Locomotion in a Perfect Fluid

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Definition - Motivations

Fuid-Structure Interaction System (FSIS)

Definition

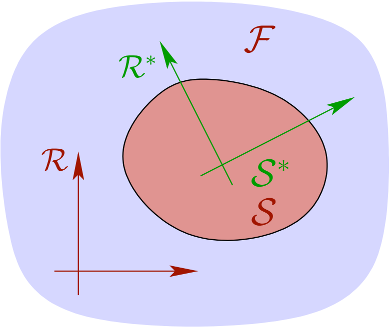
Mechanical system modeling the interactions between a rigid or deformable structure and a fluid.

Examples:

- Medicine: blood circulation in veins or arteries (which are deformable elastic structures).
- Biology: study of animal locomotion in fluids (e.g., fishes or birds but also the locomotion of microorganisms).
- Aeronautical engineering: design of planes, shape optimization of wings or fuselage.
- Nautical engineering: design of boats and submarines.

Modeling: general equations

- $\mathcal{R}$: Galilean frame.
- $\mathcal{R}^*$: moving frame.
- $\mathcal{S}$, $\mathcal{S}^*$: domain of the structure in $\mathcal{R}$ and in $\mathcal{R}^*$.
- $\mathcal{F}$, $\mathcal{F}^*$: domain of the fluid.



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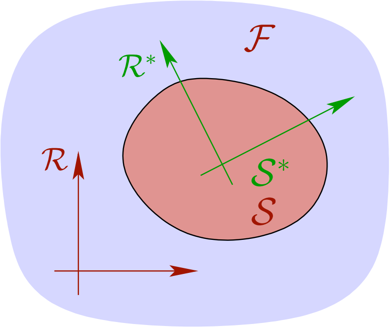

Pick up your favorite PDEs of Fluid Mechanics:

$$\frac{d}{dt}(\rho_f u) + \nabla \mathbb{T}_f = f \text{ in } \mathcal{F},$$
$$\frac{d}{dt}\rho_f = 0 \text{ in } \mathcal{F},$$

+ constitutive equation for $\mathbb{T}_f$.

- d/dt: material derivative,
- ρ_f: density of the fluid,
- u: eulerian velocity in $\mathcal{F}$,
- $\mathbb{T}_f$: stress tensor,
- f: bulk force.

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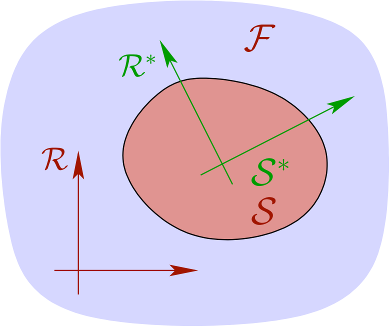

Pick up your favorite PDEs of Continuum Mechanics:

$$\frac{d}{dt}(\rho_s v) + \nabla \mathbb{T}_s = f + c \text{ in } \mathcal{S},$$
$$\frac{d}{dt}\rho_s = 0 \qquad \text{in } \mathcal{S},$$

$+$ constitutive equation for $\mathbb{T}_s$.

- d/dt: material derivative,
- ρ_s: density of the structure,
- v: eulerian velocity in $\mathcal{S}$,
- $\mathbb{T}_s$: stress tensor,
- f: bulk force,
- c: control.

Modeling: general equations

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$$\frac{d}{dt}\rho_f = 0 \text{ in } \mathcal{F} + CL \text{ for } \mathbb{T}_f,$$
$$\frac{d}{dt}(\rho_s v) + \nabla \mathbb{T}_s = f + c \text{ in } \mathcal{S},$$
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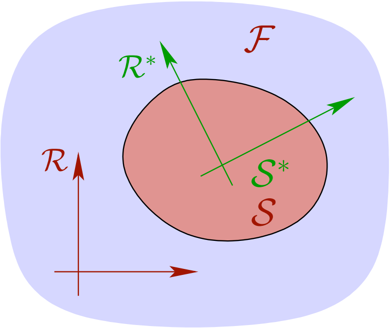

Add coupling conditions:

Modeling: general equations

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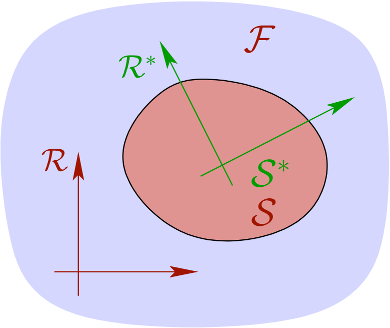

Add coupling conditions:

- Kinematic: $u = v$ (no-slip condition, viscous fluid) or $u \cdot n = v \cdot n$ (slip condition, inviscid fluid) on $\partial\mathcal{S}$.
- Dynamic: $\mathbb{T}_s\, n = \mathbb{T}_f\, n$ on $\partial\mathcal{S}$ (Newton's third law).
- Geometric: $\mathcal{F}$ and $\mathcal{S}$ are unknown (free boundary problem)...

and you obtain a FSIS.

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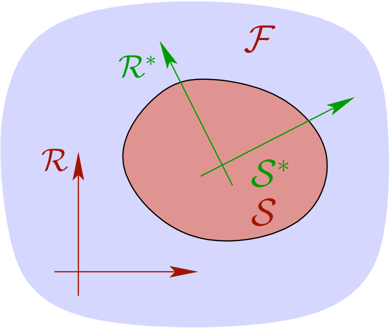

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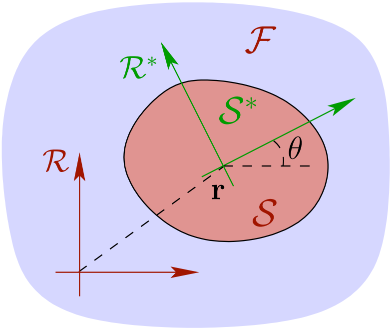
Relating issues:

- Well-posedness.
- Stability, asymptotic behavior (as $t \to +\infty$).
- Control problems: can we drive the system into prescribed states using the control c?

Particular cases: rigid solid

The structure is a rigid solid: $\mathcal{S}^*$ is fixed.



- The PDEs in $\mathcal{S}$ are replaced by Newton's laws:

$$m\ddot{\mathbf{r}} = -\int_{\partial\mathcal{S}} \mathbb{T}_f \, n \mathrm{d}\sigma + f + c_1,$$

$$I\ddot{\theta} = -\int_{\partial\mathcal{S}} \mathbb{T}_f \, n \cdot (x - \mathbf{r})^{\perp} \mathrm{d}\sigma + f + c_2,$$

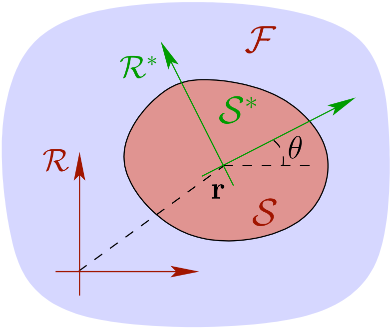
  where

  - c_1, c_2 are the controls,
  - m is the mass and I the inertia momentum of the solid,
  - $\mathbf{q} := (\mathbf{r}, \theta)$ and $\mathbb{T}_f$ are unknown.

- The domain $\mathcal{F}$ is still *unknown* in some sense.

Particular cases: shape-changing body

In this case, the shape-changes of the domain $\mathcal{S}^*$ are prescribed.



- The PDEs in $\mathcal{S}$ are still replaced by Newton's laws.
- We introduce a set of diffeomorphisms

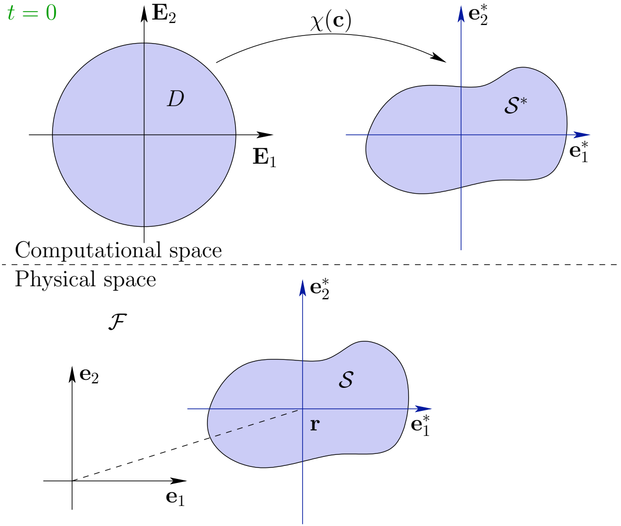
$$\chi(\mathbf{c}) : \mathbf{R}^2 \to \mathbf{R}^2,$$

  indexed by $\mathbf{c} \in \mathcal{C}$ ($\mathcal{C}$ a Banach space) and we define

$$\mathcal{S}^* := \chi(\mathbf{c})(D),$$
$$\mathcal{S} := R(\theta)\mathcal{S}^* + \mathbf{r},$$

  where
  - D is the unitary disk,
  - $R(\theta)$ the rotation of angle θ.
- The unknowns are $\mathbf{q} := (\mathbf{r}, \theta)$. The control is $\mathbf{c}$.

Example of Motion



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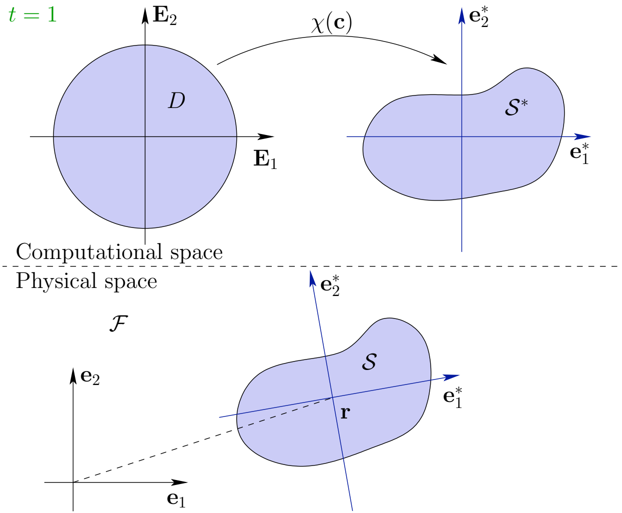

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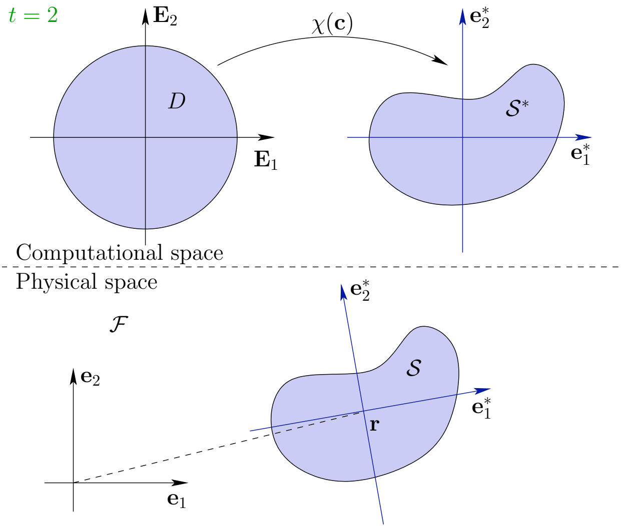

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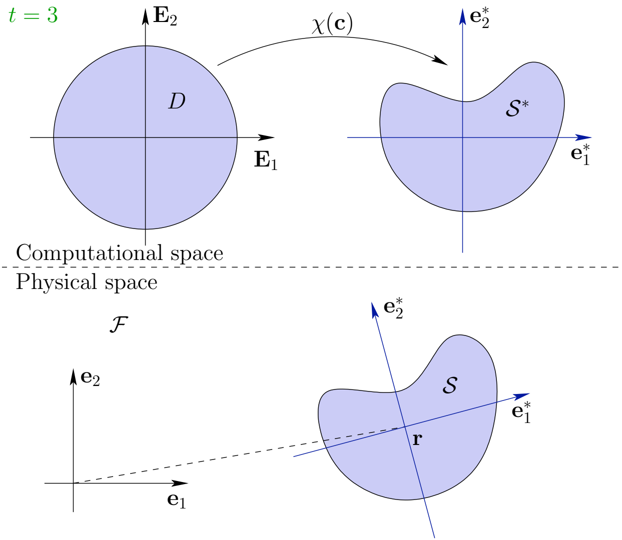

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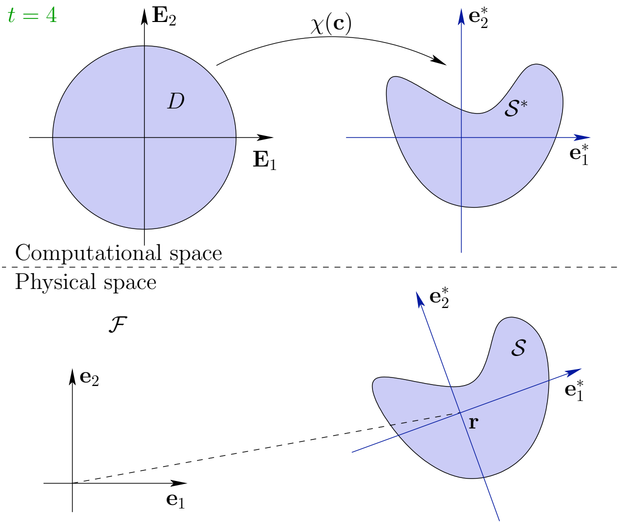

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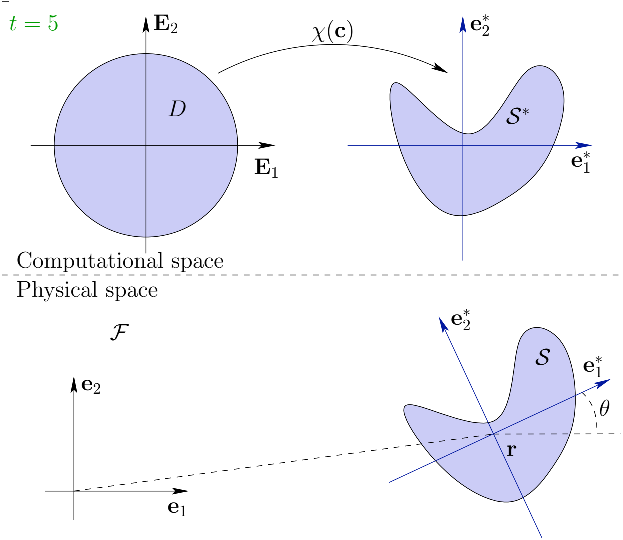

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Control by shape-changes:

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Trajectory planning:

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Control of rigid solids:

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Complete dynamics - Euler equations, potential flow

Consider a smooth control function $t \in [0,T] \mapsto \mathbf{c}(t) \in \mathcal{C}$. Denote $\mathbf{q} := (\mathbf{r}, \theta)$. Assume that $u = \nabla\varphi$ (φ is the potential function) defined by:

$$-\Delta\varphi = 0 \text{ in } \mathcal{F}, \quad \partial_n\varphi = v\cdot n \text{ on } \partial\mathcal{S}, \quad \partial_n\varphi = 0 \text{ on } \partial\mathcal{F}\setminus\partial\mathcal{S},$$

where the velocity v in $\mathcal{S}$ is given by $v = v^r + v^d$ with:

$$\begin{aligned} v^r &= \dot\theta(x-\mathbf{r})^\perp + \dot{\mathbf{r}} \quad \text{in } \mathcal{S},\\ v^d &= R(\theta)\langle\partial_{\mathbf{c}}\chi(\mathbf{c}), \dot{\mathbf{c}}\rangle(\chi(\mathbf{c})^{-1}(R(\theta)^T(x-\mathbf{r}))) \quad \text{in } \mathcal{S}. \end{aligned}$$

The pressure is given (up to and additive constant) by the Bernoulli's formula:

$$\rho_f(\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2) + p - p_0 = 0 \quad \text{in } \mathcal{F}.$$

Newton's laws drive the rigid motion of the shape-changing body:

$$m\ddot{\mathbf{r}} = -\int_{\partial\mathcal{S}} p\, n \mathrm{d}\sigma, \qquad I(\mathbf{c})\ddot\theta = -\int_{\partial\mathcal{S}} p\, n\cdot(x-\mathbf{r})^\perp \mathrm{d}\sigma.$$

The unknows are: $\mathbf{q} := (\mathbf{r},\theta)$, φ, p and $\mathcal{F}$. The control is $\mathbf{c}$.

Reduced dynamics

We denote $\mathcal{Q} := \mathbf{R}^2 \times \mathbf{R}/2\pi$, $\mathbf{q} := (\mathbf{r}, \theta) \in \mathcal{Q}$ and $\dot{\mathbf{q}} := (\dot{\mathbf{r}}, \dot{\theta}) \in \mathbf{R}^2 \times \mathbf{R}$. The dynamic can be reduced into a second order ODE (Euler-Lagrange equation):

$$\frac{d}{dt}\dot{\mathbf{q}} = \Gamma((\dot{\mathbf{q}}, \mathbf{q}), (\ddot{\mathbf{c}}, \dot{\mathbf{c}}, \mathbf{c})), \qquad (t > 0).$$

To apply the Cauchy-Lipschitz Theorem we have to study the regularity of Γ with respect to all of its variables.

It particular, we have to study the regularity of φ with respect to $\mathbf{c}$ and $\mathbf{q}$ i.e. with respect to variations of the domain $\mathcal{F}$.

The computation of p by means of the Bernoulli's formula:

$$\rho_f(\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2) + p - p_0 = 0 \quad \text{in } \mathcal{F},$$

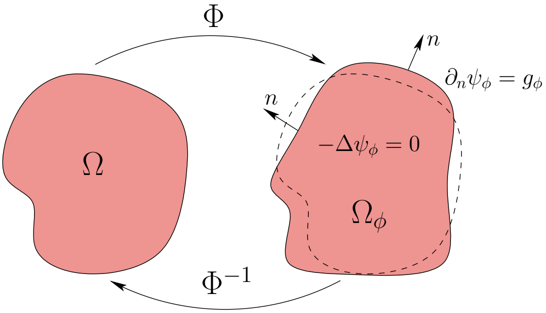
requires the computation of $\partial_{\mathbf{c}}\varphi$ and $\partial_{\mathbf{q}}\varphi$ i.e. to differentiate the solution of a BVP with respect to variations of the domain.

Shape derivative: setting of the problem

Let Ω be a connected Lipschitz continuous open set in $\mathbf{R}^2$ such that $\partial\Omega$ be compact. We consider the following Laplace equation:

$$-\Delta\psi_\phi = 0 \quad \text{in } \Omega_\phi, \qquad \partial_n\psi_\phi = g_\phi \quad \text{on } \partial\Omega_\phi,$$

where $\Phi := \mathrm{Id} + \phi$ with $\phi : \mathbf{R}^2 \to \mathbf{R}^2$, $\Omega_\phi := \Phi(\Omega)$ and $g_\phi \in L^2(\partial\Omega_\phi)$ (or $H^{-1/2}(\partial\Omega_\phi)$) is given.



- When $\phi = 0$, we denote merely g the boundary data and ψ the solution of the Laplace equation defined in Ω.
- We want to study the regularity of $\phi \mapsto \psi_\phi$.

Shape derivative: function spaces

Consider a large ball $B \subset \mathbf{R}^2$ such that $\partial\Omega \subset B$ and introduce for all $m \in \mathbf{N}$, $m \geq 1$:

$$\mathcal{D}^m(B, \mathbf{R}^2) := \{\phi \in \mathcal{C}^m(\mathbf{R}^2, \mathbf{R}^2) \; : \; \phi|_{\mathbf{R}^2 \setminus B} = 0\},$$

which is a Banach space when endowed with the norm of the Sobolev space $W^{m,\infty}(\mathbf{R}^2, \mathbf{R}^2)$:

$$\|\phi\|_{W^{m,\infty}} := \max_{\substack{\alpha \in \mathbf{N}^2 \\ |\alpha| \leq m}} \|D^\alpha \phi\|_{L^\infty},$$

$$\text{where} \quad D^\alpha \phi := \frac{\partial^{|\alpha|}\phi}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \quad \alpha := (\alpha_1, \alpha_2) \in \mathbf{N}^2, \quad |\alpha| := \alpha_1 + \alpha_2.$$

Proposition

If $\phi \in \mathcal{D}^m(B, \mathbf{R}^2)$ such that $\|\phi\|_{W^{1,\infty}} < 1$ then $\Phi := \mathrm{Id} + \phi$ is a $\mathcal{C}^m$ diffeormorphism from Ω onto Ω_ϕ.

Well-posedness

For all such ϕ and all $g_\phi \in L^2(\partial\Omega_\phi)$ such that:

$$\int_{\partial\Omega_\phi} g_\phi \,\mathrm{d}\sigma = 0,$$

the Neumann boundary value problem admits a unique (weak) solution in:

$$H^1_N(\Omega_\phi) := \{u \in L^2(\Omega_\phi) \,:\, \partial_{x_i} u \in L^2(\Omega_\phi),\, i = 1,2\}/\mathbf{R}, \quad \text{(bounded domain)},$$
$$H^1_N(\Omega_\phi) := \{u/w \in L^2(\Omega_\phi) \,:\, \partial_{x_i} u \in L^2(\Omega_\phi),\, i = 1,2\}/\mathbf{R}, \text{ (exterior domain)}.$$

where $w(x) := \sqrt{1 + |x|^2}\log(1 + |x|^2)$. The quotient means that we identify two functions when they differ only by an additive constant.

Main result

We have the equivalences:

$$v \in H^1_N(\Omega_\phi) \Longleftrightarrow v \circ \Phi \in H^1_N(\Omega),$$
$$f \in L^2(\partial\Omega_\phi) \Longleftrightarrow f \circ \Phi \in L^2(\partial\Omega).$$

We denote $\psi^*_\phi := \psi_\phi \circ \Phi$ defined in Ω for all ϕ and $g^*_\phi = g_\phi \circ \Phi$ defined in $\partial\Omega$.

Theorem

If the mapping $\phi \in \mathcal{D}^1 \mapsto g^*_\phi \in L^2(\partial\Omega)$ is of class $\mathcal{C}^k$ $(k \geq 1)$ nearby $\phi = 0$ then the mapping $\phi \in \mathcal{D}^1 \mapsto \psi^*_\phi \in H^1_N(\Omega)$ is also of class $\mathcal{C}^k$ $(k \geq 1)$ nearby $\phi = 0$.

Proof: Variational formulation

The variational formulation:

$$\int_{\Omega_\phi} \nabla\psi_\phi \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega_\phi} g_\phi v \,\mathrm{d}\sigma, \qquad \forall\, v \in H^1_N(\Omega_\phi),$$

turns, upon a change of variables, into:

$$\int_{\Omega} \nabla\psi^*_\phi \,\mathbb{A}_\phi \nabla w^T \,\mathrm{d}x = \int_{\partial\Omega} g^*_\phi w \mathrm{J}_\phi \mathrm{d}\sigma, \quad \forall\, w \in H^1_N(\Omega),$$

where:

- $\mathbb{A}_\phi = (D\Phi)^{-1}(D\Phi)^{-1T} |\det D\Phi|$,
- $\mathrm{J}_\phi = |(D\Phi)^{-1} n| |\det D\Phi|$ (tangential Jacobian).

Proof: Local inversion Theorem

Introduce the mapping:

$$\begin{array}{rcl} \Lambda : \mathcal{D}^1 \times H^1_N(\Omega) & \rightarrow & H^1_N(\Omega)' \\ (\phi, u) & \mapsto & \Lambda(\phi, u), \end{array}$$

defined by:

$$\langle \Lambda(\phi, u), w \rangle := \int_\Omega \nabla u \, \mathbb{A}_\phi \nabla w^T \, \mathrm{d}x - \int_{\partial\Omega} g^*_\phi w \mathrm{J}_\phi \mathrm{d}\sigma, \qquad \forall\, w \in H^1_N(\Omega).$$

We easily check that:

- Λ is of classe $\mathcal{C}^k$ with respect to (ϕ, u) nearby $(0, \psi)$.
- $\partial_u \Lambda(0, \psi) = \Lambda(0, \cdot)$ is an isomorphism from $H^1_N(\Omega)$ onto $H^1_N(\Omega)'$ (Lax Milgram Theorem).

We can apply the local inversion Theorem. There exists a unique function of class $\mathcal{C}^k$, $\phi \in \mathcal{D}^1 \mapsto u_\phi \in H^1_N(\Omega)$ such that $\Lambda(\phi, u) = 0 \Leftrightarrow u = u_\phi$. But we have also the equivalence: $\Lambda(\phi, u) = 0 \Leftrightarrow u = \psi^*_\phi$, because we know that the solution of the BVP is unique.

Computation of the shape derivative

What about $\phi \mapsto \psi_\phi$? We can compute directional derivatives.

- Assume that $\partial\Omega$ is of classe $\mathcal{C}^{1,1}$ (continuously differentiable with Lipschitz continuous first derivative) and that $g_\phi \in H^{1/2}(\Omega_\phi)$.
- Fix $\phi_0 \in \mathcal{D}^2$ (a direction) and choose $\phi := t\phi_0$, i.e., $\Phi := \mathrm{Id} + t\phi_0$ with $t \in \mathbf{R}$.
- Denote ψ_t in place of $\psi_{t\phi}$ and g_t^* in place of $g_{t\phi}^*$.

Then the derivative of $t \mapsto \psi_t$ at $t = 0$, denoted by ψ', is the unique solution of:

$$\begin{aligned} -\Delta\psi' &= 0 && \text{in } \Omega, \\ \partial_n\psi' &= \Delta_\sigma\psi\,(\phi_0\cdot n) - \kappa(\phi_0\cdot n)g \\ &\quad + \partial_\tau\psi\cdot\partial_\tau(\phi_0\cdot n) + (g_t^*)'|_{t=0} - \partial_\tau g\cdot(\phi_0\cdot\tau) && \text{on } \partial\Omega, \end{aligned}$$

where:

- Δ_σ is the Laplace Beltrami operator on $\partial\Omega$,
- κ is the curvature of $\partial\Omega$,
- τ is the tangent vector to $\partial\Omega$ and ∂_τ is the tangential derivative.

Well posedness of the Euler-Lagrange equation

$t \in [0,T] \mapsto \mathbf{c}(t) \in \mathcal{C}$ is a given smooth control function.

Theorem

Providing that $\mathbf{c} \in \mathcal{C} \mapsto \chi(\mathbf{c}) \in W^{1,\infty}(\mathbf{R}^2,\mathbf{R}^2)$ is of class $\mathcal{C}^k$ $(k \geq 2)$, then for any $\mathbf{q}_0 := (\mathbf{r}_0, \theta_0) \in \mathcal{Q}$ and $\dot{\mathbf{q}}_0 := (\dot{\mathbf{r}}_0, \dot{\theta}_0) \in \mathbf{R}^2 \times \mathbf{R}$, there exists one unique solution to the Cauchy problem:

$$\frac{d}{dt}\begin{pmatrix}\dot{\mathbf{q}}\\ \mathbf{q}\end{pmatrix} = \Gamma(\dot{\mathbf{q}}, \mathbf{q}, (\mathbf{c}, \dot{\mathbf{c}}, \ddot{\mathbf{c}})), \quad (t>0),$$
$$\begin{pmatrix}\dot{\mathbf{q}}(0)\\ \mathbf{q}(0)\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{q}}_0\\ \mathbf{q}_0\end{pmatrix}.$$

The solution can be continued up to $t = T$ or to the time of a contact between the body and the boundary of the fluid domain.

The problem of *contacts* or *collisions* is very challenging (asymptotic behavior of the solution of a PDE as the domain becomes singular).

Numerical simulations

All the simulations have been realized with a Matlab Toolbox (joint work with Bruno Pinçon, IECN, INRIA, Nancy):

http://bht.gforge.inria.fr/

- Ellipse in a box filled by a fluid
- Dragged down ellipse
- Two brushing ellipses
- Swimming fish
- Two swimming fishes
- Collisions

Isotropic model

We assume that the body is alone in the fluid ($\bar{\mathcal{F}} \cup \bar{\mathcal{S}} = \mathbb{R}^2$).

- As seen by an observer attached to the body, all of the positions and directions in the fluid are equivalent.
- The isotropy of the configuration allows to integrate the Euler Lagrange equation once.

The Euler Lagrange equation turns out to be a first order ODE:

$$\frac{d}{dt}\mathbf{q} = \mathcal{R}(\theta)\langle \mathbb{M}(\mathbf{c}), \dot{\mathbf{c}} \rangle, \quad (t > 0),$$

where

- $\mathcal{R}(\theta) := \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}$.
- $\dot{\mathbf{c}} \in \mathcal{C} \mapsto \langle \mathbb{M}(\mathbf{c}), \dot{\mathbf{c}} \rangle \in \mathbf{R}^3$ is a linear continuous mapping and $\mathbf{c} \in \mathcal{C} \mapsto \mathbb{M}(\mathbf{c}) \in \mathcal{L}(\mathcal{C}, \mathbf{R}^3)$ is smooth. If $\mathcal{C}$ is a finite dimensional vector space (say of dim N) then $\mathbb{M}(\mathbf{c})$ is a $3 \times N$ matrix whose entries are smooth functions in $\mathbf{c}$.

A particular model

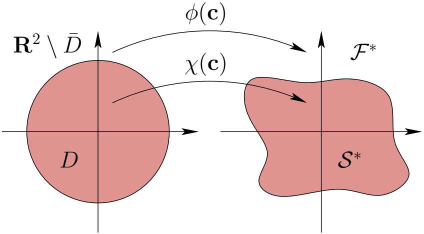
We specify:

$$\mathcal{C} = \mathcal{C}_1 := \Big\{ \mathbf{c} := (c_k)_{k\geq 1},\ c_k := a_k + ib_k \in \mathbf{C},\ \sum_{k\geq 1} k(|a_k| + |b_k|) < +\infty \Big\},$$

$$\mathcal{C}_2 := \Big\{ \mathbf{c} := (c_k)_{k\geq 1},\ c_k := a_k + ib_k \in \mathbf{C},\ \sum_{k\geq 1} k(|a_k|^2 + |b_k|^2) < +\infty \Big\},$$

and with complex notation:

$$\chi(\mathbf{c})(z) := z + \sum_{k\geq 1} c_k \bar{z}^k, \quad (z \in D), \qquad \phi(\mathbf{c})(z) := z + \sum_{k\geq 1} c_k z^{-k}, \quad (z \in \mathbf{R}^2 \setminus \bar{D}).$$



- $\phi(\mathbf{c})|_{\partial D} = \chi(\mathbf{c})|_{\partial D}$.
- When $\|\mathbf{c}\|_{\mathcal{C}_1} < 1$, $\chi(\mathbf{c})$ is a $\mathcal{C}^1$ diffeomorphism from D onto $\mathcal{S}^*$ and $\phi(\mathbf{c})$ a conformal mapping from $\mathbb{R}^2 \setminus \bar{D}$ onto $\mathcal{F}^*$.

Physical constraints

$t \in [0,T] \mapsto \mathbf{c}(t) \in \mathcal{C}$ is a given smooth control function. We denote $\dot{\mathbf{c}} = (\dot{a}_k + i\dot{b}_k)_{k\geq 1}$.
The physical constraints entail that:

$$\sum_{k\geq 1} k(\dot{a}_k a_k + \dot{b}_k b_k) = 0, \quad \text{(volume preservation)}, \tag{1}$$

$$\sum_{k\geq 1} \frac{1}{k+1}(\dot{a}_k b_k - a_k \dot{b}_k) = 0, \quad \text{(self-propelled motion)}. \tag{2}$$

The first one leads us to introduce:

$$\mathcal{E}(\mu) := \{\mathbf{c} \in \mathcal{C}_1 \ : \ \|\mathbf{c}\|_{\mathcal{C}_2} = \mu\},$$

$$\widetilde{\mathcal{E}}(\mu) := \{\mathbf{c} \in \mathcal{E}(\mu) \ : \ \|\mathbf{c}\|_{\mathcal{C}_1} < 1\} \quad \text{(non-empty iff } \mu \in]0,1[).$$

Definition (physically allowable control function)

A continuous piecewise C^1 control function $t \in [0,T] \mapsto \mathbf{c}(t) \in \mathcal{C}_1$ is said to be *physically allowable* when there exists $\mu \in]0,1[$ such that $\mathbf{c}(t) \in \widetilde{\mathcal{E}}(\mu)$ for all t and (2) holds.

Main result of controllability

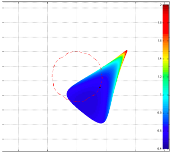
Theorem

For every $\mu^\dagger$ in $]0,1[$, for every $\varepsilon > 0$, for every reference continuous rigid motion $\mathbf{q}^\dagger : [0,T] \to \mathcal{Q}$ and for any reference continuous shape-changes $\mathbf{c}^\dagger : [0,T] \to \widetilde{\mathcal{E}}(\mu^\dagger)$, there exists a real number $\mu \in]0,1[$ and an analytic allowable control function $\mathbf{c} : [0,T] \to \widetilde{\mathcal{E}}(\mu)$ such that

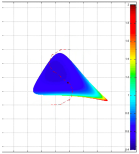
1. $\|\mathbf{c}(t) - \mathbf{c}^\dagger(t)\|_{\mathcal{C}_1} \leq \varepsilon$ for all $t \in [0,T]$;
2. The solution $\mathbf{q} : [0,T] \to \mathcal{Q}$ of the Euler-Lagrange equation with initial data $\mathbf{q}(0) = \mathbf{q}^\dagger(0)$ and control function $\mathbf{c}$ satisfies $\|\mathbf{q}(t) - \mathbf{q}^\dagger(t)\|_{\mathcal{Q}} < \varepsilon$, for all $t \in [0,T]$.

In other words: The shape-changing body can not only follow approximately any given trajectory but while undergoing approximately any prescribed shape-changes.

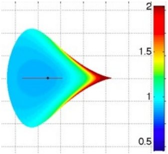
Numerical simulations



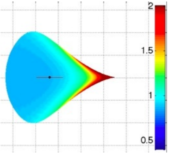
Circle



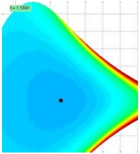
S-trajectory



Forward



Bakward



Details

Finite dimensional reduction

Let N be an integer $N \geq 1$. We define:

- $\mathcal{C}_1^N := \{\mathbf{c} \in \mathcal{C}_1 \ : \ c_k = 0 \quad \forall\, k > N\}$.
- $\mathcal{E}_N(\mu) := \mathcal{E}(\mu) \cap \mathcal{C}_1^N$.
  $\mathcal{E}_N(\mu)$ can be seen as an analytic submanifold of $\mathbf{R}^{2N}$ of dim $2N - 1$.
  $\mathcal{E}_N(\mu) \subset \mathcal{E}_{N'}(\mu)$ if $N < N'$ as immersed submanifold.

Definition (mathematically allowable control function)

A continuous piecewise C^1 control function $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{C}_1^N$ is said to be *mathematically allowable* when there exists $\mu \in]0, 1[$ such that $\mathbf{c}(t) \in \mathcal{E}_N(\mu)$ for all t and (2) holds.

amoeba

Managing the physical constraints

We define $\mathfrak{X}_N := (\mathbf{X}^j)_{1\leq j\leq n}$, a set of Lipschitz continuous vector fields on $\mathcal{C}_1$ with $\mathbf{X}^j := (X^j_k)_{k\geq 1}$, $X^j_k := x^j_k + iy^j_k$. We assume that:

1. $X^j_k = 0$ for all $k > N$,
2. $\displaystyle\sum_{k\geq 1} k(x^j_k(\mathbf{c})a_k + y^j_k(\mathbf{c})b_k) = 0$
3. $\displaystyle\sum_{k\geq 1} \frac{1}{k+1}(x^j_k(\mathbf{c})b_k - a_k y^j_k(\mathbf{c})) = 0.$

Let $\alpha_j : [0,T] \mapsto \alpha_j(t) \in \mathbf{R}$ $(1 \leq j \leq n)$ be piecewise constant functions.

Proposition

Any solution of the (well-posed) Cauchy problem $\dot{\mathbf{c}} = \sum_{j=1}^n \alpha_j \mathbf{X}^j(\mathbf{c})$ with any initial condition $\mathbf{c}_0 \in \mathcal{C}^1_N$ such that $\|\mathbf{c}_0\|_{\mathcal{C}_2} = \mu < 1$ is *mathematically allowable*.

Managing the physical constraints

We introduce $\mathfrak{Y}_N := (\mathbf{Y}^j)_{1\leq j\leq n}$, the vector fields on $\mathcal{Q}\times\mathcal{E}_N(\mu)$:

$$\mathbf{Y}^j(\theta,\mathbf{c}) = \begin{pmatrix} \mathcal{R}(\theta)\langle \mathbb{M}(\mathbf{c}), \mathbf{X}^j(\mathbf{c})\rangle \\ \mathbf{X}^j(\mathbf{c}) \end{pmatrix}.$$

The following points are worth being mentioned:

- If $\mathbf{c}\in\mathcal{C}_1^N$ then $\mathbb{M}(\mathbf{c})\circ\Pi_N = \mathbb{M}(\mathbf{c})$ where Π_N is the projection $\mathcal{C}_1\to\mathcal{C}_N^1$.
- If the vector fields $=(\mathbf{X}^j)_{1\leq j\leq n}$ are analytic then $\mathfrak{Y}_N = (\mathbf{Y}^j)_{1\leq j\leq n}$ are also analytic.

We rewrite the dynamics of the system fluid-body as:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{q}\\ \mathbf{c}\end{pmatrix} = \sum_{j=1}^n \alpha_j \mathbf{Y}^j(\theta,\mathbf{c}).$$

The new controls are now the functions α_j $(1\leq j\leq n)$.

Finite dimensional controllability result

We state a finite dimensional version of the main Theorem of controllability:

Theorem

For every $N \in \mathbf{N}$ $(N \geq 2)$, every μ in $]0,1[$, every $\varepsilon > 0$, every reference continuous rigid motion $\mathbf{q}^\dagger : [0,T] \to \mathcal{Q}$ and every reference continuous shape-changes $\mathbf{c}^\dagger : [0,T] \to \mathcal{E}_N(\mu)$, there exists a mathematically allowable analytic control function $\mathbf{c} : [0,T] \to \mathcal{E}_N(\mu)$ such that

1. $\|\mathbf{c}(t) - \mathbf{c}^\dagger(t)\|_{\mathcal{C}_1} \leq \varepsilon$ for all $t \in [0,T]$;
2. The solution $\mathbf{q} : [0,T] \to \mathcal{Q}$ of the Euler-Lagrange equations with initial data $\mathbf{q}(0) = \mathbf{q}^\dagger(0)$ and control function $\mathbf{c}$ satisfies $\|\mathbf{q}(t) - \mathbf{q}^\dagger(t)\|_{\mathcal{Q}} < \varepsilon$, for all $t \in [0,T]$.

Geometric control theory

Let M be an analytic manifold and $\mathfrak{X}$ be a set of analytic vector fields on M such that $\mathfrak{X}$ be a cone.

- **Orbit of $\mathfrak{X}$ through q ($q \in M$):** denoted by $\mathcal{O}(q)$, this set consists in all of the points of M reachable by following successively and during finite times, the curves defined by $\dot{q} = X(q), \quad (X \in \mathfrak{X})$.
- **Lie bracket:** $[X_1, X_2] = \frac{dX_2}{dq}X_1 - \frac{dX_1}{dq}X_2, \quad (X_1, X_2 \in \mathfrak{X})$.
- **Lie algebra:** $\mathrm{Lie}\,\mathfrak{X}$ is span by all of the Lie brackets of any order. $\mathrm{Lie}_q\,\mathfrak{X}$ ($q \in M$) consists of all of the vectors of $\mathrm{Lie}\,\mathfrak{X}$ at the point q.

Orbit Theorem (Nagano-Sussmann)

$\forall\, q_0 \in M$, $\mathcal{O}(q_0)$ is a connected immersed submanifold of M and for all $q \in \mathcal{O}(q_0)$, $\mathrm{Lie}_q\,\mathfrak{X} = T_q\mathcal{O}(q_0)$.

Rashevsky Chow Theorem

If $\mathrm{Lie}_q\,\mathfrak{X} = T_qM$, $\forall\, q \in M$, then $\mathcal{O}(q) = M$, $\forall\, q \in M$.

Outline of the proof: Main ideas

Remind that the dynamics is:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{q}\\ \mathbf{c}\end{pmatrix} = \sum_{j=1}^{n} \alpha_j \mathbf{Y}^j(\theta, \mathbf{c}),$$

where $\mathbf{Y}^j$ are analytic vector fields on the analytic manifold $\mathcal{Q} \times \mathcal{E}_N(\mu)$.

According to Rashevsky Chow Theorem, it is enough to find for any $N \geq 1$ a set of allowable analytic vector fields $\mathfrak{X}_N$ such that

$$\dim\left(\mathrm{Lie}_{(\mathbf{q},\mathbf{c})}\left(\mathfrak{Y}_N\right)\right) = \dim T_{(\mathbf{q},\mathbf{c})}(\mathcal{Q} \times \mathcal{E}_N(\mu)),$$

for any $(\mathbf{q}, \mathbf{c}) \in \mathcal{Q} \times \mathcal{E}_N(\mu)$.

→ Definitively too complicated.

Outline of the proof: The case $N = 2$

We proceed in several steps:

- We build a set of analytic vectors fields $\mathfrak{X}_2$ on $\mathcal{C}_1^2$ such that $\dim\left(\mathrm{Lie}_{\mathbf{c}}\left(\mathfrak{X}_2\right)\right) = \dim\left(T_{\mathbf{c}}\mathcal{E}_2(\mu)\right) = 3$ (explicit computations).
- We use the particular form of the ODE. We introduce:

$$\widehat{\mathbf{X}}^j(\mathbf{c}) := \begin{bmatrix} -\Pi_3(\langle \mathbb{M}(\mathbf{c}), \mathbf{X}^j(\mathbf{c})\rangle) \\ \mathbf{X}^j(\mathbf{c}) \end{bmatrix},$$

and we can rewrite the dynamics for $(\theta, \mathbf{c}) \in \mathbf{R}/2\pi \times \mathcal{E}_2(\mu) := \widehat{\mathcal{E}}_2(\mu)$:

$$\frac{d}{dt}\begin{pmatrix} \dot{\theta} \\ \dot{\mathbf{c}} \end{pmatrix} = \sum_{j=1}^{n} \alpha_j(t)\widehat{\mathbf{X}}^j(\mathbf{c}).$$

Computations can not be done in the general case but we can prove that:

$$\dim\left(\mathrm{Lie}_{(0,\mathbf{c}^\dagger)}\{\widehat{\mathbf{X}}^j,\, 1 \leq j \leq n\}\right) = 4,$$

for some particular $\mathbf{c}^\dagger$. Since the fields do not depend on θ, it is still true for all $(\theta, \mathbf{c}^\dagger)$. But for any $(\theta, \mathbf{c})$, there exists some $\theta^\dagger$ such that

$$\mathcal{O}((\theta, \mathbf{c})) \ni (\theta^\dagger, \mathbf{c}^\dagger).$$

Outline of the proof: the case $N = 2$

According to the Orbit Theorem:

$$\dim\left(\mathrm{Lie}_{(\theta,\mathbf{c})}\{\widehat{\mathbf{X}}^j,\, 1 \le j \le n\}\right) = 4, \quad \forall\,(\theta,\mathbf{c}).$$

Let us recall that:

$$\begin{pmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{c}}\end{pmatrix} = \sum_{j\in J}^{2} \alpha_j(t)\mathbf{Y}^j(\theta,\mathbf{c}).$$

The vector fields $\mathbf{Y}^j$ depend only on θ and $\mathbf{c}$. We compute that for $\theta = 0$ and $\mathbf{c}^\dagger$ we have:

$$\dim\left(\mathrm{Lie}_{(\mathbf{r},0,\mathbf{c}^\dagger)}\{\mathbf{Y}^j,\, 1 \le j \le n)\}\right) = 6, \quad \forall\,\mathbf{r} \in \mathbf{R}^2.$$

Arguing as previously, we next prove that:

$$\dim\left(\mathrm{Lie}_{(\mathbf{r},\theta,\mathbf{c})}(\{\mathbf{Y}^j(0,\mathbf{c}^\dagger),\, 1 \le j \le n)\}\right) = 6, \quad \forall\,(\mathbf{r},\theta,\mathbf{c}) \in \mathcal{Q}\times\mathcal{E}_2(\mu).$$

Generalization to any dimension

- For any $N > 2$, $\mathcal{E}_2(\mu) \subset \mathcal{E}_N(\mu)$ (as immersed manifold).
- Likewise, $\mathfrak{X}_2$ is a complete distribution on $\mathcal{E}_N(\mu)$.

We construct (explicitly) a sequence of analytic distributions $\mathfrak{X}_k$ on $\mathcal{E}_k(\mu)$ ($k = 3, \dots, N$) such that

$$\mathfrak{X}_{k-1} \subset \mathfrak{X}_k \quad \text{and} \quad \mathrm{Lie}_{\mathbf{c}}(\mathfrak{X}_k) = T_{\mathbf{c}}\mathcal{E}_k(\mu), \quad \forall\, \mathbf{c} \in \mathcal{E}_k(\mu).$$

We next consider the associated distributions $\widehat{\mathfrak{X}}_k$ and $\mathfrak{Y}_k$ and we check that

$$\mathrm{Lie}_{(0,\mathbf{c}^\dagger)}(\widehat{\mathfrak{X}}_2) \oplus \mathrm{span}(\widehat{\mathfrak{X}}_N) = T_{(0,\mathbf{c}^\dagger)}\widehat{\mathcal{E}}_N(\mu),$$
$$\mathrm{Lie}_{(0,0,\mathbf{c}^\dagger)}(\mathfrak{Y}_2) \oplus \mathrm{span}(\mathfrak{Y}_N) = T_{(0,0,\mathbf{c}^\dagger)}(\mathcal{Q} \times \mathcal{E}_N(\mu)).$$

We reuse the already done computations for the case $N = 2$ and we conclude as for this case.
We have proved the existence of a set of piecewise constant functions $(\alpha_j)_{1 \le j \le n}$ such that the solution $(\mathbf{q}, \mathbf{c})$ tracks the reference data $(\mathbf{q}^\dagger, \mathbf{c}^\dagger)$.
To complete the proof (analyticity of the control function), we use the density of the analytic functions for the L^1 norm in the set of measurable bounded functions.

Infinite dimensional case

For any ε, there exists an integer $N \geq 1$ and a finite dimensional control $\mathbf{c}^{\ddagger} : [0, T] \to \mathcal{E}_N(\mu^{\ddagger})$ such that

$$\|\mathbf{c}^{\ddagger}(t) - \mathbf{c}^{\dagger}(t)\|_{\mathcal{C}_1} < \varepsilon/2, \quad \forall\, t \in [0, T],$$

where we recall that $\mathbf{c}^{\dagger} : [0, T] \to \mathcal{E}(\mu^{\dagger})$ is a given reference curve.
We next apply the finite dimensional Theorem with $\mathbf{c}^{\ddagger}$ as reference curve.

Open problems relating to FSIS

Well posedness

- Motion of elastic bodies in a perfect fluid (or any other fluid).

Control problems:

- Control of a 3d (isotropic) model.
- Control of a 2d non-isotropic model, for instance with gravity or in a bounded domain.

Collisions:

- Asymptotic study nearby collision time of 2d rigid solids in a fluid.
- Modeling: find an asymptotic model for 2d colliding rigid solids.
- Collisions in a 3d perfect fluid?

Detection (inverse problems)

- Can we recover the positions and velocities of rigid solids in a fluid by measuring the potential on the boundary of the fluid domain?