Poisson approximations on the free Wigner chaos

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Abstract: We prove that an adequately rescaled sequence $\{F_n\}$ of self-adjoint operators, living inside a fixed free Wigner chaos of even order, converges in distribution to a centered free Poisson random variable with rate $\lambda > 0$ if and only if $\varphi(F_n^4) - 2\varphi(F_n^3) \to 2\lambda^2 - \lambda$ (where φ is the relevant tracial operator). This extends to a free setting some recent limit theorems by Nourdin and Peccati (2009), and provides a non-central counterpart to a result by Kemp *et al.* (2011). As a by-product of our findings, we show that Wigner chaoses of order strictly greater than 2 do not contain non-zero free Poisson random variables. Our techniques involve the so-called 'Riordan numbers', counting non-crossing partitions without singletons.

Key words: Catalan numbers; Contractions; Free Brownian motion; Free cumulants; Free Poisson distribution; Free probability; Marchenko-Pastur; Non-central limit theorems; Non-crossing partitions; Riordan numbers; Semicircular distribution; Wigner chaos.

2000 Mathematics Subject Classification: 46L54; 60H05; 60H07; 60H30.

1 Introduction

1.1 Overview

Let W be a standard Brownian motion on $\mathbb{R}_+$ and let $q \geqslant 1$ be an integer. For every deterministic symmetric function $f \in L^2(\mathbb{R}_+^q)$, we denote by $I_q^W(f)$ the multiple stochastic Wiener-Itô integral of f with respect to W. Random variables of the form $I_q^W(f)$ compose the so-called qth *Wiener chaos* associated with W. The concept of Wiener chaos roughly represents an infinite-dimensional analogous of Hermite polynomials for the one-dimensional Gaussian distribution (see e.g. [14] for an introduction to this topic).

The following two results, proved respectively in [13] and [9], provide an exhaustive characterization of normal and Gamma approximations on Wiener chaos. As in [9], we denote by $F(\nu)$ a centered random variable with the law of $2G(\nu/2)-\nu$, where $G(\nu/2)$ has a Gamma distribution with parameter $\nu/2$ (if $\nu \geqslant 1$ is an integer, then $F(\nu)$ has a centered χ^2 distribution with ν degrees of freedom).

Theorem 1.1 (A) *Let $N \sim \mathscr{N}(0,1)$, fix $q \geqslant 2$ and let $I_q^W(f_n)$ be a sequence of multiple stochastic integrals with respect to the standard Brownian motion W, where each f_n is a symmetric element of $L^2(\mathbb{R}_+^q)$ such that $E[I_q^W(f_n)^2] = q!\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = 1$. Then, the following two assertions are equivalent, as $n \to \infty$:*

(i) *$I_q^W(f_n)$ converges in distribution to N;*

(ii) *$E[I_q^W(f_n)^4] \to E[N^4] = 3$.*

(B) *Fix $\nu > 0$, and let $F(\nu)$ have the centered Gamma distribution described above. Let $q \geqslant 2$ be an even integer, and let $I_q^W(f_n)$ be a sequence of multiple stochastic integrals, where each*

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f_n *is symmetric and verifies* $E[I_q^W(f_n)^2] = E[F(\nu)^2] = 2\nu$. *Then, the following two assertions are equivalent, as* $n \to \infty$*:*

(i) $I_q^W(f_n)$ *converges in distribution to* $F(\nu)$*;*

(ii) $E[I_q(f_n)^4] - 12E[I_q(f_n)^3] \to E[F(\nu)^4] - 12E[F(\nu)^3] = 12\nu^2 - 48\nu$.

The results stated in Theorem 1.1 provide a drastic simplification of the so-called *method of moments* for probabilistic approximations, and have triggered a huge amount of applications and generalizations, involving e.g. Stein's method, Malliavin calculus, power variations of Gaussian processes, Edgeworth expansions, random matrices and universality results. See [10, 11], as well as the forthcoming monograph [12], for an overview of the most important developments. See [8] for a constantly updated web resource, with links to all available papers on the subject.

In [5], together with Kemp and Speicher, we proved an analogue of Part A of Theorem 1.1 in the framework of free probability (and free Brownian motion). Let $(\mathscr{A}, \varphi)$ be a free probability space and let $\{S(t) : t \geqslant 0\}$ be a free Brownian motion defined therein (see Section 3 for details). As shown in [3], one can define multiple integrals of the type $I_q^S(f)$, where f is a square-integrable complex kernel (to simplify the notation, throughout the paper we shall drop the suffixes q, S, and write simply $I(f) = I_q^S(f)$). Random variables of the type $I(f)$ compose the so-called *Wigner chaos* associated with S, playing in free stochastic analysis a role analogous to that of the classical Gaussian Wiener chaos (see for instance [3], where Wigner chaoses are used to develop a free version of the Malliavin calculus of variations). The following statement is the main result of [5].

Theorem 1.2 *Let s be a centered semicircular random variable with unit variance (see Definition 2.3(i)), fix an integer $q \geqslant 2$, and let $I(f_n)$ be a sequence of multiple integrals of order q with respect to the free Brownian motion S, where each f_n is a mirror symmetric (see Section 3) element of $L^2(\mathbb{R}_+^q)$ such that $\varphi[I_q(f_n)^2] = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = 1$. Then, the following two assertions are equivalent, as $n \to \infty$:*

(i) $I(f_n)$ *converges in distribution to* s*;*

(ii) $\varphi[I(f_n)^4] \to \varphi[s^4] = 2$.

The principal aim of this paper is to prove a free analogous of Part B of Theorem 1.1. As explained in Section 2, and somehow counterintuitively, the free analogous of Gamma random variables is given by free Poisson random variables (see Definition 2.3(ii), and also [6, p. 203]). The free Poisson law is also known as the *Marchenko-Pastur distribution*, arising in random matrix theory as the limit of the eigenvalue distribution of large sample covariance matrices (see e.g. Bai and Silverstein [1, Ch. 3], Hiai and Petz [4, pp. 101-103 and 130] and the references therein). The following statement is the main achievement of the present work.

Theorem 1.3 *Let $q \geqslant 2$ be an even integer. Let $Z(\lambda)$ have a centered free Poisson distribution with rate $\lambda > 0$. Let $I(f_n)$ be a sequence of multiple integrals of order q with respect to the free Brownian motion S, where each f_n is a mirror symmetric element of $L^2(\mathbb{R}_+^q)$ such that $\varphi[I_q(f_n)^2] = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \varphi[Z(\lambda)^2] = \lambda$. Then, the following two assertions are equivalent, as $n \to \infty$:*

(i) $I(f_n)$ *converges in distribution to* $Z(\lambda)$*;*

(ii) $\varphi[I(f_n)^4] - 2\varphi[I(f_n)^3] \to \varphi[Z(\lambda)^4] - 2\varphi[Z(\lambda)^3] = 2\lambda^2 - \lambda$.

One should note that the techniques involved in our proofs are different from those adopted in the previously quoted references, as they are based on a direct enumeration of contractions. These contractions emerge when iteratively applying product formulae for multiple Wigner integrals – see also [7]. One crucial point is that the moments of a free Poisson random variable can be expressed in terms of the so-called *Riordan numbers*, counting the number of non-crossing partitions without singletons (see e.g. [2]). We also stress that one cannot expect to have convergence to a non-zero free Poisson inside a Wigner chaos of odd order, since random variables inside such chaoses have all odd moments equal to zero, while one has e.g. that $\varphi[Z(\lambda)^3] = \lambda$ (see Remark 2.5(ii)).

As a consequence of Theorem 1.3, we will be able to prove the following result, stating that Wigner chaoses of order greater than 2 do not contain any non-zero Poisson random variable.

Proposition 1.4 *Let $q \geqslant 4$ be even, and let F be a non-zero random variable in the qth Wigner chaos. Then, F cannot have a free Poisson distribution.*

As pointed out in Remark 3.2 below, centered Poisson random variables with integer rate can be realized as elements of the second Wigner chaos. As a consequence, Proposition 1.4 implies that the second Wigner chaos contains random variables whose distribution is not shared by any element of higher chaoses. This result parallels the findings of [5], where it is proved that Wigner chaoses of order $\geqslant 2$ do not contain any non-zero semicircular random variable. Note that, at the present time, it is not known in general whether two non-zero random variables belonging to two distinct Wigner chaoses have necessarily different laws.

Remark 1.5 We are still far from understanding the exact structure of the free Wigner chaos. For instance, almost nothing is known about the regularity of the distributions associated with the elements of a fixed Wigner chaos. In particular, we ignore whether such laws may have atoms or are indeed absolutely continuous (as those in the classical Wiener chaos).

1.2 The free probability setting

Our main reference for free probability is the monograph by Nica and Speicher [6], to which the reader is referred for any unexplained notion or result. We shall also use a notation which is consistent with the one adopted in [5].

For the rest of the paper, we consider as given a so-called (tracial) *W^*-probability space* $(\mathscr{A}, \varphi)$, where: $\mathscr{A}$ is a von Neumann algebra of operators (with involution $X \mapsto X^*$), and φ is a unital linear functional on $\mathscr{A}$ with the properties of being *weakly continuous*, *positive* (that is, $\varphi(XX^*) \geqslant 0$ for every $X \in \mathscr{A}$), *faithful* (that is, such that the relation $\varphi(XX^*) = 0$ implies $X = 0$), and *tracial* (that is, $\varphi(XY) = \varphi(YX)$, for every $X, Y \in \mathscr{A}$).

As usual in free probability, we refer to the self-adjoint elements of $\mathscr{A}$ as *random variables*. Given a random variable X we write μ_X to indicate the *law* (or *distribution*) of X, which is defined as the unique Borel probability measure on $\mathbb{R}$ such that, for every integer $m \geqslant 0$, $\varphi(X^m) = \int_{\mathbb{R}} x^m \mu_X(dx)$ (see e.g. [6, Proposition 3.13]).

We say that the unital subalgebras $\mathscr{A}_1, ..., \mathscr{A}_n$ of $\mathscr{A}$ are *freely independent* whenever the following property holds: let $X_1, ..., X_m$ be a finite collection of elements chosen among the $\mathscr{A}_i$'s in such a way that (for $j = 1, ..., m-1$) X_j and X_{j+1} do not come from the same $\mathscr{A}_i$ and $\varphi(X_j) = 0$ for $j = 1, ..., m$; then $\varphi(X_1 \cdots X_m) = 0$. Random variables are said to be freely independent if they generate freely independent unital subalgebras of $\mathscr{A}$.

1.3 Plan

The rest of the paper is organized as follows. In Section 2 we provide a characterization of centered free Poisson distributions in terms of non-crossing partitions. Section 3 deals with free Brownian motion and Wigner chaos. Section 4 contains the proofs of the main results of the paper (that is, Theorem 1.3 and Proposition 1.4), whereas Section 5 is devoted to some auxiliary lemmas.

2 Semicircular and centered free Poisson distributions

The following definition contains most of the combinatorial objects that are used throughout the text.

Definition 2.1 (i) Given an integer $m \geqslant 1$, we write $[m] = \{1, ..., m\}$. A *partition* of $[m]$ is a collection of non-empty and disjoint subsets of $[m]$, called *blocks*, such that their union is equal to $[m]$. The cardinality of a block is called *size*. A block is said to be a *singleton* if it has size one.

(ii) A partition π of $[m]$ is said to be *non-crossing* if one cannot find integers p_1, q_1, p_2, q_2 such that: (a) $1 \leqslant p_1 < q_1 < p_2 < q_2 \leqslant m$, (b) p_1, p_2 are in the same block of π, (c) q_1, q_2 are in the same block of π, and (d) the p_i's are not in the same block of π as the q_i's. The collection of the non-crossing partitions of $[m]$ is denoted by $NC(m)$, $m \geqslant 1$.

(iii) For every $m \geqslant 1$, the quantity $C_m = |NC(m)|$, where $|A|$ indicates the cardinality of a set A, is called the mth *Catalan number*. One sets by convention $C_0 = 1$. Also, recall the explicit expression $C_m = \frac{1}{m+1}\binom{2m}{m}$.

(iv) We define the sequence $\{R_m : m \geqslant 0\}$ as follows: $R_0 = 1$, and, for $m \geqslant 1$, R_m is equal to the number of partitions in $NC(m)$ having no singletons.

(v) For every $m \geqslant 1$ and every $j = 1, ..., m$, we define $R_{m,j}$ to be the number of non-crossing partitions of $[m]$ with exactly j blocks and with no singletons. Plainly, $R_m = \sum_{j=1}^{m} R_{m,j}$. Also, when m is even, one has that $R_{m,j} = 0$ for every $j > m/2$; when m is odd, then $R_{m,j} = 0$ for every $j > (m-1)/2$.

Example 2.2 One has that:

- $R_1 = R_{1,1} = 0$, since $\{\{1\}\}$ is the only partition of $[1]$, and such a partition is composed of exactly one singleton;
- $R_2 = R_{2,1} = 1$, since the only partition of $[2]$ with no singletons is $\{\{1, 2\}\}$;
- $R_3 = R_{3,1} = 1$, since the only partition of $[3]$ with no singletons is $\{\{1, 2, 3\}\}$;
- $R_4 = 3$, since the only non-crossing partitions of $[4]$ with no singletons are $\{\{1, 2, 3, 4\}\}$, $\{\{1, 2\}, \{3, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$. This implies that $R_{4,1} = 1$ and $R_{4,2} = 2$.

The integers $\{R_m : m \geqslant 0\}$ are customarily called the *Riordan numbers*. A detailed analysis of the combinatorial properties of Riordan numbers is provided in the paper by Bernhart [2]; however, it is worth noting that the discussion to follow is self-contained, in the sense that no previous knowledge of the combinatorial properties of the sequence $\{R_m\}$ is required.

Given a random variable X, we denote by $\{\kappa_m(X) : m \geqslant 1\}$ the sequence of the *free cumulants* of X. We recall (see [6, p. 175]) that the free cumulants of X are completely determined by the following relation: for every $m \geqslant 1$

$$\varphi(X^m) = \sum_{\pi=\{b_1,\ldots,b_j\}\in NC(m)} \prod_{i=1}^{j} \kappa_{|b_i|}(X), \tag{2.1}$$

where $|b_i|$ indicates the size of the block b_i of the non-crossing partition π. It is clear from (2.1) that the sequence $\{\kappa_m(X) : m \geqslant 1\}$ completely determines the moments of X (and viceversa).

Definition 2.3 (i) The centered *semicircular distribution* of parameter $t > 0$, denoted by $S(0,t)(dx)$, is the probability distribution given by

$$S(0,t)(dx) = (2\pi t)^{-1}\sqrt{4t - x^2}dx, \ |x| < 2\sqrt{t}.$$

We recall the classical relation:

$$\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2m} S(0,t)(dx) = C_m t^m,$$

where C_m is the mth Catalan number (so that e.g. the second moment of $S(0,t)$ is t). Since the odd moments of $S(0,t)$ are all zero, one deduces from the previous relation and (2.1) (e.g. by recursion) that the free cumulants of a random variable s with law $S(0,t)$ are all zero, except for $\kappa_2(s) = \varphi(s^2) = t$.

(ii) The *free Poisson distribution* with rate $\lambda > 0$, denoted by $P(\lambda)(dx)$ is the probability distribution defined as follows: (a) if $\lambda \in (0,1]$, then $P(\lambda) = (1-\lambda)\delta_0 + \lambda\widetilde{\nu}$, and (b) if $\lambda > 1$, then $P(\lambda) = \widetilde{\nu}$, where δ_0 stands for the Dirac mass at 0. Here, $\widetilde{\nu}(dx) = (2\pi x)^{-1}\sqrt{4\lambda - (x-1-\lambda)^2}dx$, $x \in \left((1-\sqrt{\lambda})^2, (1+\sqrt{\lambda})^2\right)$. If X_λ has the $P(\lambda)$ distribution, then [6, Proposition 12.11] implies that

$$\kappa_m(X_\lambda) = \lambda, \quad m \geqslant 1. \tag{2.2}$$

From now on, we will denote by $Z(\lambda)$ a random variable having the law of $X_\lambda - \lambda 1$ (centered free Poisson distribution), where 1 is the unity of $\mathscr{A}$. Plainly, $\kappa_1[Z(\lambda)] = \varphi[Z(\lambda)] = 0$.

Note that both $S(0,t)$ and $P(\lambda)$ are compactly supported, and therefore are uniquely determined by their moments (by the Weierstrass theorem). Definition 2.3-(ii) is taken from [6, Definition 12.12]. The choice of the denomination "free Poisson" comes from the following two facts: (1) $P(\lambda)$ can be obtained as the limit of the free convolution of Bernoulli distributions (see [6, Proposition 12.11]), and (2) the classical Poisson distribution of parameter λ has (classical) cumulants all equal to λ (see e.g. [14, Section 3.3]). As recalled in the Introduction, the free Poisson law is also called the "Marchenko-Pastur distribution".

The following statement contains a characterization of the moments of $Z(\lambda)$, and shows that, when λ is integer, then $Z(\lambda)$ is the free equivalent of a classical centered χ^2 random variable with λ degrees of freedom. This last fact could alternatively be deduced from [6, Proposition 12.13], but here we prefer to provide a self-contained argument.

Proposition 2.4 *Let the notation of Definition 2.1 and Definition 2.3 prevail. Then, for every real $\lambda > 0$ and every integer $m \geqslant 1$,*

$$\varphi[Z(\lambda)^m] = \sum_{j=1}^{m} \lambda^j R_{m,j}. \tag{2.3}$$

Let $p = 1, 2, ...$ be an integer, then $Z(p)$ has the same law as $\sum_{i=1}^{p}(s_i^2 - 1)$, where $s_1, ..., s_p$ are p freely independent random variables with the $S(0,1)$ distribution, and 1 is the unit of $\mathscr{A}$.

Proof. From (2.2), one deduces that $\kappa_m[Z(\lambda)] = \lambda$ for every $m \geqslant 2$. Since $\kappa_1[Z(\lambda)] = 0$, we infer from (2.1) that

$$\varphi[Z(\lambda)^m] = \sum_{\pi=\{b_1,...,b_j\}\in NC(m)} \lambda^j \mathbf{1}_{\{\pi \text{ has no singletons}\}},$$

which immediately yields (2.3). To prove the last part of the statement, consider first the case $p = 1$, write $s = s_1$ and fix an integer $m \geqslant 2$. In order to build a non-crossing partition of $[m]$, say π, one has to perform the following three steps: (a) choose an integer $j \in \{0, ..., m\}$, denoting the number of singletons of π, (b) choose the j singletons of π among the m available integers (this can be done in exactly $\binom{m}{j}$ distinct ways), (c) build a non-crossing partition of the remaining $m - j$ integers with blocks at least of size 2 (this can be done in exactly R_{m-j} distinct ways). Since $C_0 = R_0 = 1$ and $C_1 = 1 = R_0 + R_1$, it follows that Catalan and Riordan numbers are linked by the following relation: for every $m \geqslant 0$

$$C_m = \sum_{j=0}^{m} \binom{m}{j} R_{m-j} = \sum_{j=0}^{m} \binom{m}{j} R_j, \tag{2.4}$$

where the last equality follows from $\binom{m}{j} = \binom{m}{m-j}$. By inversion, one therefore deduces that

$$R_m = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} C_j, \quad m \geqslant 0.$$

Therefore

$$\begin{aligned} \varphi[(s^2-1)^m] &= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \varphi(s^{2j}) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} C_j \\ &= R_m = \sum_{j=1}^{m} R_{m,j} = \varphi[Z(1)^m], \end{aligned}$$

from which we infer that $s^2 - 1 \overset{\text{law}}{=} Z(1)$, yielding the desired conclusion when $p = 1$. Let us now consider the general case, that is, $p \geqslant 2$. By free independence, for any $m \geqslant 2$, we have that

$$\kappa_m \left(\sum_{i=1}^{p} (s_i^2 - 1) \right) = p \times \kappa_m(s_1^2 - 1) = p \times \kappa_m(Z(1)) = p = \kappa_m(Z(p)).$$

This implies that $\sum_{i=1}^{p} s_i^2 - 1 \overset{\text{law}}{=} Z(p)$, and the proof of Proposition 2.4 is concluded.

□

Remark 2.5 (i) Relation (2.4) is well known – see e.g. [2, Section 5] for an alternate proof based on "difference triangles".

(ii) Using the last two points of Example 2.2, we deduce from (2.3) that $\varphi[Z(\lambda)^3] = \lambda R_{3,1} = \lambda$, while $\varphi[Z(\lambda)^4] = \lambda R_{4,1} + \lambda^2 R_{4,2} = \lambda + 2\lambda^2$.

3 Free Brownian motion and Wigner chaos

Our main reference for the content of this section is the paper by Biane and Speicher [3].

Definition 3.1 (L^p spaces) (i) For $1 \leqslant p \leqslant \infty$, we write $L^p(\mathscr{A}, \varphi)$ to indicate the L^p space obtained as the completion of $\mathscr{A}$ with respect to the norm $\|a\|_p = \varphi(|a|^p)^{1/p}$, where $|a| = \sqrt{a^*a}$, and $\|\cdot\|_\infty$ stands for the operator norm.

(ii) For every integer $q \geqslant 2$, the space $L^2(\mathbb{R}_+^q)$ is the collection of all complex-valued functions on $\mathbb{R}_+^q$ that are square-integrable with respect to the Lebesgue measure. Given $f \in L^2(\mathbb{R}_+^q)$, we write

$$f^*(t_1, t_2, ..., t_q) = \overline{f(t_q, ..., t_2, t_1)},$$

and we call f^* the *adjoint* of f. We say that an element of $L^2(\mathbb{R}_+^q)$ is *mirror symmetric* if

$$f(t_1, ..., t_q) = f^*(t_1, ..., t_q),$$

for almost every vector $(t_1, ..., t_q) \in \mathbb{R}_+^q$. Notice that mirror symmetric functions constitute a Hilbert subspace of $L^2(\mathbb{R}_+^q)$.

(iii) Given $f \in L^2(\mathbb{R}_+^q)$ and $g \in L^2(\mathbb{R}_+^p)$, for every $r = 1, ..., \min(q, p)$ we define the rth *contraction* of f and g as the element of $L^2(\mathbb{R}_+^{p+q-2r})$ given by

$$\begin{aligned} & f \overset{r}{\frown} g(t_1, ..., t_{p+q-2r}) \\ &= \int_{\mathbb{R}_+^r} f(t_1, ..., t_{q-r}, y_r, y_{r-1}, ..., y_1) g(y_1, y_2, ..., y_r, t_{q-r+1}, t_{p+q-2r}) dy_1 \cdots dy_r. \end{aligned} \quad (3.5)$$

One also writes $f \overset{0}{\frown} g(t_1, ..., t_{p+q}) = f \otimes g(t_1, ..., t_{p+q}) = f(t_1, ..., t_q) g(t_{q+1}, ..., t_{p+q})$. In the following, we shall use the notations $f \overset{0}{\frown} g$ and $f \otimes g$ interchangeably. Observe that, if $p = q$, then $f \overset{p}{\frown} g = \langle f, g^* \rangle_{L^2(\mathbb{R}_+^q)}$.

A *free Brownian motion* S on $(\mathscr{A}, \varphi)$ consists of: (i) a filtration $\{\mathscr{A}_t : t \geqslant 0\}$ of von Neumann sub-algebras of $\mathscr{A}$ (in particular, $\mathscr{A}_u \subset \mathscr{A}_t$, for $0 \leqslant u < t$), (ii) a collection $S = \{S(t) : t \geqslant 0\}$ of self-adjoint operators such that:

– $S(t) \in \mathscr{A}_t$ for every t;

– for every t, $S(t)$ has a semicircular distribution $S(0, t)$, with mean zero and variance t;

– for every $0 \leqslant u < t$, the 'increment' $S(t) - S(u)$ is freely independent of $\mathscr{A}_u$, and has a semicircular distribution $S(0, t-u)$, with mean zero and variance $t - u$.

For every integer $q \geqslant 1$, the collection of all random variables of the type $I_q^S(f) = I(f)$, $f \in L^2(\mathbb{R}_+^q)$, is called the qth *Wigner chaos* associated with S, and is defined according to [3, Section 5.3], namely:

– first define $I(f) = (S(b_1) - S(a_1)) \dots (S(b_q) - S(a_q))$, for every function f having the form

$$f(t_1, ..., t_q) = \prod_{i=1}^{q} \mathbf{1}_{(a_i, b_i)}(t_i), \tag{3.6}$$

where the intervals (a_i, b_i), $i = 1, ..., q$, are pairwise disjoint;

– extend linearly the definition of $I(f)$ to 'simple functions vanishing on diagonals', that is, to functions f that are finite linear combinations of indicators of the type (3.6);

– exploit the isometric relation

$$\langle I(f_1), I(f_2) \rangle_{L^2(\mathscr{A},\varphi)} = \varphi\left(I(f_1)^* I(f_2)\right) = \varphi\left(I(f_1^*) I(f_2)\right) = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}_+^q)}, \tag{3.7}$$

where f_1, f_2 are simple functions vanishing on diagonals, and use a density argument to define $I(f)$ for a general $f \in L^2(\mathbb{R}_+^q)$.

Observe that relation (3.7) continues to hold for every pair $f_1, f_2 \in L^2(\mathbb{R}_+^q)$. Moreover, the above sketched construction implies that $I(f)$ is self-adjoint if and only if f is mirror symmetric. Finally, we recall the following fundamental multiplication formula, proved in [3]. For every $f \in L^2(\mathbb{R}_+^q)$ and $g \in L^2(\mathbb{R}_+^p)$, where $q, p \geqslant 1$,

$$I(f)I(g) = \sum_{r=0}^{\min(q,p)} I(f \overset{r}{\frown} g). \tag{3.8}$$

Remark 3.2 Let $\{e_i : 1, ..., p\}$ be an orthonormal system in $L^2(\mathbb{R}_+)$. Then, the random variables $s_i = I(e_i)$, $i = 1, ..., p$, have the $S(0,1)$ distribution and are freely independent. Moreover, the product formula (3.8) implies that

$$\sum_{i=1}^{p} (s_i^2 - 1) = I\left(\sum_{i=1}^{p} e_i \otimes e_i\right),$$

and therefore that the double integral $I\left(\sum_{i=1}^{p} e_i \otimes e_i\right)$ has a centered free Poisson distribution with rate p.

4 Proof of the main results

4.1 Proof of Theorem 1.3

In the free probability setting (see e.g. [6, Definition 8.1]) convergence in distribution is equivalent to the convergence of moments, so that $I(f_n)$ converges in distribution to $Z(\lambda)$ if and only if $\varphi(I(f_n)^m) \to \varphi(Z(\lambda)^m)$, for every $m \geqslant 1$. In particular, convergence in distribution implies $\varphi(I(f_n)^4) - 2\varphi(I(f_n)^3) \to \varphi(Z(\lambda)^4) - 2\varphi(Z(\lambda)^3) = 2\lambda^2 - \lambda$.

Now assume that $\varphi[I(f_n)^4] - 2\varphi[I(f_n)^3] \to 2\lambda^2 - \lambda$. We have to show that, for every $m \geqslant 2$, $\varphi[I(f_n)^m] \to \varphi[Z(\lambda)^m]$. Iterative applications of the product formula (3.8) yield

$$I(f_n)^m = \sum_{(r_1,\dots,r_{m-1})\in A_m} I\big((\dots((f_n \overset{r_1}{\frown} f_n)\overset{r_2}{\frown} f_n)\dots f_n)\overset{r_{m-1}}{\frown} f_n\big),$$

where

$$\begin{aligned} A_m \;=\; \big\{&(r_1,\dots,r_{m-1}) \in \{0,1,\dots,q\}^{m-1} : r_2 \leqslant 2q - 2r_1,\ r_3 \leqslant 3q - 2r_1 - 2r_2, \dots, \\ & r_{m-1} \leqslant (m-1)q - 2r_1 - \dots - 2r_{m-2}\big\}. \end{aligned}$$

We deduce that

$$\varphi[I(f_n)^m] = \sum_{(r_1,\dots,r_{m-1})\in B_m} (\dots((f_n \overset{r_1}{\frown} f_n)\overset{r_2}{\frown} f_n)\dots f_n)\overset{r_{m-1}}{\frown} f_n,$$

with $B_m = \big\{(r_1,\dots,r_{m-1}) \in A_m : 2r_1 + \dots + 2r_{m-1} = mq\big\}$. We decompose B_m as follows: $B_m = D_m \cup E_m$, with $D_m = B_m \cap \{0, \frac{q}{2}, q\}^{m-1}$ and $E_m = B_m \setminus D_m$, so that

$$\begin{aligned} \varphi[I(f_n)^m] \;=\; & \sum_{(r_1,\dots,r_{m-1})\in D_m} (\dots((f_n \overset{r_1}{\frown} f_n)\overset{r_2}{\frown} f_n)\dots f_n)\overset{r_{m-1}}{\frown} f_n \\ & + \sum_{(r_1,\dots,r_{m-1})\in E_m} (\dots((f_n \overset{r_1}{\frown} f_n)\overset{r_2}{\frown} f_n)\dots f_n)\overset{r_{m-1}}{\frown} f_n. \end{aligned} \tag{4.9}$$

By the forthcoming Lemma 5.1, we have $\|f_n \overset{q/2}{\frown} f_n - f_n\| \to 0$ and $\|f_n \overset{r}{\frown} f_n\| \to 0$ for $r \in \{1,\dots,q-1\} \setminus \{\frac{q}{2}\}$. The conclusion is then obtained by observing that the first sum in (4.9) converges to $\varphi[Z(\lambda)^m]$ by Proposition 2.4 and the forthcoming Lemma 5.2, whereas the second sum converges to zero by the forthcoming Lemma 5.4. □

4.2 Proof of Proposition 1.4

Assume that $F = I(f)$, where f is a mirror symmetric element of $L^2(\mathbb{R}_+^q)$ for some even $q \geqslant 4$, and also that $\varphi[F^2] = \|f\|^2_{L^2(\mathbb{R}_+^q)} = \lambda > 0$. If F had the same law of $Z(\lambda)$, then $\varphi(F^4) - 2\varphi(F^3) = 2\lambda^2 - \lambda$, and the forthcoming Lemma 5.1 would imply that $\|f \overset{q/2}{\frown} f - f\|_{L^2(\mathbb{R}_+^q)} = 0$ and $\|f \overset{r}{\frown} f\|_{L^2(\mathbb{R}_+^{2q-2r})} = 0$ for all $r \in \{1,\dots,q-1\} \setminus \{\frac{q}{2}\}$. As shown in [5, Proof of Corollary 1.7], the relation $\|f \overset{q-1}{\frown} f\|_{L^2(\mathbb{R}_+^2)} = 0$ implies that necessarily $f = 0$, and therefore that $F = 0$. Since $\varphi[F^2] = \lambda > 0$ we have achieved a contradiction, and the proof is concluded. □

5 Ancillary lemmas

This section collects some technical results that are used in the proof of Theorem 1.3.

Lemma 5.1 *Let $q \geqslant 2$ be an even integer, and consider a sequence $\{f_n : n \geqslant 1\} \subset L^2(\mathbb{R}_+^q)$ of mirror symmetric functions such that $\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda > 0$ for every n. As $n \to \infty$, one has that*

$$\varphi[I(f_n)^4] - 2\,\varphi[I(f_n)^3] \to 2\lambda^2 - \lambda$$

if and only if $\|f_n \overset{q/2}{\frown} f_n - f_n\|_{L^2(\mathbb{R}_+^q)} \to 0$ and $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \to 0$ for all $r \in \{1, \dots, q-1\} \setminus \{\frac{q}{2}\}$.

Proof. The product formula yields

$$I(f_n)^2 - I(f_n) = \lambda + I(f_n \overset{0}{\frown} f_n) + I(f_n \overset{q/2}{\frown} f_n - f_n) + \sum_{\substack{1 \leqslant r \leqslant q-1 \\ r \neq q/2}} I(f_n \overset{r}{\frown} f_n).$$

Using the isometry property and the fact that multiple Wigner integrals of different orders are orthogonal in $L^2(\mathscr{A}, \varphi)$, we deduce that

$$\begin{aligned}
&\varphi[(I(f_n)^2 - I(f_n))^2] \\
&= \lambda^2 + \|f_n \overset{0}{\frown} f_n\|^2_{L^2(\mathbb{R}_+^{2q})} + \|f_n \overset{q/2}{\frown} f_n - f_n\|^2_{L^2(\mathbb{R}_+^q)} + \sum_{\substack{1 \leqslant r \leqslant q-1 \\ r \neq q/2}} \|f_n \overset{r}{\frown} f_n\|^2_{L^2(\mathbb{R}_+^{2q-2r})} \\
&= 2\lambda^2 + \|f_n \overset{q/2}{\frown} f_n - f_n\|^2_{L^2(\mathbb{R}_+^q)} + \sum_{\substack{1 \leqslant r \leqslant q-1 \\ r \neq q/2}} \|f_n \overset{r}{\frown} f_n\|^2_{L^2(\mathbb{R}_+^{2q-2r})},
\end{aligned}$$

and the desired conclusion follows because $\varphi[I(f_n)^2] = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda$. □

Lemma 5.2 *Let $m \geqslant 2$ be an integer, let $q \geqslant 2$ be an even integer, and recall the notation adopted in (4.9). Assume $\{f_n : n \geqslant 1\} \subset L^2(\mathbb{R}_+^q)$ is a sequence of mirror symmetric functions satisfying $\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda > 0$ for every n. If $\|f_n \overset{q/2}{\frown} f_n - f_n\|^2_{L^2(\mathbb{R}_+^q)} \to 0$ as $n \to \infty$, then*

$$\sum_{(r_1,\dots,r_{m-1}) \in D_m} (\dots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \dots f_n) \overset{r_{m-1}}{\frown} f_n \to \varphi[Z(\lambda)^m] = \sum_{j=1}^{m} \lambda^j\, R_{m,j}, \tag{5.10}$$

as $n \to \infty$.

Proof. Assume that $f_n \overset{q/2}{\frown} f_n \approx f_n$ (given two sequences $\{a_n\}$ and $\{b_n\}$ with values in some normed vector space, we write $a_n \approx b_n$ to indicate that $a_n - b_n \to 0$ with respect to the associated norm), and consider $(r_1, \dots, r_{m-1}) \in D_m$. Using the identities $f_n \overset{0}{\frown} f_n = f_n \otimes f_n$, $f_n \overset{q}{\frown} f_n = \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda$ and $f_n \overset{q/2}{\frown} f_n \approx f_n$, it is evident that

$$(\dots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \dots f_n) \overset{r_{m-1}}{\frown} f_n \to \lambda^j,$$

where j equals the number of the entries of $(r_1, \ldots, r_{m-1})$ that are equal to q. It follows that, for every $m \geqslant 2$, there exists a polynomial $w_m(\lambda)$ (independent of q) such that, for every sequence $\{f_n\}$ as in the statement,

$$\sum_{(r_1,\ldots,r_{m-1})\in D_m} (\ldots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \ldots f_n) \overset{r_{m-1}}{\frown} f_n \to w_m(\lambda).$$

Now consider the case $q = 2$ and $f_n = f = \sum_{i=1}^{p} e_i \otimes e_i$, where $p \geqslant 1$ and $\{e_i : i = 1, ..., p\}$ is an orthonormal system in $L^2(\mathbb{R}_+^q)$. The following three facts are in order: (a) $I(\sum_{i=1}^{p} e_i \otimes e_i)$ has the same law as $Z(p)$ (see Remark 3.2), (b) $\|f\|^2_{L^2(\mathbb{R}_+^2)} = p$, and (c) $f \overset{1}{\frown} f = f$. Since $E_m = \emptyset$ for $q = 2$, the previous discussion (combined with (4.9) and Proposition 2.4) yields that, for every $m \geqslant 2$, $w_m(p) = \varphi[Z(p)^m] = \sum_{j=1}^{m} p^j R_{m,j}$, for every $p = 1, 2, ...$ Since two polynomials coinciding on a countable set are necessarily equal, we deduce that $w_m(\lambda) = \varphi[Z(\lambda)^m]$ for every $\lambda > 0$.

□

Remark 5.3 By inspection of the arguments used in the proof of Lemma 5.2, one deduces that $R_m = |D_m|$, for every $m \geqslant 2$.

Lemma 5.4 *Let $m \geqslant 2$ be an integer, let $q \geqslant 2$ be an even integer, and recall the notation adopted in (4.9). Assume $\{f_n : n \geqslant 1\} \subset L^2(\mathbb{R}_+^q)$ is a sequence of mirror symmetric functions satisfying $\|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda > 0$ for every n. If $(r_1, \ldots, r_{m-1}) \in E_m$ and if $\|f_n \overset{r}{\frown} f_n\|_{L^2(\mathbb{R}_+^{2q-2r})} \to 0$ for all $r \in \{1, \ldots, q-1\} \setminus \{\frac{q}{2}\}$, then*

$$(\ldots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \ldots f_n) \overset{r_{m-1}}{\frown} f_n \to 0, \qquad \text{as } n \to \infty.$$

Proof. Let $j \in \{1, \ldots, m-1\}$ be the smallest integer such that $r_j \in \{1, \ldots, q-1\} \setminus \{\frac{q}{2}\}$. Then

$$\begin{aligned}
& \big|(\ldots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n) \ldots \overset{r_{m-1}}{\frown} f_n\big| \\
= \quad & \big|(\ldots((f_n \overset{r_1}{\frown} \ldots \overset{r_{j-1}}{\frown} f_n) \overset{r_j}{\frown} f_n)^{\overset{r_{j+1}}{\frown}} \ldots \overset{r_{m-1}}{\frown} f_n\big| \\
\approx_{(*)} \quad & C\big|(\ldots((f_n \otimes \ldots \otimes f_n) \overset{r_j}{\frown} f_n)^{\overset{r_{j+1}}{\frown}} \ldots \overset{r_{m-1}}{\frown} f_n\big| \quad \text{(using } f_n \overset{q/2}{\frown} f_n \approx f_n \text{ and } f_n \overset{q}{\frown} f_n = \lambda) \\
\leqslant_{(**)} \quad & C\,\|(f_n \otimes \ldots \otimes f_n) \otimes (f_n \otimes_{r_j} f_n)\| \times \|f_n\|^{m-j-1}_{L^2(\mathbb{R}_+^q)} \quad \text{(by Cauchy-Schwarz)} \\
= \quad & C\,\|f_n \overset{r_j}{\frown} f_n\|_{L^2\left(\mathbb{R}_+^{2q-2r_j}\right)} \quad \text{(since } \|f_n\|^2_{L^2(\mathbb{R}_+^q)} = \lambda) \\
\longrightarrow \quad & 0, \quad \text{as } n \to \infty.
\end{aligned}$$

In the previous computations, we have adopted the following conventions: (a) C is a finite constant independent of n that may change from line to line, (b) the tensor products in $(*)$ and $(**)$ have an unspecified order which depends on $(r_1, ..., r_{k-1})$, and (c) the first norm in $(**)$ refers to an appropriate $L^2(\mathbb{R}_+^s)$ space, for some s depending of the order of the tensor product.

□

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