Some issues relating to fluid-structure interaction problems

Alexandre Munnier

Université Henri Poincaré
NANCY, France

INRIA Lorraine, CORIDA team-project

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Definition - Motivations

Fuid-Structure Interaction System (FSIS)

Definition

Mechanical system modeling the interactions between a rigid or deformable structure and a fluid.

Examples:

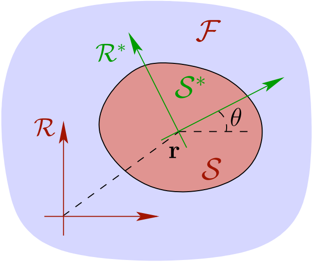
- Medicine: blood circulation in veins or arteries (which are deformable elastic structures).
- Biology: study of animal locomotion in fluids (e.g., fishes or birds but also the locomotion of microorganisms).
- Aeronautical engineering: design of planes, shape optimization of wings or fuselage.
- Nautical engineering: design of boats and submarines.

Outline

1. Modeling
2. Well-posedness
3. Asymptotic analysis: collisions
4. Control problems
5. Inverse problems

Modeling

- $\mathcal{R}$: Galilean (fixed) frame.
- $\mathcal{R}^*$: frame moving along with the solid.
- $\mathcal{S}$, $\mathcal{S}^*$: domain of the structure in $\mathcal{R}$ and in $\mathcal{R}^*$.
- $\mathcal{F}$, $\mathcal{F}^*$: domain of the fluid.



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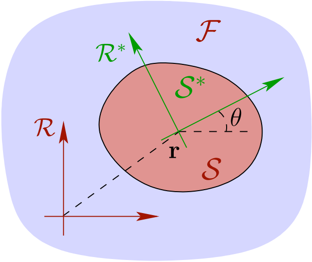

Simplest model for $\mathcal{S}$: A rigid solid

Newton's laws:

$$m\ddot{\mathbf{r}}(t) = -\int_{\partial\mathcal{S}} \mathbb{T}_f \, n \mathrm{d}\sigma,$$

$$I\ddot{\theta}(t) = -\int_{\partial\mathcal{S}} \mathbb{T}_f \, n \cdot (x - \mathbf{r})^{\perp} \mathrm{d}\sigma,$$

- $\mathbb{T}_f(t,x)$: stress tensor of the fluid,
- n is the unitary normal vector to $\partial\mathcal{S}$,
- m is the mass and I the inertia momentum of the solid.

$v(t,x) = \dot{\theta}(x - \mathbf{r})^{\perp} + \dot{\mathbf{r}}$ is the Eulerian velocity of $\mathcal{S}$.

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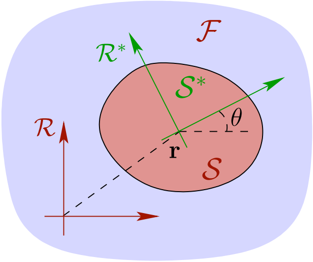


Simplest model for the fluid: Perfect fluid, potential flow.

- $\varphi(t,x)$: potential of the fluid,
- $u = \nabla\varphi$: Eulerian velocity,

$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \mathcal{F}, \\ \partial_n\varphi &= v\cdot n && \text{on } \partial\mathcal{S}, \\ \rho_f\left(\partial_t\varphi + \frac{|\nabla\varphi|^2}{2}\right) + p &= p_0 && \text{in } \mathcal{F}, \\ \mathbb{T}_f &= -p\mathrm{Id} && \text{in } \mathcal{F}, \end{aligned}$$

- $v(t,x) = \dot\theta(x-\mathbf{r})^\perp + \dot{\mathbf{r}}$ is the Eulerian velocity of $\mathcal{S}$.
- $p(t,x)$: pressure,
- ρ_f: density of the fluid,

Modeling

The complete dynamics of the fluid-solid system is:

$$\begin{aligned}
-\Delta\varphi &= 0 && \text{in } \mathcal{F}(t),\\
\partial_n\varphi &= v\cdot n && \text{on } \partial\mathcal{S}(t),\\
v &= \dot\theta(x-\mathbf{r})^\perp + \dot{\mathbf{r}} && \text{on } \partial\mathcal{S}(t),\\
p &= p_0 - \rho_f\left(\partial_t\varphi + \frac{|\nabla\varphi|^2}{2}\right) && \text{in } \mathcal{F}(t),\\
\mathbb{T}_f &= -p\mathrm{Id} && \text{in } \mathcal{F}(t),\\
m\ddot{\mathbf{r}}(t) &= -\int_{\partial\mathcal{S}(t)} \mathbb{T}_f\, n\mathrm{d}\sigma, &&\\
I\ddot\theta(t) &= -\int_{\partial\mathcal{S}(t)} \mathbb{T}_f\, n\cdot(x-\mathbf{r})^\perp\mathrm{d}\sigma. &&
\end{aligned}$$

- The domains $\mathcal{F}(t)$ depends on the position of the solid, i.e. it depends on $\theta(t)$ and $\mathbf{r}(t)$.
- Coupled PDEs-ODEs system. It can (*formally* at least) be turned into:

$$(\ddot{\mathbf{r}},\ddot\theta) = F(\dot{\mathbf{r}},\dot\theta,\mathbf{r},\theta),\quad t\geq 0.$$

Result about modeling: A.M . in J. of Nonlinear Science, 2009

Shape sensitivity analysis

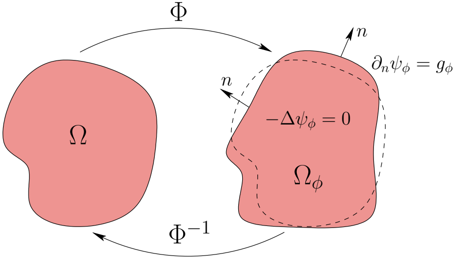
Let Ω be a smooth (C^1), connected open set in $\mathbf{R}^2$.

- $\Phi := \mathrm{Id} + \phi$ with $\phi : \mathbf{R}^2 \to \mathbf{R}^2$,
- $\Omega_\phi := \Phi(\Omega)$,
- $\phi \mapsto g_\phi \in L^2(\partial\Omega_\phi)$ (or $H^{-1/2}(\partial\Omega_\phi)$) is given.

For all ϕ, we consider the following Laplace's equation:

$$-\Delta\psi_\phi = 0 \quad \text{in } \Omega_\phi, \qquad \partial_n\psi_\phi = g_\phi \quad \text{on } \partial\Omega_\phi.$$

We want to study the regularity of $\phi \mapsto \psi_\phi$.



Changes in the topology of Ω are not allowed!

Shape derivative: main result

- Topology for ϕ: Sobolev space $W^{1,\infty}$ (uniforme convergence in $\mathbf{R}^2$ of ϕ and $\partial\phi_{x_i}$, $i = 1, 2$).
- For all ϕ in a neighborhood of 0 in $W^{1,\infty}$ and all $g_\phi \in L^2(\partial\Omega_\phi)$ such that

$$\int_{\partial\Omega_\phi} g_\phi \mathrm{d}\sigma = 0,$$

  the Laplace's equation admits a unique variational solution in the Sobolev space $H^1(\Omega_\phi)$.
- We define $g^*_\phi := g_\phi \circ \Phi$ (in $L^2(\partial\Omega)$, $\forall\phi$) and $\psi^*_\phi := \psi_\phi \circ \Phi$ (in $H^1(\Omega)$, $\forall\phi$).

Theorem

If the mapping $\phi \in W^{1,\infty} \mapsto g^*_\phi \in L^2(\partial\Omega)$ is of class $\mathcal{C}^k$ $(k \geq 1)$ (or analytic) nearby $\phi = 0$ then the mapping $\phi \in W^{1,\infty} \mapsto \psi^*_\phi \in H^1(\Omega)$ is also of class $\mathcal{C}^k$ $(k \geq 1)$ (or analytic) nearby $\phi = 0$.

Well-posedness

Consider a set of solids $\mathcal{S}_1, \ldots, \mathcal{S}_n$ in a perfect fluid. Denote by $\mathbf{r}_i$ and θ_i the center of mass and the orientation of the i-th solid.

Theorem (Math. Models Methods Appl. Sci. 2008)

If the solids are *smooth enough*, then for any initial positions $(\theta_1^\dagger, \mathbf{r}_1^\dagger, \ldots, \theta_n^\dagger, \mathbf{r}_n^\dagger)$ and velocities $(\dot\theta_1^\dagger, \dot{\mathbf{r}}_1^\dagger, \ldots, \dot\theta_n^\dagger, \dot{\mathbf{r}}_n^\dagger)$ of the solids, there exists one unique solution to the Cauchy problem:

$$(\ddot\theta_1, \ddot{\mathbf{r}}_1, \ldots, \ddot\theta_n, \ddot{\mathbf{r}}_n) = F((\dot\theta_1, \dot{\mathbf{r}}_1, \ldots, \dot\theta_n, \dot{\mathbf{r}}_n), (\theta_1, \mathbf{r}_1, \ldots, \theta_n, \mathbf{r}_n)),$$

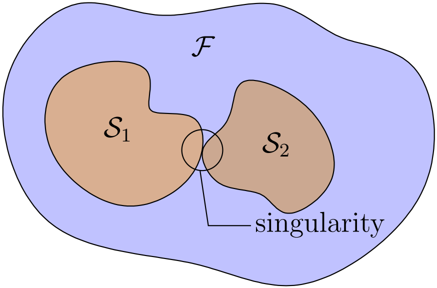
$$(\theta_1(0), \mathbf{r}_1(0), \ldots, \theta_n(0), \mathbf{r}_n(0)) = (\theta_1^\dagger, \mathbf{r}_1^\dagger, \ldots, \theta_n^\dagger, \mathbf{r}_n^\dagger),$$

$$(\dot\theta_1(0), \dot{\mathbf{r}}_1(0), \ldots, \dot\theta_n(0), \dot{\mathbf{r}}_n(0)) = (\dot\theta_1^\dagger, \dot{\mathbf{r}}_1^\dagger, \ldots, \dot\theta_n^\dagger, \dot{\mathbf{r}}_n^\dagger).$$

Moreover, the solution is analytic and can be continued either up to $t = +\infty$ or to the time of a first contact between two bodies.

Collisions: asymptotic analysis

- Does the model allow collisions? (collisions are not possible in a viscous fluid, i.e. when the fluid is modeled by Navier-Stokes equations [1].
- When two solids touch each others, the domain $\mathcal{F}$ becomes singular. It is not clear that the PDE (1) be well-posed any longer.
- Collisions correspond to changes in the topology of $\mathcal{F}$.



How behaves the potential ψ when the domain $\mathcal{F}$ undergoes topology changes?

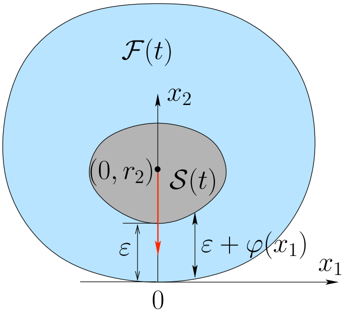
$$-\Delta\psi = 0 \quad \text{in } \mathcal{F}, \tag{1a}$$

$$\partial_n\psi = g \quad \text{on } \partial\mathcal{F}. \tag{1b}$$

[1] (Hillairet, M., Asymptot. Anal., 2005)

Collisions: asymptotic analysis

Known results



Denote by ψ_ε the potential of the fluid.

Theorem (with J. Houot, Asymptot. Anal., 2008)

If $\partial\mathcal{F}(t)$ is of class $\mathcal{C}^2$ and:

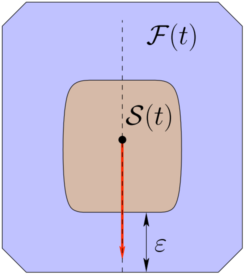
$$\lim_{x_1 \to 0} \frac{|\varphi(x_1)|}{|x_1|^3} = +\infty, \qquad \text{(H)}$$

then,

- the limit potential ψ_0 is well-defined and $\psi_\varepsilon \to \psi_0$ as $\varepsilon \to 0$.
- for any initial position $\mathbf{r} = (0, r_2)$ ($r_2 > 0$) and any initial velocity $\dot{\mathbf{r}}_0 = (0, -\dot{r}_2)$, ($\dot{r}_2 < 0$) the solid collides with the boundary in finite time.

Collisions: asymptotic analysis

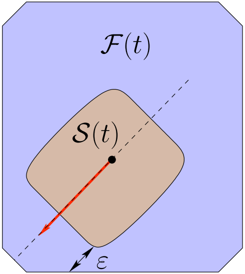
Open problems



The limit potential ψ_0 is well defined but we can not have $\psi_\varepsilon \to \psi_0$ as $\varepsilon \to 0$.

Conjecture:

No collisions (the damping effect of the fluid prevents collision).



The limit fluid domain is not connected. The limit potential ψ_0 does not exist (the limit PDE is ill-posed).

Conjecture:

No collisions.

Numerical simulations

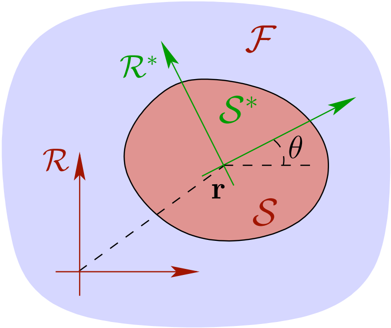
All the simulations have been realized with a Matlab Toolbox (jointed work with Bruno Pinçon, IECN, INRIA, Nancy):

`http://bht.gforge.inria.fr/`

- Ellipse in a box filled by a fluid
- Dragged down ellipse
- Two brushing ellipses
- Colliding solids

Control problem

We want to control the rigid motion of $\mathcal{S}$ by modifying its shape. The shape-changes of the domain $\mathcal{S}^*$ are prescribed.



- We introduce a set of diffeomorphisms

$$\chi(\mathbf{c}) : \mathbf{R}^2 \to \mathbf{R}^2,$$

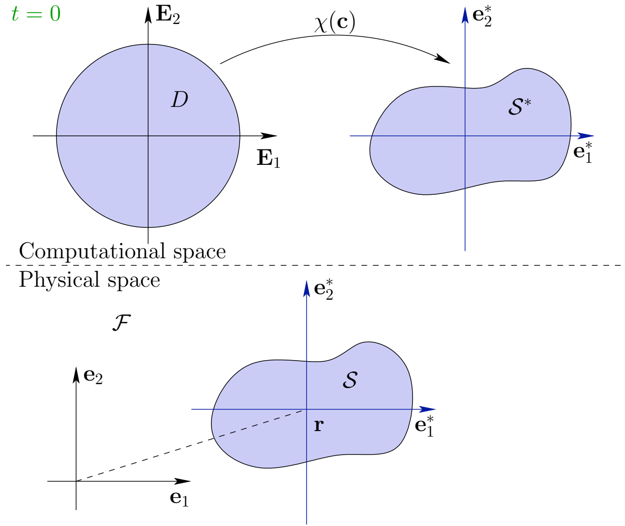
indexed by $\mathbf{c} \in \mathcal{C}$ ($\mathcal{C}$ a Banach space) and we define

$$\mathcal{S}^* := \chi(\mathbf{c})(D),$$
$$\mathcal{S} := R(\theta)\mathcal{S}^* + \mathbf{r},$$

where
  - D is the unitary disk,
  - $R(\theta)$ the rotation of angle θ.
- The unknowns are $\mathbf{r}, \theta$. The control is $\mathbf{c}$.

Example of Motion

The function $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}$ is given:



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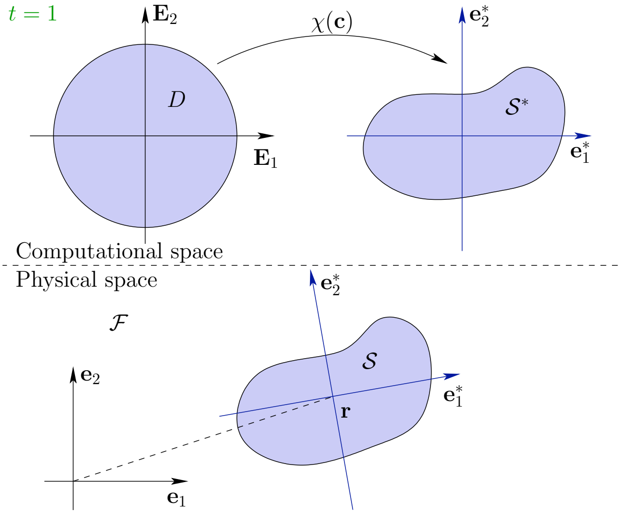

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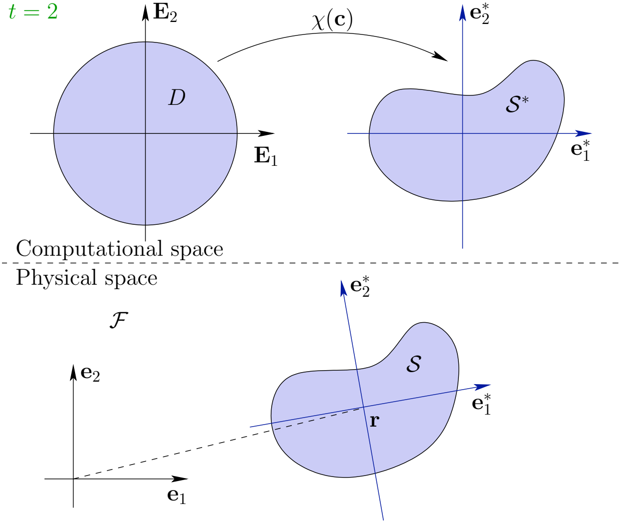

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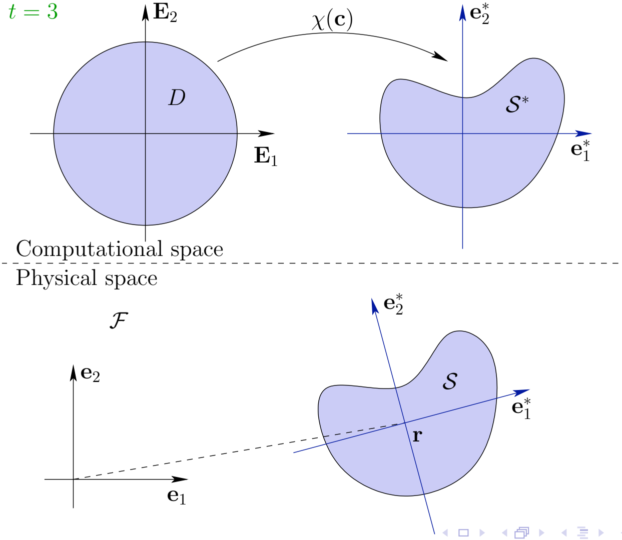

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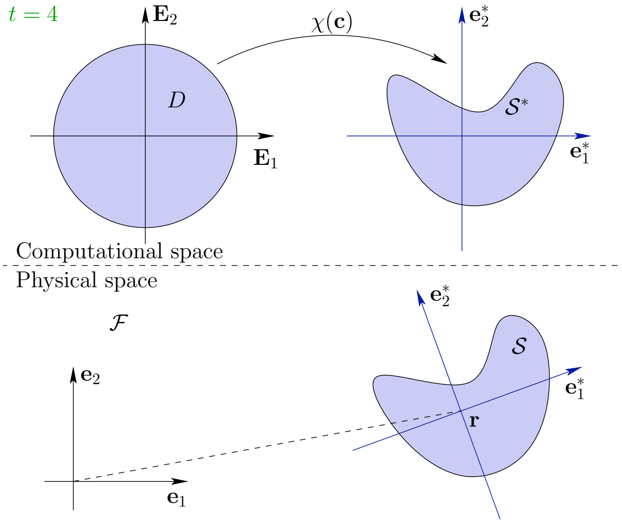

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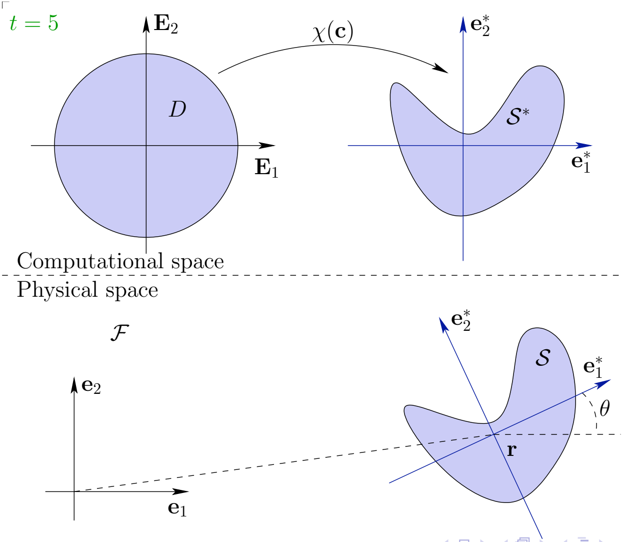


Dynamics of the self-propelled motion

The complete dynamics of the fluid-body system is:

$$
\begin{aligned}
-\Delta\varphi &= 0 && \text{in } \mathcal{F}(t),\\
\partial_n\varphi &= (v_r + v_d)\cdot n && \text{on } \partial\mathcal{S}(t),\\
v_r &= \dot\theta(x-\mathbf{r})^\perp + \dot{\mathbf{r}} && \text{on } \partial\mathcal{S}(t),\\
v_d &= R(\theta)\langle\partial_{\mathbf{c}}\chi(\mathbf{c}),\dot{\mathbf{c}}\rangle(\chi(\mathbf{c})^{-1}(R(\theta)^T(x-\mathbf{r}))) && \text{on } \partial\mathcal{S}(t),\\
p &= p_0 - \rho_f\left(\partial_t\varphi + \frac{|\nabla\varphi|^2}{2}\right) && \text{in } \mathcal{F}(t),\\
\mathbb{T}_f &= -p\mathrm{Id} && \text{in } \mathcal{F}(t),
\end{aligned}
$$

$$
m\ddot{\mathbf{r}}(t) = -\int_{\partial\mathcal{S}(t)}\mathbb{T}_f\, n\mathrm{d}\sigma,\quad I\ddot\theta(t) = -\int_{\partial\mathcal{S}(t)}\mathbb{T}_f\, n\cdot(x-\mathbf{r})^\perp\mathrm{d}\sigma.
$$

- The domains $\mathcal{F}(t)$ depends on $\theta(t)$, $\mathbf{r}(t)$ and $\mathbf{c}(t)$.
- If the body is alone in the fluid and the fluid-body system fills the whole space, the dynamics can be turned into a first order ODE:

$$
(\dot{\mathbf{r}},\dot\theta) = F(\theta,\mathbf{c},\dot{\mathbf{c}}),\quad t\geq 0.
$$

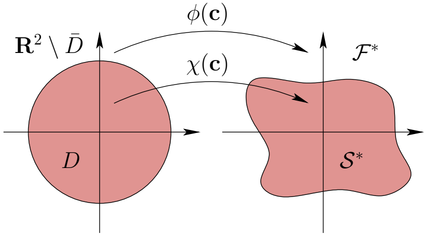
Control problem: An example

Jointed work with Thomas Chambrion (IECN, INRIA), (J. Nonlinear Sci. 2009)
We specify:

$$\mathcal{C} := \left\{ \mathbf{c} := (c_k)_{k\geq 1},\, c_k := a_k + ib_k \in \mathbf{C},\, \sum_{k\geq 1} k(|a_k| + |b_k|) < +\infty \right\},$$

and with complex notation:

$$\chi(\mathbf{c})(z) := \sum_{k\geq 1} c_k \bar{z}^k, \quad (z \in D), \qquad \phi(\mathbf{c})(z) := \sum_{k\geq 1} c_k z^{-k}, \quad (z \in \mathbf{R}^2 \setminus \bar{D}).$$



$$\phi(\mathbf{c})|_{\partial D} = \chi(\mathbf{c})|_{\partial D}.$$

Control problem: An example

Constraints on the deformations:

1. Volume conservation.
2. We assume that the shape-changes result from the work of internal forces (self-propelled motion): conservation of the linear and angular momenta.

We deduce that $\mathbf{c} \in \mathcal{E}$ (an analytic submanifold of $\mathcal{C}$) and for all $\mathbf{c} \in \mathcal{E}$, $\dot{\mathbf{c}}$ belongs to a subspace of $T_{\mathbf{c}}\mathcal{E}$ (the tangent space to $\mathcal{E}$ at $\mathbf{c}$).

Definition (physically allowable control function)

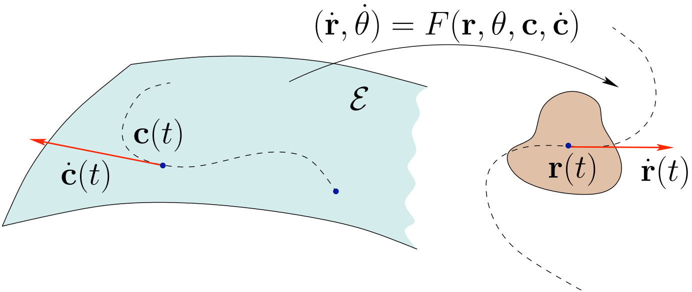
A continuous piecewise C^1 control function $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}$ satisfying these constraints is said to be *physically allowable*.

amoeba

Can the swimming body track any given trajectory by undergoing suitable shape-changes?

Control problem: abstract point of view

To any given allowable control function $t \in [0,T] \mapsto \mathbf{c}(t) \in \mathcal{E}$ corresponds a trajectory of the body in the fluid.



Is this application onto?

Control problem: Main result

Theorem

For any $\varepsilon > 0$, for every reference continuous rigid motion $(\theta^\dagger, \mathbf{r}^\dagger) : [0,T] \to \mathbf{R}/2\pi \times \mathbf{R}^2$ and for any reference continuous shape-changes $\mathbf{c}^\dagger : [0,T] \to \mathcal{E}$, there exists an analytic physically allowable control function $\mathbf{c} : [0,T] \to \mathcal{E}$ such that

1. $\|\mathbf{c}(t) - \mathbf{c}^\dagger(t)\|_{\mathcal{C}} \leq \varepsilon$ for all $t \in [0,T]$;
2. $\|\mathbf{r} - \mathbf{r}^\dagger\| + \|\theta - \theta^\dagger\| < \varepsilon$ for all $t \in [0,T]$, where $(\theta, \mathbf{r})$ is the solution of the ODE:

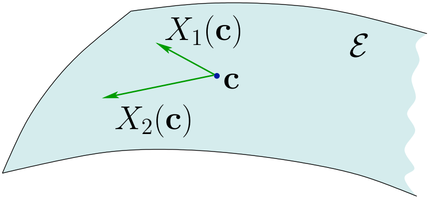
$$(\dot{\mathbf{r}}, \dot{\theta}) = F(\theta, \mathbf{c}, \dot{\mathbf{c}}),$$

with initial data $(\theta(0), \mathbf{r}(0)) = (\theta^\dagger(0), \mathbf{r}^\dagger(0))$ and control function $\mathbf{c}$.

In other words: The shape-changing body can not only follow approximately any given trajectory but while undergoing approximately any prescribed shape-changes.

Control problem: Abstract point of view

At any point $\mathbf{c} \in \mathcal{E}$, we choose some allowable directions in the tangent space $T_{\mathbf{c}}\mathcal{E}$. We define n allowable analytic vectors fields X_i $(i = 1, \dots, n)$ on $\mathcal{E}$.



We seek the control function in the form:

$$\dot{\mathbf{c}}(t) = \sum_{i=1}^{n} \alpha_i(t) X_i(\mathbf{c}(t)).$$

The new controls are the piecewise constant functions $\alpha_i : [0, T] \mapsto \mathbf{R}$. The control function $t \in [0, T] \mapsto \mathbf{c}(t)$ is always *allowable*.

Control problem: Vague idea of the proof

Because the function F is linear with respect to $\dot{\mathbf{c}}$, we can turn the dynamics into:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{r}\\ \theta\\ \mathbf{c}\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{n}\alpha_i(t)F(\mathbf{r},\theta,\mathbf{c},X_i(\mathbf{c}))\\ \sum_{i=1}^{n}\alpha_i(t)X_i(\mathbf{c})\end{pmatrix}.$$

- We no longer work on the manifold $\mathcal{E}$ but on $\mathbf{R}^2 \times \mathbf{R}/2\pi \times \mathcal{E}$.
- The controls are the functions α_i.
- The new control problem is: can we track any given *trajectory* $t \in [0,T] \mapsto (\mathbf{r}^\dagger(t), \theta^\dagger(t), \mathbf{c}^\dagger(t))$?

Geometric control theory

Let M be an analytic manifold and $\mathfrak{X}$ be a set of analytic vector fields on M such that $\mathfrak{X}$ be a cone.

- **Orbit of $\mathfrak{X}$ through q ($q \in M$):** denoted by $\mathcal{O}(q)$, this set consists in all of the points of M reachable by following successively and during finite times, the curves defined by $\dot{q} = X(q), \quad (X \in \mathfrak{X})$.
- **Lie bracket:** $[X_1, X_2] = \dfrac{dX_2}{dq} X_1 - \dfrac{dX_1}{dq} X_2, \quad (X_1, X_2 \in \mathfrak{X})$.
- **Lie algebra:** $\mathrm{Lie}\,\mathfrak{X}$ is span by all of the Lie brackets of any order. $\mathrm{Lie}_q\,\mathfrak{X}$ ($q \in M$) consists of all of the vectors of $\mathrm{Lie}\,\mathfrak{X}$ at the point q.

Orbit Theorem

$\forall\, q_0 \in M$, $\mathcal{O}(q_0)$ is a connected immersed submanifold of M and for all $q \in \mathcal{O}(q_0)$, $\mathrm{Lie}_q\,\mathfrak{X} = T_q\mathcal{O}(q_0)$.

Rashevsky Chow Theorem

If $\mathrm{Lie}_q\,\mathfrak{X} = T_q M$, $\forall\, q \in M$, then $\mathcal{O}(q) = M$, $\forall\, q \in M$.

Numerical simulations

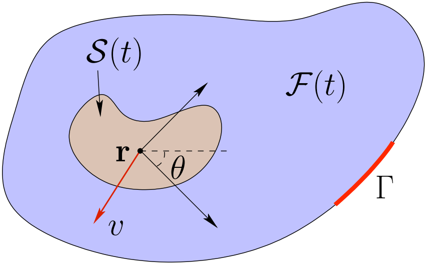
- First example
- Second example
- Moon walk 1
- Moon walk 2
- Moon walk: explanation
- Swiming fish

Control: Open problems

1. Exact controllability?
2. Generalization of this particular result: (almost) every set of 2d shape-changing bodies swimming in any domain is controllable.
3. Optimal control: how to swim the most efficiently possible.
4. Controllability results for 3d models.

Inverse problem: Detection of moving solids in a fluid.

Consider again that $\mathcal{S}$ is a rigid solid.

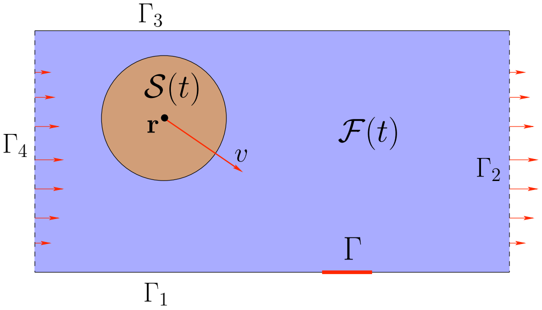


The potential ψ satisfies:

$$\begin{aligned} -\Delta\psi &= 0 && \text{in } \mathcal{F}(t), \\ \partial_n \psi &= v \cdot n && \text{on } \partial\mathcal{S}(t), \\ \partial_n \psi &= 0 && \text{on } \partial\mathcal{F}(t) \setminus \partial\mathcal{S}(t). \end{aligned}$$

The position and velocity of the solid are unknown (so $\mathcal{F}(t)$ is also unknown). We measure ψ on Γ. Can we recover $\mathbf{r}, \theta$ and $\dot{\mathbf{r}}, \dot{\theta}$?

Inverse problem: Detection of a solid in chanel.



The potential ψ satisfies:

$$
\begin{aligned}
-\Delta\psi &= 0 && \text{in } \mathcal{F}(t),\\
\partial_n\psi &= v\cdot n && \text{on } \partial\mathcal{S}(t),\\
\partial_n\psi &= 0 && \text{on } \Gamma_1\cup\Gamma_2,\\
\partial_n\psi &= v_1\cdot n && \text{on } \Gamma_2,\\
\partial_n\psi &= v_2\cdot n && \text{on } \Gamma_4.
\end{aligned}
$$

The velocities v_1 and v_2 are given. We measure ψ on Γ.

Theorem

If v_1 and v_2 are not rigid velocities, then we can recover the position $\mathbf{r}$ of the center of the ball and its velocity $v = \dot{\mathbf{r}}$.