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EXACT BOUNDARY CONTROLLABILITY FOR THE LINEAR KORTEWEG–DE VRIES EQUATION ON THE HALF-LINE*

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Abstract. This paper is concerned with the controllability of the linear Korteweg–de Vries equation on the domain $\Omega = (0, +\infty)$, the control being applied at the left endpoint $x = 0$. It is shown that the *exact* boundary controllability holds true in $L^2(0, +\infty)$ provided that the solutions are not required to be in $L^\infty(0, T, L^2(0, +\infty))$. The proof rests on a Carleman's estimate and an approximation theorem. A similar result is obtained for the heat equation and for the Schrödinger equation.

Key words. exact boundary controllability, unbounded domain, Carleman's estimate

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1. Introduction and main results.

The Korteweg–de Vries (KdV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad t \geq 0, \ x \in \Omega \subset \mathbb{R}, \tag{1.1}$$

may serve as a model for (among other things) propagation of small amplitude long water waves in a uniform channel. In this context, t is time, x is the space variable, and u stands for the deviation of the liquid's surface from the equilibrium position. The boundary (resp., internal) controllability of (1.1) has been extensively studied (see [21], [22], [19], [20], and also [16] for the Benjamin–Bona–Mahony equation) when Ω is bounded, say $\Omega = (0, L)$. The (local) exact boundary controllability of (1.1) follows in [19] from the *exact* boundary controllability of the associated linear KdV equation, namely

$$u_t + u_x + u_{xxx} = 0. \tag{1.2}$$

To date, there is no result as far as the boundary controllability of (1.1) or (1.2) on some *unbounded* domain (say $\Omega = (0, +\infty)$) is concerned. The aim of this paper is to fill this gap in providing a study of the exact boundary controllability of (1.2) on $(0, +\infty)$, which may be seen as a first step in the knowledge of the control theory for (1.1) on unbounded domains. It should be observed that the *approximate* boundary controllability of (1.2) in $L^2(0, +\infty)$ is quite easy to prove, whereas the *exact* boundary controllability requires a more sophisticated analysis, due to a lack of compactness. An enlightening example of the difference between exact and approximate (internal) controllabilities for linear PDEs in unbounded domains is provided by the following result, whose (simple) proof is sketched in the appendix.

PROPOSITION 1.1. *Consider a (real) constant coefficients differential operator* $Au = \sum_{i=0}^{n} a_i \frac{d^i u}{d x^i}$, *with domain* $\mathcal{D}(A) = \{u \in L^2(\mathbb{R});\ Au \in L^2(\mathbb{R})\}$. *Assume that*

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$n \geq 2$ (with $a_n \neq 0$) and that A generates a continuous semigroup $\big(S(t)\big)_{t\geq 0}$ on $L^2(\mathbb{R})$. Let $T>0$ and $L_1 < L_2$ be some numbers. Set

$$\mathcal{R} = \left\{ \int_0^T S(T-t)f(t,\cdot)dt; \quad f \in L^2(\mathbb{R}^2),\, \mathrm{supp}\, f \subset [0,T]\times[L_1,L_2] \right\},$$

where supp f *denotes the support of* f*. Then* $\mathcal{R}$ *is a* strict dense *subspace of* $L^2(\mathbb{R})$.

In other words, when considering *mild* solutions (in $C([0,T],L^2(\mathbb{R}))$) of the forced initial-value problem

$$\begin{cases} \frac{du}{dt} - Au &= f, \\ u(0) &= 0, \end{cases}$$

where f denotes any square integrable function supported in $[0,T]\times[L_1,L_2]$, the space $\mathcal{R}$ of all reachable states is dense in (but different from) $L^2(\mathbb{R})$. Notice that for $Au = -u_{xxx} - u_x$, letting $L_1 = -1 < L_2 = 0$ and taking the restrictions to $(0,+\infty)$ of the mild solutions, we readily infer the approximate *boundary* controllability of (1.2) in $L^2(0,+\infty)$. It turns out that the *exact* boundary controllability of (1.2) in $L^2(0,+\infty)$ also fails to be true if we restrict ourselves to solutions with bounded energy, that is, which belong to $L^\infty(0,T,L^2(0,+\infty))$. An *implicit* formulation (that is, without specification of the boundary conditions) of this fact is given in the following theorem, to be proved later in this paper.

THEOREM 1.2. *Let $T>0$. Then there exists $u_0 \in L^2(0,+\infty)$ such that if u is any function in $L^\infty(0,T,L^2(0,+\infty))$ satisfying*

$$\begin{cases} u_t + u_x + u_{xxx} &= 0 \quad \text{in } \mathcal{D}'\big((0,T)\times(0,+\infty)\big), \\ u_{|t=0} &= u_0, \end{cases} \tag{1.3}$$

then $u_{|t=T} \neq 0$.

(Notice that $u_{|t=0}$ and $u_{|t=T}$ are meaningful in $H^{-3}(0,+\infty)$ for any $u \in L^\infty(0,T, L^2(0,+\infty))$ satisfying (1.3): Indeed, such a function belongs to the space $W^{1,\infty}(0,T, H^{-3}(0,+\infty))$.) Theorem 1.2 tells us that even the (boundary) *null-controllability* fails to be true for solutions with bounded energy. Nevertheless, when the bounded energy condition ($u \in L^\infty(0,T,L^2(0,+\infty))$) is dropped, the exact boundary controllability of KdV holds true, as is shown in the following theorem, which is the main result of this paper.

THEOREM 1.3. *Let T,ϵ,b be positive numbers, with $\epsilon < \frac{T}{2}$. Let $L^2((0,+\infty), e^{-2bx}dx)$ denote the space of (class of) measurable functions $u:(0,+\infty)\to\mathbb{R}$ such that $\int_0^{+\infty} u^2(x)e^{-2bx}\,dx < \infty$. Let $u_0 \in L^2(0,+\infty)$ and $u_T \in L^2((0,+\infty),e^{-2bx}dx)$. Then there exists a function*

$$u \in L^2_{loc}\big([0,T]\times[0,+\infty)\big) \cap C\big([0,\epsilon],L^2(0,+\infty)\big) \cap C\big([T-\epsilon,T],L^2((0,+\infty),e^{-2bx}dx)\big)$$

which solves

$$\begin{cases} u_t + u_x + u_{xxx} &= 0 \quad \text{in } \mathcal{D}'\big((0,T)\times(0,+\infty)\big), \\ u_{|t=0} &= u_0, \\ u_{|t=T} &= u_T. \end{cases} \tag{1.4}$$

Let us make some comments.

1. The proof of Theorem 1.3 combines Fursikov–Imanuvilov's approach (see [4]) for the boundary controllability of the Burgers equation on bounded domains (which

is based on a global Carleman's estimate) and, for the extension to some unbounded domain, Rosay's clever proof of Malgrange–Ehrenpreis's theorem (see [18]), which uses an approximation theorem. Roughly speaking, the approximation theorem allows us to modify a sequence of solutions of $u_t + u_x + u_{xxx} = f$, defined on an increasing sequence of domains, in such a way that it converges (strongly) in $L^2_{loc}(\mathbb{R}^2)$. It should be emphasized that our approach allows us to consider initial and final states in *different* spaces of functions, thus exploiting an *asymmetric* property of the KdV equation, namely the (forward) wellposedness of (1.2) in the asymmetric space $L^2(\mathbb{R}, e^{2bx}dx)$ for any $b > 0$ (see [9]). Notice that we may require that $u \in C([T-\epsilon, T], L^2(0,+\infty))$ if u_T is also assumed to be in $L^2(0,+\infty)$.

2. As in [2] and [13], the formulation of the previous boundary controllability result is *implicit*. Nevertheless, setting $h_0 = u_{|x=0}$, $h_1 = u_{x|x=0}$, and $h_2 = u_{xx|x=0}$, it may be seen that $h_0, h_1, h_2 \in H^{-1}(0,T)$ and, thanks to Holmgren's uniqueness theorem, that u is the *only* solution (in the same space as above) of the initial-value boundary problem

$$\begin{cases} u_t + u_x + u_{xxx} &= 0 \ \text{ in } \mathcal{D}'\big((0,T)\times(0,+\infty)\big), \\ u_{|x=0} &= h_0, \ u_{x|x=0} = h_1, \ u_{xx|x=0} = h_2, \\ u_{|t=0} &= u_0. \end{cases}$$

Moreover u satisfies $u_{|t=T} = u_T$.

3. The method described in item 1 applies also to many other linear PDEs for which the characteristic hyperplanes take the form $\{t = \text{Const.}\}$: For instance, the heat equation $u_t - \Delta u = 0$ and the Schrödinger equation $iu_t + \Delta u = 0$ are concerned. (See section 5.)

The paper is outlined as follows. The proof of Theorem 1.2 is given in section 2. It rests on a duality argument and on the behavior of the traces $u_{x|x=0}$, $u_{xx|x=0}$ of exponential solutions for (1.2) with the boundary condition $u_{|x=0} = 0$. A global Carleman's estimate for the KdV equation (which is subsequently used) is stated and proved in section 3. The proof of Theorem 1.3 is given in section 4, together with the proof of the approximation theorem (Lemma 4.4). In the last section we sketch the proof of similar results for the heat equation and the Schrödinger equation.

From now on, for the sake of brevity, we shall write P for the operator $(\partial/\partial t) + (\partial/\partial x) + (\partial^3/\partial x^3)$.

2. Proof of Theorem 1.2.

The proof of Theorem 1.2 rests on the following key result.

LEMMA 2.1. *There exists a family $(v^\lambda)_{\lambda>0}$ of functions in $\cap_{n\geq 0} C^\infty([0,T], H^n(0,+\infty))$ such that for every $\lambda > 0$*

$$P\,v^\lambda = 0 \quad \textit{in } (0,T)\times(0,+\infty), \tag{2.1}$$

$$v^\lambda{}_{|x=0} = 0 \quad \textit{on } (0,T), \tag{2.2}$$

$$\|v^\lambda{}_{|t=0}\|_{L^2(0,+\infty)} = 1, \tag{2.3}$$

and

$$\|v^\lambda_{x\,|x=0}\|_{H^n(0,T)} + \|v^\lambda_{xx\,|x=0}\|_{H^n(0,T)} \to 0 \ \textit{as } \lambda \to 0 \ (\textit{for every } n \geq 1). \tag{2.4}$$

Proof. Let us consider the operator $Av := -v_{xxx} - v_x$ with domain

$$D(A) = H^3(0,+\infty) \cap H^1_0(0,+\infty) \subset L^2(0,+\infty).$$

Then A generates a strongly continuous semigroup $\big(S(t)\big)_{t\geq 0}$ on $L^2(0,+\infty)$, and we shall search for v^λ in the form of an exponential solution:

$$v^\lambda(t,\cdot) = S(t)v^\lambda_0 = e^{-\lambda t}v^\lambda_0,$$

where $v^\lambda_0 \in D(A)$ solves $Av^\lambda_0 = -\lambda v^\lambda_0$ and $\lambda \in (0,+\infty)$. The roots of the equation $-z^3 - z = -\lambda$ may be written in the form $r, -\frac{r}{2} \pm i\mu$, where $0 < r \sim \lambda$ as $\lambda \to 0^+$ and $\mu = \big(1 + \frac{3}{4}r^2\big)^{\frac{1}{2}}$. Let $w^\lambda_0(x) := \Im m\,\big(e^{(-\frac{r}{2}+i\mu)x}\big) = e^{-\frac{r}{2}x}\sin(\mu x)$ (hence $w^\lambda_0 \in D(A)$ and $Aw^\lambda_0 = -\lambda w^\lambda_0$). Easy calculations give

$$\|w^\lambda_0\|_{L^2(0,+\infty)} = \left(\frac{2\mu^2}{r(r^2+4\mu^2)}\right)^{\frac{1}{2}}.$$

Set $c_\lambda := (\frac{r(r^2+4\mu^2)}{2\mu^2})^{\frac{1}{2}}$, $v^\lambda_0 := c_\lambda w^\lambda_0$, and $v^\lambda(t,x) := e^{-\lambda t}v^\lambda_0(x)$. Since we know that (2.1)–(2.3) are true, it remains to prove (2.4). Obviously $v^\lambda_x(t,0) = c_\lambda \mu e^{-\lambda t}$ and $v^\lambda_{xx}(t,0) = -c_\lambda \mu\, r\, e^{-\lambda t}$, and since $c_\lambda \to 0$ as $\lambda \to 0^+$, (2.4) follows. ◻

It will result from the next lemma that the traces $u_{|x=0}$, $u_{x|x=0}$, and $u_{xx|x=0}$ of a bounded energy solution $u = u(t,x)$ of (1.2) belong to the dual space to $H^1(0,T)$ (which is not to be confused with $H^{-1}(0,T) = H^1_0(0,T)'$).

LEMMA 2.2. *Let T and L be positive numbers and let $u \in L^\infty(0,T,L^2(0,L))$ be such that $Pu = 0$ in $\mathcal{D}'\big((0,T)\times(0,L)\big)$. Then $u \in H^3(0,L,H^1(0,T)')$ and we have for some constant $C = C(L,T) > 0$*

$$\begin{aligned}\|u(\cdot,0)\|_{H^1(0,T)'} + \|u_x(\cdot,0)\|_{H^1(0,T)'} + \|u_{xx}(\cdot,0)\|_{H^1(0,T)'}\\ \leq C\|u\|_{L^\infty(0,T,L^2(0,L))}.\end{aligned} \tag{2.5}$$

Proof. Since $u_t = -(u_{xxx} + u_x) \in L^2(0,T,H^{-3}(0,L))$, we see that $u \in H^1(0,T, H^{-3}(0,L))$; hence for every $f \in H^1(0,T,H^3_0(0,L))$

$$\int_0^T \langle u_t, f\rangle\, dt = -\int_0^T\int_0^L uf_t\, dxdt + [\langle u,f\rangle]^T_{t=0}, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing $\langle\cdot,\cdot\rangle_{H^{-3}(0,L),H^3_0(0,L)}$. Since $u \in C([0,T], H^{-3}(0,L)) \cap L^\infty(0,T,L^2(0,L))$, u, as a function of t, is weakly continuous in $L^2(0,L)$. Hence

$$\begin{aligned}\left|\int_0^T \langle u_t,f\rangle\, dt\right| &\leq \|u\|_{L^2((0,T)\times(0,L))}\cdot\|f_t\|_{L^2((0,T)\times(0,L))}\\ &\quad+\|u\|_{L^\infty(0,T,L^2(0,L))}\cdot\big(\|f(0,\cdot)\|_{L^2(0,L)} + \|f(T,\cdot)\|_{L^2(0,L)}\big)\\ &\leq C_1\|u\|_{L^\infty(0,T,L^2(0,L))}\cdot\|f\|_{L^2(0,L,H^1(0,T))}\end{aligned} \tag{2.7}$$

for some constant $C_1 = C_1(T,L) > 0$. Since $H^1(0,T,H^3_0(0,L))$ is dense in $L^2(0,L, H^1(0,T))$, we infer from (2.7) that $u_t \in L^2(0,L,H^1(0,T)')$. Integrating three times with respect to (w.r.t.) x in the equation

$$(u_{xx} + u)_x = -u_t \qquad \text{in } \mathcal{D}'(0,L,H^1(0,T)')$$

($u_t \in L^2(0,L,H^1(0,T)')$ being given by (2.6)), we deduce that $u \in H^3(0,L,H^1(0,T)')$ and (2.5) follows. □

We now proceed to the proof of Theorem 1.2. Arguing by contradiction, we assume that for every $u_0 \in L^2(0,+\infty)$ there exists a function $u \in L^\infty(0,T,L^2(0,+\infty))$ such that $P\,u = 0$ in $\mathcal{D}'((0,T)\times(0,+\infty))$, $u_{|t=0} = u_0$, and $u_{|t=T} = 0$. Let E denote the space

$$\{u \in L^\infty(0,T,L^2(0,+\infty)),\ P\,u = 0 \text{ in } \mathcal{D}'((0,T)\times(0,+\infty)) \text{ and } u_{|t=T} = 0\},$$

endowed with the norm $\|u\|_E = \|u\|_{L^\infty(0,T,L^2(0,+\infty))}$. It is a Banach space, since

$$\|u_{|t=T}\|_{H^{-3}(0,+\infty)} \le C\|u\|_{H^1(0,T,H^{-3}(0,+\infty))} \le C'\|u\|_E$$

for all $u \in E$ and some constants $C, C' > 0$. Also, the linear map $\Lambda : u \in E \mapsto u_{|t=0} \in H^{-3}(0,+\infty)$ is continuous. Actually, thanks to [14, Lem. 8.1], Λ takes values in $L^2(0,+\infty)$ and we readily infer from the closed graph theorem that Λ is continuous as a map from E into $L^2(0,+\infty)$. Let $N = \ker\Lambda$, let $\tilde{E}$ stand for the quotient space of E by N, and let π denote the natural projection of E onto $\tilde{E}$. Then $\tilde{E}$ is a Banach space for the norm $\|\pi(u)\|_{\tilde{E}} := \inf_{w\in\pi(u)}\|w\|_E$, and the induced map $\tilde{\Lambda} : \tilde{E} \to L^2(0,+\infty)$ (defined by $\tilde{\Lambda}(\pi(u)) = \Lambda(u)$ for any $u \in E$) has a continuous inverse by the open mapping theorem. For every $\lambda > 0$ we pick $u^\lambda \in E$ such that $\pi(u^\lambda) = \tilde{\Lambda}^{-1}(v^\lambda(0,\cdot))$ (with v^λ as in Lemma 2.1) and

$$\|u^\lambda\|_E \le 2\|\pi(u^\lambda)\|_{\tilde{E}} \le 2\|\tilde{\Lambda}^{-1}\|. \tag{2.8}$$

Let L be a positive number. Integrations by part in

$$\int_0^T\int_0^L P(u^\lambda)v^\lambda\,dxdt = 0$$

result in

$$-\int_0^L v^\lambda(0,x)^2\,dx + \left[\langle u^\lambda_{xx} + u^\lambda, v^\lambda\rangle - \langle u^\lambda_x, v^\lambda_x\rangle + \langle u^\lambda, v^\lambda_{xx}\rangle\right]_{x=0}^L = 0, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ denotes here the duality pairing $\langle\cdot,\cdot\rangle_{H^1(0,T)',H^1(0,T)}$. Since

$$\begin{aligned}&\|u^\lambda(\cdot,L)\|_{H^1(0,T)'} + \|u^\lambda_x(\cdot,L)\|_{H^1(0,T)'} + \|u^\lambda_{xx}(\cdot,L)\|_{H^1(0,T)'}\\ &\quad\le C\|u\|_{L^\infty(0,T,L^2(L,L+1))}\\ &\quad\le C\|u\|_{L^\infty(0,T,L^2(0,+\infty))}\end{aligned}$$

(where $C = C(1,T)$ is as in Lemma 2.2) and since $v^\lambda(\cdot,L)$, $v^\lambda_x(\cdot,L)$ and $v^\lambda_{xx}(\cdot,L) \to 0$ in $H^1(0,T)$ as $L \to +\infty$, letting $L \to +\infty$ in (2.9) and using (2.2)–(2.3) we get

$$1 = \int_0^{+\infty} v^\lambda(0,x)^2\,dx = \langle u^\lambda_x, v^\lambda_x\rangle_{|x=0} - \langle u^\lambda, v^\lambda_{xx}\rangle_{|x=0}.$$

Hence, by (2.5) and (2.8) (also with $C = C(1,T)$)

$$1 \le 2C\|\tilde{\Lambda}^{-1}\|\left(\|v^\lambda_{x\,|x=0}\|_{H^1(0,T)} + \|v^\lambda_{xx\,|x=0}\|_{H^1(0,T)}\right). \tag{2.10}$$

Letting $\lambda \to 0$ in (2.10) and using (2.4) we get a contradiction. The proof of Theorem 1.2 is complete. □

3. A Carleman's estimate. Let T and L be positive numbers. Set

$$\mathcal{Z} = \{q \in C^3([0,T]\times[-L,L]);\ q(t,\pm L) = q_x(t,\pm L) = q_{xx}(t,\pm L) = 0 \text{ for } 0 \le t \le T\}.$$

This section is devoted to the proof of the following global Carleman's estimate for the KdV equation.

PROPOSITION 3.1. *There exists a smooth positive function ψ on $[-L,L]$ (which depends on L) and there exist constants $s_0 = s_0(L,T)$ and $C = C(L,T)$ such that for all $s \ge s_0$ and all $q \in \mathcal{Z}$*

$$\begin{aligned}\int_0^T\int_{-L}^L \left\{\frac{s^5}{t^5(T-t)^5}|q|^2 + \frac{s^3}{t^3(T-t)^3}|q_x|^2 + \frac{s}{t(T-t)}|q_{xx}|^2\right\} e^{-\frac{2s\psi(x)}{t(T-t)}}\,dxdt \\ \le C\int_0^T\int_{-L}^L |q_t + q_x + q_{xxx}|^2 e^{-\frac{2s\psi(x)}{t(T-t)}}\,dxdt.\end{aligned} \tag{3.1}$$

Proof. Let $\psi = \psi(x)$ be a positive function (to be specified later) of class C^3 in $[-L,L]$ and let $\varphi(t,x) := \frac{\psi(x)}{t(T-t)}$. Let q be given in $\mathcal{Z}$ and let $s > 0$. Set $u := e^{-s\varphi}q$ and $w := e^{-s\varphi}P(e^{s\varphi}u)$. We readily get

$$w = Au + Bu_x + Cu_{xx} + u_{xxx} + u_t, \tag{3.2}$$

with

$$\begin{aligned} A &:= s(\varphi_t + \varphi_x + \varphi_{xxx}) + 3s^2\varphi_x\varphi_{xx} + (s\varphi_x)^3, \\ B &:= 1 + 3s\varphi_{xx} + 3(s\varphi_x)^2, \\ C &:= 3s\varphi_x. \end{aligned}$$

Set $M_1(u) := u_t + u_{xxx} + Bu_x$ and $M_2(u) := Au + Cu_{xx}$. We deduce the following inequality:

$$2\iint M_1(u)\,M_2(u) \le \iint \big(M_1(u) + M_2(u)\big)^2 = \iint w^2. \tag{3.3}$$

(Here and in what follows, the integrals are extended to $(0,T)\times(-L,L)$.) To compute the integral in the left-hand side of (3.3) we perform integrations by part w.r.t. x or t. We readily get

$$\iint M_1(u)Au = -\frac{1}{2}\iint (A_t + A_{xxx} + (AB)_x)u^2 + \frac{3}{2}\iint A_x u_x^2 \tag{3.4}$$

and

$$\iint (u_{xxx} + Bu_x)Cu_{xx} = -\frac{1}{2}\iint C_x u_{xx}^2 - \frac{1}{2}\iint (BC)_x u_x^2. \tag{3.5}$$

Finally, using (3.2),

$$\begin{aligned}\iint u_t C u_{xx} &= -\iint C_x u_t u_x - \iint C u_{tx} u_x \\ &= \iint C_x\big(Au + Bu_x + Cu_{xx} + u_{xxx} - w\big)u_x + \frac{1}{2}\iint C_t u_x^2 \\ &= -\frac{1}{2}\iint (C_x A)_x u^2 + \frac{1}{2}\iint (2BC_x - (CC_x)_x + C_{xxx} + C_t)u_x^2 \\ &\quad - \iint C_x u_{xx}^2 - \iint C_x w u_x.\end{aligned} \tag{3.6}$$

Combining (3.4), (3.5), and (3.6) we get

$$
\begin{aligned}
2\iint M_1(u)M_2(u) &= -\iint \big(A_t + A_{xxx} + (AB)_x + (C_xA)_x\big)u^2 \\
&\quad + \iint \big(3A_x - (BC)_x + 2BC_x - (CC_x)_x + C_{xxx} + C_t\big)u_x^2 \qquad (3.7)\\
&\quad -3\iint C_x u_{xx}^2 - 2\iint C_x w u_x.
\end{aligned}
$$

If ϵ is any number in $(0,1)$, then by the Cauchy–Schwarz inequality

$$2\iint C_x w u_x \le \epsilon \iint C_x^2 u_x^2 + \epsilon^{-1}\iint w^2.$$

Hence, setting

$$
\begin{aligned}
D &:= -\big(A_t + A_{xxx} + (AB)_x + (C_xA)_x\big),\\
E &:= 3A_x + BC_x - B_xC - (CC_x)_x + C_{xxx} + C_t - \epsilon C_x^2,\\
F &:= -3C_x
\end{aligned}
$$

and using (3.3), (3.7) we get

$$\iint Du^2 + \iint Eu_x^2 + \iint Fu_{xx}^2 \le (1+\epsilon^{-1})\iint w^2. \qquad (3.8)$$

The function ψ will be chosen in such a way that D, E, and F are positive. Clearly

$$
\begin{aligned}
D &= -(AB)_x + \frac{1}{t^4(T-t)^4}O(s^4) \qquad (\text{as } s \to +\infty)\\
&= -\big(3(s\varphi_x)^5\big)_x + \frac{O(s^4)}{t^4(T-t)^4}\\
&= -15s^5\frac{\psi'(x)^4\psi''(x)}{t^5(T-t)^5} + \frac{O(s^4)}{t^4(T-t)^4}.
\end{aligned}
$$

It follows that for s large enough, if

$$|\psi'(x)| > 0 \text{ and } \psi''(x) < 0 \text{ for } x \in [-L, L], \qquad (3.9)$$

we have

$$D \ge C_1 \frac{s^5}{t^5(T-t)^5} \qquad (3.10)$$

for some constant $C_1 > 0$. On the other hand, expanding E in a series of powers of s, it is easy to see that there is no term in s^3 (because of cancellations) and that

$$
\begin{aligned}
E &= 9s^2\big((1-\epsilon)\varphi_{xx}^2 - \varphi_x\varphi_{xxx}\big) + \frac{O(s)}{t^2(T-t)^2}\\
&= 9s^2\frac{(1-\epsilon)\psi''(x)^2 - \psi'(x)\psi'''(x)}{t^2(T-t)^2} + \frac{O(s)}{t^2(T-t)^2}.
\end{aligned}
$$

Hence for s large enough, if

$$(1-\epsilon)\psi''(x)^2 - \psi'(x)\psi'''(x) > 0 \quad \text{for all } x \in [-L, L], \qquad (3.11)$$

we get for some constant $C_2 > 0$

$$E \geq C_2 \frac{s^2}{t^2(T-t)^2}. \tag{3.12}$$

Finally, for some constant $C_3 > 0$

$$F = -\frac{9\psi''(x)s}{t(T-t)} \geq C_3 \frac{s}{t(T-t)} \tag{3.13}$$

provided that (3.9) holds true. Now pick some smooth positive function ψ on $[-L, L]$ such that (3.9) and (3.11) are fulfilled for some $\epsilon > 0$. (For instance, picking any ϵ in $(0,1)$, $\psi(x) = -x^2 + (2L+1)(x+2L)$ is convenient.) We infer from (3.8), (3.10), (3.12), and (3.13) that, for s large enough,

$$\iint \left\{ \frac{s^5}{t^5(T-t)^5} u^2 + \frac{s^2}{t^2(T-t)^2} u_x^2 + \frac{s}{t(T-t)} u_{xx}^2 \right\} \leq C_4 \iint w^2 \tag{3.14}$$

for some constant $C_4 > 0$. Actually (3.14) may be slightly improved by observing that

$$\begin{aligned} \iint \frac{s^3}{t^3(T-t)^3} u_x^2 &= -\iint \frac{s^3}{t^3(T-t)^3} u u_{xx} \\ &\leq \frac{1}{2} \left(\iint \frac{s^5}{t^5(T-t)^5} u^2 + \iint \frac{s}{t(T-t)} u_{xx}^2 \right) \\ &\leq \frac{C_4}{2} \iint w^2 \end{aligned}$$

(thanks to (3.14)); hence, for s large enough,

$$\iint \left\{ \frac{s^5}{t^5(T-t)^5} u^2 + \frac{s^3}{t^3(T-t)^3} u_x^2 + \frac{s}{t(T-t)} u_{xx}^2 \right\} \leq \frac{3}{2} C_4 \iint w^2. \tag{3.15}$$

Replacing u with $e^{-s\varphi} q$ in (3.15) we readily get (3.1) for some constant $C > 0$ and s large enough. The proof of Proposition 3.1 is complete. □

Corollary 3.2. *Let $L > 0$ and let $f = f(t,x)$ be any function in $L^2(\mathbb{R}_t \times (-L,L)_x)$ such that* $\operatorname{supp} f \subset [t_1, t_2] \times (-L, L)$*, where $-\infty < t_1 < t_2 < \infty$. Then for every $\epsilon > 0$ there exist a positive number $C = C(L, t_1, t_2, \epsilon)$ (C does not depend on f) and a function $v \in L^2(\mathbb{R} \times (-L, L))$ such that*

$$v_t + v_x + v_{xxx} = f \ \text{ in } \mathcal{D}'(\mathbb{R} \times (-L, L)), \tag{3.16}$$

$$\operatorname{supp} v \subset [t_1 - \epsilon, t_2 + \epsilon] \times (-L, L), \tag{3.17}$$

$$\|v\|_{L^2(\mathbb{R} \times (-L,L))} \leq C \|f\|_{L^2(\mathbb{R} \times (-L,L))}. \tag{3.18}$$

Proof. Applying a translation w.r.t. time if needed, we may assume without loss of generality that $0 = t_1 - \epsilon < t_1 < t_2 < t_2 + \epsilon =: T$. We readily infer from (3.1) that for some constants $k, C_1 > 0$ and for every $q \in \mathcal{Z}$

$$\int_0^T \int_{-L}^L |q|^2 e^{-\frac{k}{t(T-t)}} \, dxdt \leq C_1 \int_0^T \int_{-L}^L |Pq|^2 \, dxdt. \tag{3.19}$$

Thus the bilinear form

$$(p,q) := \int_0^T \int_{-L}^{L} Pp\, Pq\, dxdt$$

is a scalar product on $\mathcal{Z}$. Let H denote the completion of $\mathcal{Z}$ for $(\cdot,\cdot)$. Obviously $|q|^2 e^{-\frac{k}{t(T-t)}}$ is integrable on $(0,T)\times(-L,L)$ if $q\in H$, and (3.19) holds true as well. On the other hand the linear form

$$l(q) := -\int_0^T \int_{-L}^{L} fq\, dxdt$$

is well defined and continuous on H. Indeed, using (3.19) and the assumption on the support of f, we get

$$\int_0^T \int_{-L}^{L} |fq|\, dxdt = \int_{t_1}^{t_2} \int_{-L}^{L} |fq|\, dxdt \le C_2 \|f\|_{L^2\big((t_1,t_2)\times(-L,L)\big)} \cdot (q,q)^{\frac{1}{2}} \tag{3.20}$$

for some constant $C_2>0$. It follows from the Riesz representation theorem that there exists a unique $p\in H$ such that

$$\text{for all } q\in H \quad (p,q) = l(q). \tag{3.21}$$

We set $v := P(p) \in L^2\big((0,T)\times(-L,L)\big)$. Taking $q\in\mathcal{D}\big((0,T)\times(-L,L)\big)$ as a test function in (3.21) we get

$$\langle P^*(v), q\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)} = \langle -f, q\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)},$$

where $Q=(0,T)\times(-L,L)$ and $P^*=-P$ is the (formal) adjoint to the operator P. Hence $Pv=f$ in $\mathcal{D}'(Q)$. Notice that $v\in H^1(0,T,H^{-3}(-L,L))$, since v and $v_t = f - v_{xxx} - v_x$ belong to $L^2(0,T,H^{-3}(-L,L))$; hence $v_{|t=0}$ and $v_{|t=T}$ are meaningful in $H^{-3}(-L,L)$. Now let $q\in\mathcal{Z}\subset H^1(0,T,H_0^3(-L,L))$. It follows from (3.21) that

$$\begin{aligned}
-\int_0^T \int_{-L}^{L} fq\, dxdt &= \int_0^T \int_{-L}^{L} v(q_t+q_x+q_{xxx})\, dxdt \\
&= -\int_0^T \langle v_t + v_x + v_{xxx}, q\rangle\, dt + [\langle v,q\rangle]_{t=0}^T \\
&= -\int_0^T \int_{-L}^{L} fq\, dxdt + [\langle v,q\rangle]_{t=0}^T,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing $\langle\cdot,\cdot\rangle_{H^{-3}(-L,L),H_0^3(-L,L)}$. Since $q_{|t=0}$ and $q_{|t=T}$ may be arbitrarily chosen in $\mathcal{D}(-L,L)$, we infer that $v_{|t=0}=v_{|t=T}=0$ in $H^{-3}(-L,L)$. Extending v by setting $v(t,x)=0$ for $(t,x)\notin(0,T)\times(-L,L)$, we see that (3.16), (3.17), and (3.18) hold true (with $C=C_2$). □

Remark 1. Using [19, Thm. 1.2] instead of Proposition 3.1, one may prove that the result in Corollary 3.2 also holds true for $\epsilon=0$ and that the weight $e^{-\frac{k}{t(T-t)}}$ may be dropped in the integral term of the left-hand side of (3.19). Nevertheless, the proof given here is direct and shorter, and it leads to a self-contained paper. Moreover, this proof also works for the heat equation (see below the proof of Theorem 5.2).

4. The proof of Theorem 1.3. For the comfort of the reader, we first give an outline of the proof of Theorem 1.3. In the first step, we show that we are finished if, for any $f \in L^2_{loc}(\mathbb{R}^2)$ with support in some strip $[0,T]\times\mathbb{R}$ and any $\epsilon > 0$, there exists a function $u \in L^2_{loc}(\mathbb{R}^2)$ supported in $[-\epsilon, T+\epsilon]\times\mathbb{R}$, which solves $P\,u = f$. This problem has already been solved when the whole domain $\mathbb{R}^2$ is replaced by $\mathbb{R}_t \times (-n,n)_x$, $n \geq 1$. (See Corollary 3.2.) At this stage, we are given a sequence of solutions of $P\,u = f$, which are defined on an increasing sequence of domains and are supported in $[-\epsilon, T+\epsilon]\times\mathbb{R}$. To ensure the convergence of this sequence in $L^2_{loc}(\mathbb{R}^2)$, we need an approximation theorem (Lemma 4.4), which differs from the one in [18] by a careful control on the growth of the support in time. Two technical lemmas (namely, Lemmas 4.2 and 4.3) are needed to prove the approximation theorem. The final step is a standard Mittag–Leffler's procedure.

Let $u_0 \in L^2(0,+\infty)$ and $u_T \in L^2\big((0,+\infty), e^{-2bx}dx\big)$. It is well known (see [9]) that the operator $Av = -v_{xxx}$ with domain $H^3(\mathbb{R})$ (resp., $\{v \in L^2(\mathbb{R}, e^{2bx}dx),\ Av \in L^2(\mathbb{R}, e^{2bx}dx)\}$) generates a continuous semigroup on $L^2(\mathbb{R})$ (resp., $L^2(\mathbb{R}, e^{2bx}dx)$). Thanks to the standard change of functions

$$v(t,x) = u(t,t+x) \tag{4.1}$$

we easily get two functions $u_1(t,x)$, $u_2(t,x)$ such that $u_1 \in C([0,T], L^2(\mathbb{R}))$, $u_2 \in C([0,T], L^2(\mathbb{R}, e^{2bx}dx))$, and

$$Pu_1 = Pu_2 = 0 \quad \text{on } (0,T)\times\mathbb{R},$$

$$u_1(0,x) = \begin{cases} u_0(x) & \text{for a.e. } x > 0, \\ 0 & \text{for a.e. } x < 0, \end{cases} \qquad u_2(0,x) = \begin{cases} u_T(-x) & \text{for a.e. } x < 0, \\ 0 & \text{for a.e. } x > 0. \end{cases}$$

Now set $\tilde{u}_2(t,x) = u_2(T-t,-x)$. Obviously $P\,\tilde{u}_2 = 0$ and $\tilde{u}_{2_{|t=T}} = \tilde{u}_T$ on $(0,+\infty)$. Let ϵ' be any number in $(\epsilon, \frac{T}{2})$ and let $\varphi \in C^\infty([0,T])$ be such that $\varphi(t) = 1$ for $t \leq \epsilon'$ and $\varphi(t) = 0$ for $t \geq T - \epsilon'$. The change of functions

$$u(t,x) = \varphi(t)u_1(t,x) + \big(1-\varphi(t)\big)\tilde{u}_2(t,x) + w(t,x)$$

transforms (1.4) into

$$\begin{cases} P\,w &= \frac{d\varphi}{dt}\,(\tilde{u}_2 - u_1) \quad \text{in } \mathcal{D}'\big((0,T)\times(0,+\infty)\big), \\ w_{|t=0} &= w_{|t=T} = 0 \quad \text{on } (0,+\infty). \end{cases}$$

Setting $f(t,x) = \varphi'(t)\big(\tilde{u}_2(t,x) - u_1(t,x)\big)$, it is clear that we are finished if the following result is proved.

PROPOSITION 4.1. *Let $f = f(t,x)$ be any function in $L^2_{loc}(\mathbb{R}^2)$ such that*

$$\operatorname{supp} f \subset [t_1, t_2]\times\mathbb{R}$$

where $0 < t_1 < t_2 < T$. Let $\epsilon \in (0, \min(t_1, T-t_2))$. Then there exists $u \in L^2_{loc}(\mathbb{R}^2)$ such that

$$P\,u = f \ \text{ in } \mathcal{D}'(\mathbb{R}^2) \quad \text{and} \quad \operatorname{supp} u \subset [t_1-\epsilon, t_2+\epsilon]\times\mathbb{R}. \tag{4.2}$$

Remark 2. The question whether Proposition 4.1 remains valid with $\epsilon = 0$ is open. Notice that the answer is negative for the heat equation. (See Remark 3 below.)

As in [18] the proof of Proposition 4.1 rests on an approximation theorem (Lemma 4.4), which in turn is obtained as a consequence of two preliminary lemmas. In what follows S_L will denote the unitary group in $L^2(-L,L)$ generated by the operator $Au = -u_{xxx} - u_x$ with domain

$$\mathcal{D}(A) = \{u \in H^3(-L,L);\ u(-L) = u(L),\ u_x(-L) = u_x(L),\ u_{xx}(-L) = u_{xx}(L)\}\cdot$$

Set $e_n(x) = \frac{1}{\sqrt{2L}} e^{in\frac{\pi}{L}x}$ for $n \in \mathbb{Z}$. e_n is an eigenvector for A associated with the eigenvalue $\omega_n = i\lambda_n$, with

$$\lambda_n = \left(n\frac{\pi}{L}\right)^3 - n\frac{\pi}{L}\cdot \tag{4.3}$$

If u_0 is any complex-valued function in $L^2(-L,L)$, decomposed as $u_0 = \sum_{n\in\mathbb{Z}} c_n e_n$, we have for every $t \in \mathbb{R}$

$$S_L(t)\,u_0 = \sum_{n\in\mathbb{Z}} e^{i\lambda_n t} c_n e_n. \tag{4.4}$$

We are now ready to state the first lemma, which may be seen as a preliminary version to the approximation theorem.

Lemma 4.2. *Let l_1, l_2, L, t_1, t_2, T be numbers such that $0 < l_1 < l_2 < L$ and $0 < t_1 < t_2 < T$. Let $u \in L^2\left((0,T)\times(-l_2,l_2)\right)$ be such that*

$$P\,u = 0 \quad \text{in } (0,T)\times(-l_2,l_2) \qquad \text{and} \quad \operatorname{supp} u \subset [t_1,t_2]\times(-l_2,l_2). \tag{4.5}$$

Let $\delta > 0$ with $2\delta < \min(t_1, T-t_2)$ and $\eta > 0$ be given. Then there exist $v_1, v_2 \in L^2(-L,L)$ and $v \in L^2\left((0,T)\times(-L,L)\right)$ such that

$$P\,v = 0 \quad \text{in } (0,T)\times(-L,L), \tag{4.6}$$

$$v(t,\cdot) = S_L(t - t_1 + 2\delta)v_1 \quad \text{for} \quad t_1 - 2\delta < t < t_1 - \delta, \tag{4.7}$$

$$v(t,\cdot) = S_L(t - t_2 - \delta)v_2 \quad \text{for} \quad t_2 + \delta < t < t_2 + 2\delta, \tag{4.8}$$

$$\|v - u\|_{L^2((t_1-2\delta,t_2+2\delta)\times(-l_1,l_1))} < \eta. \tag{4.9}$$

Roughly speaking, (4.7)–(4.8) mean that for $t \in (t_1-2\delta, t_1-\delta)\cup(t_2+\delta, t_2+2\delta)$ v satisfies (in addition to $P\,v = 0$) the boundary conditions $v(-L) = v(L)$, $v_x(-L) = v_x(L)$, and $v_{xx}(-L) = v_{xx}(L)$.

Proof of Lemma 4.2. Set $Q = (0,T)\times(-L,L)$, $Q_\delta = (t_1-2\delta, t_2+2\delta)\times(-l_1,l_1)$. Smoothing u by convolution and multiplying the regularized function by a cut-off function (of x), we easily get a function $u' \in \mathcal{D}(\mathbb{R}^2)$ such that

$$\begin{cases} \operatorname{supp} u' \subset [t_1-\delta, t_2+\delta]\times[-l_2,l_2], \\ P\,u' = 0 \ \text{ in } (0,T)\times(-l_1,l_1) \text{ and} \\ \|u'-u\|_{L^2\left((0,T)\times(-l_1,l_1)\right)} < \frac{\eta}{2}. \end{cases} \tag{4.10}$$

Let

$$\mathcal{E} = \{v \in L^2(Q);\ \exists v_1, v_2 \in L^2(-L,L) \text{ s.t. } (4.6),\ (4.7),\ \text{and } (4.8) \text{ hold true}\}.$$

The lemma is proved if we may find $v \in \mathcal{E}$ such that $\|v - u'\|_{L^2(Q_\delta)} < \frac{\eta}{2}$. We are finished if we prove $u' \in \overline{\mathcal{E}} = \mathcal{E}^{\perp\perp}$, where the closure and the orthogonal complement are taken in the space $L^2(Q_\delta)$. Fix a function $g \in \mathcal{E}^\perp \subset L^2(Q_\delta)$. Before proving $(u', g)_{L^2(Q_\delta)} = 0$, we begin with the following claim.

Claim 1. *Let* $\mathcal{T} = \{\varphi \in C^\infty(\mathbb{R}^2);\ \text{supp}\ \varphi \subset [t_1 - \delta, t_2 + \delta] \times \mathbb{R}\}$. *Then there exists* $C > 0$ *such that*

$$\text{for all } \varphi \in \mathcal{T} \quad |(\varphi, g)_{L^2(Q_\delta)}| \le C\|P\varphi\|_{L^2(Q)}. \tag{4.11}$$

Proof of Claim 1. Let $\varphi \in \mathcal{T}$, and set $\psi(t) := \int_0^t S_L(t-\tau)P\varphi(\tau)\, d\tau$ for $0 \le t \le T$; that is, ψ is the (strong) solution of the following boundary initial-value problem:

$$\begin{aligned} P\psi &= P\varphi \qquad \text{in } Q,\\ \psi(t,-L) &= \psi(t,L),\\ \psi_x(t,-L) &= \psi_x(t,L),\\ \psi_{xx}(t,-L) &= \psi_{xx}(t,L),\\ \psi(0,\cdot) &= 0. \end{aligned}$$

Clearly $v := \psi - \varphi \in \mathcal{E}$ ((4.7)–(4.8) hold true with $v_1 = 0$, $v_2 = \psi(t_2 + \delta)$); hence $(\psi - \varphi, g)_{L^2(Q_\delta)} = 0$. On the other hand, it is clear that

$$\text{for all } t \in [0,T] \quad \|\psi(t)\|_{L^2(-L,L)} \le \|P\varphi\|_{L^1(0,t,L^2(-L,L))} \le \sqrt{T}\|P\varphi\|_{L^2(Q)};$$

hence

$$|(\varphi, g)|_{L^2(Q_\delta)} = |(\psi, g)|_{L^2(Q_\delta)} \le T\|g\|_{L^2(Q_\delta)} \cdot \|P\varphi\|_{L^2(Q)}.$$

This completes the proof of Claim 1. We now proceed to the next claim.

Claim 2. *There exists a function* $w \in L^2(Q)$ *such that*

$$\text{for all } \varphi \in \mathcal{T} \quad (\varphi, g)_{L^2(Q_\delta)} = (P\varphi, w)_{L^2(Q)}. \tag{4.12}$$

Proof of Claim 2. Let $\mathcal{Z} := \{(P\varphi)_{|Q};\ \varphi \in \mathcal{T}\}$. Notice first that for any $\zeta \in \mathcal{Z}$, if $\zeta = (P\varphi_1)_{|Q} = (P\varphi_2)_{|Q}$ for two functions $\varphi_1, \varphi_2 \in \mathcal{T}$, then $\varphi_1 - \varphi_2 \in \mathcal{E}$; hence $(\varphi_1 - \varphi_2, g)_{L^2(Q_\delta)} = 0$. It follows that the (linear) map $\Lambda : \zeta \in \mathcal{Z} \mapsto (\varphi, g)_{L^2(Q_\delta)} \in \mathbb{R}$ (if $\zeta = (P\varphi)_{|Q}$, $\varphi \in \mathcal{T}$) is well defined. Let H denote the closure of $\mathcal{Z}$ in $L^2(Q)$. We infer from (4.11) that Λ may be extended to H in such a way that Λ is a continuous linear form on H. It follows from Riesz representation theorem that there exists $w \in H$ such that $\Lambda(\zeta) = (\zeta, w)_{L^2(Q)}$ for all $\zeta \in H$. Then (4.12) holds true.

We are now ready to prove $(u', g)_{L^2(Q_\delta)} = 0$. Extend g and w on $\mathbb{R}^2$ to $\tilde{g}$, $\tilde{w}$ by setting

$$\begin{aligned} \tilde{g}(t,x) &= 0 \quad \text{for } (t,x) \in \mathbb{R}^2 \setminus Q_\delta,\\ \tilde{w}(t,x) &= 0 \quad \text{for } (t,x) \in \mathbb{R}^2 \setminus Q. \end{aligned}$$

Set $\Omega = (t_1 - \delta, t_2 + \delta) \times \mathbb{R}$ and let $\varphi \in \mathcal{D}(\Omega) \subset \mathcal{T}$. Obviously

$$(\varphi, g)_{L^2(Q_\delta)} = (\varphi, \tilde{g})_{L^2(\Omega)} \quad \text{and} \quad (P\varphi, w)_{L^2(Q)} = (P\varphi, \tilde{w})_{L^2(\Omega)};$$

hence it follows from (4.12) that

$$\langle P^* \tilde{w}, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle \tilde{g}, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Thus

$$P^* \tilde{w} = \tilde{g} \;\text{ in }\; \mathcal{D}'(\Omega)$$

and

$$P^* \tilde{w} = 0 \;\text{ for }\; t_1 - \delta < t < t_2 + \delta \;\text{ and }\; |x| > l_1.$$

Since

$$\tilde{w}(t,x) = 0 \;\text{ for } t_1 - \delta < t < t_2 + \delta \text{ and } |x| > L$$

we infer from Holmgren's uniqueness theorem (see [5, Thm. 8.6.8]) that

$$\tilde{w}(t,x) = 0 \text{ for } t_1 - \delta < t < t_2 + \delta \;\text{ and }\; |x| > l_1. \tag{4.13}$$

Applying (4.12) to $u' \in \mathcal{T}$ and using (4.10), (4.13) we get

$$\begin{aligned}(u', g)_{L^2(Q_\delta)} &= (P\,u', w)_{L^2(Q)} \\ &= (P\,u', w)_{L^2\left((t_1-\delta, t_2+\delta)\times(-l_1,l_1)\right)} \\ &= 0.\end{aligned}$$

The proof of Lemma 4.2 is complete. □

Next result is an observability result.

LEMMA 4.3. *Let l, L, T be positive numbers such that $l < L$. Then there exists a constant $C > 0$ such that for every $u_0 \in L^2(-L,L)$, if u denotes $S_L(\cdot)u_0$, we have*

$$\|u_0\|_{L^2(-L,L)} \le C\|u\|_{L^2\left((0,T)\times(-l,l)\right)}. \tag{4.14}$$

(Hence

$$\|u\|_{L^2((0,T)\times(-L,L))} \le \sqrt{T}C\|u\|_{L^2((0,T)\times(-l,l))}.) \tag{4.15}$$

Proof. Pick $T' \in (0, \frac{T}{2})$ and $\gamma > \frac{\pi}{T'}$. Let $N \in \mathbb{N}$ be such that

$$\lambda_N - \lambda_{-N} = 2\lambda_N \ge \gamma \;\text{ and } (n \in \mathbb{Z},\; |n| \ge N) \Rightarrow \lambda_{n+1} - \lambda_n \ge \gamma.$$

By Ingham's inequality (see [7]) there exists a constant $C^{T'} > 0$ such that for every sequence $(a_n)_{|n|\ge N}$ of complex numbers, with $a_n = 0$ for $|n|$ large enough, the following inequality holds true:

$$\sum_{|n|\ge N} |a_n|^2 \le C^{T'} \int_{-T'}^{T'} \left| \sum_{|n|\ge N} a_n e^{-i\lambda_n t}\right|^2 dt. \tag{4.16}$$

Let $\mathcal{Z}_n := \mathrm{Span}(e_n)$ for $n \in \mathbb{Z}$ and $\mathcal{Z} = \bigoplus_{n\in\mathbb{Z}} \mathcal{Z}_n \subset L^2(-L,L)$. We define a seminorm p in $\mathcal{Z}$ by

$$p(u) := \left(\int_{-l}^{l} |u(x)|^2\, dx\right)^{\frac{1}{2}} \quad \text{for } u \in \mathcal{Z}.$$

p is clearly a norm in each $\mathcal{Z}_n$. On the other hand, if $u_0 \in \mathcal{Z} \cap (\bigoplus_{|n|<N} \mathcal{Z}_n)^\perp$ (i.e., u_0 may be written in the form $u_0 = \sum_{|n|\geq N} c_n e_n$ with $c_n = 0$ for $|n|$ large enough), then applying (4.16) (with $a_n = \frac{c_n}{\sqrt{2L}} e^{i(\lambda_n T' + n\frac{\pi}{L}x)}$) and integrating w.r.t. x on $(-l, l)$, we get

$$2l \sum_{|n|\geq N} \frac{|c_n|^2}{2L} \leq C^{T'} \int_{-l}^{l} \int_0^{2T'} \left| \sum_{|n|\geq N} e^{i\lambda_n \tau} c_n e_n(x) \right|^2 d\tau dx;$$

hence, by Fubini's theorem,

$$\|u_0\|^2_{L^2(-L,L)} \leq \frac{L}{l} C^{T'} \int_0^{2T'} p\big(S_L(\tau)u_0\big)^2 \, d\tau.$$

Finally, for any $u_0 \in L^2(-L, L)$, we have

$$\int_0^{2T'} p\big(S_L(\tau)u_0\big)^2 \, d\tau \leq \|S_L(\cdot)u_0\|^2_{L^2\left((0,2T')\times(-L,L)\right)} = 2T'\|u_0\|^2_{L^2(-L,L)}.$$

Since $T > 2T'$, it follows from [10, Thm. 5.2] that there exists a constant $C > 0$ such that (4.14) holds true for all $z_0 \in \mathcal{Z}$. We get (4.14) for all $u_0 \in L^2(-L, L)$ by a density argument. □

We now proceed to the proof of the following approximation theorem, which differs from the one in [18] by an additional property on the support.

Lemma 4.4. *Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let t_1, t_2, T be numbers such that $0 < t_1 < t_2 < T$. Let $u \in L^2\big((0, T) \times (-n, n)\big)$ be such that $P u = 0$ in $(0, T) \times (-n, n)$ and* supp *$u \subset [t_1, t_2] \times (-n, n)$. Let $0 < \epsilon < \min(t_1, T - t_2)$. Then there exists $v \in L^2\big((0, T) \times (-n - 1, n + 1)\big)$ such that*

(4.17) $$P v = 0 \quad \text{in } (0, T) \times (-n - 1, n + 1),$$

(4.18) $$\text{supp } v \subset [t_1 - \epsilon, t_2 + \epsilon] \times (-n - 1, n + 1),$$

(4.19) $$\|v - u\|_{L^2\left((0,T)\times(-n+1,n-1)\right)} < \epsilon.$$

Proof. Let $\eta > 0$ (to be chosen later). By Lemma 4.2, applied with $L = n + 1$, there exists $\tilde{v} \in L^2\big((0, T) \times (-n - 1, n + 1)\big)$ such that

$$P \tilde{v} = 0 \quad \text{in} \quad (0, T) \times (-n - 1, n + 1),$$

(4.20) $$\tilde{v}(t, \cdot) = S_{n+1}\left(t - t_1 + \frac{\epsilon}{2}\right) v_1 \quad \text{for} \quad t_1 - \frac{\epsilon}{2} < t < t_1 - \frac{\epsilon}{4},$$

(4.21) $$\tilde{v}(t, \cdot) = S_{n+1}\left(t - t_2 - \frac{\epsilon}{4}\right) v_2 \quad \text{for} \quad t_2 + \frac{\epsilon}{4} < t < t_2 + \frac{\epsilon}{2}$$

for some $v_1, v_2 \in L^2(-n - 1, n + 1)$ and

$$\|\tilde{v} - u\|_{L^2\left((t_1-\frac{\epsilon}{2},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)} < \eta.$$

In order that (4.18) be fulfilled, we multiply $\tilde{v}$ by a cut-off function. Let $\varphi \in \mathcal{D}(0,T)$ be such that $0 \le \varphi \le 1$, $\varphi(t) = 1$ for all $t \in [t_1 - \frac{\epsilon}{4}, t_2 + \frac{\epsilon}{4}]$ and $\text{supp}\,(\varphi) \subset [t_1 - \frac{\epsilon}{2}, t_2 + \frac{\epsilon}{2}]$. Set $\bar{v}(t,x) = \varphi(t)\tilde{v}(t,x)$. It follows that

$$\text{supp }\bar{v} \subset \left[t_1 - \frac{\epsilon}{2}, t_2 + \frac{\epsilon}{2}\right] \times (-n-1, n+1)\cdot$$

Hence

$$\begin{aligned}
&\|\bar{v} - u\|_{L^2\left((0,T)\times(-n+1,n-1)\right)} \\
&\quad = \|\bar{v} - u\|_{L^2\left((t_1-\frac{\epsilon}{2},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)} \\
&\quad \le \|(\varphi - 1)\tilde{v}\|_{L^2\left((t_1-\frac{\epsilon}{2},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)} \\
&\qquad + \|\tilde{v} - u\|_{L^2\left((t_1-\frac{\epsilon}{2},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)}.
\end{aligned}$$

Since $\text{supp }u \subset [t_1, t_2] \times (-n, n)$ and $\varphi(t) = 1$ for $t_1 - \frac{\epsilon}{4} \le t \le t_2 + \frac{\epsilon}{4}$, we get

$$\begin{aligned}
&\|(\varphi - 1)\tilde{v}\ \|^2_{L^2\left((t_1-\frac{\epsilon}{2},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)} \\
&\quad \le \|\tilde{v}\|^2_{L^2\left(\{(t_1-\frac{\epsilon}{2},t_1-\frac{\epsilon}{4})\cup(t_2+\frac{\epsilon}{4},t_2+\frac{\epsilon}{2})\}\times(-n+1,n-1)\right)} \\
&\quad \le \|\tilde{v} - u\|^2_{L^2\left((t_1-\frac{\epsilon}{2},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)} \\
&\quad < \eta^2\cdot
\end{aligned} \tag{4.22}$$

Hence

$$\|\bar{v} - u\|_{L^2\left((0,T)\times(-n+1,n-1)\right)} \le 2\eta\cdot \tag{4.23}$$

Finally $P\,\bar{v} = \frac{d\varphi}{dt}\,\tilde{v}$ in $(0,T) \times (-n-1, n+1)$; hence

$$\begin{aligned}
&\|P\,\bar{v}\ \|^2_{L^2\left((0,T)\times(-n-1,n+1)\right)} \\
&\quad \le \|\tfrac{d\varphi}{dt}\|^2_{L^\infty(0,T)} \cdot \|\tilde{v}\|^2_{L^2\left(\{(t_1-\frac{\epsilon}{2},t_1-\frac{\epsilon}{4})\cup(t_2+\frac{\epsilon}{4},t_2+\frac{\epsilon}{2})\}\times(-n-1,n+1)\right)}.
\end{aligned}$$

Since (4.20), (4.21) hold true, we infer from Lemma 4.3 that there exists a constant $C = C(n, \epsilon) > 0$ such that

$$\|\tilde{v}\|_{L^2\left((t_1-\frac{\epsilon}{2},t_1-\frac{\epsilon}{4})\times(-n-1,n+1)\right)} \le C\|\tilde{v}\|_{L^2\left((t_1-\frac{\epsilon}{2},t_1-\frac{\epsilon}{4})\times(-n+1,n-1)\right)}$$

and also

$$\|\tilde{v}\|_{L^2\left((t_2+\frac{\epsilon}{4},t_2+\frac{\epsilon}{2})\times(-n-1,n+1)\right)} \le C\|\tilde{v}\|_{L^2\left((t_2+\frac{\epsilon}{4},t_2+\frac{\epsilon}{2})\times(-n+1,n-1)\right)}.$$

Hence, by (4.22),

$$\|P\,\bar{v}\|_{L^2\left((0,T)\times(-n-1,n+1)\right)} \le C\|\frac{d\varphi}{dt}\|_{L^\infty(0,T)}\,\eta\cdot \tag{4.24}$$

We finally modify $\bar{v}$ in order that (4.17) be satisfied. By Corollary 3.2 there exist a constant $C' > 0$ (which depends on n, t_1, t_2, and ϵ) and a function $w \in L^2\big((0,T) \times (-n-1, n+1)\big)$ such that

$$\begin{gathered}
P\,w = P\,\bar{v} \ \text{ in } (0,T) \times (-n-1, n+1), \\
\text{supp }w \subset [t_1 - \epsilon, t_2 + \epsilon] \times (-n-1, n+1),
\end{gathered}$$

and also

$$\|w\|_{L^2\left((0,T)\times(-n-1,n+1)\right)} \le C'\|P\,\bar v\|_{L^2\left((0,T)\times(-n-1,n+1)\right)}. \tag{4.25}$$

Set $v := \bar v - w$. Then (4.17) and (4.18) are obvious, and we infer from (4.23), (4.24), and (4.25) that

$$\|v-u\|_{L^2\left((0,T)\times(-n+1,n-1)\right)} \le \left(2 + CC'\left\|\frac{d\varphi}{dt}\right\|_{L^\infty(0,T)}\right)\eta.$$

Hence (4.19) holds true provided that η is small enough. □

We now turn to the proof of Proposition 4.1, which is carried out as in [18].

Proof of Proposition 4.1. Let $(t_1^n)_{n\ge 2}$ and $(t_2^n)_{n\ge 2}$ be two sequences of numbers such that

$$\text{for all } n\ge 2 \qquad t_1-\epsilon < t_1^{n+1} < t_1^n < t_1 < t_2 < t_2^n < t_2^{n+1} < t_2+\epsilon. \tag{4.26}$$

We construct (by induction on n) a sequence $(u_n)_{n\ge 2}$ of functions such that, for every $n\ge 2$,

$$u_n \in L^2\big((0,T)\times(-n,n)\big), \tag{4.27}$$

$$\text{supp } u_n \subset [t_1^n, t_2^n]\times(-n,n), \tag{4.28}$$

$$P\,u_n = f \quad\text{in}\quad (0,T)\times(-n,n), \tag{4.29}$$

and, if $n>2$,

$$\|u_n - u_{n-1}\|_{L^2\left((0,T)\times(-n+2,n-2)\right)} < 2^{-n}. \tag{4.30}$$

u_2 is given by Corollary 3.2. Now let $n\ge 2$ and assume that $u_2,\dots,u_n$ have been constructed in such a way that (4.27)–(4.30) hold true. By Corollary 3.2 there exists $w\in L^2\big((0,T)\times(-n-1,n+1)\big)$ such that

$$\text{supp } w\subset[t_1^2,t_2^2]\times(-n-1,n+1) \quad\text{and}\quad P\,w=f \quad\text{in}\quad (0,T)\times(-n-1,n+1).$$

Since $P(u_n-w)=0$ in $(0,T)\times(-n,n)$ and

$$\text{supp}\,(u_n - w_{|(0,T)\times(-n,n)}) \subset [t_1^n,t_2^n]\times(-n,n),$$

with $t_1^{n+1} < t_1^n < t_2^n < t_2^{n+1}$, it follows from Lemma 4.4 that there exists a function $v\in L^2\big((0,T)\times(-n-1,n+1)\big)$ such that

$$\text{supp } v\subset[t_1^{n+1},t_2^{n+1}]\times(-n-1,n+1), \quad P\,v=0 \text{ in } (0,T)\times(-n-1,n+1)$$

and also

$$\|v-(u_n-w)\|_{L^2\left((0,T)\times(-n+1,n-1)\right)} < 2^{-n-1}.$$

We set $u_{n+1} := v+w$. Then (4.27)–(4.30) are fulfilled. Extending the u_n's by setting $u_n(t,x)=0$ for $(t,x)\notin(0,T)\times(-n,n)$, we infer from (4.30) that the sequence $(u_n)_{n\ge2}$ converges in $L^2_{loc}(\mathbb{R}^2)$ towards a function u such that

$$\text{supp } u \subset [t_1-\epsilon, t_2+\epsilon]\times\mathbb{R}$$

by (4.26) and (4.28). Finally $P\,u=f$ in $\mathbb{R}^2$, because of (4.29). This completes the proof of Proposition 4.1 and also the proof of Theorem 1.3. □

5. The heat equation and the Schrödinger equation. In this section, we are concerned with the control of the heat equation and of the Schrödinger equation in unbounded domains. Let us first briefly discuss the controllability of the heat equation

$$u_t - \Delta u = 0, \tag{5.1}$$

where $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$, $N \geq 1$. By Proposition 1.1, the (boundary or internal) *approximate* controllability of (5.1) in unbounded domains is obvious. Notice that this result is still valid (but not so obvious) for a *semilinear* heat equation; see [23]. As far as the boundary null-controllability of (5.1) is concerned, it has been proved in [17] that no (nontrivial) function in $\mathcal{D}(\Omega)$ (the space of test functions) can be driven to 0 when $\Omega = \mathbb{R}^N_+ := \{(x', x_N) :\ x' \in \mathbb{R}^{N-1},\ x_N > 0\}$ and solutions of (5.1) are taken in some *transposition* sense. However, if *all* the solutions of (5.1) are taken into consideration, then the null-controllability is recovered, thanks to the following result by Jones (see [8], [13]).

THEOREM 5.1. *Let $g \in C^0(\mathbb{R}^N)$ and $T > 0$. Then there exists a function $u \in C^0([0,T] \times \mathbb{R}^N)$ which solves* (5.1) *in the distributional sense for $t > 0$ and satisfies $u_{|t=0} = g$, $u_{|t=T} = 0$.*

Notice that, as it has been pointed out in [13], the boundary control problem is solved once and for all without reference to any *specific* domain or set of boundary conditions. Theorem 5.1 is derived in [8] from the existence, for any $\epsilon > 0$, of a fundamental solution of the heat equation which is supported in the strip $[0, \epsilon] \times \mathbb{R}^N$. A result close to Theorem 5.1 may be proved along the same lines as in section 4.

THEOREM 5.2. *Let $\big(S(t)\big)_{t\geq 0}$ denote the continuous semigroup on $L^2(\mathbb{R}^N)$ generated by the operator $Au = \Delta u$ with domain $H^2(\mathbb{R}^N)$. Let T, ϵ be positive numbers with $\epsilon < \frac{T}{2}$ and let $u_0, u_1 \in L^2(\mathbb{R}^N)$. Then there exists a function*

$$u \in L^2_{loc}\big([0,T] \times \mathbb{R}^N)\big) \cap C\big([0,\epsilon] \cup [T-\epsilon, T], L^2(\mathbb{R}^N)\big)$$

which solves

$$\begin{cases} u_t - \Delta u &= 0 \qquad \text{in } \mathcal{D}'\big((0,T) \times \mathbb{R}^N\big), \\ u_{|t=0} &= u_0, \\ u_{|t=T} &= S(T)u_1. \end{cases} \tag{5.2}$$

Proof. Set, for any $L > 0$, $\Omega_L := (-L, L)^N$. Let P_1 denote the operator $\frac{\partial}{\partial t} - \Delta$. Let $\epsilon' \in (\epsilon, \frac{T}{2})$ and let $\varphi \in C^\infty([0,T])$ be such that $\varphi(t) = 1$ for $t \leq \epsilon'$ and $\varphi(t) = 0$ for $t \geq T - \epsilon'$. The change of functions

$$u(t,\cdot) = \varphi(t) S(t) u_0 + \big(1 - \varphi(t)\big) S(t) u_1 + w(t,\cdot)$$

transforms (5.2) into

$$\begin{cases} P_1 w &= \frac{d\varphi}{dt} S(t)(u_1 - u_0) \quad \text{ in } \mathcal{D}'\big((0,T) \times \mathbb{R}^N\big), \\ w_{|t=0} &= w_{|t=T} = 0. \end{cases}$$

Once again, it is clear that we are finished if Proposition 4.1 holds true with P_1 instead of P. The estimate (see [3, Lem. 5.2], [4, Thm. 4.1])

$$\exists k > 0,\ \exists C > 0 \text{ s.t.} \quad \int_0^T \int_{\Omega_L} |q|^2 e^{-\frac{k}{t(T-t)}} \, dxdt \leq C \int_0^T \int_{\Omega_L} |q_t + \Delta q|^2 \, dxdt$$

for any $q \in C^2([0,T]\times[-L,L]^N)$ such that $q(t,x)=\partial_n q(t,x)=0$ for $(t,x)\in[0,T]\times\partial\Omega_L$ shows that Corollary 3.2 holds true with P_1 instead of P. The other key ingredient in the proof of Proposition 4.1, namely the internal observability result (Lemma 4.3), may be found in the literature (see [12, Cor. 2]). If $\big(S_L(t)\big)_{t\geq 0}$ now denotes the continuous *semigroup* on $L^2(\Omega_L)$ generated by the operator $Au=\Delta u$ with domain $H^2_{per}(\Omega_L)$, then the proofs of Lemmas 4.2 and 4.4 and of Proposition 4.1 are word for word the same as those given above for the KdV equation. □

Remark 3. For the heat equation, the results in Corollary 3.2 and in Proposition 4.1 are no longer true if we do $\epsilon=0$. Indeed, if we assume that Corollary 3.2 is true for the one-dimensional heat equation with $\epsilon=0$ (for any f), then an argument similar to the one used in the proof of Theorem 1.2 shows that for some constant $C>0$ we have

$$\int_0^T\int_{-L}^{L}|q|^2\,dxdt\leq C\int_0^T\int_{-L}^{L}|q_t-q_{xx}|^2\,dxdt \tag{5.3}$$

for any $q\in C^2([0,T]\times[-L,L])$ such that $q(t,\pm L)=q_x(t,\pm L)=0$. Let E be the classical fundamental solution of the one-dimensional heat equation, namely

$$E(t,x)=\begin{cases}(4\pi t)^{-\frac{1}{2}}\exp(-\frac{x^2}{4t}) & \text{if } t>0,\ x\in\mathbb{R},\\ 0 & \text{otherwise.}\end{cases}$$

Let $\rho\in\mathcal{D}(-L,L)$ be such that $\rho(x)=1$ for $|x|<\frac{L}{2}$. Set $q(t,x)=\rho(x)\frac{\partial E}{\partial x}(t,x)$ for $x\in(-L,L)$ and $t>0$. Direct computations show that $\|q\|_{L^2((\epsilon,T+\epsilon)\times(-L,L))}\to+\infty$ as $\epsilon\to 0^+$, whereas $\|q_t-q_{xx}\|_{L^2((\epsilon,T+\epsilon)\times(-L,L))}=O(1)$, contradicting (5.3).

We now turn to the Schrödinger equation

$$iu_t+\Delta u=0. \tag{5.4}$$

For the sake of simplicity, we restrict ourselves to the one-dimensional case (i.e., $N=1$). The following result is proved in the same way as Theorem 1.3.

THEOREM 5.3. *Let T,ϵ be positive numbers with $\epsilon<\frac{T}{2}$ and let $u_0,u_T\in L^2(\mathbb{R})$. Then there exists a function*

$$u\in L^2_{loc}\big([0,T]\times\mathbb{R}\big)\cap C\big([0,\epsilon]\cup[T-\epsilon,T],L^2(\mathbb{R})\big)$$

which solves

$$\begin{cases} iu_t+u_{xx} &=0 \quad \text{in } \mathcal{D}'\big((0,T)\times\mathbb{R}\big),\\ u_{|t=0} &=u_0,\\ u_{|t=T} &=u_T.\end{cases} \tag{5.5}$$

Proof. We shall write P_2 for the operator $\frac{\partial}{\partial t}-i\frac{\partial^2}{\partial x^2}$. Let $\big(S(t)\big)_{t\in\mathbb{R}}$ denote the unitary group on $L^2(\mathbb{R})$ generated by the operator $Au=iu_{xx}$ with domain $H^2(\mathbb{R})$. Let ϵ' and φ be as in the proof of Theorem 5.2. The change of functions

$$u(t,\cdot)=\varphi(t)S(t)u_0+\big(1-\varphi(t)\big)S(t-T)u_T+w(t,\cdot)$$

transforms (5.5) into

$$\begin{cases} P_2\,w &=\frac{d\varphi}{dt}\,S(t)\big(S(-T)u_T-u_0\big) \quad \text{in } \mathcal{D}'\big((0,T)\times\mathbb{R}\big),\\ w_{|t=0} &=w_{|t=T}=0.\end{cases}$$

We are again led to prove Proposition 4.1, but with P_2 instead of P. Let S_L denote here the unitary group on $L^2(-L,L)$ generated by the operator $Au = iu_{xx}$ with domain $\{u \in H^2(-L,L);\ u(-L) = u(L),\ u_x(-L) = u_x(L)\}$. Let $e_n(x) = \frac{1}{\sqrt{2L}} e^{in\frac{\pi}{L}x}$ and $\lambda_n = (n\frac{\pi}{L})^2$ for $n \in \mathbb{Z}$. If $u_0 \in L^2(-L,L)$ is decomposed as $u_0 = \sum_{n\in\mathbb{Z}} c_n e_n$, then $S(t)u_0 = \sum_{n\in\mathbb{Z}} e^{-i\lambda_n t} c_n e_n$ for all $t \in \mathbb{R}$.

A proof of Corollary 3.2 using a controllability result in the literature instead of a Carleman's estimate is as follows. Let $w(t) := \int_{t_1}^t S_L(t-\tau) f(\tau)\, d\tau \in L^2(-L,L)$ for $t_1 \le t \le t_2$. By [15, Thm. 1.2] (the result being in fact true for any final state $y_T \in L^2(\Omega)$ instead of 0) there exists some (internal) control function $h \in L^2((t_1,t_2)\times(L,L+1))$ such that the solution $y \in C([t_1,t_2], L^2(-L,L+1))$ of

$$\begin{cases} y_t - iy_{xx} &= h\chi_{(L,L+1)} \quad \text{in } (t_1,t_2)\times(-L,L+1),\\ y &= 0 \quad \text{on } (t_1,t_2)\times\{-L,L+1\},\\ y_{|t=t_1} &= 0 \end{cases}$$

satisfies $y_{|t=t_2} = -w(t_2)$ on $(-L,L)$. Clearly, the function

$$v(t) := \begin{cases} w(t) + y(t) & \text{if } t_1 \le t \le t_2,\\ 0 & \text{otherwise} \end{cases}$$

fulfills $v_t - iv_{xx} = f$ in $\mathcal{D}'(\mathbb{R}\times(-L,L))$, (3.17) (with $\epsilon = 0$) and (3.18).

As for the KdV equation the proof of Lemma 4.3 rests on Ingham's inequality and on [10, Thm. 5.2]. Here $\mathcal{Z} = \bigoplus_{n\in\mathbb{N}} \mathcal{Z}_n$, with $\mathcal{Z}_0 = \mathrm{Span}(e_0)$ and $\mathcal{Z}_n = \mathrm{Span}(e_n, e_{-n})$ for $n \ge 1$. To properly handle the left-hand side in Ingham's estimate

$$\sum_{n\ge N} |a_n|^2 \le C^{T'} \int_{-T'}^{T'} \left| \sum_{n\ge N} a_n e^{-i\lambda_n t} \right|^2 dt \tag{5.6}$$

(which is applied with $a_n = (c_n e_n + c_{-n} e_{-n}) e^{-i\lambda_n T'}$) it is sufficient to observe that the estimate

$$\|c_n e_n + c_{-n} e_{-n}\|^2_{L^2(-l,l)} \ge \frac{1}{2}\big(\|c_n e_n\|^2_{L^2(-l,l)} + \|c_{-n} e_{-n}\|^2_{L^2(-l,l)}\big)$$

holds true provided that $|n|$ is large enough, which implies (for N large enough)

$$\int_{-l}^{l} \sum_{n\ge N} |c_n e_n + c_{-n} e_{-n}|^2\, dx \ge \frac{l}{2L} \sum_{|n|\ge N} |c_n|^2 = \frac{l}{2L} \|u_0\|^2_{L^2(-L,L)}.$$

The rest of the proof of Lemma 4.3 and of Proposition 4.1 is as above for the KdV equation. ∎

Appendix. Proof of Proposition 1.1.

We argue by contradiction and assume that $\mathcal{R} = L^2(\mathbb{R})$. Consider the map

$$\Lambda : f \in L^2\big((0,T)\times(L_1,L_2)\big) \to \int_0^T S(T-t)\tilde{f}(t,\cdot)dt \in L^2(\mathbb{R})$$

(where $\tilde{f}$ is the prolongation of f by 0 on $\mathbb{R}^2$). Let $N = \ker(\Lambda)$. Then the restriction of Λ to the orthogonal complement of N in $L^2\big((0,T)\times(L_1,L_2)\big)$ is a one-to-one continuous linear map which is onto $L^2(\mathbb{R})$; hence its inverse $(\Lambda_{|_{N^\perp}})^{-1}$ is continuous.

Let $f \in \mathcal{D}\big((0,T)\times(L_1,L_2)\big)$ and let $w_T \in D(A^*)$, where A^* denotes the adjoint of the operator A. (Clearly $A^*w = \sum_{i=0}^{n}(-1)^i a_i \frac{d^i w}{dx^i}$.) Recall that A^* generates the continuous semigroup $\big(S^*(t)\big)_{t\geq 0}$ on $L^2(\mathbb{R})$. Set $w(t) := S^*(T-t)w_T$ for $t \in [0,T]$. Then w solves

$$\begin{cases} \frac{dw}{dt} & = -A^*w, \\ w_{|_{t=T}} & = w_T. \end{cases}$$

Let $u(t) = \int_0^t S(t-s)\tilde{f}(s,\cdot)ds$. Integrating by part in

$$\int_0^T \int_{\mathbb{R}} \left(\frac{dw}{dt} + A^*w\right) u\, dxdt = 0,$$

we get

$$\text{(A.1)} \qquad \int_0^T \int_{L_1}^{L_2} f(t,x)w(t,x)\,dxdt = \int_{\mathbb{R}} w_T(x)u(T,x)\,dx.$$

The same equation holds true (by density) for $f \in L^2\big((0,T)\times(L_1,L_2)\big)$ and $w_T \in L^2(\mathbb{R})$. Letting $f = (\Lambda_{|_{N^\perp}})^{-1}(w_T)$, where w_T is any function in $L^2(\mathbb{R})\setminus\{0\}$, we get

$$\|w_T\|^2_{L^2(\mathbb{R})} \leq \|f\|_{L^2\left((0,T)\times(L_1,L_2)\right)} \cdot \|w\|_{L^2\left((0,T)\times(L_1,L_2)\right)};$$

hence

$$\|w_T\|_{L^2(\mathbb{R})} \leq \|(\Lambda_{|_{N^\perp}})^{-1}\| \cdot \|w\|_{L^2\left((0,T)\times(L_1,L_2)\right)}.$$

Replacing w_T by $w_T(\cdot+n)$ (and also $w(t,x)$ by $w(t,x+n)$), we get

$$\|w_T\|_{L^2(\mathbb{R})} = \|w_T(\cdot+n)\|_{L^2(\mathbb{R})} \leq \|(\Lambda_{|_{N^\perp}})^{-1}\| \cdot \|w\|_{L^2\left((0,T)\times(L_1+n,L_2+n)\right)}.$$

Letting $n\to\infty$, we get (by Lebesgue's theorem) $w_T = 0$, a contradiction. Thus $\mathcal{R} \neq L^2(\mathbb{R})$. Now let $w_T \in \mathcal{R}^\perp$. We infer from (A.1) that

$$\int_0^T \int_{L_1}^{L_2} f(t,x)w(t,x)\,dxdt = 0$$

for all $f \in L^2\big((0,T)\times(L_1,L_2)\big)$; hence $w_{|_{(0,T)\times(L_1,L_2)}} = 0$. Since $n \geq 2$ (with $a_n \neq 0$) it follows from Holmgren's uniqueness theorem that $w = 0$ in $(0,T)\times\mathbb{R}$; hence $w_T = 0$, and we infer that $\mathcal{R}$ is dense in $L^2(\mathbb{R})$.

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