

# Controllability Results for a Fish-like Swimming Body

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# Definition - Motivations

## Fluid-Structure Interaction System (FSIS)

### Definition

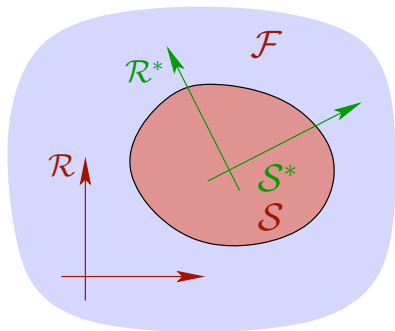
Mechanical system modeling the interactions between a rigid or deformable structure and a fluid.

### Examples:

- Medicine: blood circulation in veins or arteries (which are deformable elastic structures).
- Biology: study of animal locomotion in fluids (e.g., fishes or birds but also the locomotion of microorganisms).
- Aeronautical engineering: design of planes, shape optimization of wings or fuselage.
- Nautical engineering: design of boats and submarines.

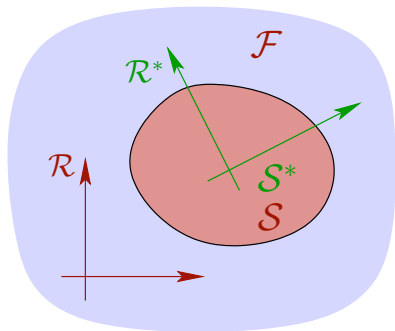
# Modeling: general equations

- $\mathcal{R}$ : Galilean frame.
- $\mathcal{R}^*$ : moving frame.
- $\mathcal{S}, \mathcal{S}^*$ : domain of the structure in  $\mathcal{R}$  and in  $\mathcal{R}^*$ .
- $\mathcal{F}, \mathcal{F}^*$ : domain of the fluid.



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Pick your favorite PDEs of Fluid Mechanics:

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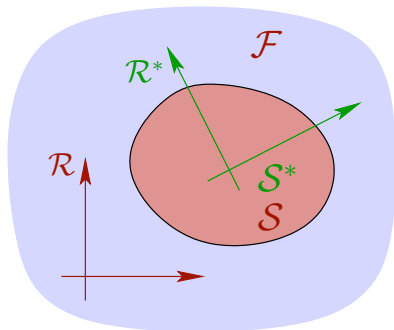
$$\frac{d}{dt}\rho_f = 0 \text{ in } \mathcal{F},$$

+ constitutive law for  $\mathbb{T}_f$ .

- $d/dt$ : material derivative,
- $\rho_f$ : density of the fluid,
- $u$ : eulerian velocity in  $\mathcal{F}$ ,
- $\mathbb{T}_f$ : stress tensor,
- $f$ : bulk force.

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Pick your favorite PDEs of Continuum Mechanics:

$$\begin{aligned} \frac{d}{dt}(\rho_s v) + \nabla \mathbb{T}_s &= f + c \quad \text{in } \mathcal{S}, \\ \frac{d}{dt} \rho_s &= 0 \quad \text{in } \mathcal{S}, \end{aligned}$$

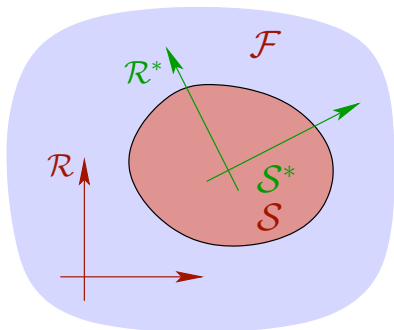
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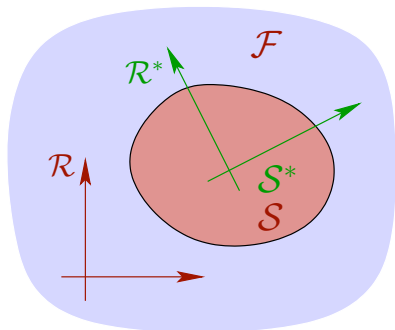
$$\begin{aligned} \frac{d}{dt}(\rho_f u) + \nabla \mathbb{T}_f &= f \text{ in } \mathcal{F}, \\ \frac{d}{dt}\rho_f &= 0 \text{ in } \mathcal{F} + CL \text{ for } \mathbb{T}_f, \\ \frac{d}{dt}(\rho_s v) + \nabla \mathbb{T}_s &= f + c \text{ in } \mathcal{S}, \\ \frac{d}{dt}\rho_s &= 0 \text{ in } \mathcal{S} + CL \text{ for } \mathbb{T}_s. \end{aligned}$$

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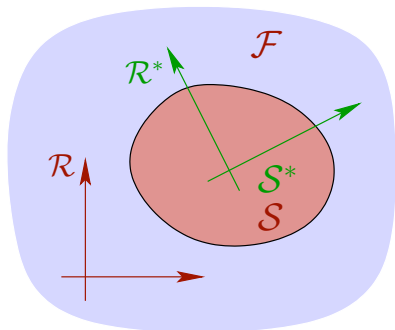
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- Kinematic:  $u = v$  (no-slip condition, viscous fluid) or  $u \cdot n = v \cdot n$  (slip condition, inviscid fluid) on  $\partial\mathcal{S}$ .
- Dynamic:  $\mathbb{T}_s n = \mathbb{T}_f n$  on  $\partial\mathcal{S}$  (Newton's third law).
- Geometric:  $\mathcal{F}$  and  $\mathcal{S}$  are unknown (free boundary problem)...

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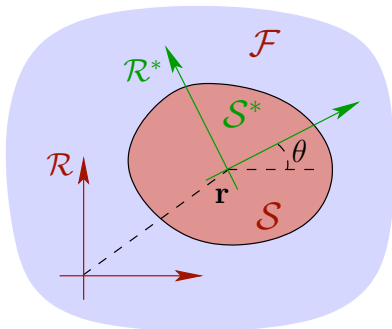
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Relating issues:

- Well-posedness.
- Stability, asymptotic behavior (as  $t \rightarrow +\infty$ ).
- Control problems: can we drive the system into prescribed states using the control  $c$ ?

## Particular cases: rigid solid

The structure is a rigid solid:  $\mathcal{S}^*$  is fixed.



- The PDEs in  $\mathcal{S}$  are replaced by Newton's laws:

$$m\ddot{\mathbf{r}} = - \int_{\partial\mathcal{S}} \mathbb{T}_f n d\sigma + \mathbf{f} + \mathbf{c}_1,$$

$$I\ddot{\theta} = - \int_{\partial\mathcal{S}} \mathbb{T}_f n \cdot (\mathbf{x} - \mathbf{r})^\perp d\sigma + \mathbf{f} + \mathbf{c}_2,$$

where

- $\mathbf{c}_1, \mathbf{c}_2$  are the controls,
- $m$  is the mass and  $I$  the inertia momentum of the solid,
- $\mathbf{q} := (\mathbf{r}, \theta)$  and  $\mathbb{T}_f$  are unknown.
- The domain  $\mathcal{F}$  is still *unknown* in some sense.

## Particular cases: shape-changing body

In this case, the shape-changes of the domain  $\mathcal{S}^*$  are prescribed.

- The PDEs in  $\mathcal{S}$  are still replaced by Newton's laws.
- We introduce a set of diffeomorphisms

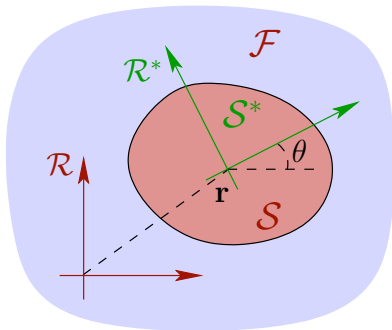
$$\chi(\mathbf{c}) : \mathbf{R}^2 \rightarrow \mathbf{R}^2,$$

indexed by  $\mathbf{c} \in \mathcal{C}$  ( $\mathcal{C}$  a Banach space) and we define

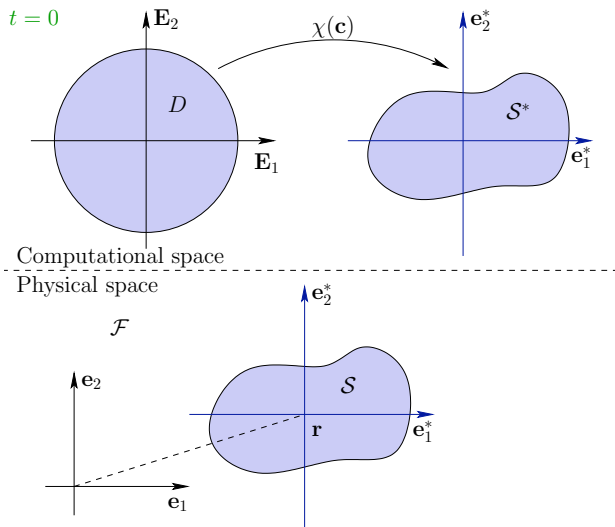
$$\begin{aligned}\mathcal{S}^* &:= \chi(\mathbf{c})(D), \\ \mathcal{S} &:= R(\theta)\mathcal{S}^* + \mathbf{r},\end{aligned}$$

where

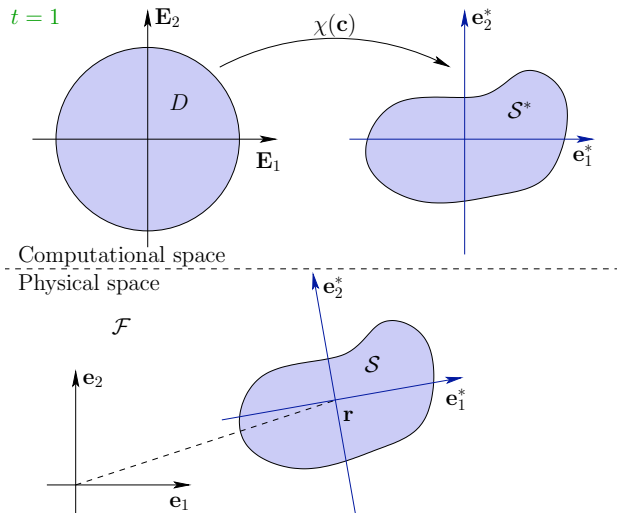
- $D$  is the unitary disk,
- $R(\theta)$  the rotation of angle  $\theta$ .
- The unknowns are  $\mathbf{q} := (\mathbf{r}, \theta)$ . The control is  $\mathbf{c}$ .



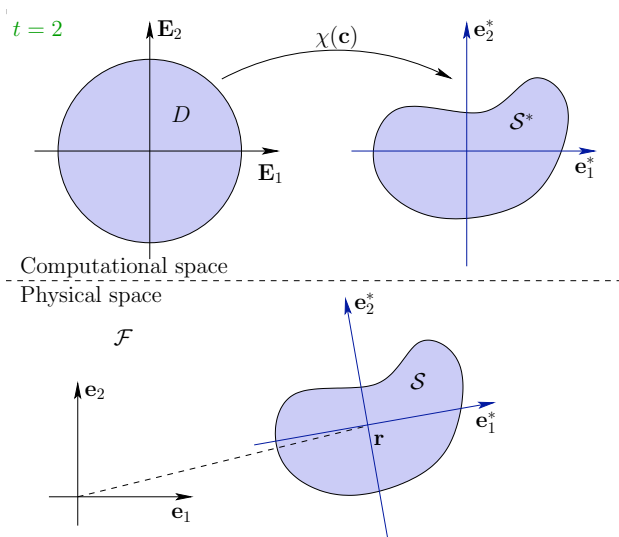
# Example of Motion



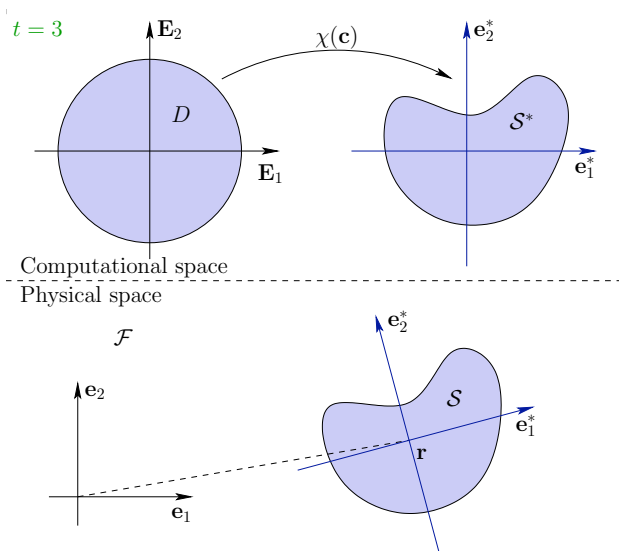
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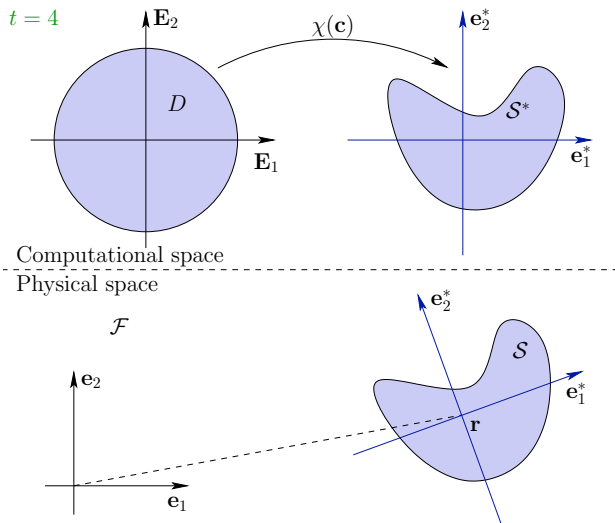
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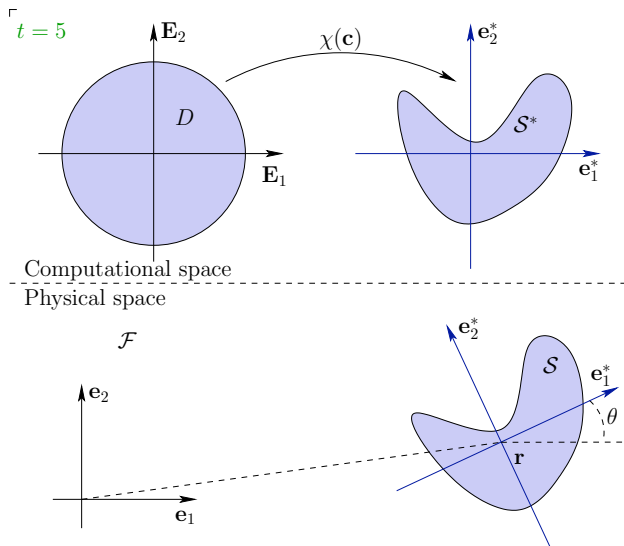
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# Some References

## Control by shape-changes:



F. Alouges, A. Lefebvre and A. DeSimone, *Optimal Strokes for Low Reynolds Number Swimmers: An example*, J. of. Nonlinear Science, 2008.

## Trajectory planning:



R. Mason, *Fluid Locomotion and Trajectory Planning for Shape-Changing Robots*, Ph.D. Thesis, California Institute of Technology, 2003.



J.-B Melli, J.-B. C. W. Rowley et D. S. Rufat, *Motion planning for an articulated body in a perfect planar fluid*, SIAM J. Appl. Dyn. Syst., 2006.

## Control of rigid solids:



J. San Martín, T. Takahashi et M. Tucsnak, *A control theoretic approach to the swimming of microscopic organisms*, Quart. Appl. Math., 2007.



T. Chambrion and M. Sigalotti, *Tracking control for an ellipsoidal submarine driven by Kirchhoff's laws*, IEEE Trans. Automat. Control, 2008.

# Complete dynamics - Euler's equation, potential flow

Consider a smooth control function  $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}$ . Denote  $\mathbf{q} := (\mathbf{r}, \theta)$ . Assume that  $u = \nabla \varphi$  ( $\varphi$  is the potential function) defined by:

$$-\Delta \varphi = 0 \text{ in } \mathcal{F}, \quad \partial_n \varphi = v \cdot n \text{ on } \partial \mathcal{S}, \quad \partial_n \varphi = 0 \text{ on } \partial \mathcal{F} \setminus \partial \mathcal{S},$$

where the velocity  $v$  in  $\mathcal{S}$  is given by  $v = v^r + v^d$  with:

$$v^r = \dot{\theta}(x - \mathbf{r})^\perp + \dot{\mathbf{r}} \quad \text{in } \mathcal{S},$$

$$v^d = R(\theta) \langle \partial_{\mathbf{c}} \chi(\mathbf{c}), \dot{\mathbf{c}} \rangle (\chi(\mathbf{c})^{-1} (R(\theta)^T (x - \mathbf{r}))) \quad \text{in } \mathcal{S}.$$

The pressure is given (up to and additive constant) by Bernoulli's formula:

$$\rho_f (\partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2) + p - p_0 = 0 \quad \text{in } \mathcal{F}.$$

Newton's laws drive the rigid motion of the shape-changing body:

$$m \ddot{\mathbf{r}} = - \int_{\partial \mathcal{S}} p n d\sigma, \quad I(\mathbf{c}) \ddot{\theta} = - \int_{\partial \mathcal{S}} p n \cdot (x - \mathbf{r})^\perp d\sigma.$$

The unknowns are:  $\mathbf{q} := (\mathbf{r}, \theta)$ ,  $\varphi$ ,  $p$  and  $\mathcal{F}$ . The control is  $\mathbf{c}$ .

## Reduced dynamics

We denote  $\mathcal{Q} := \mathbf{R}^2 \times \mathbf{R}/2\pi$ ,  $\mathbf{q} := (\mathbf{r}, \theta) \in \mathcal{Q}$  and  $\dot{\mathbf{q}} := (\dot{\mathbf{r}}, \dot{\theta}) \in \mathbf{R}^2 \times \mathbf{R}$ . The dynamics can be reduced into a second order ODE (Euler-Lagrange equation):

$$\frac{d}{dt} \dot{\mathbf{q}} = \Gamma((\dot{\mathbf{q}}, \mathbf{q}), (\ddot{\mathbf{c}}, \dot{\mathbf{c}}, \mathbf{c})), \quad (t > 0).$$

To apply Cauchy-Lipschitz Theorem we have to study the regularity of  $\Gamma$  with respect to all of its variables.

In particular, we have to study the regularity of  $\varphi$  with respect to  $\mathbf{c}$  and  $\mathbf{q}$  i.e. with respect to variations of the domain  $\mathcal{F}$ .

In particular, the computation of  $p$  with Bernoulli's formula:

$$\rho_f (\partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2) + p - p_0 = 0 \quad \text{in } \mathcal{F},$$

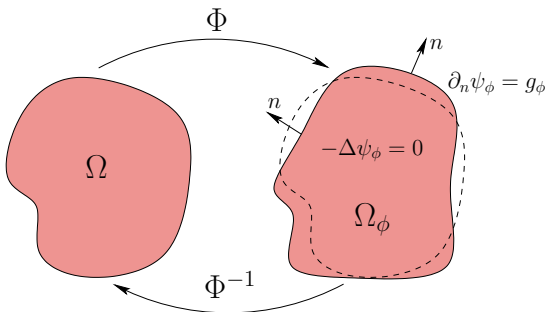
requires the computation of  $\partial_{\mathbf{c}} \varphi$  and  $\partial_{\mathbf{q}} \varphi$  i.e. to differentiate the solution of a BVP with respect to variations of the domain.

## Shape derivative: setting of the problem

Let  $\Omega$  be a connected Lipschitz continuous open set in  $\mathbf{R}^2$  such that  $\partial\Omega$  be compact. We consider the following Laplace's equation:

$$-\Delta\psi_\phi = 0 \quad \text{in } \Omega_\phi, \quad \partial_n\psi_\phi = g_\phi \quad \text{on } \partial\Omega_\phi,$$

where  $\Phi := \text{Id} + \phi$  with  $\phi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ,  $\Omega_\phi := \Phi(\Omega)$  and  $g_\phi \in L^2(\partial\Omega_\phi)$  (or  $H^{-1/2}(\partial\Omega_\phi)$ ) is given.



- When  $\phi = 0$ , we denote merely  $g$  the boundary data and  $\psi$  the solution of the Laplace's equation defined in  $\Omega$ .
- We wish to study the regularity of  $\phi \mapsto \psi_\phi$ .

# Shape derivative: function spaces

Consider a large ball  $B \subset \mathbf{R}^2$  such that  $\partial\Omega \subset B$  and introduce for all  $m \in \mathbf{N}$ ,  $m \geq 1$ :

$$\mathcal{D}^m(B, \mathbf{R}^2) := \{\phi \in \mathcal{C}^m(\mathbf{R}^2, \mathbf{R}^2) : \phi|_{\mathbf{R}^2 \setminus B} = 0\},$$

which is a Banach space when endowed with the norm of the Sobolev space  $W^{m,\infty}(\mathbf{R}^2, \mathbf{R}^2)$ :

$$\|\phi\|_{W^{m,\infty}} := \max_{\substack{\alpha \in \mathbf{N}^2 \\ |\alpha| \leq m}} \|D^\alpha \phi\|_{L^\infty},$$

$$\text{where } D^\alpha \phi := \frac{\partial^{|\alpha|} \phi}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \quad \alpha := (\alpha_1, \alpha_2) \in \mathbf{N}^2, \quad |\alpha| := \alpha_1 + \alpha_2.$$

## Proposition

If  $\phi \in \mathcal{D}^m(B, \mathbf{R}^2)$  such that  $\|\phi\|_{W^{1,\infty}} < 1$  then  $\Phi := \text{Id} + \phi$  is a  $\mathcal{C}^m$  diffeomorphism from  $\Omega$  onto  $\Omega_\phi$ .

# Well-posedness

For all such  $\phi$  and all  $g_\phi \in L^2(\partial\Omega_\phi)$  such that:

$$\int_{\partial\Omega_\phi} g_\phi \, d\sigma = 0,$$

the Neumann boundary value problem admits a unique (weak) solution in:

$$H_N^1(\Omega_\phi) := \{u \in L^2(\Omega_\phi) : \partial_{x_i} u \in L^2(\Omega_\phi), i = 1, 2\} / \mathbf{R}, \quad (\text{bounded domain}),$$

$$H_N^1(\Omega_\phi) := \{u/w \in L^2(\Omega_\phi) : \partial_{x_i} u \in L^2(\Omega_\phi), i = 1, 2\} / \mathbf{R}, \quad (\text{exterior domain}).$$

where  $w(x) := \sqrt{1 + |x|^2} \log(1 + |x|^2)$ . The quotient means that we identify two functions when they differ only by an additive constant.

# Main result

We have the equivalences:

$$\begin{aligned} v \in H_N^1(\Omega_\phi) &\iff v \circ \Phi \in H_N^1(\Omega), \\ f \in L^2(\partial\Omega_\phi) &\iff f \circ \Phi \in L^2(\partial\Omega). \end{aligned}$$

We denote  $\psi_\phi^* := \psi_\phi \circ \Phi$  defined in  $\Omega$  for all  $\phi$  and  $g_\phi^* = g_\phi \circ \Phi$  defined in  $\partial\Omega$ .

## Theorem

If the mapping  $\phi \in \mathcal{D}^1 \mapsto g_\phi^* \in L^2(\partial\Omega)$  is of class  $\mathcal{C}^k$  ( $k \geq 1$ ) nearby  $\phi = 0$  then the mapping  $\phi \in \mathcal{D}^1 \mapsto \psi_\phi^* \in H_N^1(\Omega)$  is also of class  $\mathcal{C}^k$  ( $k \geq 1$ ) nearby  $\phi = 0$ .

# Proof: Variational formulation

The variational formulation:

$$\int_{\Omega_\phi} \nabla \psi_\phi \cdot \nabla v \, dx = \int_{\partial\Omega_\phi} g_\phi v \, d\sigma, \quad \forall v \in H_N^1(\Omega_\phi),$$

turns, upon a change of variables, into:

$$\int_{\Omega} \nabla \psi_\phi^* \mathbb{A}_\phi \nabla w^T \, dx = \int_{\partial\Omega} g_\phi^* w J_\phi \, d\sigma, \quad \forall w \in H_N^1(\Omega),$$

where:

- $\mathbb{A}_\phi = (D\Phi)^{-1} (D\Phi)^{-1T} |\det D\Phi|$ ,
- $J_\phi = |(D\Phi)^{-1} n| |\det D\Phi|$  (tangential Jacobian).

# Proof: Local inversion Theorem

Introduce the mapping:

$$\begin{aligned} \Lambda : \mathcal{D}^1 \times H_N^1(\Omega) &\rightarrow H_N^1(\Omega)' \\ (\phi, u) &\mapsto \Lambda(\phi, u), \end{aligned}$$

defined by:

$$\langle \Lambda(\phi, u), w \rangle := \int_{\Omega} \nabla u \mathbb{A}_{\phi} \nabla w^T \, dx - \int_{\partial\Omega} g_{\phi}^* w J_{\phi} \, d\sigma, \quad \forall w \in H_N^1(\Omega).$$

We easily check that:

- $\Lambda$  is of class  $\mathcal{C}^k$  with respect to  $(\phi, u)$  nearby  $(0, \psi)$ .
- $\partial_u \Lambda(0, \psi) = \Lambda(0, \cdot)$  is an isomorphism from  $H_N^1(\Omega)$  onto  $H_N^1(\Omega)'$  (Lax Milgram Theorem).

We can apply the local inversion Theorem. There exists a unique function of class  $\mathcal{C}^k$ ,  $\phi \in \mathcal{D}^1 \mapsto u_{\phi} \in H_N^1(\Omega)$  such that  $\Lambda(\phi, u) = 0 \Leftrightarrow u = u_{\phi}$ . But we have also the equivalence:  $\Lambda(\phi, u) = 0 \Leftrightarrow u = \psi_{\phi}^*$ , because we know that the solution of the BVP is unique.

# Computation of the shape derivative

What about  $\phi \mapsto \psi_\phi$ ? We can compute directional derivatives.

- Assume that  $\partial\Omega$  is of class  $\mathcal{C}^{1,1}$  (continuously differentiable with Lipschitz continuous first derivative) and that  $g_\phi \in H^{1/2}(\Omega_\phi)$ .
- Fix  $\phi_0 \in \mathcal{D}^2$  (a direction) and choose  $\phi := t\phi_0$ , i.e.,  $\Phi := \text{Id} + t\phi_0$  with  $t \in \mathbf{R}$ .
- Denote  $\psi_t$  in place of  $\psi_{t\phi}$  and  $g_t^*$  in place of  $g_{t\phi}^*$ .

Then the derivative of  $t \mapsto \psi_t$  at  $t = 0$ , denoted by  $\psi'$ , is the unique solution of:

$$\begin{aligned}
 -\Delta\psi' &= 0 && \text{in } \Omega, \\
 \partial_n\psi' &= \Delta_\sigma\psi(\phi_0 \cdot n) - \kappa(\phi_0 \cdot n)g \\
 &\quad + \partial_\tau\psi \cdot \partial_\tau(\phi_0 \cdot n) + (g_t^*)'|_{t=0} - \partial_\tau g \cdot (\phi_0 \cdot \tau) && \text{on } \partial\Omega,
 \end{aligned}$$

where:

- $\Delta_\sigma$  is the Laplace Beltrami operator on  $\partial\Omega$ ,
- $\kappa$  is the curvature of  $\partial\Omega$ ,
- $\tau$  is the tangent vector to  $\partial\Omega$  and  $\partial_\tau$  is the tangential derivative.

# Well posedness of the Euler-Lagrange equation

$t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}$  is a given smooth control function.

## Theorem

Providing that  $\mathbf{c} \in \mathcal{C} \mapsto \chi(\mathbf{c}) \in W^{1,\infty}(\mathbf{R}^2, \mathbf{R}^2)$  is of class  $\mathcal{C}^k$  ( $k \geq 2$ ), then for any  $\mathbf{q}_0 := (\mathbf{r}_0, \theta_0) \in \mathcal{Q}$  and  $\dot{\mathbf{q}}_0 := (\dot{\mathbf{r}}_0, \dot{\theta}_0) \in \mathbf{R}^2 \times \mathbf{R}$ , there exists one unique solution to the Cauchy problem:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{pmatrix} &= \Gamma(\dot{\mathbf{q}}, \mathbf{q}, (\mathbf{c}, \dot{\mathbf{c}}, \ddot{\mathbf{c}})), \quad (t > 0), \\ \begin{pmatrix} \dot{\mathbf{q}}(0) \\ \mathbf{q}(0) \end{pmatrix} &= \begin{pmatrix} \dot{\mathbf{q}}_0 \\ \mathbf{q}_0 \end{pmatrix}. \end{aligned}$$

The solution can be continued up to  $t = T$  or to the time of a contact between the body and the boundary of the fluid domain.

The problem of *contacts* or *collisions* is very challenging (asymptotic behavior of the solution of a PDE as the domain becomes singular).

# Numerical simulations

All the simulations have been realized with a Matlab Toolbox (jointed work with Bruno Pinçon, IECN, INRIA, Nancy):

<http://bht.gforge.inria.fr/>

- Ellipse in a box filled by a fluid
- Dragged down ellipse
- Two brushing ellipses
- Swimming fish
- Two swimming fishes

# Isotropic model

We assume that the body is alone in the fluid ( $\bar{\mathcal{F}} \cup \bar{\mathcal{S}} = \mathbb{R}^2$ ).

- As seen by an observer attached to the body, all of the positions and directions in the fluid are equivalent.
- The isotropy of the configuration allows to integrate the Euler Lagrange equation once.

The Euler Lagrange equation turns out to be a first order ODE:

$$\frac{d}{dt} \mathbf{q} = \mathcal{R}(\theta) \langle \mathbb{M}(\mathbf{c}), \dot{\mathbf{c}} \rangle, \quad (t > 0),$$

where

- $\mathcal{R}(\theta) := \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}$ .
- $\dot{\mathbf{c}} \in \mathcal{C} \mapsto \langle \mathbb{M}(\mathbf{c}), \dot{\mathbf{c}} \rangle \in \mathbf{R}^3$  is a linear continuous mapping and  $\mathbf{c} \in \mathcal{C} \mapsto \mathbb{M}(\mathbf{c}) \in \mathcal{L}(\mathcal{C}, \mathbf{R}^3)$  is smooth. If  $\mathcal{C}$  is a finite dimensional vector space (say of dim  $N$ ) then  $\mathbb{M}(\mathbf{c})$  is a  $3 \times N$  matrix whose entries are smooth functions in  $\mathbf{c}$ .

# A particular model

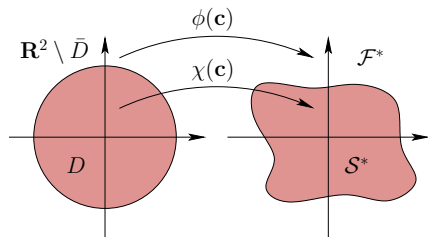
We specify:

$$\mathcal{C} = \mathcal{C}_1 := \left\{ \mathbf{c} := (c_k)_{k \geq 1}, c_k := a_k + ib_k \in \mathbf{C}, \sum_{k \geq 1} k(|a_k| + |b_k|) < +\infty \right\},$$

$$\mathcal{C}_2 := \left\{ \mathbf{c} := (c_k)_{k \geq 1}, c_k := a_k + ib_k \in \mathbf{C}, \sum_{k \geq 1} k(|a_k|^2 + |b_k|^2) < +\infty \right\},$$

and with complex notation:

$$\chi(\mathbf{c})(z) := \sum_{k \geq 1} c_k \bar{z}^k, \quad (z \in D), \quad \phi(\mathbf{c})(z) := \sum_{k \geq 1} c_k z^{-k}, \quad (z \in \mathbf{R}^2 \setminus \bar{D}).$$



- $\phi(\mathbf{c})|_{\partial D} = \chi(\mathbf{c})|_{\partial D}$ .
- When  $\|\mathbf{c}\|_{\mathcal{C}_1} < 1$ ,  $\chi(\mathbf{c})$  is a  $\mathcal{C}^1$  diffeomorphism from  $D$  onto  $S^*$  and  $\phi(\mathbf{c})$  a conformal mapping from  $\mathbf{R}^2 \setminus \bar{D}$  onto  $\mathcal{F}^*$ .

## Physical constraints

$t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}$  is a given smooth control function. We denote  $\dot{\mathbf{c}} = (\dot{a}_k + i\dot{b}_k)_{k \geq 1}$ .

Physical constraints entail that:

$$\sum_{k \geq 1} k(\dot{a}_k a_k + \dot{b}_k b_k) = 0, \quad (\text{volume preservation}), \quad (1)$$

$$\sum_{k \geq 1} \frac{1}{k+1} (\dot{a}_k b_k - a_k \dot{b}_k) = 0, \quad (\text{self-propelled motion}). \quad (2)$$

The first one leads us to introduce:

$$\mathcal{E}(\mu) := \{\mathbf{c} \in \mathcal{C}_1 : \|\mathbf{c}\|_{\mathcal{C}_2} = \mu\},$$

$$\tilde{\mathcal{E}}(\mu) := \{\mathbf{c} \in \mathcal{E}(\mu) : \|\mathbf{c}\|_{\mathcal{C}_1} < 1\} \quad (\text{non-empty iff } \mu \in ]0, 1[).$$

### Definition (physically allowable control function)

A continuous piecewise  $C^1$  control function  $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}_1$  is said to be *physically allowable* when there exists  $\mu \in ]0, 1[$  such that  $\mathbf{c}(t) \in \tilde{\mathcal{E}}(\mu)$  for all  $t$  and (2) holds.

# Main result of controllability

## Theorem

For every  $\mu^\dagger$  in  $]0, 1[$ , for every  $\varepsilon > 0$ , for every reference continuous rigid motion  $\mathbf{q}^\dagger : [0, T] \rightarrow \mathcal{Q}$  and for any reference continuous shape-changes  $\mathbf{c}^\dagger : [0, T] \rightarrow \tilde{\mathcal{E}}(\mu^\dagger)$ , there exists a real number  $\mu \in ]0, 1[$  and an analytic allowable control function  $\mathbf{c} : [0, T] \rightarrow \tilde{\mathcal{E}}(\mu)$  such that

- ①  $\|\mathbf{c}(t) - \mathbf{c}^\dagger(t)\|_{\mathcal{C}_1} \leq \varepsilon$  for all  $t \in [0, T]$ ;
- ② The solution  $\mathbf{q} : [0, T] \rightarrow \mathcal{Q}$  of the Euler-Lagrange equation with initial data  $\mathbf{q}(0) = \mathbf{q}^\dagger(0)$  and control function  $\mathbf{c}$  satisfies  $\|\mathbf{q}(t) - \mathbf{q}^\dagger(t)\|_{\mathcal{Q}} < \varepsilon$ , for all  $t \in [0, T]$ .

In other words: The shape-changing body can not only follow approximately any given trajectory but while undergoing approximately any prescribed shape-changes.

# Finite dimensional reduction

Let  $N$  be an integer  $N \geq 1$ . We define:

- $\mathcal{C}_1^N := \{\mathbf{c} \in \mathcal{C}_1 : c_k = 0 \quad \forall k > N\}$ .
- $\mathcal{E}_N(\mu) := \mathcal{E}(\mu) \cap \mathcal{C}_1^N$ .  
 $\mathcal{E}_N(\mu)$  can be seen as an analytic submanifold of  $\mathbf{R}^{2N}$  of  $\dim 2N - 1$ .  
 $\mathcal{E}_N(\mu) \subset \mathcal{E}_{N'}(\mu)$  if  $N < N'$  as immersed submanifold.

## Definition (mathematically allowable control function)

A continuous piecewise  $C^1$  control function  $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}_1^N$  is said to be *mathematically allowable* when there exists  $\mu \in ]0, 1[$  such that  $\mathbf{c}(t) \in \mathcal{E}_N(\mu)$  for all  $t$  and (2) holds.

amoeba

# Managing the physical constraints

We define  $\mathfrak{X}_N := (\mathbf{X}^j)_{1 \leq j \leq n}$ , a set of Lipschitz continuous vector fields on  $\mathcal{C}_1$  with  $\mathbf{X}^j := (X_k^j)_{k \geq 1}$ ,  $X_k^j := x_k^j + iy_k^j$ . We assume that:

- ①  $X_k^j = 0$  for all  $k > N$ ,
- ②  $\sum_{k \geq 1} k(x_k^j(\mathbf{c})a_k + y_k(\mathbf{c})b_k) = 0$
- ③  $\sum_{k \geq 1} \frac{1}{k+1}(x_k(\mathbf{c})b_k - a_k y_k(\mathbf{c})) = 0$ .

Let  $\alpha_j : [0, T] \mapsto \alpha_j(t) \in \mathbf{R}$  ( $1 \leq j \leq n$ ) be piecewise constant functions.

## Proposition

Any solution of the (well-posed) Cauchy problem  $\dot{\mathbf{c}} = \sum_{j=1}^n \alpha_j \mathbf{X}^j(\mathbf{c})$  with any initial condition  $\mathbf{c}_0 \in \mathcal{C}_N^1$  such that  $\|\mathbf{c}_0\|_{\mathcal{C}_2} = \mu < 1$  is *mathematically allowable*.

## Managing the physical constraints

We introduce  $\mathfrak{Y}_N := (\mathbf{Y}^j)_{1 \leq j \leq n}$ , the vector fields on  $\mathcal{Q} \times \mathcal{E}_N(\mu)$ :

$$\mathbf{Y}^j(\theta, \mathbf{c}) = \begin{pmatrix} \mathcal{R}(\theta) \langle \mathbb{M}(\mathbf{c}), \mathbf{X}^j(\mathbf{c}) \rangle \\ \mathbf{X}^j(\mathbf{c}) \end{pmatrix}.$$

The following points are worth being mentioned:

- If  $\mathbf{c} \in \mathcal{C}_1^N$  then  $\mathbb{M}(\mathbf{c}) \circ \Pi_N = \mathbb{M}(\mathbf{c})$  where  $\Pi_N$  is the projection  $\mathcal{C}_1 \rightarrow \mathcal{C}_1^1$ .
- If the vector fields  $(\mathbf{X}^j)_{1 \leq j \leq n}$  are analytic then  $\mathfrak{Y}_N = (\mathbf{Y}^j)_{1 \leq j \leq n}$  are also analytic.

We rewrite the dynamics of the system fluid-body as:

$$\frac{d}{dt} \begin{pmatrix} \mathbf{q} \\ \mathbf{c} \end{pmatrix} = \sum_{j=1}^n \alpha_j \mathbf{Y}^j(\theta, \mathbf{c}).$$

The new controls are now the functions  $\alpha_j$  ( $1 \leq j \leq n$ ).

# Finite dimensional controllability result

We state a finite dimensional version of the main Theorem of controllability:

## Theorem

For every  $N \in \mathbf{N}$  ( $N \geq 2$ ), every  $\mu$  in  $]0, 1[$ , every  $\varepsilon > 0$ , every reference continuous rigid motion  $\mathbf{q}^\dagger : [0, T] \rightarrow \mathcal{Q}$  and every reference continuous shape-changes  $\mathbf{c}^\dagger : [0, T] \rightarrow \mathcal{E}_N(\mu)$ , there exists a mathematically allowable analytic control function  $\mathbf{c} : [0, T] \rightarrow \mathcal{E}_N(\mu)$  such that

- ①  $\|\mathbf{c}(t) - \mathbf{c}^\dagger(t)\|_{\mathcal{C}_1} \leq \varepsilon$  for all  $t \in [0, T]$ ;
- ② The solution  $\mathbf{q} : [0, T] \rightarrow \mathcal{Q}$  of the Euler-Lagrange equations with initial data  $\mathbf{q}(0) = \mathbf{q}^\dagger(0)$  and control function  $\mathbf{c}$  satisfies  $\|\mathbf{q}(t) - \mathbf{q}^\dagger(t)\|_{\mathcal{Q}} < \varepsilon$ , for all  $t \in [0, T]$ .

# Geometric control theory

Let  $M$  be an analytic manifold and  $\mathfrak{X}$  be a set of analytic vector fields on  $M$  such that  $\mathfrak{X}$  be a cone.

- **Orbit of  $\mathfrak{X}$  through  $q$  ( $q \in M$ ):** denoted by  $\mathcal{O}(q)$ , this set consists in all of the points of  $M$  reachable by following successively and during finite times, the curves defined by  $\dot{q} = X(q)$ , ( $X \in \mathfrak{X}$ ).
- **Lie bracket:**  $[X_1, X_2] = \frac{dX_2}{dq}X_1 - \frac{dX_1}{dq}X_2$ , ( $X_1, X_2 \in \mathfrak{X}$ ).
- **Lie algebra:** **Lie  $\mathfrak{X}$**  is span by all of the Lie brackets of any order.  
**Lie $_q \mathfrak{X}$**  ( $q \in M$ ) consists of all of the vectors of Lie  $\mathfrak{X}$  at the point  $q$ .

## Orbit Theorem

$\forall q_0 \in M$ ,  $\mathcal{O}(q_0)$  is a connected immersed submanifold of  $M$  and for all  $q \in \mathcal{O}(q_0)$ ,  $\text{Lie}_q \mathfrak{X} = T_q \mathcal{O}(q_0)$ .

## Rashevsky Chow Theorem

If  $\text{Lie}_q \mathfrak{X} = T_q M$ ,  $\forall q \in M$ , then  $\mathcal{O}(q) = M$ ,  $\forall q \in M$ .

# Outline of the proof: Main ideas

Remind that the dynamics is:

$$\frac{d}{dt} \begin{pmatrix} \mathbf{q} \\ \mathbf{c} \end{pmatrix} = \sum_{j=1}^n \alpha_j \mathbf{Y}^j(\theta, \mathbf{c}),$$

where  $\mathbf{Y}^j$  are analytic vector fields on the analytic manifold  $\mathcal{Q} \times \mathcal{E}_N(\mu)$ .

According to Rashevsky Chow Theorem, it is enough to find for any  $N \geq 1$  a set of allowable analytic vector fields  $\mathfrak{X}_N$  such that

$$\dim(\text{Lie}_{(\mathbf{q}, \mathbf{c})}(\mathfrak{X}_N)) = \dim T_{(\mathbf{q}, \mathbf{c})}(\mathcal{Q} \times \mathcal{E}_N(\mu)),$$

for any  $(\mathbf{q}, \mathbf{c}) \in \mathcal{Q} \times \mathcal{E}_N(\mu)$ .

→ Definitively too complicated.

## Outline of the proof: The case $N = 2$

We proceed in several steps:

- We build a set of analytic vectors fields  $\mathfrak{X}_2$  on  $\mathcal{C}_1^2$  such that  $\dim(\text{Lie}_{\mathbf{c}}(\mathfrak{X}_2)) = \dim(T_{\mathbf{c}}\mathcal{E}_2(\mu)) = 3$  (explicit computations).
- We use the particular form of the ODE. We introduce:

$$\widehat{\mathbf{X}}^j(\mathbf{c}) := \begin{bmatrix} -\Pi_3(\langle \mathbb{M}(\mathbf{c}), \mathbf{X}^j(\mathbf{c}) \rangle) \\ \mathbf{X}^j(\mathbf{c}) \end{bmatrix},$$

and we can rewrite the dynamics for  $(\theta, \mathbf{c}) \in \mathbf{R}/2\pi \times \mathcal{E}_2(\mu) := \widehat{\mathcal{E}}_2(\mu)$ :

$$\frac{d}{dt} \begin{pmatrix} \dot{\theta} \\ \dot{\mathbf{c}} \end{pmatrix} = \sum_{j=1}^n \alpha_j(t) \widehat{\mathbf{X}}^j(\mathbf{c}).$$

Computations cannot be done in the general case but we can prove that:

$$\dim(\text{Lie}_{(0, \mathbf{c}^\dagger)}\{\widehat{\mathbf{X}}^j, 1 \leq j \leq n\}) = 4,$$

for some particular  $\mathbf{c}^\dagger$ . Since the fields do not depend on  $\theta$ , it is still true for all  $(\theta, \mathbf{c}^\dagger)$ . But for any  $(\theta, \mathbf{c})$ , there exists some  $\theta^\dagger$  such that

$$\mathcal{O}((\theta, \mathbf{c})) \ni (\theta^\dagger, \mathbf{c}^\dagger).$$

# Outline of the proof: the case $N = 2$

According to the Orbit Theorem:

$$\dim(\text{Lie}_{(\theta, \mathbf{c})}\{\widehat{\mathbf{X}}^j, 1 \leq j \leq n\}) = 4, \quad \forall (\theta, \mathbf{c}).$$

Let us recall that:

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{c}} \end{pmatrix} = \sum_{j \in J}^2 \alpha_j(t) \mathbf{Y}^j(\theta, \mathbf{c}).$$

The vector fields  $\mathbf{Y}^j$  depend only on  $\theta$  and  $\mathbf{c}$ . We compute that for  $\theta = 0$  and  $\mathbf{c}^\dagger$  we have:

$$\dim(\text{Lie}_{(\mathbf{r}, 0, \mathbf{c}^\dagger)}\{\mathbf{Y}^j, 1 \leq j \leq n\}) = 6, \quad \forall \mathbf{r} \in \mathbf{R}^2.$$

Arguing as previously, we next prove that:

$$\dim(\text{Lie}_{(\mathbf{r}, \theta, \mathbf{c})}(\{\mathbf{Y}^j(0, \mathbf{c}^\dagger), 1 \leq j \leq n\})) = 6, \quad \forall (\mathbf{r}, \theta, \mathbf{c}) \in \mathcal{Q} \times \mathcal{E}_2(\mu).$$

## Generalization to any dimension

- For any  $N > 2$ ,  $\mathcal{E}_2(\mu) \subset \mathcal{E}_N(\mu)$  (as immersed manifold).
- Likewise,  $\mathfrak{X}_2$  is a complete distribution on  $\mathcal{E}_N(\mu)$ .

We construct (explicitly) a sequence of analytic distributions  $\mathfrak{X}_k$  on  $\mathcal{E}_k(\mu)$  ( $k = 3, \dots, N$ ) such that

$$\mathfrak{X}_{k-1} \subset \mathfrak{X}_k \quad \text{and} \quad \text{Lie}_{\mathbf{c}}(\mathfrak{X}_k) = T_{\mathbf{c}}\mathcal{E}_k(\mu), \quad \forall \mathbf{c} \in \mathcal{E}_k(\mu).$$

We next consider the associated distributions  $\widehat{\mathfrak{X}}_k$  and  $\mathfrak{Y}_k$  and we check that

$$\begin{aligned} \text{Lie}_{(0, \mathbf{c}^\dagger)}(\widehat{\mathfrak{X}}_2) \oplus \text{span}(\widehat{\mathfrak{X}}_N) &= T_{(0, \mathbf{c}^\dagger)}\widehat{\mathcal{E}}_N(\mu), \\ \text{Lie}_{(0, 0, \mathbf{c}^\dagger)}(\mathfrak{Y}_2) \oplus \text{span}(\mathfrak{Y}_N) &= T_{(0, 0, \mathbf{c}^\dagger)}(\mathcal{Q} \times \mathcal{E}_N(\mu)). \end{aligned}$$

We reuse the already done computations for the case  $N = 2$  and we conclude as for this case.

We have proved the existence of a set of piecewise constant functions  $(\alpha_j)_{1 \leq j \leq n}$  such that the solution  $(\mathbf{q}, \mathbf{c})$  tracks the reference data  $(\mathbf{q}^\dagger, \mathbf{c}^\dagger)$ . To complete the proof (analyticity of the control function), we use the density of the analytic functions for the  $L^1$  norm in the set of measurable bounded functions.

# Infinite dimensional case

For any  $\varepsilon$ , there exists an integer  $N \geq 1$  and a finite dimensional control  $\mathbf{c}^\ddagger : [0, T] \rightarrow \mathcal{E}_N(\mu^\ddagger)$  such that

$$\|\mathbf{c}^\ddagger(t) - \mathbf{c}^\dagger(t)\|_{C_1} < \varepsilon/2, \quad \forall t \in [0, T],$$

where we recall that  $\mathbf{c}^\dagger : [0, T] \rightarrow \mathcal{E}(\mu^\dagger)$  is a given reference curve.

We next apply the finite dimensional Theorem with  $\mathbf{c}^\ddagger$  as reference curve.

# Open problems relating to FSIS

## Well posedness

- Motion of elastic bodies in a perfect fluid (or any other fluid).

## Control problems:

- Control of a 3d (isotropic) model.
- Control of a 2d non-isotropic model, for instance with gravity or in a bounded domain.

## Collisions:

- Asymptotic study nearby collision time of 2d rigid solids in a fluid.
- Modeling: find an asymptotic model for 2d colliding rigid solids.
- Collisions in a 3d perfect fluid?

## Detection (inverse problems)

- Can we recover the positions and velocities of rigid solids in a fluid by measuring the potential on the boundary of the fluid domain?