Journal of Mathematical Systems, Estimation, and Control
Vol. 4, No. 1, 1994, pp. 67–84

A Relation Between Continuous Time-Varying and Discontinuous Feedback Stabilization*

Jean-Michel Coron   Lionel Rosier

Abstract

We prove that a control system which can be asymptotically stabilized *in the Filippov sense* by means of a discontinuous feedback law can be asymptotically stabilized by means of a continuous periodic time-varying feedback law. If, moreover, the control system is affine then it can be asymptotically stabilized by means of a continuous feedback law.

Key words: asymptotic stabilization, discontinuous feedback law, time-varying feedback law

AMS Subject Classifications: 93D15, 93C10

1 Introduction

Let Σ be the control system

$$\Sigma \ : \ \dot{x} = f(x,u) \tag{1.1}$$

where $u \in \mathbb{R}^m$ is the control, $x \in \mathbb{R}^n$ the state, and f a map from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$ such that

$$f \in C^0(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n). \tag{1.2}$$

Throughout this paper we assume

$$f(0,0) = 0. \tag{1.3}$$

It has been proved by R.W. Brockett in [4] that the local controllability of Σ does not imply that Σ can be asymptotically stabilized by means of a

*Received July 7, 1992; received in final form November 30, 1992.

continuous feedback law. To get round this problem two main strategies have been proposed:

(i) Asymptotic stabilization by means of a discontinuous feedback law -see e.g. the pioneer work by H. Sussmann [18] as well as [11], [15], [1], and [5], and the references therein,

(ii) Asymptotic stabilization by means of a continuous periodic time-varying feedback law -see e.g. the pioneer works by E. Sontag and H. Sussmann [19] and by C. Samson [17] as well as [6], [7], and the references therein.

The main goal of this paper is to show a relation between these two strategies. Before stating this relation we need to specify the meaning of asymptotic stability for a system $\dot{x} = X(x)$ where X is a discontinuous vector field. Many definitions are possible but, following H. Hermes [13], it seems natural to adopt

Definition 1.1. *Let $X \in L^{\infty}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Then 0 is a locally asymptotically stable point of $\dot{x} = X(x)$ if*

(i) *for any $\epsilon \geq 0$ there exists $\delta > 0$ such that any solution,* in the Filippov sense, *of $\dot{x} = X(x)$ which is defined at $t = 0$ and is maximal, is defined on $[0, +\infty)$ if $|x(0)| < \delta$ and moreover*

$$|x(0)| < \delta \Longrightarrow |x(t)| < \epsilon, \quad \forall t \in [0, +\infty), \tag{1.4}$$

(ii) *there exists η in $(0,+\infty]$ such that any maximal solution,* in the Filippov sense, *of $\dot{x} = X(x)$ which is defined at $t = 0$ is defined on $[0,+\infty)$ if $|x(0)| < \eta$ and moreover*

$$|x(0)| < \eta \Longrightarrow (x(t) \to 0 \text{ as } t \to \infty). \tag{1.5}$$

Furthermore, if in (ii) *one can take $\eta = +\infty$, then 0 is a globally asymptotically stable point of $\dot{x} = X(x)$.*

Let us recall that a solution in the Filippov sense of $\dot{x} = X(x)$ on an interval I is (see [10]) a locally absolutely continuous map from I into $\mathbb{R}^n$ such that

$$\dot{x}(t) \in F(x(t)) \text{ for almost all } t \text{ in } I \tag{1.6}$$

with

$$F(x) := \bigcap_{\epsilon > 0} \bigcap_{|N| = 0} \overline{\text{conv}} X((x + \epsilon B) \backslash N), \tag{1.7}$$

where B is the unit ball of $\mathbb{R}^n$, and, for a set A, $|A|$ is the Lebesgue measure of 2 and $\overline{\text{conv}} A$ is the smaller closed convex set containing A. A maximal solution (in the Filippov sense) of $\dot{x} = X(x)$ is a solution x on some interval

I such that there exists no solution defined on an interval which contains strictly I and which is equal to x on I.

Note that, if X is continuous,

$$F(y) = \{X(y)\} \tag{1.8}$$

and therefore in this case our definition of asymptotic stability coincide with the usual one.

We now define "asymptotically stabilizable by means of a discontinuous feedback law (resp. periodic time-varying continuous feedback law)".

Definition 1.2. *System Σ is locally (resp. globally) asymptotically stabilizable by means of a discontinuous feedback law if there exists*

$$u \in L^{\infty}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$$

such that

$$\text{Essential Sup } \{|u(x)|;\ |x| < \epsilon\} \to 0 \text{ as } \epsilon \to 0, \tag{1.9}$$

$$\begin{cases} 0 \text{ is a locally (resp. globally) asymptotically} \\ \qquad\qquad \text{stable point of } \dot{x} = f(x, u(x)). \end{cases} \tag{1.10}$$

Definition 1.3. *System Σ is locally (resp. globally) asymptotically stabilizable by means of a continuous time-varying feedback law of period T if there exists $u \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that*

$$\begin{gathered} u(x, t + T) = u(x, t) \ \ \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(0, t) = 0 \ \ \forall t \in \mathbb{R}, \end{gathered} \tag{1.11}$$

$$\begin{cases} 0 \text{ is a locally (resp. globally) asymptotically} \\ \qquad\qquad \text{stable point of } \dot{x} = f(x, u(x, t)). \end{cases} \tag{1.12}$$

The reason for considering in Definition 1.1 solutions in the Filippov sense is, as explained in [13], the following one: the feedback law $u(x(t))$ is determined after making a measurement of the state $x(t)$ at time t; of course this measurement gives only an approximation of $x(t)$: there is an "error" $e(t)$ between $x(t)$ and its measurement. A direct consequence of [13;Lemma 3] is

Proposition 1.4. *Assume that f is locally Lipschitzian with respect to x. Let $x : [0, T] \to \mathbb{R}^n$ be a Filippov solution of $\dot{x} = f(x, u(x))$ where $u \in L^{\infty}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$. Let ϵ be a positive real number. Then there exist $e \in L^{\infty}((0, T); \mathbb{R}^m)$ and an absolutely continuous function $y : [0, T] \to \mathbb{R}^n$ such that*

$$|e(t)| \leq \epsilon \text{ for all } t \text{ in } (0, T), \tag{1.13}$$

$$\dot{y}(t) = f(y(t), u(y(t) + e(t))) \text{ for almost all } t \text{ in } (0,T), \quad (1.14)$$

$$y(0) = x(0), \quad (1.15)$$

and

$$|y(t) - x(t)| \leq \epsilon \text{ for all } t \text{ in } [0,T]. \quad (1.16)$$

Proof: Let $\delta > 0$. By [13;Lemma 3] there exists $\omega \in L^\infty((0,T);\mathbb{R}^m)$, and an absolutely continuous function $z : [0,T] \to \mathbb{R}^n$ such that

$$\dot{z}(t) = f(z(t) + \omega(t),\ u(z(t) + \omega(t))) \text{ for almost all } t \text{ in } (0,T), \quad (1.17)$$

$$z(0) = x(0), \quad (1.18)$$

and

$$|z(t) - x(t)| \leq \delta, \text{ and } |\omega(t)| \leq \delta \text{ for all } t \text{ in } [0,T]. \quad (1.19)$$

Let y be the maximal solution of

$$\dot{y} = f(y, u(z(t) + \omega(t))),\ y(0) = x(0). \quad (1.20)$$

By standard estimates on ordinary differential equations (see e.g. [12;II Thm 3.2]) we have, if δ is small enough

$$y \text{ is defined on } [0,T], \quad (1.21)$$

(1.16), (1.13), and (1.14) with e.g. $e(t) = z(t) + \omega(t) - y(t)$.

Proposition 1.4 justifies our definition of asymptotically stabilization by means of a discontinuous feedback law, and our main result is

Theorem 1.5. *Assume that $\dot{x} = f(x,u)$ can be locally (resp. globally) asymptotically stabilized by means of a discontinuous feedback law. Then, for any $T > 0$, $\dot{x} = f(x,u)$ can be locally (resp. globally) stabilized by means of a continuous time-varying feedback law of period T; if, moreover, $\dot{x} = f(x,u)$ is an affine system (i.e. $f(x,u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$), then $\dot{x} = f(x,u)$ can be locally (resp. globally) asymptotically stabilized by means of a continuous feedback law* (independent of $t : u = u(x)$).

Remark 1.6. There are completely controllable affine systems which are globally asymptotically stabilized by means of a continuous periodic time varying feedback law which cannot be locally asymptotically stabilized by means of a continuous feedback law (e.g. $\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$, $\dot{x}_3 = x_1 u_2 - x_2 u_1$; see [17] or [6]). By Theorem 1.5 these systems cannot be locally asymptotically stabilized by means of a discontinuous feedback law. Note that this is not in contradiction with [18] (or [5] and [M] for the above example): indeed our definition of asymptotic stability is more restrictive than the one used in [18], [5] and [M] (see e.g. the definition of "steers M to p" in [18,p.42]; in particular we do not have any "exit rule" E on the singular set of u in our definition of asymptotic stability).

In Section 2 we will give the proof of Theorem 1.5. In Section 3 we will show that, in opposition to smooth stabilization, $\dot{x} = f(x,u)$ may be asymptotically stabilized by means of a discontinuous feedback law even if $\dot{x} = f(x,y)$, $\dot{y} = v$ cannot be asymptotically stabilized by means of a discontinuous feedback law.

2 Proof of Theorem 1.5

For simplicity we prove only the global statement (the proof of the local statement is similar). Our first lemma is

Lemma 2.1. *Assume that $\dot{x} = f(x,u)$ can be globally asymptotically stabilized by means of a discontinuous feedback law. Then there exists $V \in C^1(\mathbb{R}^n; [0,+\infty))$ such that*

$$V(x) \to +\infty \text{ as } |x| \to +\infty, \tag{2.1}$$

$$\forall x \neq 0 \;\; \exists u \in \mathbb{R}^m \text{ s.t. } f(x,u)\cdot\nabla V(x) < 0, \tag{2.2}$$

and, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x| < \delta \Longrightarrow (\exists u \in \mathbb{R}^m \text{ with } |u| < \epsilon \text{ and } f(x,u)\cdot\nabla V(x) < 0). \tag{2.3}$$

Let us remark that Theorem 1.5 in the affine case follows from Lemma 2.1 and a theorem due to A. Artstein [2;Thm 5.2].

Let us also remark that the converse of Lemma 2.1 is true. Indeed if the conclusion of Lemma 2.1 holds, then there exist $V \in C^1(\mathbb{R}^n; [0,+\infty))$ satisfying (2.1), $\delta \in C^0(\mathbb{R}^n \setminus \{0\}; (0,+\infty))$, and $u \in L^\infty_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ satisfying (1.9) and such that, for all x in $\mathbb{R}^n \setminus \{0\}$, $f(x,u(x))\cdot\nabla V(x) \leq -\delta(x)$ from which it follows, as in the regular case, that 0 is a globally asymptotically stable point of $\dot{x} = f(x,u(x))$.

Proof of Lemma 2.1. Let $u \in L^\infty_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ satisfying (1.9) be such that 0 is a globally asymptotically stable point of $\dot{x} = f(x,u(x))$. Multiplying, if necessary, $f(x,u(x))$ by $\rho(x)$ where ρ is a suitable function in $C^0(\mathbb{R}^n; [0,+\infty))$ which is positive on $\mathbb{R}^n \setminus \{0\}$, we may assume, without loss of generality, that

$$|f(x,u(x+e))| \leq \text{Min}(1,|x|/2) \;\; \forall (x,e) \in \mathbb{R}^n \times \mathbb{R}^m \text{ with } |e| \leq 1. \tag{2.4}$$

Using [16;Thm 2], we get

Lemma 2.2. *There exist* V^* *in* $C^0(\mathbb{R}^n;[0,+\infty))$ *and* η in $C^0(\mathbb{R}^n;[0,+\infty))$ *such that*

$$V^*(x) \to +\infty \text{ as } |x| \to +\infty, \tag{2.5}$$

$$V^*(x) = 0 \Longleftrightarrow x = 0, \tag{2.6}$$

$$\eta(x) = 0 \Longleftrightarrow x = 0, \tag{2.7}$$

and

$$V^*(x(1)) \leq V^*(x(0)) - \eta(x(0)) \tag{2.8}$$

for any Filippov solution of $\dot{x} = f(x, u(x))$ *defined on* [0,1].

Let, for a measurable set A, $|A|$ its Lebesgue measure. Let $B(x,r)$ be the open ball of radius r centered at x. For δ in $C^0(\mathbb{R}^n;[0,1])$ such that

$$\delta(x) = 0 \Longleftrightarrow x = 0 \tag{2.9}$$

let $X_\delta : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$X_\delta(x) = \frac{1}{|B(x,\delta(x))|} \int_{B(x,\delta(x))} f(x, u(y))\, dy \quad \text{if } x \neq 0, \tag{2.10}$$

$$X_\delta(0) = 0. \tag{2.11}$$

Note that by (2.4), (2.9), (2.10) and (2.11)

$$X_\delta \in C^0(\mathbb{R}^n; \{x \in \mathbb{R}^n; |x| \leq 1\}). \tag{2.12}$$

Our next lemma is

Lemma 2.3. *There exists* δ *in* $C^0(\mathbb{R}^n;[0,1])$ *satisfying* (2.9) *such that*

$$0 \textit{ is a globally asymptotically stable point of } \dot{x} = X_\delta(x). \tag{2.13}$$

Lemma 2.1 is a corollary of Lemma 2.3. Indeed, by an inverse of Lyapunov's second theorem due to J. Kurzweil [14,Thm 7] there exists $V \in C^\infty(\mathbb{R}^n;[0,+\infty))$ satisfying (2.1) such that

$$X_\delta(x) \cdot \nabla V(x) < 0 \quad \forall x \in \mathbb{R}^n \backslash \{0\}; \tag{2.14}$$

it remains only to notice that (2.2) and (2.3) follow from (1.9), (2.9) and (2.10).

We now prove Lemma 2.3. For $i \in \mathbb{Z}$ let

$$D_i = \{x \in \mathbb{R}^n;\ \ i \leq |x| \leq i+1\} \text{ if } i > 0, \tag{2.15}$$
$$D_i = \left\{x \in \mathbb{R}^n;\ \ 2^{i-1} \leq |x| \leq 2^i\right\} \text{ if } i \leq 0, \tag{2.16}$$

By (2.4) and (2.10), for any δ in $C^0(\mathbb{R}^n;[0,1])$, any i in $\mathbb{Z}$, and any maximal solution of $\dot{x} = X_\delta(x)$,

$$x(0) \in D_i \Longrightarrow (x(t) \in D_{i-1} \cup D_i \cup D_{i+1} \quad \forall t \in [0,1]). \tag{2.17}$$

Note also that

$$X_\delta = f_\delta + g_\delta \tag{2.18}$$

with

$$f_\delta(x) = \frac{1}{|B(x,\delta(x))|}\int_{B(x,\delta(x))} f(y,u(y))\,dy \tag{2.19}$$

and

$$g_\delta(x) = \frac{1}{|B(x,\delta(x))|}\int_{B(x,\delta(x))} \left(f(x,u(y)) - f(y,u(y))\right)dy. \tag{2.20}$$

Arguing by contradiction, we get, using (2.7), (2.8), (2.17), (2.18), (2.19), (2.20) and a compactness theorem due to A. Filippov [10,Thm 3] that for any i in $\mathbb{Z}$ there exists $\delta_i > 0$ such that, if $\delta(x) \leq \delta_i \ \ \forall x \in D_{i-1} \cup D_i \cup D_{i+1}$ and if $x(\cdot)$ is any maximal solution of $\dot{x} = X_\delta(x)$ then

$$(x(0) \in D_i) \Rightarrow (V^*(x(1)) \leq V^*(x(0)) - \frac{1}{2}\eta(x(0))\,). \tag{2.21}$$

Therefore (recall $|X_\delta| \leq 1$) any δ in $C^0(\mathbb{R}^n;[0,1])$ satisfying (2.9) and

$$\delta(x) \leq \text{Min}(\delta_{i-1},\delta_i,\delta_{i+1}) \quad \forall x \in D_i,\ \forall i \in \mathbb{Z} \tag{2.22}$$

has the properties required by Lemma 2.3.

Remark 2.4. We need Lemma 2.2 only to prove Lemma 2.3. One can obtain also Lemma 2.3 by using Theorem A.2 of the appendix instead of Lemma 2.2.

In the nonaffine case Theorem 1.5 is a consequence of Lemma 2.1 and of

Proposition 2.5. *Assume that the conclusion of Lemma 2.1 holds. Then, for any positive T, $\dot{x} = f(x,u)$ can be globally asymptotically stabilized by means of a continuous $T-$periodic time-varying feedback law.*

Remark 2.6. a) Proposition 2.5 has been previously proved by E. Sontag and H. Sussmann in [19;(3.5)] when $n = 1$ (the periodicity of the stabilizing time-varying feedback law is not stated in [19] but can be easily obtained from the proof of [19;(3.5)]). b) It follows from [2;Thm 4.3] and Proposition 2.5 that, if $\dot{x} = f(x, u)$ can be stabilized by means of a relaxed feedback law which is continuous at 0 (and vanishes at zero), it can be, for any positive T, globally asymptotically stabilized by means of a continuous $T-$periodic time-varying feedback law.

Proof of Proposition 2.5. Clearly by changing the scale of time if necessary we may assume

$$T = 2. \tag{2.23}$$

Our first lemma follows from [2].

Lemma 2.7. *Under the assumptions of Proposition* 2.5 *there exists* v : $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ *such that*

$$v \in C^\infty((\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}; \mathbb{R}^m) \cap C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$$

$$v(x, s+1) = v(x, s) \quad \forall (x, s) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.24}$$

$$v(0, s) = 0 \quad \forall s \in \mathbb{R}, \tag{2.25}$$

and

$$\nabla V(x) \cdot \int_0^1 f(x, v(x, s)) ds < 0 \;\; \forall x \in \mathbb{R}^n \backslash \{0\}. \tag{2.26}$$

Proof of Lemma 2.7. With the notations of [2;p.1167] (with $W = \mathbb{R}^n \backslash \{0\}$) we define $\tilde{v}$: $(\mathbb{R}^n \backslash \{0\}) \times [0, 1) \to \mathbb{R}^m$ in the following way: let x be a point belonging to the simplex $[z_{i_1}, ..., z_{i_k}]$ with $i_1 < \cdots < i_k$; in particular

$$x = \sum_{j=1}^{k} t_{i_j} z_{i_j} \tag{2.27}$$

with $t_{i_j} \in [0, 1]$ $\forall j \in [1, k]$ and

$$\sum_{j=1}^{k} t_{i_j} = 1; \tag{2.28}$$

For $t \in \left[\sum_{j=1}^{\ell-1} t_{i_j}, \sum_{j=1}^{\ell} t_{i_j}\right)$ (with the convention $\sum_{j=1}^{0} t_{i_j} = 0$) and $\ell \in [1, k]$ let

$$\tilde{v}(x, t) = u_{i_\ell}. \tag{2.29}$$

Using (2.3) we may require that if $z_{i_q} \to 0$ as $q \to \infty$ then $u_{i_q} \to 0$ as $q \to \infty$. Therefore

$$\text{Sup}\,\{|\tilde{v}(x,t)|\,;\ t \in [0,1)\} \to 0 \text{ as } x \to 0. \tag{2.30}$$

By construction

$$\nabla V(x) \cdot \int_0^1 f(x, \tilde{v}(x,t))dt < 0 \quad \forall x \in \mathbb{R}^n \backslash \{0\}. \tag{2.31}$$

It remains only to extend $\tilde{v}$ to all $\mathbb{R}^n \times \mathbb{R}$ by requiring (2.24) and (2.25) and then to smooth it to get v.

We now go back to the proof of Proposition 2.5. Let v be as in Lemma 2.7 and let for $\varepsilon > 0$

$$X_\varepsilon(x,t) = f(x, v(x, t/\varepsilon)).$$

Let K and K' be two compact sets of $\mathbb{R}^n \backslash \{0\}$ such that $K \subset$ Interior(K'). Let

$$0 < \delta = \text{Min}\left\{-\nabla V(x) \cdot \int_0^1 f(x, v(x,t))dt;\ x \in K'\right\}. \tag{2.32}$$

Then we have

Lemma 2.8. *There exists $\varepsilon_0 > 0$ such that for all ε in $(0, \varepsilon_0]$*

$$(\dot{x} = X_\varepsilon(x,t),\ x(0) \in K) \Rightarrow (x(t) \in K',\ \forall t \in [0, \varepsilon]), \tag{2.33}$$

$$(\dot{x} = X_\varepsilon(x,t),\ x(0) \in K) \Rightarrow (V(x(\varepsilon)) \le V(x(0)) - (\delta/2)\varepsilon). \tag{2.34}$$

Proof of Lemma 2.8. Clearly there exists $\varepsilon_1 > 0$ such that, for $\varepsilon \in (0, \varepsilon_1]$, (2.33) holds. We now assume that $\varepsilon \in (0, \varepsilon_1]$ and let x be a solution of $\dot{x} = X_\varepsilon(x,t)$ with $x(0) \in K$. We have

$$V(x(\varepsilon)) - V(x(0)) = \int_0^\varepsilon \nabla V(x(t)) \cdot f(x(t), v(x(t), t/\varepsilon))dt, \tag{2.35}$$

$$|\nabla V(x(t)) - \nabla V(x(0))| \le C(\varepsilon), \tag{2.36}$$

$$|f(x(t), v(x(t), t/\varepsilon)) - f(x(0), v(x(0), t/\varepsilon))| \le C(\varepsilon), \tag{2.37}$$

where $C(\varepsilon)$ is a constant independent of $x(0)$ in K and t in $[0, \varepsilon)$ and satisfies

$$\lim_{\varepsilon \to 0} C(\varepsilon) = 0. \tag{2.38}$$

From (2.32), (2.35), (2.36), (2.37), and (2.38) we get (2.34) if $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 \in (0, \varepsilon_1]$ small enough.

Let $r = (r_i;\ i \in \mathbb{Z})$ be a sequence of positive real numbers such that

$$r_i < r_{i+1} \quad \forall i \in \mathbb{Z}, \tag{2.39}$$

$$\lim_{i \to +\infty} r_i = +\infty, \tag{2.40}$$

$$\lim_{i \to -\infty} r_i = 0. \tag{2.41}$$

Replacing V by $V - V(0)$ we may assume

$$V(0) = 0. \tag{2.42}$$

For two real numbers a and b let

$$D(a,b) = \{x \in \mathbb{R}^n;\ a \leq V(x) \leq b\}. \tag{2.43}$$

Let $\mu = (\mu_i; i \in \mathbb{Z})$ be the sequence of positive real numbers defined by

$$0 < \mu_i = \mathrm{Min}\left\{-\nabla V(x) \cdot \int_0^1 f(x, v(x,t))dt;\quad x \in D(r_i, r_{i+1})\right\}. \tag{2.44}$$

Let $a = (a_i; i \in \mathbb{Z})$ be a sequence of positive real numbers such that

$$r_i + 2a_i < r_{i+1} - 2a_{i+1} \quad \forall i \in \mathbb{Z} \tag{2.45}$$

and, for a sequence, let $A \in C^0((0,+\infty);[0,1])$ be defined by

$$A(s) = |s - r_i|/a_i \text{ if } s \in [r_i - a_i, r_i + a_i]$$

$$A(s) = 1 \text{ if } s \in [r_i + a_i, r_{i+1} - a_{i+1}]. \tag{2.46}$$

For a sequence of positive real numbers $\varepsilon = (\varepsilon_i; i \in \mathbb{Z})$ we define a continuous time-varying feedback law w on $\mathbb{R}^n \times [0,1]$ by

$$w(x,t) = v(x, t/\varepsilon_i)A(V(x)) \text{ if } x \in D(r_i, r_{i+1}) \tag{2.47}$$

$$w(0,t) = 0. \tag{2.48}$$

If $\bar{\varepsilon} = (\bar{\varepsilon}_i; i \in \mathbb{Z})$ is another sequence of positive real numbers we will say that $\varepsilon \leq \bar{\varepsilon}$ if

$$\varepsilon_i \leq \bar{\varepsilon}_i, \quad \forall i \in \mathbb{Z}. \tag{2.49}$$

Then, using Lemma 2.8, one easily gets

Lemma 2.9. *Let* $a = (a_i; i \in \mathbb{Z})$ *be a sequence of positive real numbers such that* (2.45) *holds. Then there exists a sequence* $\bar{\varepsilon}$ *of positive real numbers such that if* $\varepsilon \leq \bar{\varepsilon}$ *then any maximal solution of* $\dot{x} = f(x, w(x,t))$ *defined at* 0 *is defined on* $[0,1]$ *and satisfies for all* i *in* $\mathbb{Z}$ *and for all* t *in* $[0,1]$

$$V(x(0)) \in [r_{i-1}, r_i] \Longrightarrow V(x(t)) \leq V(x(0)) + 2\text{Max}(a_{i-1}, a_i)\ , \qquad (2.50)$$

$$V(x(0)) \leq r_i - 2a_i \Longrightarrow$$
$$V(x(1)) \leq \text{Max}(r_{i-1} + 2a_{i-1}, V(x(0)) - (\mu_i/2)). \qquad (2.51)$$

Let now $r' = (r'_i; i \in \mathbb{Z})$ be the sequence of positive real numbers defined by

$$r'_i = \frac{r_i + r_{i+1}}{2}. \qquad (2.52)$$

For two sequences a', ε' of positive real numbers we define as before but, with r'_i instead of r_i, a time-varying feedback law $w' \in C^0(\mathbb{R}^n \times [0,1]; \mathbb{R}^m)$. Finally let $\bar{u} : \mathbb{R}^n \times [0,2] \to \mathbb{R}^m$ be defined by

$$\bar{u}(x,t) = w(x,t) \text{ if } t \in [0,1], \qquad (2.53)$$
$$\bar{u}(x,t) = w'(x,t-1) \text{ if } t \in (1,2]. \qquad (2.54)$$

Note that

$$\bar{u} \in \mathrm{C}^0(\mathbb{R}^n \times ([0,2] \backslash \{1\}); \mathbb{R}^m), \qquad (2.55)$$

$$\text{Sup}\,\{|\bar{u}(x,t)|\,; t \in [0,2]\} \to 0 \text{ as } x \to 0. \qquad (2.56)$$

One easily checks using Lemma 2.9 that if a and a' are small enough (i.e. $a_i + a'_i \leq \eta_i\ \forall i \in \mathbb{Z}$ for a suitable choice of $\eta_i > 0$) then, if ε and ε' are small enough, any maximal solution of $\dot{x} = f(x, \bar{u}(x,t))$ defined at $t = \tau \in \{0,1\}$ is defined on $[\tau, 2]$ and satisfies

$$V(x(t)) \leq 2V(x(\tau)) \quad \forall t \in [\tau, 2], \qquad (2.57)$$

$$V(x(2)) \leq V(x(0)) - \bar{\theta}(x(0)) \text{ if } \tau = 0, \qquad (2.58)$$

where $\bar{\theta} \in C^0(\mathbb{R}^n; [0, +\infty))$ satisfies

$$\bar{\theta}(x) = 0 \Longleftrightarrow x = 0. \qquad (2.59)$$

Let $\eta \in C^0(\mathbb{R}^n; [0,1/2])$ with $\eta(x) = 0 \Longleftrightarrow x = 0$. We define $u : \mathbb{R}^n \times [0,2] \to \mathbb{R}^m$ by $u(0,t) = 0$ for all t in $[0,2]$, and for $x \neq 0$

(i) if $t \leq 1 - \eta(x)$ or $t \in [1, 2 - \eta(x)]$

$$u(x,t) = \bar{u}(x,t), \tag{2.60}$$

(ii) if $t \in [1 - \eta(x), 1]$

$$u(x,t) = \frac{1-t}{\eta(x)} w(x,t) + \frac{\eta(x) + t - 1}{\eta(x)} w'(x,0), \tag{2.61}$$

(iii) if $t \in [2 - \eta(x), 2]$

$$u(x,t) = \frac{2-t}{\eta(x)} w'(x,t) + \frac{t - 2 + \eta(x)}{\eta(x)} w(x,0). \tag{2.62}$$

Note that

$$u \in C^0(\mathbb{R}^n \times [0,2]; \mathbb{R}^m), \tag{2.63}$$

$$u(x,0) = u(x,2), \quad \forall x \in \mathbb{R}^n. \tag{2.64}$$

We extend u to all $\mathbb{R}^n \times \mathbb{R}$ by requiring

$$u(x, t+2) = u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{2.65}$$

By (2.64) and (2.65)

$$u \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m). \tag{2.66}$$

Let $D_k = \{x \in \mathbb{R}^n; 3^{k-1} \leq V(x) \leq 3^k\}$ for $k \in \mathbb{Z}$. From the definition of u we get that there exists a sequence $\varepsilon = (\varepsilon_k; k \in \mathbb{Z})$ of positive real numbers such that, if

$$\eta(x) \leq \varepsilon_k \quad \forall x \in D_k \quad \forall k \in \mathbb{Z}, \tag{2.67}$$

then any maximal solution of $\dot{x} = f(x, u(x,t))$ defined at $t = 0$ which satisfies $V(x(0)) \in [3^{k-1}, 3^{k+1}]$ is defined on $[0,2]$ and satisfies

$$V(x(t)) \leq 3V(x(0)) \quad \forall t \in [0,2], \tag{2.68}$$

$$V(x(2)) < V(x(0)). \tag{2.69}$$

Indeed this can be easily seen by noticing that, given k in $\mathbb{Z}$, there exists $\delta = \delta(k) > 0$ such that for every solution $x \ : \ [0,2] \to \cup_{l \leq k+2} D_l$ of $\dot{x} = f(x, \bar{u}(x,t))$ and every solution $\tilde{x} \ : \ [1,2] \to \cup_{l \leq k+2} D_l$ of the same equation then

$$|x(1) - \tilde{x}(1)| < \delta \Longrightarrow V(\tilde{x}(2)) < V(x(0)) - \frac{1}{2}\theta(x(0)). \tag{2.70}$$

We now choose $\eta \in C^0(\mathbb{R}^n; [0, 1/2])$ satisfying (2.67) for all k in $\mathbb{Z}$. Then 0 is globally asymptotically stable point of $\dot{x} = f(x, u(x,t))$.

3 Stabilisation

STABILISATION BY MEANS OF DISCONTINUOUS FEEDBACK LAWS AND INTEGRATOR

Let us recall first a result proved independently in [3] and [20] (see also [8] for related results): if the smooth (e.g. of class C^1) system $\dot{x} = f(x, u)$ can be locally asymptotically stabilized by means of a smooth (e.g. of class C^1) feedback law then $\dot{x} = f(x, y)$, $\dot{y} = v$ can be locally asymptotically stabilized by means of a smooth (e.g. of class C^0) feedback law.

In this section we give an example showing that, even if f is analytic, this no longer is true if we replace smooth feedback laws by discontinuous feedback laws. Our example will be with $n = 2$, $m = 1$. Note that by a theorem due to W.P. Dayawansa and C.F. Martin [9] there is no such example with $n = 1$, $m = 1$. Our example is given in the following

Proposition 3.1. *The system $\dot{x}_1 = u(\in \mathbb{R})$, $\dot{x}_2 = {x_1}^2 - 2x_1u^2$ can be locally asymptotically stabilized by means of a discontinuous feedback law. But the system $\dot{x}_1 = y$, $\dot{x}_2 = {x_1}^2 - 2x_1y^2$, $\dot{y} = v$ cannot be locally asymptotically stabilized by means of a discontinuous feedback law. In fact it cannot also be locally asymptotically stabilized by means of a time-varying feedback law.*

Remark 3.2. If f is only continuous there are systems with $n = m = 1$ such that $\dot{x} = f(x, u)$ can be locally asymptotically stabilized by means of a continuous feedback law and such that $\dot{x} = f(x, y)$, $\dot{y} = v$ cannot be asymptotically stabilized by means of a feedback law (time-varying or discontinuous). This is the case for example if $f(x, u) = x^{\frac{1}{3}} - u$: 0 is a globally asymptotically stable point of $\dot{x} = f(x, u(x))$ with $u(x) = 2x^{\frac{1}{3}}$, but, if $V(x, y) = x - y^2$, $\dot{x} = x^{\frac{1}{3}} - y$, $\dot{y} = v$ then $\dot{V} = x^{\frac{1}{3}} - y - 2yv \geq (y^2)^{\frac{1}{3}} - y - 2yv \geq 0$ if $V \geq 0, |v| \leq 1$, and $|y|$ small; this shows that $\dot{x} = f(x, y)$, $\dot{y} = v$ cannot be locally asymptotically stabilized by means of a feedback law (time-varying or discontinuous).

Proof of Proposition 3.1. We first prove that $\dot{x}_1 = u$, $\dot{x}_2 = {x_1}^2 - 2x_1u^2$ can be locally asymptotically stabilized by means of a discontinuous feedback law. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$u(x_1, x_2) = (x_2 - 2x_1)^{\frac{1}{3}} \text{ if } |x_2 - x_1| \geq ({x_2}^2 + {x_1}^2)^{\frac{33}{32}}, \tag{3.1}$$

$$u(x_1, x_2) = |x_2|^{\frac{1}{3}} \text{ if } |x_2 - x_1| \leq ({x_2}^2 + {x_1}^2)^{\frac{33}{32}}. \tag{3.2}$$

Note that (1.9) holds. Let $V \in C^1(\mathbb{R}^2; \mathbb{R})$ be defined by

$$V(x_1, x_2) = x_1^2 - x_1x_2 + {x_2}^2. \tag{3.3}$$

Note that

$$V(x_1, x_2) > V(0) \text{ if } (x_1, x_2) \neq (0,0). \tag{3.4}$$

Moreover straightforward computations show that, with $X(x) = (u(x), {x_1}^2 - 2x_1u^2)$ $(= f(x, u(x)))$,

$$X(x) \cdot \nabla V(x) \leq -({x_1}^2 + {x_2}^2)^2 \text{ if } |x| \text{ is small enough} \tag{3.5}$$

which shows that 0 is a locally asymptotically stable point of $\dot{x} = f(x, u(x))$.

We now prove that $\dot{x}_1 = y$, $\dot{x}_2 = {x_1}^2 - 2x_1y^2$, $\dot{y} = v$ cannot be locally asymptotically stabilized by means of (discontinuous or time-varying) feedback law. Let $H \in C^1(\mathbb{R}^3, \mathbb{R})$ be defined by

$$H(x_1, x_2, y) = x_2 + y{x_1}^2. \tag{3.6}$$

Then

$$\dot{H} = {x_1}^2(1 + v). \tag{3.7}$$

Therefore $\dot{H} \geq 0$ if $v \geq -1$ which gives the desired statement (note that $(0,0,0)$ is in the closure of $\{(x,y); H(x,y) > 0\}$).

Appendix

Robustness of Asymptotical Stability for Discrete-time Systems

It is sometimes useful to study the discrete-time system associated with an ordinary differential equation

$$\dot{x} = f(x) \tag{A.1}$$

in the following way: if f is locally Lipschitzian, we define $F(x)$ to be the point reached from x at time 1, that is $F(x) = x(1)$ where $x(\cdot)$ is the solution of $\dot{x} = f(x)$ such that $x(0) = x$. The discrete-time system we consider is then

$$x_{n+1} = F(x_n). \tag{A.2}$$

When (A.1) is an asymptotically stable system, then (A.2) is also an asymptotically stable system. If f is now only assumed to be continuous, F must be regarded as a set-valued map, since there exist in general many points reached from x in time 1. Our aim is to prove, under some general hypotheses—which are satisfied when F is constructed from f as above—that asymptotic stability for systems of the type

$$x_{n+1} \in F(x_n). \tag{A.3}$$

is a robust property, i.e: if we perturb slightly such systems the asymptotic stability property is not lost, (even if F is not the Poincaré map of an ordinary differential equation.)

From now on we consider an open set $G \subset \mathbb{R}^n$, containing the origin, a continuous map $\omega : G \rightarrow \mathbb{R}^+$, such that $\omega(x) = 0 \iff x = 0$ and $\omega(x) \rightarrow +\infty$ as $x \rightarrow \partial G$, and a set-valued map $F : x \in G \mapsto F(x) \subset G$ such that

(i) $\forall x \in G$, $F(x)$ is a nonempty compact set;

(ii) F is upper semicontinuous (usc): given $x_0 \in G$ and an open set $\Omega \supset F(x_0)$, we may find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow F(x) \subset \Omega$.

It may be noted that (i) together with (ii) are equivalent to

(iii) For every nonempty compact subset K, the set $\{(x,y) \in G^2; x \in K, y \in F(x)\}$ is a nonempty compact set.

For $A \subset G$, $F(A)$ denotes the set $\cup_{x \in A} F(x)$ and we define, for $x \in G$, the sequence $(F^n(x))_{n \geq 0}$ by the natural following induction: $F^0(x) = \{x\}$, $F^1(x) = F(x)$, and $F^n(x) = F(F^{n-1}(x)) = \cup_{y \in F^{n-1}(x)} F(y)$ for $n \geq 1$. The following proposition is an immediate translation of a result obtained by Kurzweil in [14].

Proposition A.1. *Let $F : G \rightarrow \mathcal{P}(G)$ be a set-valued map satisfying (i), (ii) and such that $F(0) = \{0\}$. Then the following two properties are equivalent:*

$\mathcal{P}_1$: *0 is an asymptotically stable (on G) point for the system (A.3), that is*

a) $\forall \epsilon, \exists \delta$ *such that* $\omega(x) < \delta \Rightarrow \forall n, \ \omega(F^n(x)) \subset [0, \epsilon]$;

b) all trajectories of (A.3) tend to 0 as $n \rightarrow +\infty$.

$\mathcal{P}_2$: *There exist an increasing function $B : (0, +\infty) \rightarrow (0, +\infty)$, with $\lim_{\beta \rightarrow 0} B(\beta) = 0$ and a function $T : (0, +\infty)^2 \rightarrow (0, +\infty)$ such that $\forall \epsilon > 0, \forall \beta > 0$, for any trajectory $(x_n)_{n \geq 0}$ of (A.3), with $\omega(x_0) \leq \beta$, we have*

c) $\omega(x_n) \leq B(\beta) \ \forall n \geq 0$.

d) $\omega(x_n) \leq \epsilon \ \forall n \geq T(\beta, \epsilon)$.

We now state the main result of this appendix, which can be used instead of Lemma 2.2 to prove Lemma 2.3.

Theorem A.2. *Assume that $\mathcal{P}_1$ holds, then there exists δ in $C^0(G; [0, +\infty))$ such that*

i) $\delta > 0$ *on* $G \setminus \{0\}$;

ii) $\delta(0) = 0$;

iii) $F(x) + B(0, \delta(x)) \subset G, \quad \forall x \in G \setminus \{0\}$;

iv) 0 is an asymptotically stable (on G) point for the system

$$x_{n+1} \in F(x_n) + B(0, \delta(x_n)).$$

Theorem A.2 is an immediate consequence of

Lemma A.3. *Let $\beta > 0$ and $N \in \mathbb{N}$ be such that $\omega(x) \leq \beta \Rightarrow \omega(F^n(x)) \subset [0, \frac{\beta}{4}]$ $\forall n \geq N$. Then there exists $\delta > 0$ such that, for any x_0 and any sequence $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N$ in G satisfying: $\omega(x_0) \leq \beta$ and $\hat{x}_{n+1} \in F(\hat{x}_n) + B(0, \delta)$ for $0 \leq n \leq N-1$, we have*

$$\begin{cases} \omega(\hat{x}_N) < \dfrac{\beta}{2} \\ \omega(\hat{x}_n) < 2B(\beta) \quad \text{for } n = 0, 1, \ldots, N. \end{cases} \tag{A.4}$$

($B(\cdot)$ is the function given by $\mathcal{P}_2$.)

Proof of Lemma A.3. Since the set-valued map F is usc, for any subset K of G and any neighborhood V of $F(K)$, there exists a neighborhood U of K such that $F(U) \subset V$. We apply inductively this fact to the sequence of compact sets $F^{N-1}(x_0), F^{N-2}(x_0), \ldots, F(x_0), \{x_0\}$ for proving the existence of δ_{x_0} such that, if $\tilde{x}_0 \in B(x_0, \delta_{x_0})$ and $\tilde{x}_{n+1} \in F(\tilde{x}_n) + B(0, \delta_{x_0})$ for all n in $[0, N-1]$, then

$$d(\tilde{x}_n, F^n(x_0)) < \min(B(\beta), \frac{\beta}{4}) \quad 0 \leq n \leq N. \tag{A.5}$$

Using the compactness of $\{x; \omega(x) \leq \beta\}$ the lemma follows.

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LABORATOIRE D'ANALYSE NUMÉRIQUE, UNIVERSITÉ PARIS-SUD, BÂTIMENT 425, 92405 ORSAY, FRANCE

CMLA, ENS CACHAN, 61, AV. DU PRÉSIDENT WILSON, 94235 CACHAN, FRANCE

Communicated by Clyde F. Martin