C$, then $f(x_0) > 0$ and, by the uniqueness, $T = \{\varphi(t, x_0),\ t \in \mathbb{R}\} \subset [0, 1] \setminus C$. Moreover, since φ is continuous, T is an interval. Now, by construction, V is constant on each interval which is contained in $[0, 1] \setminus C$.

We have so proved that V is a Liapunov function for (1).

To complete the proof, assume by contradiction that there exists a locally Lipschitz continuous Liapunov function $W(x)$ for (1). The composite function $W(\varphi(t))$ is absolutely continuous for each solution φ of (1). Hence, the assertion "$W(\varphi(t))$ is non-increasing for each solution φ" is equivalent to

$$\underline{D^+} W(x, f(x)) \leqslant 0 \quad \forall x \in \mathbb{R} \tag{2}$$

where $\underline{D^+}$ denotes the lower right directional Dini derivative (see [9, p. 347]). Let $x \in [0, 1] \setminus C$, so that $f(x) > 0$. Since the Dini derivative is positively homogeneous with respect to the direction, (2) yields

$$\underline{D^+} W(x, 1) = \underline{D^+} W(x) \leqslant 0.$$

Since $\text{meas}(C) = 0$, we conclude that $\underline{D^+} W(x) \leqslant 0$ a.e. on $[0, 1]$. This implies that $W(x)$ is nonincreasing in $[0, 1]$ [7, p. 207]. But $W(0) = 0$. Hence, we have to conclude that $W(x) \leqslant 0$ for $x \in [0, 1]$. On the other hand, in order to be a Liapunov function, it is required that $W(x) > 0$ for $x \neq 0$. Thus, we have a contradiction and the statement is finally proved. □

Next, we present necessary and sufficient conditions for the existence of more regular Liapunov functions. Together with the continuous function $f(x)$ which defines the right-hand side of (1), we consider the following subsets of the half-line $[0, +\infty)$:

$$N^+ = \{x \geqslant 0\colon f(x) \leqslant 0\} \quad \text{and}$$
$$P^+ = \{x \geqslant 0\colon f(x) > 0\}.$$

Since $f(0)=0$, N^+ is closed and P^+ open (possibly, $P^+ = \emptyset$). Moreover, $N^+ = [0, +\infty) \setminus P^+$. Analogously, we consider the following subsets of $(-\infty, 0]$:

$$N^- = \{x \leqslant 0\colon f(x) \geqslant 0\}, \quad P^- = \{x \leqslant 0\colon f(x) < 0\}.$$

Theorem 2. *Let $n = 1$ and let f be continuous. The following statements are equivalent*:

(i) *there exists a locally absolutely continuous Liapunov function for* (1),

(ii) *there exists a (globally) Lipschitz continuous Liapunov function for* (1),

(iii) $\forall \varepsilon > 0$ *one has*

$$\text{meas}(N^+ \cap (0,\varepsilon)) > 0$$

$$\textit{and} \quad \text{meas}(N^- \cap (-\varepsilon, 0)) > 0. \tag{3}$$

Proof. First we prove that (iii) $\Rightarrow$ (ii). Let $N = N^+ \cup N^-$ and let

$$V(x) = \begin{cases} \int_0^x \chi_N(\xi)\,\mathrm{d}\xi & \text{if } x \geqslant 0, \\ -\int_0^x \chi_N(\xi)\,\mathrm{d}\xi & \text{if } x < 0, \end{cases}$$

where χ_N is the characteristic function of the set N (i.e., $\chi_N(\xi) = 1$ if $\xi \in N$, and $\chi_N(\xi) = 0$ elsewhere). It is clear that $V(0) = 0$, and the condition implies that $V(x) > 0$ for $x \neq 0$. Since $|\chi_N(\xi)| \leqslant 1$, we easily see that $V(x)$ is globally Lipschitz continuous. Indeed, for each pair x_1, x_2,

$$|V(x_2) - V(x_1)| = \left| \int_{x_1}^{x_2} \text{sgn}(\xi)\chi_N(\xi)\,\mathrm{d}\xi \right| \leqslant |x_2 - x_1|.$$

Let $W^{1,p}(a,b)$ denote the usual Sobolev space, i.e. $W^{1,p}(a,b) = \{u \in L^p(a,b),\ u' \in L^p(a,b)\}$, with $-\infty \leqslant a < b \leqslant +\infty$ and $1 \leqslant p \leqslant \infty$, and let $W^{1,p}_{\text{loc}}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R},\ u_{|(-r,r)} \in W^{1,p}(-r,r) \text{ for any } r > 0\}$. Recall that $W^{1,1}_{\text{loc}}(\mathbb{R})$ coincides with the space of locally absolutely continuous functions on $\mathbb{R}$ and that $W^{1,\infty}_{\text{loc}}(\mathbb{R})$ coincides with the space of locally Lipschitz continuous functions on $\mathbb{R}$. Thus, we have that $V \in W^{1,p}_{\text{loc}}(\mathbb{R})$ for each p ($1 \leqslant p \leqslant \infty$). In particular, $\text{sgn}(x)\chi_N(x)$ is some (measurable) representative of the generalized derivative $V' \in L^\infty(\mathbb{R})$. To prove that V is non-increasing along the solutions of (1), we need the following chain rule, to be found in [4].

Lemma 1. *Let $h = g \circ f$, where $f \in W^{1,1}(0,T)$ and $g \in W^{1,1}_{\text{loc}}(\mathbb{R})$. If h has bounded variation on $[0,T]$, then $h \in W^{1,1}(0,T)$ and $h'(t) = g'(f(t))f'(t)$ for a.e. $t \in [0,T]$.*

Let $\varphi(\cdot)$ be any trajectory of (1) on some interval $[0,T]$. Then V, $\varphi(\cdot)$ and $V \circ \varphi$ are Lipschitz continuous, so it follows from Lemma 1 that for a.e. $t \in [0,T]$

$$\frac{\mathrm{d}V(\varphi(t))}{\mathrm{d}t} = V'(\varphi(t))\varphi'(t) = \text{sgn}(\varphi(t))\chi_N(\varphi(t))f(\varphi(t)) \leqslant 0.$$

Hence,

$$V(\varphi(T)) - V(\varphi(0)) = \int_0^T \frac{\mathrm{d}V(\varphi(t))}{\mathrm{d}t}\,\mathrm{d}t \leqslant 0,$$

as required. We now prove that (i) implies (iii). Let $V \in W^{1,1}_{\text{loc}}(\mathbb{R})$ be a Liapunov function for (1) and assume that for some $\varepsilon > 0$ it may happen that $\text{meas}(N^+ \cap (0,\varepsilon)) = 0$. Hence $f(x) \geqslant 0$ for all $x \in [0,\varepsilon]$ and $\text{meas}\{x \in [0,\varepsilon],\ f(x) = 0\} = 0$. Let $\Omega = P^+ \cap (0,\varepsilon)$. Ω is an open set with measure ε. Write $\Omega = \bigcup_{i \geqslant 1}(a_i, b_i)$, a union of disjoint open intervals. Pick any $i \geqslant 1$. It is easy to construct a solution φ of (1) such that for some $-\infty \leqslant t_i^- < t_i^+ \leqslant +\infty$ we have $\varphi((t_i^-, t_i^+)) = (a_i, b_i)$. Since $\varphi \in W^{1,\infty}(t_i^-, t_i^+)$ and $V \circ \varphi$ is non-increasing and bounded (hence $V \circ \varphi$ has bounded variation on (t_i^-, t_i^+)), it follows from Lemma 1 that $V \circ \varphi \in W^{1,1}_{\text{loc}}(t_i^-, t_i^+)$ and that for a.e. $t \in (t_i^-, t_i^+)$,

$$\frac{\mathrm{d}V \circ \varphi}{\mathrm{d}t} = V'(\varphi(t))f(\varphi(t)).$$

On the other hand, since $V \circ \varphi$ is non-increasing, we have

$$\frac{\mathrm{d}V \circ \varphi}{\mathrm{d}t} \leqslant 0$$

a.e., hence $V'(\varphi(t)) \leqslant 0$ for a.e. $t \in (t_i^-, t_i^+)$ (recall that $f > 0$ on (a_i, b_i)). Since the function φ (of class C^1) maps a zero measure set into a zero measure set (see [5, p. 30]) we infer that $V' \leqslant 0$ a.e. in each (a_i, b_i) and a.e. in $(0,\varepsilon)$ (since $\text{meas}((0,\varepsilon) \setminus \Omega) = 0$). Then $V(x) \leqslant 0$ for all $0 \leqslant x \leqslant \varepsilon$, which is impossible.

The implication (ii) $\Rightarrow$ (i) is trivial. The proof is complete. □

Note that (ii) of Theorem 1 is indeed a consequence of Theorem 2. We address now the problem of the existence of smooth Liapunov functions.

Theorem 3. *Let $n = 1$ and let f be continuous. The following statements are equivalent*:

(i) *there exist a C^1 Liapunov function*,

(ii) *there exist a C^∞ Liapunov function*,

(iii) $\forall \varepsilon > 0$ *one has*

$$\text{Int}\, N^+ \cap (0,\varepsilon) \neq \emptyset \quad \textit{and} \quad \text{Int}\, N^- \cap (-\varepsilon, 0) \neq \emptyset \tag{4}$$

(*here*, Int *denotes the set of interior points*).

Proof. We prove first that (iii) implies (ii). Set, as before, $N = N^+ \cup N^-$ and $P = P^+ \cup P^-$. Of course, $\operatorname{Int} N = \mathbb{R} \setminus \operatorname{Clos} P$.

According to [5, p. 17], it is possible to find a function $g^+(x)$ of class C^∞ such that $g^+(x) > 0$ for $x \in \operatorname{Int} N^+ \setminus \{0\}$, and $g^+(x) = 0$ for $x \in \operatorname{Clos} P^+ \cup (-\infty, 0]$. By the same argument, we can also take $g^-(x)$ in such a way that $g^-(x) < 0$ for $x \in \operatorname{Int} N^- \setminus \{0\}$, and $g^-(x) = 0$ for $x \in \operatorname{Clos} P^- \cup [0, +\infty)$. Note that by construction, all the derivatives of both g^+ and g^- vanish at $x = 0$.

We define

$$V(x) = \begin{cases} \int_0^x g^+(\xi)\,\mathrm{d}\xi & \text{if } x \geqslant 0, \\ \int_0^x g^-(\xi)\,\mathrm{d}\xi & \text{if } x < 0. \end{cases}$$

Obviously $V(0)=0$, and we have $V(x) > 0$ for $x \neq 0$ because of assumption (iii). Moreover, the derivative of V is given by

$$V'(x) = \begin{cases} g^+(x) & \text{if } x \geqslant 0, \\ g^-(x) & \text{if } x < 0. \end{cases} \tag{5}$$

Note that (5) is valid also for $x = 0$. From (5) we deduce that $V(x)$ is of class C^∞ on the whole of $\mathbb{R}$. Finally, the derivative of V with respect to (1) is

$$\dot{V}(x) = \begin{cases} g^+(x)f(x) & \text{if } x \geqslant 0, \\ g^-(x)f(x) & \text{if } x < 0 \end{cases}$$

which is less of or equal to zero for each x.

The implication (ii) ⇒ (i) is evident. Thus, it remains to prove that (i) ⇒ (iii).

Assume that for some $\varepsilon > 0$, one has

$$\operatorname{Int} N^+ \cap (0, \varepsilon) = \emptyset$$

or, equivalently,

$$[0, \varepsilon] \subseteq \operatorname{Clos} P^+. \tag{6}$$

If V is a C^1 Liapunov function, we must have

$$\dot{V}(x) = V'(x) f(x) \leqslant 0.$$

We conclude that $V'(x) \leqslant 0$ for $x \in P^+$ and also (by (6)) for $x \in [0, \varepsilon]$ and this is impossible since $V(0)=0$ and $V(x) > 0$ for $x > 0$. In a similar way, we see that $\operatorname{Int} N^- \cap (-\varepsilon, 0) \neq \emptyset$. □

Comparing conditions (3) and (4), allows us to focus on the gap between those one-dimensional stable systems which admit smooth Liapunov functions and those systems which admit only a Lipschitz continuous one. For the sake of completeness, an explicit example is given below.

Theorem 4. *There exists a function* $f : \mathbb{R} \to \mathbb{R}$ *of class* C^∞ *with the following properties*:

(i) *the differential equation* (1) *has a Lipschitz continuous Liapunov function*,

(ii) *there exists no Liapunov function of class* C^1 *for* (1).

Proof. Let $\{q_n\}_{n \geqslant 0}$ be a numbering of $\mathbb{Q}^*$, and let $\{r_n\}_{n \geqslant 0}$ be a sequence of positive numbers such that

$$\forall n \geqslant 0, \quad 0 < r_n < 2^{-(n+3)} |q_n|. \tag{7}$$

Let $\Omega = \bigcup_{n \geqslant 0} (q_n - r_n, q_n + r_n)$, and $F = \mathbb{R} \setminus \Omega$. Let finally f be any function $f \in C^\infty$ such that $f(x) \geqslant 0$ for all $x \in \mathbb{R}$ and $f(x) = 0$ if and only if $x \in F$.

A Lipschitz continuous Liapunov function for this system can be defined as shown in Theorem 2. To this purpose, we only have to prove that for each $\varepsilon > 0$, $\operatorname{meas}(N^+ \cap (0, \varepsilon)) = \operatorname{meas}(F \cap (0, \varepsilon)) > 0$. Indeed, the other condition $\operatorname{meas}(N^- \cap (-\varepsilon, 0)) > 0$ is clearly fulfilled since $N^- = (-\infty, 0]$.

This is equivalent to proving that $\operatorname{meas}(P^+ \cap (0, \varepsilon)) = \operatorname{meas}(\Omega \cap (0, \varepsilon)) < \varepsilon$. First of all we observe that

$$\begin{aligned} &\operatorname{meas}(\Omega \cap (0, \varepsilon)) \\ &\quad = \operatorname{meas}\left(\bigcup_{n \geqslant 0} (q_n - r_n, q_n + r_n) \cap (0, \varepsilon) \right) \\ &\quad \leqslant \sum_{n=0}^{\infty} \operatorname{meas}((q_n - r_n, q_n + r_n) \cap (0, \varepsilon)). \end{aligned}$$

We claim that $(q_n - r_n, q_n + r_n) \cap (0, \varepsilon) \neq \emptyset$ only if $0 < q_n < \frac{8}{7}\varepsilon$. Indeed, if $q_n < 0$, since $r_n < |q_n|/2^{n+3} \leqslant |q_n|/8$, we get $q_n + r_n < q_n - q_n/8 = \frac{7}{8} q_n < 0$; hence $(q_n - r_n, q_n + r_n) \cap (0, \varepsilon) = \emptyset$, a contradiction. Thus, $q_n > 0$ and $q_n - r_n > \frac{7}{8} q_n > 0$, and $(q_n - r_n, q_n + r_n) \cap (0, \varepsilon) \neq \emptyset$ implies $\varepsilon > q_n - r_n$, hence $q_n < \varepsilon + r_n < \varepsilon + q_n/8$ and $q_n < \frac{8}{7}\varepsilon$ as required.

We infer that

$$\begin{aligned} \operatorname{meas}(\Omega \cap (0, \varepsilon)) &\leqslant \sum_{n \geqslant 0;\ 0 < q_n < 8\varepsilon/7} 2 r_n \\ &\leqslant \sum_{n \geqslant 0;\ 0 < q_n < 8\varepsilon/7} \frac{q_n}{2^{n+2}} \\ &\leqslant \frac{8}{7}\varepsilon \sum_{n \geqslant 0} \frac{1}{2^{n+2}} = \frac{4}{7}\varepsilon < \varepsilon. \end{aligned}$$

To see that a C^1 Liapunov function cannot exist we use Theorem 3. Indeed, Ω contains a dense set, so that the interior of F must be empty. □

Remark 1. If f is real analytic and stable, then $V(x)=x^2$ is a Liapunov function. Indeed, either $f(x)$ vanishes everywhere, or there exists $\varepsilon>0$ such that $xf(x)<0$ for $x\in(-\varepsilon,0)\cup(0,\varepsilon)$. On the other hand, it is not difficult to construct a function $f\in C^\infty$ for which there exists a C^∞ Liapunov function but not an analytic one. Take for instance

$$f(x)=\begin{cases}0 & \text{if } x=0,\\ e^{-1/x^2}\sin\frac{1}{x} & \text{if } x\neq 0.\end{cases}$$

Remark 2. The material presented in the first two sections of this paper concerns systems in dimension one; it could be objected that the one-dimensional case has a poor interest, since it is atypical and topologically contrary to higher dimensions. However, we notice that, starting by a counterexample in dimension one, we can construct counterexamples in arbitrary dimension. Indeed, it is easily checked that if $V(x)$ is a Liapunov function for (1), then $V(x_1)+x_2^2+\cdots+x_n^2$ is a Liapunov function for the system

$$\begin{aligned}\dot{x}_1&=f(x_1),\\ \dot{x}_2&=0,\\ &\vdots\\ \dot{x}_n&=0.\end{aligned}\tag{8}$$

Vice versa, if $V(x_1,\dots,x_n)$ is a Liapunov function for (8), then $V(x,0,\dots,0)$ is a Liapunov function for (1). Moreover, if $V(x)$ is a Liapunov function for (1) and $\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$, then the radial function $V(|\boldsymbol{x}|)$ is a Liapunov function for the radial vector field $f(|\boldsymbol{x}|)\boldsymbol{x}/|\boldsymbol{x}|$, where $|\boldsymbol{x}|$ is the Euclidean norm of $\boldsymbol{x}$.

3. The two-dimensional case

In this section we show that Theorem 3 cannot be generalized to the case $n>1$.

Theorem 5. *Let $p\in\mathbb{N}^*$. Then there exists a smooth vector field f on $\mathbb{R}^2$ such that*
(1) *the origin is a stable equilibrium point for the system $\dot{x}=f(x)$;*
(2) *there exists a Liapunov function $V=V(x)$ of class C^p;*
(3) *there does not exist any Liapunov function of class C^{p+1}.*

Proof. Let the function h be defined by $h(r)=\exp(-1/r^2)$ for $r>0$ and $h(r)=0$ for $r\leqslant 0$. Pick any positive function g of class C^∞ on $[0,2\pi]$ and such that $g^{(i)}(0)=g^{(i)}(2\pi)$ for all $i\leqslant p$ (with $g'(0)=0$) but $g^{(p+1)}(0)\neq g^{(p+1)}(2\pi)$. Extend g on $\mathbb{R}$ to get a 2π-periodic function. Let R denote the (linear) rotation of angle $+\pi/2$, i.e. $R(u_1,u_2)=(-u_2,u_1)$. Define the field f together with the Liapunov function V in polar coordinates by

$$V(r,\theta)=h(r)\,g(\theta),\quad r\geqslant 0,\ \theta\in[0,2\pi],$$

$$f(r,\theta)=h(\theta)\,h(2\pi-\theta)\,R(\nabla V),\quad r\geqslant 0,\ \theta\in[0,2\pi].$$

Since $h(r)$ and all its derivatives vanish at $r=0$, we see that $V(x)$ (i.e., V rewritten in Cartesian coordinates) is of class C^p everywhere (even at the origin) and that $f(x)$ is of class C^∞ everywhere (even at the origin and along the ray $\theta=0$). To complete the proof, assume by contradiction that there exists a Liapunov function W of class C^{p+1} for (1). Let $\tilde{f}=R(\nabla V)$; clearly, $\tilde{f}$ is a locally Lipschitz continuous vector field. Since $(\nabla W,\tilde{f})=(h(\theta)h(2\pi-\theta))^{-1}(\nabla W,f)\leqslant 0$ for $0<\theta<2\pi$, and hence also $(\nabla W,\tilde{f})\leqslant 0$ for $\theta=0$ by continuity, we infer that W is also a Liapunov function for the system $\dot{x}=\tilde{f}(x)$. Since W is of class C^{p+1}, with $p+1\geqslant 2>\dim(\mathbb{R}^2)-\dim(\mathbb{R}^1)$, it follows from Sard's theorem (see [5, p. 34]) that for a.e. $\alpha\in\mathbb{R}$, $\nabla W\neq 0$ on the set $M=W^{-1}(\alpha)$. Using some standard connectness argument we may choose a number $\bar{r}>0$ such that, if we set $\bar{x}=(\bar{x}_1,0)=(\bar{r},0)$ and $\alpha=W(\bar{x})$, then $\nabla W\neq 0$ on $M=W^{-1}(\alpha)$. Let $\varphi(t)$ denote the solution of the Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\tilde{f}(x),\qquad x(0)=\bar{x}.$$

Clearly φ and $W\circ\varphi$ are periodic functions, hence the (nonincreasing) function $W\circ\varphi$ takes a constant value, namely α. On the other hand

$$0=\left.\frac{\mathrm{d}W\circ\varphi}{\mathrm{d}t}\right|_{t=0}=(\nabla W(\bar{x}),\tilde{f}(\bar{x}))=(\nabla W(\bar{x}),R(\nabla V(\bar{x}))),$$

hence $\nabla W(\bar{x})=\lambda\nabla V(\bar{x})=\lambda(h'(\bar{r})g(0),0)$ for some $\lambda\neq 0$. We conclude that $(\partial W/\partial r)(\bar{r},0)\neq 0$. Now we infer from the implicit function theorem that there exists a function $\rho(\theta)$ of class C^{p+1} in a neighborhood of 0, such that $\rho(0)=\bar{r}$ and such that for every (r,θ) in a neighborhood of $(\bar{r},0)$,

$$W(r,\theta)=\alpha\Leftrightarrow r=\rho(\theta).$$

If $(r(t), \theta(t))$ denote the polar coordinates of $\varphi(t)$, then for $|t|$ small enough

$$r(t) = \rho(\theta(t)) \quad \text{and}$$
$$h(r(t))\, g(\theta(t)) = V(\varphi(t)) = V(\bar{x}).$$

Indeed, V is a first integral for $\tilde{f}$. Hence $g(\theta) = (V(\bar{x})/h(\rho(\theta)))$ if $|\theta|$ is small enough, and we conclude that g is of class C^{p+1} near 0, which contradicts the assumptions. □

To complete the picture, we remark that there exist systems in $\mathbb{R}^2$ defined by an analytic vector field which admit a C^∞ Liapunov function but not an analytic one. Indeed, it is well known that there exists an analytic (more precisely, polynomial) vector field with an equilibrium position of center type which does not have analytic first integrals [10], that is functions $u(x)$ taking a constant value along every trajectory. The following simple argument shows that such a system possesses not even an analytic Liapunov function. Assume that $V(x)$ is a Liapunov function which is not a first integral in a neighborhood of the origin. Hence, there must exist a nontrivial trajectory $\varphi(t)$ such that $\nabla V(\varphi(0)) \cdot f(\varphi(0)) < 0$. This means that there exist $t_1 < 0$ and $t_2 > 0$ such that

$$V(\varphi(t_1)) > V(\varphi(0)) > V(\varphi(t_2)).$$

Since the system has a center at the origin, $\varphi(t)$ is periodic. Accordingly, there exists $t_3 > t_2$ such that $V(\varphi(t_1)) = V(\varphi(t_3))$. In other words, $V(\varphi(t))$ is increasing on the interval $[t_2, t_3]$ and hence, V is not a Liapunov function.

The existence of C^∞ first integrals for a system with a center configuration and a C^∞ right-hand side follows from [6].

As a final remark, we point out that for $n > 1$, it seems to be not even possible to characterize systems which admit regular Liapunov functions in terms of topological properties or of the measure of certain sets. Indeed, there exist pairs of systems (say (S_1) and (S_2)) of form (1) with $n = 2$ which are topologically equivalent and such that (S_1) has a globally Lipschitz continuous Liapunov function but not a C^1 one, while (S_2) has a polynomial Liapunov function. Such examples will be reported elsewhere.

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