ASYMPTOTIC INDEPENDENCE OF MULTIPLE WIENER-ITÔ INTEGRALS AND THE RESULTING LIMIT LAWS

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Abstract. We characterize the asymptotic independence of multiple Wiener-Itô integrals. As a consequence of this characterization, we derive the celebrated fourth moment theorem of Nualart and Peccati and other related results on the multivariate convergence of multiple Wiener-Itô integrals, that involve Gaussian and non Gaussian limits.

1. Introduction

Let $B = (B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion, $q \geqslant 1$ be an integer, and let f be a symmetric element of $L^2(\mathbb{R}_+^q)$. Denote by $I_q(f)$ the q-tuple Wiener-Itô integral of f with respect to B. It is well know that multiple Wiener-Itô integrals of different orders are uncorrelated but not necessarily independent. In an important paper [11], Üstünel and Zakai gave the following characterization of the independence of multiple Wiener-Itô integrals.

Theorem 1.1 (Üstünel-Zakai). *Let $p, q \geqslant 1$ be integers and let $f \in L^2(\mathbb{R}_+^p)$ and $g \in L^2(\mathbb{R}_+^q)$ be symmetric. Then, random variables $I_p(f)$ and $I_q(g)$ are independent if and only if*

$$\int_{\mathbb{R}_+} f(x_1,\ldots,x_{p-1},u)g(x_p,\ldots,x_{p+q-2},u)\,du = 0 \quad \text{in } L^2(\mathbb{R}_+^{p+q-2}). \tag{1.1}$$

Rosiński and Samorodnitsky [10] observed that multiple Wiener-Itô integrals are independent if and only if their squares are uncorrelated:

$$I_p(f) \perp\!\!\!\perp I_q(g) \iff \mathrm{Cov}(I_p(f)^2, I_q(g)^2) = 0. \tag{1.2}$$

This condition can be viewed as a generalization of the usual covariance criterion for the independence of jointly Gaussian random variables (the case of $p = q = 1$).

In the seminal paper [6], Nualart and Peccati discovered the following surprising central limit theorem.

Theorem 1.2 (Nualart-Peccati). *Let $F_n = I_q(f_n)$, where $q \geqslant 2$ is fixed and $f_n \in L^2(\mathbb{R}_+^q)$ are symmetric. Assume also that $E[F_n^2] = 1$ for all n. Then convergence in distribution of*

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(F_n) to the standard normal law is equivalent to convergence of the fourth moment. That is, as $n \to \infty$,

$$F_n \overset{\text{law}}{\to} N(0,1) \quad \Longleftrightarrow \quad E[F_n^4] \to 3. \tag{1.3}$$

Shortly afterwards, Peccati and Tudor [7] established a multidimensional extension of Theorem 1.2. Since the publication of these two important papers, many improvements and developments on this theme have been considered. In particular, Nourdin and Peccati [3] extended Theorem 1.2 to the case when the limit of F_n's is a centered gamma distributed random variable. We refer the reader to the forthcoming book [4] for further information and details of the above results.

In this paper we establish an asymptotic version of Theorem 1.1 from which Theorem 1.2 of Nualart and Peccati follows, as do some other related limit theorems. Heuristic argument linking Theorem 1.1 and Theorem 1.2 was given by Rosiński [9, pages 3–4], while addressing a question of Albert Shiryaev. Namely, let F and G be two independent and identically distributed random variables with fourth moment and unit variance. The link comes via a simple formula

$$\frac{1}{2}\mathrm{Cov}\big((F+G)^2,(F-G)^2\big) = E[F^4]-3$$

as well as the celebrated Bernstein's theorem that asserts that F and G are Gaussian if and only if $F+G$ and $F-G$ are independent (cf. (1.2)). A rigorous argument to carry through this idea is based on a characterization of the asymptotic independence of multiple Wiener-Itô integrals, which is much more difficult to handle than the plain independence, and may also be of an independent interest. The covariance between the squares of multiple Wiener-Itô integrals plays again the pivotal role.

Theorem 3.1, characterizing the asymptotic moment-independence of multiple Wiener-Itô integrals, is the main result of this work. As a consequence of this result, we deduce the fourth moment theorem of Nualart and Peccati [6] in Theorem 4.1 and its multidimensional extension due to Peccati and Tudor [7] in Theorem 4.2. Furthermore, we obtain a new multidimensional extension of a theorem of Nourdin and Peccati [3] in Theorem 4.4, and give another new result on the bivariate convergence of multiple Wiener-Itô integrals in Theorem 4.5.

The paper is organized as follows. In Section 2 we list some basic facts on Gaussian analysis and prove some lemmas that are used in the present work. Section 3 is devoted to the main result. It includes the proof of the main result and remarks showing that certain conclusions true in the case of independence cannot be extended to the asymptotic independence. Applications are given in Section 4.

2. Preliminaries

We will give here some basic elements of Gaussian analysis that are in the foundations of the present work. The reader is referred to the monograph by Nualart [5] for further details and ommited proofs.

Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geqslant 1$ let $\mathfrak{H}^{\otimes q}$ be the qth tensor product of $\mathfrak{H}$ and denote by $\mathfrak{H}^{\odot q}$ the associated qth symmetric tensor product. We write $X = \{X(h), h \in \mathfrak{H}\}$ to indicate an isonormal Gaussian process over $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. This means that X is a centered Gaussian family, whose covariance is given in terms of the inner product of $\mathfrak{H}$ by $E\left[X(h)X(g)\right] = \langle h, g\rangle_{\mathfrak{H}}$. We also assume that $\mathcal{F}$ is generated by X.

For every $q \geqslant 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables of the type $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right)$. We write by convention $\mathcal{H}_0 = \mathbb{R}$. For any $q \geqslant 1$, the mapping $I_q(h^{\otimes q}) = H_q(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ equipped with the modified norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$ and the qth Wiener chaos $\mathcal{H}_q$. For $q = 0$ we write $I_0(c) = c$, $c \in \mathbb{R}$.

It is well known (Wiener chaos expansion) that $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. Therefore, any square integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ admits the following chaotic expansion

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.4}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by F. For every $q \geqslant 0$ we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega, \mathcal{F}, P)$ is as in (2.4), then $J_q F = I_q(f_q)$ for every $q \geqslant 0$.

Let $\{e_k,\, k \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.5}$$

Notice that $f \otimes_r g$ is not necessarily symmetric: we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g\rangle_{\mathfrak{H}^{\otimes q}}$. In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the q-tuple Wiener-Itô integral of f. In this case, (2.5) can be written as

$$\begin{aligned}(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) &= \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r) \\ &\quad \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).\end{aligned}$$

We have

$$\|f \otimes_r g\|^2 = \langle f \otimes_{p-r} f, g \otimes_{q-r} g\rangle \quad \text{for } r = 0, \ldots, p \wedge q, \tag{2.6}$$

where $\langle\cdot\rangle$ ($\|\cdot\|$, respectively) stands for inner product (the norm, respectively) in an appropriate tensor product space $\mathfrak{H}^{\otimes s}$. Also, the following *multiplication formula* holds: if

$f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g), \tag{2.7}$$

where $f\widetilde{\otimes}_r g$ denotes the symmetrization of $f \otimes_r g$.

We conclude these preliminaries by three useful lemmas, that will be needed throughout the sequel.

Lemma 2.1.

(i) *Multiple Wiener-Itô integral has all moments satisfying the following hypercontractivity-type inequality*

$$\big[E|I_p(f)|^r\big]^{1/r} \leq (r-1)^{p/2}\big[E|I_p(f)|^2\big]^{1/2}, \quad r \geqslant 2. \tag{2.8}$$

(ii) *If a sequence of distributions of $\{I_p(f_n)\}_{n\geq 1}$ is tight, then*

$$\sup_n E|I_p(f_n)|^r < \infty \quad \textit{for every } r > 0. \tag{2.9}$$

Proof. (i) Inequality (2.8) is well known and corresponds e.g. to [4, Corollary 2.8.14].
(ii) Combining (2.8) for $r = 4$ with Paley's inequality we get for every $\theta \in (0, 1)$

$$P\big(|I_p(f)|^2 > \theta E|I_p(f)|^2\big) \geq (1-\theta)^2 \frac{\big(E|I_p(f)|^2\big)^2}{E|I_p(f)|^4} \geq (1-\theta)^2 9^{-p}. \tag{2.10}$$

By the assumption, there is an $M > 0$ such that $P(|I_p(f_n)|^2 > M) < 9^{-p-1}$, $n \geq 1$. By (2.10) with $\theta = 2/3$ and all n, we have

$$P\big(|I_p(f_n)|^2 > M\big) < 9^{-p-1} \leq P\big(|I_p(f_n)|^2 > (2/3)E|I_p(f_n)|^2\big).$$

As a consequence, $E|I_p(f_n)|^2 \leq (3/2)M$. Applying (2.8) we conclude (2.9). □

Lemma 2.2.

(1) *Let $p, q \geqslant 1$, $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$. Then*

$$\|f\widetilde{\otimes} g\|^2 = \frac{p!q!}{(p+q)!}\sum_{r=0}^{p\wedge q}\binom{p}{r}\binom{q}{r}\|f \otimes_r g\|^2, \tag{2.11}$$

(2) *Let $q \geqslant 1$ and $f_1, f_2, f_3, f_4 \in \mathfrak{H}^{\odot q}$. Then*

$$(2q)!\langle f_1 \widetilde{\otimes} f_2, f_3 \widetilde{\otimes} f_4\rangle = \sum_{r=1}^{q-1} q!^2\binom{q}{r}^2\langle f_1\otimes_r f_3, f_4\otimes_r f_2\rangle + q!^2\big(\langle f_1, f_3\rangle\langle f_2, f_4\rangle + \langle f_1, f_4\rangle\langle f_2, f_3\rangle\big). \tag{2.12}$$

(3) *Let $q \geqslant 1$, $f \in \mathfrak{H}^{\odot(2q)}$ and $g \in \mathfrak{H}^{\odot q}$. We have*

$$\langle f \widetilde{\otimes}_q f, g \widetilde{\otimes} g\rangle = \frac{2q!^2}{(2q)!}\langle f \otimes_q f, g \otimes g\rangle + \frac{q!^2}{(2q)!}\sum_{r=1}^{q-1}\binom{q}{r}^2\langle f \otimes_r g, g \otimes_r f\rangle. \tag{2.13}$$

Proof. Without loss of generality, we suppose throughout the proof that $\mathfrak{H}$ is equal to $L^2(A,\mathcal{A},\mu)$, where $(A,\mathcal{A})$ is a measurable space and μ is a σ-finite measure without atoms.

(1) Let σ be a permutation of $\{1,\dots,p+q\}$ (this fact is written in symbols as $\sigma\in\mathfrak{S}_{p+q}$). If $r\in\{0,\dots,p\wedge q\}$ denotes the cardinality of $\{1,\dots,p\}\cap\{\sigma(p+1),\dots,\sigma(p+q)\}$, then it is readily checked that r is also the cardinality of $\{p+1,\dots,p+q\}\cap\{\sigma(1),\dots,\sigma(p)\}$ and that

$$\begin{aligned}&\int_{A^{p+q}} f(t_1,\dots,t_p)g(t_{p+1},\dots,t_{p+q})f(t_{\sigma(1)},\dots,t_{\sigma(p)})g(t_{\sigma(p+1)},\dots,t_{\sigma(p+q)})d\mu(t_1)\dots d\mu(t_{p+q})\\ =&\int_{A^{p+q-2r}}(f\otimes_r g)(x_1,\dots,x_{p+q-2r})^2 d\mu(x_1)\dots d\mu(x_{p+q-2r})=\|f\otimes_r g\|^2. \qquad (2.14)\end{aligned}$$

Moreover, for any fixed $r\in\{0,\dots,p\wedge q\}$, there are $p!\binom{p}{r}q!\binom{q}{r}$ permutations $\sigma\in\mathfrak{S}_{p+q}$ such that $\{1,\dots,p\}\cap\{\sigma(p+1),\dots,\sigma(p+q)\}=r$. (Indeed, such a permutation is completely determined by the choice of: (a) r distinct elements $y_1,\dots,y_r$ of $\{p+1,\dots,p+q\}$; (b) $p-r$ distinct elements $y_{r+1},\dots,y_p$ of $\{1,\dots,p\}$; (c) a bijection between $\{1,\dots,p\}$ and $\{y_1,\dots,y_p\}$; (d) a bijection between $\{p+1,\dots,p+q\}$ and $\{1,\dots,p+q\}\backslash\{y_1,\dots,y_p\}$.) Now, observe that the symmetrization of $f\otimes g$ is given by

$$f\widetilde{\otimes}g(t_1,\dots,t_{p+q})=\frac{1}{(p+q)!}\sum_{\sigma\in\mathfrak{S}_{p+q}} f(t_{\sigma(1)},\dots,t_{\sigma(p)})g(t_{\sigma(p+1)},\dots,t_{\sigma(p+q)}).$$

Therefore, using (2.14), we can write

$$\begin{aligned}\|f\widetilde{\otimes}g\|^2=\langle f\otimes g,f\widetilde{\otimes}g\rangle&=\frac{1}{(p+q)!}\sum_{\sigma\in\mathfrak{S}_{p+q}}\int_{A^{p+q}} f(t_1,\dots,t_p)g(t_{p+1},\dots,t_{p+q})\\ &\qquad\times f(t_{\sigma(1)},\dots,t_{\sigma(p)})g(t_{\sigma(p+1)},\dots,t_{\sigma(p+q)})d\mu(t_1)\dots d\mu(t_{p+q})\\ =\ &\frac{1}{(p+q)!}\sum_{r=0}^{p\wedge q}\|f\otimes_r g\|^2\mathrm{Card}\{\sigma\in\mathfrak{S}_{p+q}:\ \{1,\dots,p\}\cap\{\sigma(p+1),\dots,\sigma(p+q)\}=r\}.\end{aligned}$$

and (2.11) follows.

(2) We proceed analogously. Indeed, we have

$$\begin{aligned}&\langle f_1\widetilde{\otimes}f_2,f_3\widetilde{\otimes}f_4\rangle=\langle f_1\otimes f_2,f_3\widetilde{\otimes}f_4\rangle\\ =\ &\frac{1}{(2q)!}\sum_{\sigma\in\mathfrak{S}_{2q}}\int_{A^{2q}} f_1(t_1,\dots,t_q)f_2(t_{q+1},\dots,t_{2q})\\ &\qquad\times f_3(t_{\sigma(1)},\dots,t_{\sigma(q)})f_4(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})d\mu(t_1)\dots d\mu(t_{2q})\\ =\ &\frac{1}{(2q)!}\sum_{r=0}^{q}\langle f_1\otimes_r f_3,f_4\otimes_r f_2\rangle\,\mathrm{Card}\{\sigma\in\mathfrak{S}_{2q}:\ \{\sigma(1),\dots,\sigma(q)\}\cap\{1,\dots,q\}=r\},\end{aligned}$$

from which we deduce (2.12).

(3) We have

$$\begin{aligned}(g\,\widetilde{\otimes}\,g)(t_1,\ldots,t_{2q}) &= \frac{1}{(2q)!}\sum_{\sigma\in\mathfrak{S}_{2q}} g(t_{\sigma(1)},\ldots,t_{\sigma(q)})g(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)})\\ &= \frac{1}{(2q)!}\sum_{r=0}^{q}\sum_{\substack{\sigma\in\mathfrak{S}_{2q}\\ \{\sigma(1),\ldots,\sigma(q)\}\cap\{1,\ldots,q\}=r}} g(t_{\sigma(1)},\ldots,t_{\sigma(q)})g(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)}),\end{aligned}$$

and

$$(f\otimes_q f)(t_1,\ldots,t_{2q}) = \int_{A^q} f(t_1,\ldots,t_q,x_1,\ldots,x_q)f(x_1,\ldots,x_q,t_{q+1},\ldots,t_{2q})d\mu(x_1)\ldots d\mu(x_q),$$

so that

$$\begin{aligned}&\langle f\,\widetilde{\otimes}_q\,f, g\,\widetilde{\otimes}\,g\rangle = \langle f\otimes_q f, g\,\widetilde{\otimes}\,g\rangle\\ =\;& \frac{1}{(2q)!}\sum_{r=0}^{q}\langle f\otimes_r g, g\otimes_r f\rangle \mathrm{Card}\{\sigma\in\mathfrak{S}_{2q} : \{\sigma(1),\ldots,\sigma(q)\}\cap\{1,\ldots,q\}=r\}\\ =\;& \frac{1}{(2q)!}\sum_{r=0}^{q}\binom{q}{r}^2 q!^2\langle f\otimes_r g, g\otimes_r f\rangle\\ =\;& \frac{q!^2}{(2q)!}\langle f\otimes_q g, g\otimes_q f\rangle + \frac{q!^2}{(2q)!}\langle f\otimes g, g\otimes f\rangle + \frac{1}{(2q)!}\sum_{r=1}^{q-1}\binom{q}{r}^2 q!^2\langle f\otimes_r g, g\otimes_r f\rangle.\end{aligned}$$

Since $\langle f\otimes_q g, g\otimes_q f\rangle = \langle f\otimes g, g\otimes f\rangle = \langle f\otimes_q f, g\otimes g\rangle$, the desired conclusion (2.13) follows. □

Lemma 2.3 (Generalized Cauchy-Schwarz Inequality). *Assume that $\mathfrak{H} = L^2(A,\mathcal{A},\mu)$, where $(A,\mathcal{A})$ is a measurable space equipped with a σ-finite measure μ. For any integer $M\geqslant 1$, put $[M]=\{1,...,M\}$. Also, for every element $\mathbf{z}=(z_1,...,z_M)\in A^M$ and every nonempty set $c\subset[M]$, let $\mathbf{z}_c$ denote the element of $A^{|c|}$ (where $|c|$ is the cardinality of c) obtained by deleting from $\mathbf{z}$ the entries with index not contained in c. (For instance, if $M=5$ and $c=\{1,3,5\}$, then $\mathbf{z}_c=(z_1,z_3,z_5)$.) Let*

- (α) *$C,q\geqslant 2$ be integers, and let $c_1,...,c_q$ be nonempty subsets of $[C]$ such that each element of $[C]$ appears in exactly two of the c_i's (this implies that $\bigcup_i c_i=[C]$ and $\sum_i|c_i|=2C$);*
- (β) *let $h_1,...,h_q$ be functions such that $h_i\in L^2(\mu^{|c_i|}):=L^2(A^{|c_i|},\mathcal{A}^{|c_i|},\mu^{|c_i|})$ for every $i=1,...,q$ (in particular, each h_i is a function of $|c_i|$ variables).*

Then

$$\Big|\int_{A^C}\prod_{i=1}^{q}h_i(\mathbf{z}_{c_i})\,\mu^C(d\mathbf{z}_{[C]})\Big| \leqslant \prod_{i=1}^{q}\|h_i\|_{L^2(\mu^{|c_i|})}. \tag{2.15}$$

Moreover, if $c_0 := c_j \cap c_k \neq \emptyset$ *for some* $j \neq k$, *then*

$$\Big| \int_{A^C} \prod_{i=1}^{q} h_i(\mathbf{z}_{c_i})\, \mu^C(d\mathbf{z}_{[C]}) \Big| \leqslant \|h_j \otimes_{c_0} h_k\|_{L^2(\mu^{|c_j \triangle c_k|})} \prod_{i \neq j,k}^{q} \|h_i\|_{L^2(\mu^{|c_i|})}, \tag{2.16}$$

where

$$h_j \otimes_{c_0} h_k(\mathbf{z}_{c_j \triangle c_k}) = \int_{A^{|c_0|}} h_j(\mathbf{z}_{c_j}) h_k(\mathbf{z}_{c_k})\, \mu^{|c_0|}(d\mathbf{z}_{c_0}).$$

(Notice that $h_j \otimes_{c_0} h_k = h_j \otimes_{|c_0|} h_k$ *when* h_j *and* h_k *are symmetric.)*

Proof. In the case $q = 2$, (2.15) is just the Cauchy-Schwarz inequality and (2.16) is an equality. Assume that (2.15)–(2.16) hold for at most $q-1$ functions and proceed by induction. Among the sets $c_1, \dots, c_q$ at least two, say c_j and c_k, have nonempty intersection. Set $c_0 := c_j \cap c_k$, as above. Since c_0 does not have common elements with c_i for all $i \neq j, k$, by Fubini's theorem

$$\int_{A^C} \prod_{i=1}^{q} h_i(\mathbf{z}_{c_i})\, \mu^C(d\mathbf{z}_{[C]}) = \int_{A^{C-|c_0|}} h_j \otimes_{c_0} h_k(\mathbf{z}_{c_j \triangle c_k}) \prod_{i \neq j,k}^{q} h_i(\mathbf{z}_{c_i})\, \mu^{C-|c_0|}(d\mathbf{z}_{[C]\setminus c_0}). \tag{2.17}$$

Observe that every element of $[C] \setminus c_0$ belongs to exactly two of the $q-1$ sets: $c_j \triangle c_k$, c_i, $i \neq j, k$. Therefore, by the induction assumption, (2.15) implies (2.16), provided $c_j \triangle c_k \neq \emptyset$. When $c_j = c_k$, we have $h_j \otimes_{c_0} h_k = \langle h_j, h_k \rangle$ and (2.16) follows from (2.15) applied to the product of $q-2$ functions in (2.17). This proves (2.16), which in turn yields (2.15) by the Cauchy-Schwarz inequality. The proof is complete. □

3. The main result

Our main result gives conditions for the asymptotic independence among the components of random vectors consisting of multiple Wiener-Itô integrals.

Theorem 3.1. *Let* $d \geqslant 2$, *and let* $q_1, \dots, q_d$ *be positive integers. Consider vectors*

$$(F_{1,n}, \dots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \dots, I_{q_d}(f_{d,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. *Assume that for some random vector* $(U_1, \dots, U_d)$,

$$(F_{1,n}, \dots, F_{d,n}) \overset{\text{law}}{\to} (U_1, \dots, U_d) \quad \text{as } n \to \infty. \tag{3.18}$$

Then U_i*'s admit moments of all orders and the following three conditions are equivalent:*

(a) $E[U_1^{k_1} \dots U_d^{k_d}] = E[U_1^{k_1}] \dots E[U_d^{k_d}]$ *for all* $k_1, \dots, k_d \in \mathbb{N}$;

(b) $\lim_{n\to\infty} \operatorname{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *for all* $i \neq j$;

(c) $\lim_{n\to\infty} \|f_{i,n} \otimes_r f_{j,n}\| = 0$ *for all* $i \neq j$ *and all* $r = 1, \dots, q_i \wedge q_j$;

Moreover, if the distribution of each U_i *is determined by its moments, then (a) is equivalent to that*

(d) $U_1, \dots, U_d$ *are independent.*

Corollary 3.2. *Under the notation of Theorem 3.1, suppose that for each* $i = 1, \dots, d$,

$$F_{n,i} \overset{\text{law}}{\to} U_i, \quad \text{as } n \to \infty,$$

where U_i *are independent random variables whose distributions are determined by their moments. Then the joint convergence* (3.18) *holds provided one of the equivalent conditions (b) or (c) of Theorem 3.1 is satisfied.*

Proof: Since the sequence $\{(F_{1,n}, \dots, F_{d,n})\}_{n\geq 1}$ is tight, it is relatively compact in law. Let $(V_1, \dots, V_d)$ be a random vector such that

$$(F_{1,n_k}, \dots, F_{d,n_k}) \overset{\text{law}}{\to} (V_1, \dots, V_d)$$

as $n_k \to \infty$ along a subsequence. Since for each i, $V_i \overset{\text{law}}{=} U_i$,

$$(V_1, \dots, V_d) \overset{\text{law}}{=} (U_1, \dots, U_d)$$

by Theorem 3.1(d). Thus (3.18) holds. ☐

Remark 3.3.

(1) Recall that a probability distribution having all moments is called determinate if there is no any other distribution having the same sequence of moments. The well-known Carleman's condition provides a useful sufficient criterion for the determinacy of measures on the real line. On the other hand, a relevant fact in our context is that the cube of a normal distribution is indeterminate, see Berg [1].

(2) A natural question arises, whether condition (a) of Theorem 3.1 may imply (d) under a weaker condition than that the determinacy of one-dimensional marginals. The following example, shown to us by Jean Bertoin, demonstrates that in general such implication is false.

Consider two different probability distributions μ and ν on $\mathbb{R}$ having all moments finite and equal (thus they are indeterminate). Consider random variables $X, X' \sim \mu$, $Y, Y' \sim \nu$, and s with Bernoulli distribution with parameter $1/2$. Assume that s, X, X', Y, Y' are independent. Set $(U_1, U_2) = (X, X')$ if $s = 0$ and $(U_1, U_2) = (Y, Y')$ if $s = 1$. It is easy to check that $E[U_1^m U_2^n] = E[U_1^m]E[U_2^n]$ for all $m, n \in \mathbb{N}$. Moreover, for every $u \in \mathbb{R}$ we have

$$P(U_1 > u, U_2 > u) = \frac{1}{2}\big[P(X > u)^2 + P(Y > u)^2\big],$$

$$P(U_1 > u) = P(U_2 > u) = \frac{1}{2}\big[P(X > u) + P(Y > u)\big],$$

and thus

$$P(U_1 > u, U_2 > u) - P(U_1 > u)P(U_2 > u) = \frac{1}{4}\big[P(X > u) - P(Y > u)\big]^2.$$

Since $\mu \neq \nu$, the last term is positive for some u, showing that U_1, U_2 are dependent.

(3) Assume that $d = 2$ (for simplicity). In this case, (c) becomes $\|f_{1,n} \otimes_r f_{2,n}\| \to 0$ for all $r = 1, \dots, q_1 \wedge q_2$. In view of Theorem 1.1 of Üstünel and Zakai, one may expect that (c) could be replaced by a weaker condition (c'): $\|f_{1,n} \otimes_1 f_{2,n}\| \to 0$.

However, the latter is false. To see it, consider a sequence $f_n \in \mathfrak{H}^{\odot 2}$ such that $\|f_n\|^2 = \frac{1}{2}$ and $\|f_n \otimes_1 f_n\| \to 0$. By Theorem 4.1 below, $F_n := I_2(f_n) \overset{\text{law}}{\to} U \sim N(0,1)$. Putting $f_{1,n} = f_{2,n} = f_n$, we observe that (c') holds but (a) does not, as $(I_2(f_{1,n}), I_2(f_{2,n})) \overset{\text{law}}{\to} (U, U)$.

(4) Taking into account that assumptions (γ) and (δ) of forthcoming Theorems 4.1 and 4.3 are equivalent, it is natural to wonder whether assumption (c) of Theorem 3.1 is equivalent to its symmetrized version

$$(c') \quad \lim_{n\to\infty} \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\| = 0 \text{ for all } i \neq j \text{ and all } r = 1, \dots, q_i \wedge q_j.$$

The answer is actually negative in general, as is shown by the following simple counterexample. Indeed, let $f_1, f_2 : [0,1]^2 \to \mathbb{R}$ be symmetric functions given by

$$f_1(s,t) = \begin{cases} -1 & s,t \in [0,1/2] \\ 1 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_2(s,t) = \begin{cases} -1 & s,t \in (1/2,1] \\ 1 & \text{elsewhere.} \end{cases}$$

Then $\langle f_1, f_2 \rangle = 0$ and

$$(f_1 \otimes_1 f_2)(s,t) = \begin{cases} -1 & \text{if } s \in [0,1/2] \text{ and } t \in (1/2,1] \\ 1 & \text{if } t \in [0,1/2] \text{ and } s \in (1/2,1] \\ 0 & \text{elsewhere,} \end{cases}$$

so that $f_1 \widetilde{\otimes}_1 f_2 \equiv 0$ and $\|f_1 \otimes_1 f_2\| = \sqrt{2}$.

Proof of Theorem 3.1. From the assumption (3.18) and Lemma 2.1 we infer that for every $i = 1, \dots, d$ and $r > 0$

$$\sup_n E|F_{i,n}|^r < \infty. \tag{3.19}$$

It follows that U_i's have moments of all orders.

Step 1: (**a**) → (**b**). This is immediate because from (**a**) and (3.19) we have

$$\mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = E[F_{i,n}^2 F_{j,n}^2] - E[F_{i,n}^2]E[F_{j,n}^2] \to E[U_i^2 U_j^2] - E[U_i^2]E[U_j^2] = 0$$

as $n \to \infty$.

Step 2: (**b**) → (**c**). Fix $i, j \in \{1, \dots, d\}$ and $n \geqslant 1$. By (2.7) we have

$$F_{i,n} F_{j,n} = \sum_{r=0}^{q_i \wedge q_j} r! \binom{q_i}{r} \binom{q_j}{r} I_{q_i+q_j-2r}(f_{i,n} \widetilde{\otimes}_r f_{j,n}),$$

which yields

$$E[F_{i,n}^2 F_{j,n}^2] = \sum_{r=0}^{q_i \wedge q_j} r!^2 \binom{q_i}{r}^2 \binom{q_j}{r}^2 (q_i + q_j - 2r)! \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2.$$

Moreover,

$$E[F_{i,n}^2]E[F_{j,n}^2]=q_i!q_j!\|f_{i,n}\|^2\|f_{j,n}\|^2.$$

Applying (2.11) to the second equality below, we evaluate $\mathrm{Cov}(F_{i,n}^2,F_{j,n}^2)$ as follows:

$$\begin{aligned}\mathrm{Cov}(F_{i,n}^2,F_{j,n}^2)&=(q_i+q_j)!\|f_{i,n}\widetilde{\otimes}f_{j,n}\|^2-q_i!q_j!\|f_{i,n}\|^2\|f_{j,n}\|^2 &(3.20)\\&\quad+\sum_{r=1}^{q_i\wedge q_j}r!^2\binom{q_i}{r}^2\binom{q_j}{r}^2(q_i+q_j-2r)!\|f_{i,n}\widetilde{\otimes}_r f_{j,n}\|^2\\&=q_i!q_j!\sum_{r=1}^{q_i\wedge q_j}\binom{q_i}{r}\binom{q_j}{r}\|f_{i,n}\otimes_r f_{j,n}\|^2+\sum_{r=1}^{q_i\wedge q_j}r!^2\binom{q_i}{r}^2\binom{q_j}{r}^2(q_i+q_j-2r)!\|f_{i,n}\widetilde{\otimes}_r f_{j,n}\|^2\\&\geqslant\max_{r=1,\dots,q_i\wedge q_j}\|f_{i,n}\otimes_r f_{j,n}\|^2. &(3.21)\end{aligned}$$

This bound yields the desired conclusion.

Step 3: (**c**) → (**a**). It is easy to check that (**a**) is equivalent to: for every $k_1,\dots,k_d\in\mathbb{N}\backslash\{0\}$ and $J\subset\{1,\dots,d\}$

$$E\prod_{i\in J}\big(U_i^{k_i}-E[U_i^{k_i}]\big)=0. \tag{3.22}$$

To show (3.22) it suffices to consider only $J=\{1,\dots,d\}$ (taking smaller sets J correspond to lowering the order d). Therefore, it is enough to prove that (**c**) implies

$$E\prod_{i=1}^{d}\big(U_i^{k_i}-E[U_i^{k_i}]\big)=0$$

or, equivalently, that

$$\lim_{n\to\infty}E\prod_{i=1}^{d}\big(I_{q_i}(f_{i,n})^{k_i}-E[I_{q_i}(f_{i,n})^{k_i}]\big)=0. \tag{3.23}$$

When $k\geqslant 2$ is a given integer, a suitable iteration of the product formula (2.7) leads to

$$I_q(f)^k=\sum_{\mathbf{r}\in C_{q,k}}a(q,k,\mathbf{r})I_{kq-2|\mathbf{r}|}\big((\dots((f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_{r_2}f)\dots)\widetilde{\otimes}_{r_{k-1}}f\big), \tag{3.24}$$

where $\mathbf{r}=(r_1,\dots,r_{k-1})$, $|\mathbf{r}|=r_1+\dots+r_{k-1}$, $a(q,k,\mathbf{r})$ are positive constants, and

$$C_{q,k}=\big\{\mathbf{r}\in\{0,\dots,q\}^{k-1}:\ r_2\leqslant 2q-2r_1,\ \dots,\ r_{k-1}\leqslant(k-1)q-2r_1-\dots-2r_{k-2}\big\}.$$

Since the expectation of a Wiener-Itô integral of positive order is 0, from (3.24) we get

$$I_q(f)^k-E[I_q(f)^k]=\sum_{\mathbf{r}\in D_{q,k}}a(q,k,\mathbf{r})I_{kq-2|\mathbf{r}|}\big((\dots((f\widetilde{\otimes}_{r_1}f)\widetilde{\otimes}_{r_2}f)\dots)\widetilde{\otimes}_{r_{k-1}}f\big), \tag{3.25}$$

where

$$D_{q,k}=\big\{\mathbf{r}\in C_{q,k}:\ kq-2r_1-\dots-2r_{k-1}>0\big\}.$$

Therefore,

$$\prod_{i=1}^{d}\big(I_{q_i}(f_{i,n})^{k_i}-E[I_{q_i}(f_{i,n})^{k_i}]\big)=\sum_{\mathbf{r}^1\in D_{q_1,k_1}}\cdots\sum_{\mathbf{r}^d\in D_{q_d,k_d}}\prod_{i=1}^{d}a(q_i,k_i,\mathbf{r}^i)I_{k_iq_i-2|\mathbf{r}^i|}\big(h_{i,n}\big),$$

with $h_{i,n}=(\dots((f_{i,n}\widetilde{\otimes}_{r^i_1}f_{i,n})\widetilde{\otimes}_{r^i_2}f_{i,n})\dots)\widetilde{\otimes}_{r^i_{k_i-1}}f_{i,n}$. To prove (3.23) it is enough to show that the expectation of each product of stochastic integrals given above tends to 0. Every such product of integrals has a form

$$\prod_{i=1}^{d}I_{p_i}(h_{i,n})=\sum_{\mathbf{s}\in S}b(\mathbf{p},d,\mathbf{s})I_{|\mathbf{p}|-2|\mathbf{s}|}\big((\dots((h_{1,n}\widetilde{\otimes}_{s_1}h_{2,n})\widetilde{\otimes}_{s_2}h_{3,n})\dots)\widetilde{\otimes}_{s_{d-1}}h_{d,n}\big), \tag{3.26}$$

where $p_i=k_iq_i-2|\mathbf{r^i}|>0$ and $h_{i,n}\in\mathfrak{H}^{\odot p_i}$ are as above, $\mathbf{p}=(p_1,\dots,p_d)$, $\mathbf{s}=(s_1,\dots,s_{d-1})\in\prod_{i=1}^{d-1}\{0,\dots,p_{i+1}\}$,

$$S=\{\mathbf{s}:s_2\le p_1-2s_1,\dots,s_{d-1}\le(p_1+\dots+p_{d-1})-2(s_1+\dots+s_{d-2})\},$$

$b(\mathbf{p},d,\mathbf{s})$ are constants, and $|\mathbf{p}|$, $|\mathbf{s}|$ denote the sums of components in the respective vectors. Since the expectation of a Wiener-Itô integral of positive order is 0, we get from (3.26)

$$E\prod_{i=1}^{d}I_{p_i}(h_{i,n})=\sum_{\mathbf{s}\in S_0}b(\mathbf{p},d,\mathbf{s})\big\langle(\dots((h_{1,n}\widetilde{\otimes}_{s_1}h_{2,n})\widetilde{\otimes}_{s_2}h_{3,n})\dots)\,,\,h_{d,n}\big\rangle,$$

where

$$S_0=\{\mathbf{s}\in S:2|\mathbf{s}|=|\mathbf{p}|\}.$$

Notice that $S_0=\emptyset$ when $|\mathbf{p}|$ is odd, in which case the above sum is 0 by convention. Therefore, to prove (3.23) it is enough to show that for each $\mathbf{s}\in S_0$

$$\big\langle(\dots((h_{1,n}\widetilde{\otimes}_{s_1}h_{2,n})\widetilde{\otimes}_{s_2}\dots)\widetilde{\otimes}_{s_{d-2}}h_{d-1,n})\,,\,h_{d,n}\big\rangle\to 0\quad\text{as } n\to\infty, \tag{3.27}$$

where $|\mathbf{p}|$ is even, $p_i>0$ for $i=1.\dots,d$, and $h_{i,n}\in\mathfrak{H}^{\odot p_i}$ are given above.

Without loss of generality we may and do assume that $\mathfrak{H}$ is equal to $L^2(\mu):=L^2(A,\mathcal{A},\mu)$, where $(A,\mathcal{A})$ is a measurable space and μ is a σ-finite measure without atoms. We begin with some general observations for which we need to recall notation of Lemma 2.3.

For every integer $M\geqslant 1$, put $[M]=\{1,...,M\}$. Also, for every element $\mathbf{z}=(z_1,...,z_M)\in A^M$ and every nonempty set $c\subset[M]$, we denote by $\mathbf{z}_c$ the element of $A^{|c|}$ (where $|c|$ is the cardinality of c) obtained by deleting from $\mathbf{z}$ the entries with index not contained in c. (For instance, if $M=5$ and $c=\{1,3,5\}$, then $\mathbf{z}_c=(z_1,z_3,z_5)$.)

We begin with a general observation on the structure of symmetrized contractions. Let $g_i\in L^2(\mu^{l_i})$ be symmetric functions, $l_i\ge 1$, $i=1,\dots,m$. Let

$$G=(\dots((g_1\widetilde{\otimes}_{t_1}g_2)\widetilde{\otimes}_{t_2}\dots)\widetilde{\otimes}_{t_{m-1}}g_{m-1})\widetilde{\otimes}_{t_{m-1}}g_m \tag{3.28}$$

where $t_1,\dots,t_{m-1}$ are nonnegative integers such that the contractions are well defined; put $|\mathbf{l}|=l_1+\dots+l_m$ and $\mathbf{t}=t_1+\dots+t_{m-1}$. Then G is a linear combination of functions ϕ

given up to a fixed permutation of its of $|\mathbf{l}|-2|\mathbf{t}|$ variables by

$$\phi(\mathbf{z}_{J_1}) = \int_{A^{|J_2|}} g_1(\mathbf{z}_{b_1})\cdots g_m(\mathbf{z}_{b_m})\,\mu^{|J_2|}(d\mathbf{z}_{J_2}) \tag{3.29}$$

where for $M=|\mathbf{l}|-|\mathbf{t}|$, the sets $b_i\subset[M]$ satisfy the following conditions: $|b_i|=l_i$, $|b_j\cap\bigcup_{i=1}^{j-1}b_i|=t_{j-1}$, and $b_i\cap b_j\cap b_k=\emptyset$ for $1\le i<j<k\le m$. It follows that $\bigcup_{i=1}^{m}b_i=[M]$, so that each element of $[M]$ belongs exactly to one or two sets b_i's. In (3.29) we have

$$J_k=\{j\in[M] : j \text{ is in exactly } k \text{ sets } b_i\},$$

$[M]=J_1\cup J_2$.

We will now show that for all $1\le i<j\le d$ and $1\le s\le p_i\wedge p_j$

$$\lim_{n\to\infty}\|h_{i,n}\otimes_s h_{j,n}\|=0 \tag{3.30}$$

It is enough to consider $i=1$, $j=2$. From (3.29) we infer that $h_{1,n}\otimes_s h_{2,n}$ is a linear combination of functions ψ given up to a fixed permutation of its of p_1+p_2-2s variables by

$$\psi(\mathbf{z}_{U_1}) = \int_{A^{|U_2|}} f_{1,n}(\mathbf{z}_{b_1})\cdots f_{1,n}(\mathbf{z}_{b_{k_1}})f_{2,n}(\mathbf{z}_{b_{k_1+1}})\cdots f_{2,n}(\mathbf{z}_{b_{k_1+k_2}})\,\mu^{|U_2|}(d\mathbf{z}_{U_2}), \tag{3.31}$$

where, for $i=1,\dots,k_1$, the sets $b_i\subset[M_1]$ are constructed as in (3.29) for $g_i=f_{1,n}$ and $M_1=p_1+|\mathbf{r}^{\mathbf{1}}|$. Thus each $l\in[M_1]$ belongs to exactly one or two sets b_i, which results in a partition $[M_1]=J_1\cup J_2$, as above. Fix $S\subset J_1$ with $|S|=s$ and let $K=S\cup\{M_1+1,\dots,M_1+M_2-s\}$. For $i=k_1+1,\dots,k_1+k_2$, the sets $b_i\subset K$ are constructed analogously to (3.29) for the case $g_i=f_{2,n}$ and $M=M_2:=p_2+|\mathbf{r}^{\mathbf{2}}|$. This yields a partition $K=K_1\cup K_2$, where K_j is the set of elements of K belonging to exactly j of b_i's, $k_1+1\le i\le k_1+k_2$. It is required that $S\subset K_1$, so that $S\subset J_1\cap K_1$. Finally, put $U_2=J_2\cup S\cup K_2$ and $U_1=[M_1+M_2-s]\setminus U_2$.

We may now apply Lemma 2.3 to the right hand side of (3.31). Indeed, each element of U_2 belongs to exactly two sets $c_i:=b_i\cap U_2$, $i=1,\dots,k_1+k_2$. Since S is nonempty, there exist $i\le k_1<j$ such that $c_0:=c_i\cap c_j\neq\emptyset$. Therefore, by (2.15) we have

$$|\psi(\mathbf{z}_{U_1})|^2 \leqslant |f_{1,n}\otimes_{|c_0|} f_{2,n}(\mathbf{z}_{b_i\triangle b_j})|^2 \prod_{l\le k_1,\, l\neq i}\int_{A^{|c_l|}} |f_{1,n}(\mathbf{z}_{b_l})|^2\,\mu^{|c_l|}(d\mathbf{z}_{c_l})$$

$$\times \prod_{l>k_1,\, l\neq j}\int_{A^{|c_l|}} |f_{2,n}(\mathbf{z}_{b_l})|^2\,\mu^{|c_l|}(d\mathbf{z}_{c_l}).$$

Since $b_i\triangle b_j$, $b_l\setminus c_l$, $1\le l\le k_1+k_2$, $l\neq i,j$ form a partition of U_1, additional integration gives

$$\int_{A^{|U_1|}} |\psi(\mathbf{z}_{U_1})|^2\,\mu^{|c_l|}(d\mathbf{z}_{c_l}) \le \|f_{1,n}\otimes_{|c_0|} f_{2,n}\|^2_{L^2(\mu^{|b_i\triangle b_j|})}\|f_{1,n}\|^{2k_1-2}_{L^2(\mu^{q_1})}\|f_{2,n}\|^{2k_2-2}_{L^2(\mu^{q_2})}.$$

By assumption **(c)** of the theorem and (3.19), $\|\psi(\mathbf{z}_{U_1})\|_{L^2(\mu^{|U_1|})}\to 0$ as $n\to\infty$, which concludes the proof of (3.30).

Finally, we will show (3.27). We can write the inner product in (3.27) as

$$H = (\dots((h_{1,n}\widetilde{\otimes}_{s_1} h_{2,n})\widetilde{\otimes}_{s_2}\dots)\widetilde{\otimes}_{s_{d-2}} h_{d-1,n})\widetilde{\otimes}_{s_{d-1}} h_{d,n}.$$

Therefore, by (3.29) H is a linear combination of integrals of the form

$$\int_J h_{1,n}(\mathbf{z}_{b_1})\cdots h_{d,n}(\mathbf{z}_{b_d})\,\mu^{|J|}(d\mathbf{z}_J),$$

where $b_i \subset [M]$, with $M = |\mathbf{p}| - |\mathbf{s}| > 0$, $|b_i| = p_i > 0$. Since $|\mathbf{p}| = 2|\mathbf{s}|$, each element of $[M]$ belongs to exactly two b_i's. That is, $J_1 = \emptyset$ and $J_2 = J$ in the notation of (3.29). Since $|\mathbf{s}| > 0$, there must be $i < j$ such that $b_0 := b_i \cap b_j \neq \emptyset$. By Lemma 2.3, (3.19) and (3.30)

$$\Big|\int_J h_{1,n}(\mathbf{z}_{b_1})\cdots h_{d,n}(\mathbf{z}_{b_d})\,\mu^{|J|}(d\mathbf{z}_J)\Big| \le \|h_{i,n}\otimes_{|b_0|} h_{j,n}\|_{L^2(\mu^{|b_i\triangle b_j|})}\prod_{k\neq i,j}\|h_{k,n}\|_{L^2(\mu^{p_k})}\to 0$$

as $n\to\infty$. This concludes the proof of the implication (**c**) → (**a**).

Step 4: (**a**) ↔ (**d**). If the distribution of each U_i is determinate, then by [8], the distribution of the random vector $\mathbf{U} = (U_1,\dots,U_d)$ is determined by its joint moments. Assuming (**a**), these joint moments of $\mathbf{U}$ are the same as if the U_i's were independent. The determinacy implies (**d**). The converse implication is obvious.

The proof of Theorem 3.1 is complete. □

Remark 3.4. Condition (**a**) of Theorem 3.1 can also be stated in terms of cumulants. Recall that the joint cumulant of random variables $X_1,\dots,X_n$ is defined by

$$\kappa(X_1,\dots,X_n) = (-i)^n\frac{\partial^n}{\partial t_1\cdots\partial t_n}\log E[e^{i(t_1X_1+\cdots+t_nX_n)}]\Big|_{t_1=0,\dots,t_n=0},$$

provided $E|X_1\cdots X_n| < \infty$. When all X_i are equal to X, $\kappa(X,\dots,X) = \kappa_n(X)$, the usual nth cumulant of X. Using [2] it is easy to deduce that condition (**a**) of Theorem 3.1 is equivalent to

(a') *for all integers* $1\le j_1<\cdots<j_k\le d$, $k\ge 2$, *and* $m_1,\dots,m_k\ge 1$

$$\kappa(\underbrace{U_{j_1},\dots,U_{j_1}}_{m_1},\dots,\underbrace{U_{j_k},\dots,U_{j_k}}_{m_k}) = 0. \tag{3.32}$$

4. APPLICATIONS

4.1. The fourth moment theorem of Nualart-Peccati.

We can give a short proof of the difficult and surprising part (implication $(\beta)\to(\alpha)$) of the fourth moment theorem of Nualart and Peccati [6], that we restate here for a convenience.

Theorem 4.1 (Nualart-Peccati). *Let* (F_n) *be a sequence of the form* $F_n = I_q(f_n)$, *where* $q\geqslant 2$ *is fixed and* $f_n\in\mathfrak{H}^{\odot q}$. *Assume moreover that* $E[F_n^2] = q!\|f_n\|^2 = 1$ *for all* n. *Then, as* $n\to\infty$, *the following four conditions are equivalent:*

(α) $F_n \overset{\text{law}}{\to} N(0,1)$;
(β) $E[F_n^4] \to 3$;
(γ) $\|f_n \otimes_r f_n\| \to 0$ *for all* $r = 1, \dots, q-1$;
(δ) $\|f_n \widetilde{\otimes}_r f_n\| \to 0$ *for all* $r = 1, \dots, q-1$.

Proof of (β) $\to$ (α). Since the sequence (F_n) is bounded in $L^2(\Omega)$ by the assumption, it is tight, and so, is relatively compact in law. Hence, in order to show (α) it suffices to prove that the limit of any converging subsequence is $N(0,1)$. Suppose that, for a subsequence (n_k), we have that $F_{n_k} \to Y$ in law as $k \to \infty$, and let G_n be an independent copy of F_n of the form $G_n = I_q(g_n)$ with $f_n \otimes_1 g_n = 0$. This can easily be done by extending the underlying isonormal process to the direct sum $\mathfrak{H} \oplus \mathfrak{H}$. Then have

$$(F_{n_k} + G_{n_k}, F_{n_k} - G_{n_k}) = \left(I_p\left(f_{n_k} + g_{n_k}\right), I_p\left(f_{n_k} - g_{n_k}\right)\right) \overset{\text{law}}{\to} (Y+Z, Y-Z)$$

as $k \to \infty$, where Z stands for an independent copy of Y. Since

$$\mathrm{Cov}[(F_{n_k} + G_{n_k})^2, (F_{n_k} - G_{n_k})^2] = 2E[F_{n_k}^4] - 6 \to 0,$$

(β) implies condition (b) of Theorem 3.1. Consequently, from condition (a') given in (3.32)

$$\kappa(\underbrace{Y+Z, \dots, Y+Z}_{m_1}, \underbrace{Y-Z, \dots, Y-Z}_{m_2}) = 0 \quad \text{for all } m_1, m_2 \geq 1.$$

Taking $n \geq 3$ we get

$$\begin{aligned} 0 &= \kappa(\underbrace{Y+Z, \dots, Y+Z}_{n-2}, Y-Z, Y-Z) \\ &= \kappa(\underbrace{Y, \dots, Y}_{n}) + \kappa(\underbrace{Z, \dots, Z}_{n}) = 2\kappa_n(Y), \end{aligned}$$

where we used the multilinearity of κ and the fact that Y and Z are independent. Since $\kappa_1(Y) = EY = 0$, $\kappa_2(Y) = \mathrm{Var}(Y) = 1$, and $\kappa_n(Y) = 0$ when $n \geq 3$, Y has a standard normal distribution. □

4.2. **Recovering a result by Peccati and Tudor.**

By combining Theorems 3.1 and 4.1 we immediately obtain the following multidimensional extension of Theorem 4.1 proved by Peccati and Tudor [7].

Theorem 4.2 (Peccati-Tudor). *Let* $d \geqslant 2$, *and let* $q_1, \dots, q_d$ *be positive integers. Consider vectors*

$$(F_{1,n}, \dots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \dots, I_{q_d}(f_{d,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. *Assume moreover that, for all* $i, j = 1, \dots, d$,

$$E[F_{i,n}F_{j,n}] \to \delta_{ij} \quad \textit{(Kronecker symbol).} \tag{4.33}$$

Then, as $n \to \infty$, *the following two conditions are equivalent:*

(i) $F_{i,n} \overset{\text{law}}{\to} N(0,1)$ *for each* $i = 1, \ldots, d$;

(ii) $(F_{1,n}, \ldots, F_{d,n}) \overset{\text{law}}{\to} N_d(0, I_d)$.

Proof. Of course, only $(i) \to (ii)$ has to be shown. If (i) holds, then Theorem 4.1(γ) implies that $\|f_{i,n} \otimes_r f_{i,n}\| \to 0$ for all $r = 1, \ldots, q_i - 1$ and all $i = 1, \ldots, d$. Observe that

$$\|f_{i,n} \otimes_r f_{j,n}\|^2 = \langle f_{i,n} \otimes_{q_i - r} f_{i,n}, f_{j,n} \otimes_{q_j - r} f_{j,n} \rangle \,,$$

see (2.6). By Cauchy-Schwarz inequality this implies condition (c) of Theorem 3.1 for all r, q_i, q_j, except when $r = q_i = q_j$. But in the latter case,

$$q_i! \|f_{i,n} \otimes_r f_{j,n}\| = q_i! \big|\langle f_{i,n}, f_{j,n} \rangle\big| = \big|E[F_{i,n} F_{j,n}]\big| \to 0$$

by our assumption (4.33). Thus condition (c) of Theorem 3.1 holds and Corollary 3.2 concludes the proof. □

4.3. **A multivariate version of the convergence towards χ^2.**

Here we will prove a multivariate extension of a result of Nourdin and Peccati [3]. Such an extension was an open problem as far as we know.

Throughout this paper $G(\nu)$ will denote a random variable distributed according to the centered χ^2 distribution with $\nu > 0$ degrees of freedom. When $\nu > 0$ is an integer, then $G(\nu) \overset{\text{law}}{=} \sum_{i=1}^{\nu} (N_i^2 - 1)$, where $N_1, \ldots, N_\nu$ are independent $N(0,1)$ random variables. In general, $G(\nu)$ is a centered gamma random variable with a shape parameter $\nu/2$ and scale parameter 2. Using the well-known Carleman's condition, it is a routine exercise to check that the law of $G(\nu)$ is determinate (in the sense of Remark 3.4(1)) for any $\nu > 0$.

We begin by recalling the Nourdin and Peccati [3] theorem.

Theorem 4.3 (Nourdin-Peccati). *Fix* $\nu > 0$ *and let* $G(\nu)$ *be as above. Let* $q \geqslant 2$ *be an even integer, and let* $I_q(f_n)$ *be such that* $E[I_q(f_n)^2] = E[G(\nu)^2] = 2\nu$. *Set*

$$c_q = \frac{1}{(q/2)! \binom{q-1}{q/2-1}^2} = \frac{4}{(q/2)! \binom{q}{q/2}^2}.$$

Then, the following four assertions are equivalent, as $n \to \infty$:

(α) $I_q(f_n) \overset{\text{law}}{\to} G(\nu)$;

(β) $E[I_q(f_n)^4] - 12E[I_q(f_n)^3] \to E[G(\nu)^4] - 12E[G(\nu)^3] = 12\nu^2 - 48\nu$;

(γ) $\|f_n \widetilde{\otimes}_{q/2} f_n - c_q \times f_n\| \to 0$, *and* $\|f_n \otimes_r f_n\| \to 0$ *for every* $r = 1, ..., q-1$ *such that* $r \neq q/2$;

(δ) $\|f_n \widetilde{\otimes}_{q/2} f_n - c_q \times f_n\| \to 0$, *and* $\|f_n \widetilde{\otimes}_r f_n\| \to 0$ *for every* $r = 1, ..., q-1$ *such that* $r \neq q/2$.

Our multivariate extension is in the spirit of Theorem 4.2.

Theorem 4.4. *Let $d \geqslant 2$, let $\nu_1, \ldots, \nu_d$ be positive integers, and let $q_1, \ldots, q_d \geqslant 2$ be even integers. Consider vectors*

$$(F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$, and assume that $\lim_{n\to\infty} E[F_{i,n}^2] = 2\nu_i$ for every i. Assume further that:

(i) $\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *whenever $q_i = q_j$ and $i \neq j$;*

(ii) $\lim_{n\to\infty} E[F_{i,n}^2 F_{j,n}] = 0$ *whenever $q_j = 2q_i$.*

Let $G(\nu_1), \ldots, G(\nu_d)$ be independent random variables having centered χ^2 distributions with $\nu_1, \ldots, \nu_d$ degrees of freedom, respectively. Then, as $n \to \infty$, the following two conditions are equivalent:

(a) $F_{i,n} \overset{\text{law}}{\to} G(\nu_i)$ *for each $i = 1, \ldots, d$;*

(b) $(F_{1,n}, \ldots, F_{d,n}) \overset{\text{law}}{\to} (G(\nu_1), \ldots, G(\nu_d))$.

Proof. We only need to prove that $(a) \to (b)$. Since the distribution of each $G(\nu_i)$ is determined by its moments, by Corollary 3.2 it is enough to show that condition (c) of Theorem 3.1 holds.

Fix $1 \leqslant i \neq j \leqslant d$ as well as $1 \leqslant r \leqslant q_i \wedge q_j$. Switching i and j if necessary, assume that $q_i \leqslant q_j$. From Theorem 4.3(γ) we get that $f_{k,n} \otimes_r f_{k,n} \to 0$ for each $1 \leqslant k \leqslant d$ and every $1 \leqslant r \leqslant q_k - 1$, except when $r = q_k/2$. Using the identity

$$\|f_{i,n} \otimes_r f_{j,n}\|^2 = \langle f_{i,n} \otimes_{q_i - r} f_{i,n}, f_{j,n} \otimes_{q_j - r} f_{j,n}\rangle \tag{4.34}$$

(see (2.6)) together with the Cauchy-Schwarz inequality we infer that condition (c) of Theorem 3.1 holds for all values of r, i and j, except of the cases: $r = q_i = q_j$, $r = q_i/2 = q_j/2$, and $r = q_i = q_j/2$. Assumption (i) together with (3.21) show that $f_{i,n} \otimes_r f_{j,n} \to 0$ for all $1 \leqslant r \leqslant q_i = q_j$. Thus, it remains to verify condition (c) of Theorem 3.1 when $r = q_i = q_j/2$. Lemma 2.2 (identity (2.13) therein) yields

$$\begin{aligned}&\langle f_{j,n} \widetilde{\otimes}_{q_i} f_{j,n}, f_{i,n} \widetilde{\otimes} f_{i,n}\rangle \\ = \;& \frac{2q_i!^2}{q_j!}\langle f_{j,n} \otimes_{q_i} f_{j,n}, f_{i,n} \otimes f_{i,n}\rangle + \frac{q_i!^2}{q_j!}\sum_{s=1}^{q_i-1}\binom{q_i}{s}^2 \langle f_{j,n} \otimes_s f_{i,n}, f_{i,n} \otimes_s f_{j,n}\rangle.\end{aligned}$$

Using (4.34) and Theorem 4.3 and a reasoning as above, it is straightforward to show that the sum $\sum_{s=1}^{q_i-1}\binom{q_i}{s}^2\langle f_{j,n} \otimes_s f_{i,n}, f_{i,n} \otimes_s f_{j,n}\rangle$ tends to zero as $n \to \infty$. On the other hand, the condition on the q_i-th contraction in Theorem 4.3(δ) yields that $f_{j,n} \widetilde{\otimes}_{q_i} f_{j,n} - c_{q_j} f_{j,n} \to 0$ as $n \to \infty$. Moreover, we have

$$\langle f_{j,n}, f_{i,n} \widetilde{\otimes} f_{i,n}\rangle = \frac{1}{q_j!}E[F_{j,n}F_{i,n}^2],$$

which tends to zero by assumption (ii). All these facts together imply that $\langle f_{j,n} \otimes_{q_i} f_{j,n}, f_{i,n} \otimes f_{i,n}\rangle \to 0$ as $n \to \infty$. Using (4.34) for $r = q_i$ we get $f_{i,n} \otimes_{q_i} f_{j,n} \to 0$, showing that condition (c) of Theorem 3.1 holds true in the last remaining case. The proof of the theorem is complete. □

4.4. **Bivariate convergence.**

Theorem 4.5. *Let $p \geq q$ be positive integers. Consider $(F_n, G_n) = \big(I_p(f_n), I_q(g_n)\big)$, $n \geqslant 1$, with $f_n \in \mathfrak{H}^{\odot p}$ and $g_n \in \mathfrak{H}^{\odot q}$. Suppose that as $n \to \infty$*

$$F_n \overset{\text{law}}{\to} N \quad \textit{and} \quad G_n \overset{\text{law}}{\to} V, \tag{4.35}$$

where $N \sim N(0,1)$, V is determinate, and N, V are independent. If $E[F_nG_n] \to 0$ (which trivially holds when $p \neq q$), then

$$(F_n, G_n) \overset{\text{law}}{\to} (N, V) \tag{4.36}$$

jointly, as $n \to \infty$.

Proof. We will show that condition (c) of Theorem 3.1 holds. By (2.9) we may and do assume that $E[F_n^2] = 1$ for all n. By Theorem 4.1(γ), $\|f_n \otimes_r f_n\| \to 0$ for all $r = 1, \dots, p-1$. Observe that

$$\|f_n \otimes_r g_n\|^2 = \langle f_n \otimes_{p-r} f_n, g_n \otimes_{q-r} g_n\rangle$$

so that $\|f_n \otimes_r g_n\| \to 0$ for $1 \leq r \leq p \wedge q = q$, except possibly when $r = p = q$. But in this latter case,

$$p!\|f_n \otimes_r g_n\| = p!\big|\langle f_n, g_n\rangle\big| = \big|E[F_nG_n]\big| \to 0$$

by the assumption. Corollary 3.2 concludes the proof. □

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