

Swimming in a perfect fluid

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- 1 Introduction
 - Motivations
 - History
 - Systems of fluid-structure interactions
- 2 Modeling
 - Kinematics of the fish
 - Fluid modeling
 - Fish dynamics
 - Complete Newtonian system
 - Potential flow
 - Analytic Mechanics
 - Lagrangian system
- 3 Existence and uniqueness of solutions
 - Shape sensitivity analysis
- 4 Collisions in a fluid
 - Viscous fluid, perfect fluid
 - Collisions in a perfect fluid
- 5 Numerical simulations (with B. Pinçon, IECN, INRIA)
 - Computation of the potentials
 - Numerical scheme

What *nature* has done

Some examples of fishes' velocities

Harring		6 <i>km/h</i>
Pike		6 <i>km/h</i>
Carp		6 <i>km/h</i>
Mackerel		11 <i>km/h</i>
Tuna		80 <i>km/h</i>
Swordfish		96 <i>km/h</i>

Swimming capabilities far superior to what can be achieved with a propeller (efficiency, maneuverability and stealth).

Pioneering works

Biohydrodynamics

Hydrodynamical mechanisms related to animal locomotion in fluids (microorganisms, fishes, aquatic mammals, birds...).

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- Pettigrew (1873): studied the locomotion of the atlantic salmons.
- E.J. Marey (1895): used movie technics to study eels locomotion.
- Housay (1912): kinematic studies of eels.
- Breder (1926): classification and nomenclature of the swimming types.
- James Gray (1930): kinematic bases of aquatic locomotion, *Dolphins' paradox*.

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Quantitative and descriptive theories done largely by zoologists and engineers

Biohydrodynamics in the 20th century

First half of the 20th century:

- Few theoretical studies.
- Based on very simplified models.

An important exception:

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Second half of the 20th century: mathematicians developed theories allowing qualitative studies:

 G. Taylor, *Analysis of the swimming of microscopic organisms*, Proc. R. Soc. Lond., 1951.

 J. Lighthill, *Mathematical biofluidodynamics*, Soc. Ind. and Appl. Math., 1975.

 S. Childress, *Mechanics of swimming and flying*, Cambridge University Press, 1981.

(Selected) contemporary references

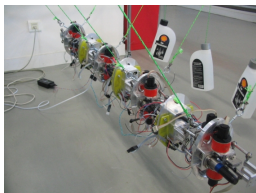
-  G. P. Galdi, *On the steady self-propelled motion of a body in a viscous incompressible fluid*, Arch. Ration. Mech. Anal., 1999.
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-  J. San Martin, J. F. Scheid, T. Takahashi et M. Tucsnak, *An initial and boundary problem modeling fish-like swimming*, Arch. Ration. Mech. Anal., 2008.
-  J. Houot et A. M., *On the motion and collisions of rigid bodies in an ideal fluid*, Asymptot. Anal., 2008.
-  A. M., *On the self displacement of deformable bodies in a potential fluid flow*, Math. Model. Num. Anal., 2008.
-  F. Alouges, A. Lefebvre et A. DeSimone, *Optimal Strokes for Low Reynolds Number Swimmers: An example*, J. of. Nonlinear Science, 2008.

Experimental approaches, robotic

Essex University - Department of Computer Science



École des Mines de Nantes



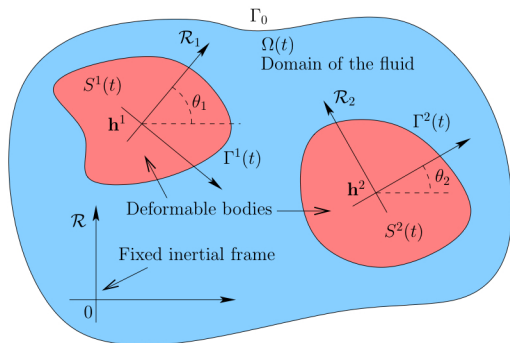
University of Washington (UWMFAUS)



Modeling of fluid-structure interactions systems

Notation, unknowns

Consider two deformable bodies immersed in a fluid.



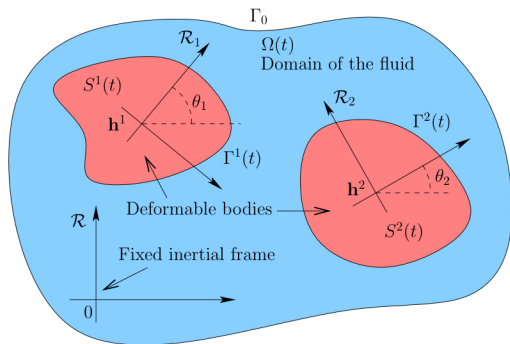
Notation:

- $\Omega(t)$: domain of the fluid.
- $S^1(t), S^2(t)$: domains of the immersed bodies.
- Γ_0 : fixed boundary.
- $\Gamma^1(t), \Gamma^2(t)$: moving boundaries.
- $\mathcal{R}_1, \mathcal{R}_2$: moving frames.
- \mathcal{R} : inertial fixed frame.
- $\mathbf{h}_1, \mathbf{h}_2$: centers of mass.
- θ_1, θ_2 : orientations (in \mathcal{R}).

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The unknowns are: $\mathbf{h}_1, \mathbf{h}_2, \theta_1, \theta_2$ and $\Gamma^1(t)/\mathcal{R}_1$ (the shape of $S^1(t)$ in the moving frame) and $\Gamma^2(t)/\mathcal{R}_2$.

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Coupled dynamics

- 1 Dynamics of the fluid (equations of Fluid Mechanics: Navier-Stokes, Stokes, Euler...):

System of PDEs in $\Omega(t)$.

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Unknowns domains: **free boundary problems**.

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- Coupling through the boundary conditions of the PDEs (kinematic coupling).
- Coupling through the right hand side terms of the ODEs (first and second Newton's laws).

Swimming: an heuristic explanation

- The fish is not able to move its center of mass under the action of inner forces. It is just able to change its shape.
- The net locomotion results from the transfer of momentum between the fish and the fluid.

Assume that the system fish-fluid is starting from rest.

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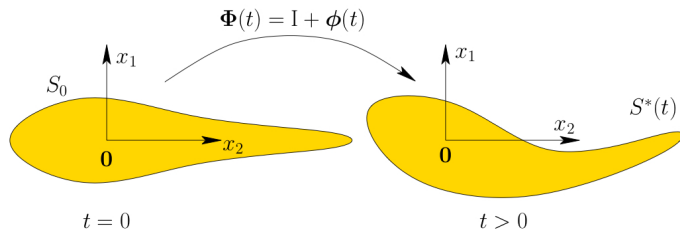
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- 3 The fish get back momentum from the fluid what allows its net locomotion.

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Deformations of the fish

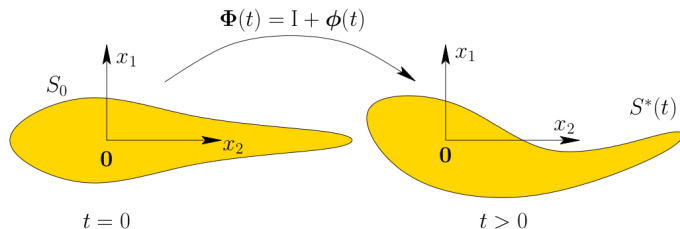
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- Deformations are described in the frame attached to the fish:

$$\phi : t \in [0, \infty) \mapsto \phi(t) \in C^m(\mathbb{R}^2, \mathbb{R}^2), (m \geq 1) \text{ of class } C^k, k \geq 2.$$

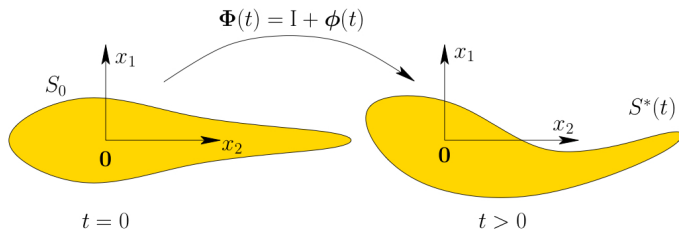


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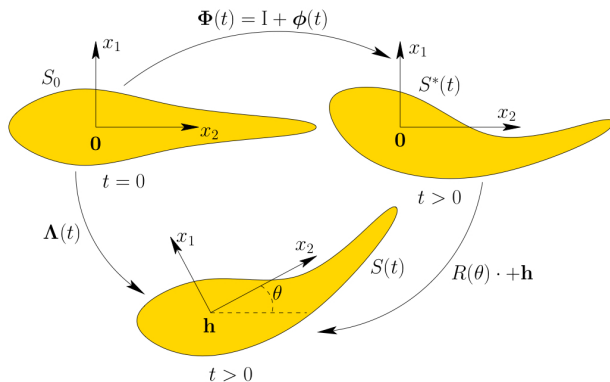
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- ϕ compactly supported and $\|\phi(t)\|_{L^\infty(\mathbb{R}^2)} + \|D\phi(t)\|_{L^\infty(\mathbb{R}^2)} < 1$ for all $t > 0$. Thus $\Phi(t) = I + \phi(t)$ is a C^2 diffeomorphism for all $t > 0$.



Deformations of the fish: complete description



- S_0 : fish at time $t = 0$, $S^*(t) := \Phi(t, S_0)$ (deformed fish in the local frame).
- $R(\theta)$: rotation matrix of angle θ .
- $S(t) := R(\theta)S^*(t) + \mathbf{h}$: fish at time $t > 0$ in the local frame.
- The unknowns are: θ et \mathbf{h} .

Velocity of deformation and rigid velocity

Let $x(t)$ be a point attached to the fish:

$$\begin{aligned} x(t) &= \mathbf{\Lambda}(t, x) \\ &= R(\theta(t))\phi(t, x) + \mathbf{h}. \end{aligned}$$

Differentiating with respect to time, we obtain the **Eulerian velocity**:

$$\mathbf{v}(t, x) = R(\theta)\dot{\phi}(t, \mathbf{\Lambda}^{-1}(t, x)) + \dot{\theta}(x - \mathbf{h})^\perp + \dot{\mathbf{h}}.$$

It can be decomposed into:

- $\mathbf{v}^r(t, x) = \dot{\theta}(x - \mathbf{h})^\perp + \dot{\mathbf{h}}$: the (unknown) **rigid velocity**.
- $\mathbf{v}^d(t, x) = R(\theta)\dot{\phi}(t, \mathbf{\Lambda}^{-1}(t, x))$: the (prescribed) **velocity of deformation**.

Physical constraints for the deformation

Local conservation of mass:

$$\rho_S(t, x) = \rho_S^0(\mathbf{\Lambda}^{-1}(t, x)) |\det[D\mathbf{\Lambda}^{-1}(t, x)]|,$$

- ρ_S^0 is the density in S_0 ,
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Deformations are due to inner forces only. It entails the conservation of linear and angular momenta:

$$\int_{S(t)} \rho_S(t, x) \mathbf{v}^d(t, x) \, dx = 0, \quad \int_{S(t)} \rho_S(t, x) (x - \mathbf{h}) \cdot \mathbf{v}^d(t, x) \, dx = 0.$$

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Volume conservation (for bounded domains):

$$\int_{S(t)} \operatorname{div}(\mathbf{v}^d(t, x)) dx = 0.$$

Choice of the fluid model

The fluid model depends on the **Reynolds number**: $Re = \rho_F u_c L_c / \mu$ (ρ_F fluid density, μ fluid viscosity, u_c mean fluid velocity and L_c characteristic length).

At any time $t > 0$, we denote $u(t)$ the Eulerian fluid velocity and $p(t)$ the pressure defined in $\Omega(t)$.

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- **Microorganisms** (ameba)

Very small Reynolds number ($Re \sim 10^{-6}$).

We choose Stokes' equations.

$$\rho_F \partial_t u(t) - \mu \Delta u(t) + \nabla p(t) = F,$$

$$\operatorname{div}(u(t)) = 0.$$

$$\text{Stress tensor: } \mathbb{T} = -p\mathbf{I} + \mu(Du(t) + Du(t)^T).$$



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- Common simplification: steady Stokes' equations.

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Choice of the fluid model

- **Small fishes ($\sim 1\text{cm}$), plankton**

We compute that $Re \sim 10^3$.

We choose Navier-Stokes' equations:

$$\begin{aligned}\rho_F[\partial_t u(t) + (u(t) \cdot \nabla)u(t)] - \mu\Delta u(t) + \nabla p(t) &= F, \\ \operatorname{div}(u(t)) &= 0.\end{aligned}$$



Stress tensor: $\mathbb{T} = -p\mathbf{I} + \mu(Du(t) + Du(t)^T)$.

- **Classical fishes, aquatic mammals**

We have $Re \sim 10^6$.

We choose Euler's equations:

$$\begin{aligned}\rho_F[\partial_t u(t) + (u(t) \cdot \nabla)u(t)] + \nabla p(t) &= F, \\ \operatorname{div}(u(t)) &= 0.\end{aligned}$$



Stress tensor: $\mathbb{T} = -p\mathbf{I}$.

Boundary and initial conditions

For the well-posedness of the problem, we add boundary conditions:

- On Γ_0 (the fixed boundary)
 - For Stokes' and Navier-Stokes' equations, nonslip condition: $u(t) = 0$.
 - For Euler's equations, slip condition: $u(t) \cdot n = 0$.
- On $\Gamma(t)$ (boundary of the fish)
 - For Stokes et Navier-Stokes: $u(t) = v(t)$ ($v(t)$ is the velocity of the fish).
 - For Euler: $u(t) \cdot n = v(t) \cdot n$.

Other conditions are possible (work in progress of D. Bucur and Matthieu Bonnivard (Chambéry)).

Initial condition (when required): $u(0) = u_0$.

Dynamics of the fish

Considered as a rigid body

We denote

- m the mass of the fish,
- $I(t) = \int_{S(t)} \rho_F |x - \mathbf{h}|^2 dx$ its inertia momentum.

Newton's laws tell us that:

$$m\ddot{\mathbf{h}} = - \int_{\Gamma(t)} \mathbb{T}n \, d\Gamma + F,$$

$$(I(t)\dot{\theta})' = - \int_{\Gamma(t)} \mathbb{T}n \cdot (x - \mathbf{h})^\perp \, d\Gamma + C.$$

We add initial data for these ODEs :

$$\mathbf{h}(0) = \mathbf{h}_0, \dot{\mathbf{h}}_0 = \mathbf{h}_0, \theta(0) = \theta_0, \dot{\theta}(0) = \theta_0.$$

Complete system for the system fluid-fish

With Euler's equations

$$\begin{aligned}
 \rho_F [\partial_t \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t)] + \nabla p(t) &= F && \text{in } \Omega(t), \quad t > 0, \\
 \operatorname{div}(\mathbf{u}(t)) &= 0 && \text{in } \Omega(t), \quad t > 0, \\
 \mathbf{u}(t) \cdot \mathbf{n} &= 0 && \text{on } \Gamma_0, \quad t > 0, \\
 \mathbf{u}(t) \cdot \mathbf{n} &= \mathbf{v}(t) \cdot \mathbf{n} && \text{on } \Gamma(t), \quad t > 0, \\
 \mathbf{v}(t) &= \dot{\theta}(t)(x - \mathbf{h})^\perp + \dot{\mathbf{h}}(t) + v^d(t) && \text{in } S(t), \quad t > 0, \\
 m \ddot{\mathbf{h}} &= - \int_{\Gamma(t)} \mathbb{T} \mathbf{n} \, d\Gamma + F, && t > 0, \\
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 \end{aligned}$$

We add initial data:

$$\mathbf{h}(0) = \mathbf{h}_0, \quad \dot{\mathbf{h}}(0) = \dot{\mathbf{h}}_0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0 \quad \text{et} \quad u(0) = u_0.$$

The unknowns are \mathbf{h} , θ , \mathbf{u} and p .

Perfect fluid with potential flow

We seek a solution of Euler's equations under the form:

$$u(t) = \nabla\varphi(t),$$

where $\varphi(t)$ is the *fluid potential*. Euler's equations and boundary conditions yield, at any time $t > 0$:

$$\begin{aligned} -\Delta\varphi(t) &= 0 && \text{in } \Omega(t), \\ \partial_n\varphi(t) &= 0 && \text{on } \Gamma_0, \\ \partial_n\varphi(t) &= v(t) \cdot n && \text{on } \Gamma(t). \end{aligned}$$

We add a decreasing condition at infinity (for unbounded domains) or any condition that ensures the uniqueness of the solution for bounded domains.

The pressure is given by **Bernoulli's formula**:

$$\rho_F(\partial_t\varphi(t) + \frac{1}{2}|\nabla\varphi(t)|^2) + p(t) - p_0(t) = 0.$$

$\partial_t\varphi(t)$: derivative of the solution with respect to time \rightarrow the boundary conditions and the domain both depend on time.

Potential decomposition, Kirchhoff's law

From the decomposition $v(t) = v^r(t) + v^d(t)$ on $\Gamma(t)$ with

$$v^r(t) = \dot{\mathbf{h}} + \dot{\theta}(x - \mathbf{h})^\perp,$$

we deduce likewise the decomposition of the potential (Kirchhoff's law):

$$\varphi(t) = \dot{h}_1 \varphi_1^r(t) + \dot{h}_2 \varphi_2^r(t) + \dot{\theta} \varphi_3^r(t) + \varphi^d(t),$$

where the *elementary potentials* φ_i^r ($i = 1, 2, 3$) et φ^d are harmonic in $\Omega(t)$ and satisfy the Neumann boundary conditions:

$$\begin{aligned} \partial_n \varphi_1^r(t) &= n_1 & \text{on } \Gamma(t), & \quad \partial_n \varphi_1^r(t) = 0 \text{ on } \Gamma_0, \\ \partial_n \varphi_2^r(t) &= n_2 & \text{on } \Gamma(t), & \quad \partial_n \varphi_2^r(t) = 0 \text{ on } \Gamma_0, \\ \partial_n \varphi_3^r(t) &= (\mathbf{x} - \mathbf{h})^\perp \cdot \mathbf{n} & \text{on } \Gamma(t), & \quad \partial_n \varphi_3^r(t) = 0 \text{ on } \Gamma_0, \\ \partial_n \varphi^d(t) &= \mathbf{v}^d(t) \cdot \mathbf{n} & \text{on } \Gamma(t), & \quad \partial_n \varphi^d(t) = 0 \text{ on } \Gamma_0. \end{aligned}$$

+ decreasing conditions at infinity (unbounded domains) + condition for uniqueness (bounded domains).

Cancellation of time, generalized coordinates (1/2)

Time appears only through the variables ϕ, θ et $\mathbf{h} = (h_1, h_2)$ and their derivatives $\dot{\phi}, \dot{\theta}$ et $\dot{\mathbf{h}} = (\dot{h}_1, \dot{h}_2)$.

We introduce the *generalized variables*:

$$\mathbf{q} = (h_1, h_2, \theta), \dot{\mathbf{q}} = (\dot{h}_1, \dot{h}_2, \dot{\theta}), \phi \text{ and } \dot{\phi}.$$

All of the quantities can be rewritten by means of the generalized variables:

- $\Omega(\mathbf{q}, \phi) = \Omega(t)$: the fluid domain,
- $S(\mathbf{q}, \phi) = S(t)$: the domain of the fish,
- $\Gamma(\mathbf{q}, \phi) = \partial S(\mathbf{q}, \phi)$: the boundary of the fish,
- $I(\phi) = I(t)$: the inertia momentum of the fish,
- $\varphi(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = \varphi(t)$: the fluid potential,
- $u(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = u(t)$: the fluid Eulerian velocity,
- $v^r(\mathbf{q}, \phi, \dot{\mathbf{q}})$ and $v^d(\mathbf{q}, \phi, \dot{\phi})$: the rigid velocity and the velocity of deformation of the fish.

Cancellation of time, generalized coordinates (2/2)

The boundary conditions of the elementary potentials can be rewritten:

$$\begin{aligned}
 \partial_n \varphi_1^r(\mathbf{q}, \phi) &= n_1 & \text{sur } \Gamma(\mathbf{q}, \phi), & \quad \partial_n \varphi_1^r(\mathbf{q}, \phi) = 0 \text{ on } \Gamma_0, \\
 \partial_n \varphi_2^r(\mathbf{q}, \phi) &= n_2 & \text{on } \Gamma(\mathbf{q}, \phi), & \quad \partial_n \varphi_2^r(\mathbf{q}, \phi) = 0 \text{ on } \Gamma_0, \\
 \partial_n \varphi_3^r(\mathbf{q}, \phi) &= (\mathbf{x} - \mathbf{h})^\perp \cdot \mathbf{n} & \text{on } \Gamma(\mathbf{q}, \phi), & \quad \partial_n \varphi_3^r(\mathbf{q}, \phi) = 0 \text{ on } \Gamma_0, \\
 \partial_n \varphi^d(\mathbf{q}, \phi) &= \mathbf{v}^d(\mathbf{q}, \phi, \dot{\phi}) \cdot \mathbf{n} & \text{on } \Gamma(\mathbf{q}, \phi), & \quad \partial_n \varphi^d(\mathbf{q}, \phi) = 0 \text{ on } \Gamma_0.
 \end{aligned}$$

Next, we denote

$$\varphi^r(\mathbf{q}, \phi) = (\varphi_1^r(\mathbf{q}, \phi), \varphi_2^r(\mathbf{q}, \phi), \varphi_3^r(\mathbf{q}, \phi)),$$

and $[D\varphi(\mathbf{q}, \phi)]$ its Jacobian matrix.

Important remark

The fluid has no own degree of freedom left. At any time, it is completely determined by the position and velocity of the fish.

Lagrangian of the system fluid-fish

Kinetic energy

Lagrangian Mechanics

Lagrangian formalism allows the system fluid-fish to be handled in its integrity.

Computation of the Lagrangian for the system fluid-fish:

- Kinetic energy of the fish:

$$E_P(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = \frac{1}{2}(m|\dot{\mathbf{h}}|^2 + I(\phi)|\dot{\theta}|^2 + \int_{S_0} \rho_S^0 |\dot{\phi}|^2 dx).$$

- Kinetic energy of the fluid:

$$E_F(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = \frac{1}{2} \int_{\Omega(\mathbf{q}, \phi)} \rho_F |\nabla \varphi(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi})|^2 dx.$$

- We get the expression of the Lagrangian:

$$\mathcal{L}(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = E_P(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) + E_F(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}).$$

Lagrangian of the system fluid-fish

Mass matrices

The Lagrangian is a quadratic form in $(\dot{\mathbf{q}}, \dot{\phi})$ whose coefficients depend on (\mathbf{q}, ϕ) . Let us introduce:

Mass matrices

Mass matrix of the fish: $[M_P(\phi)] = \text{diag}(m, m, I(\phi))$.

Added mass matrix:

$$[M_F(\mathbf{q}, \phi)] = \rho_F \int_{\Omega(\mathbf{q}, \phi)} [D\varphi^r][D\varphi^r]^T dx.$$

Virtual mass matrix of the system: $[M(\mathbf{q}, \phi)] = [M_P(\phi)] + [M_F(\mathbf{q}, \phi)]$.

The entries of $[M_F(\mathbf{q}, \phi)]$ read:

$$\rho_F \int_{\Omega(\mathbf{q}, \phi)} \nabla \varphi_i^r \cdot \nabla \varphi_j^r dx = \rho_F \frac{1}{2} \left(\int_{\Gamma(\mathbf{q}, \phi)} \frac{\partial \varphi_i^r}{\partial n} \varphi_j^r d\Gamma + \int_{\Gamma(\mathbf{q}, \phi)} \frac{\partial \varphi_j^r}{\partial n} \varphi_i^r d\Gamma \right).$$

Lagrangian of the system fluid-fish

Final expression

Let us define as well:

$$\langle \mathbf{M}^{r,d}(\mathbf{q}, \phi), \dot{\phi} \rangle = \rho_F \int_{\Omega(\mathbf{q}, \phi)} \nabla \varphi^d(\mathbf{q}, \phi, \dot{\phi}) [D\varphi^r(\mathbf{q}, \phi)]^T dx,$$

$$\langle M^d(\mathbf{q}, \phi), \dot{\phi}, \dot{\phi} \rangle = \rho_F \int_{\Omega(\mathbf{q}, \phi)} |\nabla \varphi^d(\mathbf{q}, \phi, \dot{\phi})|^2 dx.$$

The expression of the Lagrangian turns into:

$$\mathcal{L}(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = \frac{1}{2} \dot{\mathbf{q}} \cdot [M(\mathbf{q}, \phi)] \dot{\mathbf{q}} + \langle \mathbf{M}^{r,d}(\mathbf{q}, \phi), \dot{\phi} \rangle \cdot \dot{\mathbf{q}} + \frac{1}{2} \langle M^d(\mathbf{q}, \phi), \dot{\phi}, \dot{\phi} \rangle.$$

Equations of motion (Euler-Lagrange equations):

Least action principle:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}(\mathbf{q}, \phi, \dot{\mathbf{q}}, \dot{\phi}) = 0.$$

Equations of motion (Lagrangian approach)

Since:

$$\frac{d}{dt} = \ddot{\mathbf{q}}\partial_{\dot{\mathbf{q}}} + \dot{\mathbf{q}}\partial_{\mathbf{q}} + \ddot{\phi}\partial_{\dot{\phi}} + \dot{\phi}\partial_{\phi},$$

we formally obtain the following equation:

$$\begin{aligned} & \langle [M_P(\mathbf{q}, \phi)] + [M_F(\mathbf{q}, \phi)], \ddot{\mathbf{q}}, \cdot \rangle \\ & + \langle \partial_{\mathbf{q}}[M_F(\mathbf{q}, \phi)], \dot{\mathbf{q}}, \dot{\mathbf{q}}, \cdot \rangle - \frac{1}{2} \langle \partial_{\mathbf{q}}[M_F(\mathbf{q}, \phi)], \cdot, \dot{\mathbf{q}}, \dot{\mathbf{q}} \rangle \\ & + \langle \partial_{\phi}[M_P(\mathbf{q}, \phi)], \dot{\phi}, \dot{\mathbf{q}}, \cdot \rangle + \langle \partial_{\phi}[M_F(\mathbf{q}, \phi)], \dot{\phi}, \dot{\mathbf{q}}, \cdot \rangle \\ & + \langle \partial_{\mathbf{q}}\mathbf{M}^{r,d}(\mathbf{q}, \phi), \dot{\mathbf{q}}, \dot{\phi}, \cdot \rangle - \langle \partial_{\mathbf{q}}\mathbf{M}^{r,d}(\mathbf{q}, \phi), \cdot, \dot{\phi}, \dot{\mathbf{q}} \rangle \\ & + \langle \mathbf{M}^{r,d}(\mathbf{q}, \phi), \ddot{\phi}, \cdot \rangle + \langle \partial_{\phi}\mathbf{M}^{r,d}(\mathbf{q}, \phi), \dot{\phi}, \dot{\phi}, \cdot \rangle - \frac{1}{2} \langle \partial_{\mathbf{q}}M^d(\mathbf{q}, \phi), \cdot, \dot{\phi}, \dot{\phi} \rangle = 0. \end{aligned}$$

It is a system of nonlinear second order ODEs in \mathbf{q} .

Remark

The derivatives with respect to \mathbf{q} and ϕ are derivatives with respect to the position and shape of the fish: they are called *shape* derivatives.

Bodies hydrodynamically decoupled

A commonly admitted assumption.

- Only for unbounded domains.
- Each one of the immersed bodies behaves as if being alone in the fluid: the mass matrices are **constant**.

The assumption is relevant when the bodies are **very far** one from the others.

It is not relevant any more when:

- The fluid is partially or totally bounded.
- The bodies (or the fishes) are closed (see the numerical simulations).

Short list of questions

- ① Is the Lagrangian system well-posed (i.e., do all of the terms make sense)?
→ regularity of the mass matrices with respect to the domain.
- ② Are both systems (Lagrangian and Newtonian) equivalent?
- ③ Does Cauchy-Lipschitz Theorem apply to the Lagrangian system? (also a problem of regularity of the mass matrices with respect to the domain).
- ④ What about numerical simulations?

Regularity of the mass matrices with respect to \mathbf{q} et ϕ has to be investigated. All of the elements of the mass matrices have the form:

$$\int_{\Omega(\mathbf{q}, \phi)} \rho_F \nabla \varphi_1(\mathbf{q}, \phi) \cdot \nabla \varphi_2(\mathbf{q}, \phi) dx,$$

where $\varphi_1(\mathbf{q}, \phi)$ and $\varphi_2(\mathbf{q}, \phi)$ solve Neumann boundary value problems in $\Omega(\mathbf{q}, \phi)$.

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Shape derivative

Data of the problem

- Ω is a open, bounded, connected and Lipschitz continuous set and $\Gamma = \partial\Omega$.
- $\mathcal{D} = \{\phi \in C^1(\mathbb{R}^2, \mathbb{R}^2) \text{ s. t. } \|\phi\|_{L^\infty} + \|[D\phi]\|_{L^\infty} < 1\}$ and $\Phi = I + \phi$.
- $\Omega(\phi) = \Phi(\Omega)$, $\Gamma(\phi) = \Phi(\Gamma)$.
- $g_1(\phi), g_2(\phi) \in L^2(\Gamma(\phi))$, $\int_{\Gamma(\phi)} g_k(\phi, \mathbf{x}) \, d\Gamma = 0$.

Neumann boundary value problem

$\varphi_1(\phi), \varphi_2(\phi) \in H_N^1(\Omega(\phi))$ are the weak solutions of:

$$-\Delta\varphi_k(\phi) = 0 \text{ in } \Omega(\phi), \quad \partial_n\varphi_k(\phi) = g_k(\phi) \text{ on } \Gamma(\phi).$$

What is the regularity of:

$$J(\phi) = \int_{\Omega(\phi)} \nabla\varphi_1(\phi, x) \cdot \nabla\varphi_2(\phi, x) \, dx \quad ?$$

Main result

Theorem

Assume that

- 1 Ω is an open, connected, Lipschitz continuous set.
- 2 The application $\phi \in \mathcal{D} \mapsto g_i(\phi, \Phi(\cdot)) \in L^2(\Gamma)$ ($i = 1, 2$) is of class C^k ($1 \leq k \leq +\infty$) nearby $\phi = 0$.

Then, the application:

$$\phi \in \mathcal{D} \mapsto J(\phi) \in \mathbb{R},$$

is also of class C^k nearby $\phi = 0$.

The analytic case can be treated as well:

Theorem

If, in the preceding Theorem, the functions $\phi \in \mathcal{D} \mapsto g_i(\phi, \Phi(\cdot)) \in L^2(\Gamma)$ ($i = 1, 2$) are analytic, then also is: $\phi \in \mathcal{D} \mapsto J(\phi) \in \mathbb{R}$.

Sketch of the proof

Change of variables

We make the change of variables in the variational formulation:

$$\int_{\Omega(\phi)} \nabla \varphi_1(\phi, \mathbf{x}) \cdot \nabla v(\mathbf{x}) \, dx = \int_{\Gamma(\phi)} g_1(\phi, \mathbf{x}) v(\mathbf{x}) \, d\Gamma, \quad \forall v \in H_N^1(\Omega(\phi)),$$

to work on a fixed domain:

$$\int_{\Omega} \nabla \tilde{\varphi}_1(\phi, \mathbf{x}) [A(\phi)] \nabla \tilde{v}^T(\mathbf{x}) \, dx = \int_{\Gamma} \tilde{g}_1(\phi, \mathbf{x}) \tilde{v}(\mathbf{x}) \text{Jac}_{\Gamma}(\phi, \mathbf{x}) \, d\Gamma,$$

$$\forall \tilde{v} \in H_N^1(\Omega).$$

In this expression, we have set:

- $[A(\phi)] = [D\Phi][D\Phi]^T / |\det([D\Phi])|$
- $\tilde{\varphi}_1(\phi, \cdot) = \varphi_1(\phi, \Phi(\cdot))$ and $\tilde{v} = v \circ \Phi$.
- $\text{Jac}_{\Gamma}(\phi, \mathbf{x}) = |[D\Phi]^{-1}n| |\det[D\Phi]|$ (tangential Jacobian).
- $\tilde{g}_1 = g_1 \circ \Phi$.

Sketch of the proof

Implicit functions theorem

Let us introduce the function:

$$\begin{aligned} \mathcal{A} : \mathcal{D} \times H_N^1(\Omega) &\rightarrow H_N^1(\Omega)^* \\ (\phi, \tilde{\varphi}) &\mapsto \mathcal{A}(\phi, \tilde{\varphi}), \end{aligned}$$

where we have defined:

$$\langle \mathcal{A}(\phi, \tilde{\varphi}), \varphi \rangle = \int_{\Omega} \nabla \tilde{\varphi}(x) \cdot [A(\phi)] \nabla \varphi(x) \, dx - \int_{\Gamma} \tilde{g}_1(\phi, x) \varphi(x) \text{Jac}_{\Gamma}(\phi, x) \, d\Gamma.$$

The implicit functions theorem applies to \mathcal{A} at the point $(0, \varphi_1)$: there exists a neighborhood \mathcal{D}_0 of 0 in \mathcal{D} and a function $\phi \in \mathcal{D}_0 \mapsto \tilde{\varphi}(\phi) \in H_N^1(\Omega)$ of class C^k such that $\tilde{\varphi}(0) = \varphi_1$ and $\mathcal{A}(\phi, \tilde{\varphi}(\phi)) = 0$ for all $\phi \in \mathcal{D}_0$.

The conclusion arises after rewriting J as:

$$J(\phi) = \int_{\Omega} \nabla \tilde{\varphi}_1(\phi, x) [A(\phi)] \nabla \tilde{\varphi}_2^T(\phi, x) \, dx.$$

Existence and uniqueness of a solution

Let us go back to the swimming fish problem:

Theorem (A. M., 2007)

Assume that Ω is a Lipschitz continuous open set and that $t \in [0, \infty) \mapsto \phi(t) \in \mathcal{D}$ is of class C^k ($k \geq 2$) (respect. analytic) then the equations of motion that govern the motion of the fish are well-posed (all of the terms make sense). Cauchy-Lipschitz Theorem applies: for any initial data $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$, there exists a unique solution of class C^k (respect. analytic), local in time.

With energy estimates, we show that the solution is in fact global in time, up to an hypohetic collision (between two bodies or with the fluid boundary).




Corollary

The free motion of solids in a fluid is analytic.


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Collisions are not always possible

In a viscous fluid, collisions are (theoretically) not possible:

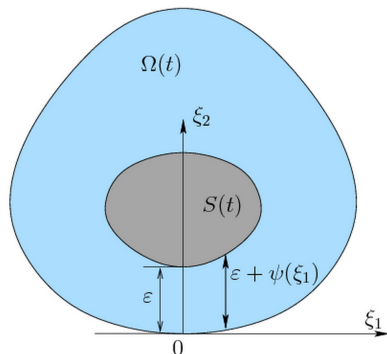
-  E. Zuazua et J.L. Vásquez, *Lack of collision in a simplified 1D model for fluid-solid interaction*, Arch. Ration. Mech. Anal., 1999.
-  Hesla, T. I., *Collisions of smooth bodies in viscous fluids, a mathematical investigation*, Phd thesis, University of Minnesota, 2005.
-  *Lack of collision between solid bodies in a 2D incompressible viscous flow*, Comm. Partial Differential Equations, 2007.

This result is no longer true in a perfect fluid:

-  J. Houot et A. M., *On the motion and collisions of rigid bodies in an ideal fluid*, Asymptot. Anal., 2007.

Collisions in a perfect fluid

Consider the symmetric configuration on the figure below:



Theorem (A. M., 2007)

If $\partial\Omega(t)$ is of class C^2 and:

$$\lim_{\xi_1 \rightarrow 0} \frac{|\psi(\xi_1)|}{|\xi_1|^3} = +\infty. \quad (\text{H})$$

Then, for any initial position $\mathbf{q}_0 = (0, h_2, 0)$ ($h_2 > 0$) and any initial velocity $\dot{\mathbf{q}}_0 = (0, -\dot{h}_2, 0)$, the solid does collide the boundary in finite time and with non zero relative velocity.

Conjecture

Collisions are not possible any more when (H) does not hold.

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Numerical simulations (with B. Pinçon, 2008)

The Neumann boundary value problem set in Ω (bounded or exterior domain) with smooth boundary Γ :

$$-\Delta u = 0 \text{ in } \Omega, \quad \partial_n u = g \text{ on } \Gamma,$$

can be reformulated under the form of an integral equation:

- For all $y \in \Omega$:

$$u(y) - \frac{1}{2\pi} \int_{\Gamma} \frac{(x-y)}{|x-y|^2} \cdot n(x) u(x) d\Gamma = -\frac{1}{2\pi} \int_{\Gamma} \ln|x-y| g(x) d\Gamma.$$

- Pour tout $y \in \Gamma$:

$$u(y) - \frac{1}{\pi} \int_{\Gamma} \frac{(x-y)}{|x-y|^2} \cdot n(x) u(x) d\Gamma = -\frac{1}{\pi} \int_{\Gamma} \ln|x-y| g(x) d\Gamma.$$

We use the latter formula to compute the potentials on Γ .

Regularity of the kernels

We introduce $t \in [0, 2\pi[\mapsto \gamma(t) \in \Gamma$ a parameterization of the boundary and the kernels:

$$K(t, s) = \frac{(\gamma(s) - \gamma(t))}{|\gamma(s) - \gamma(t)|^2} \cdot n(\gamma(s)) |\gamma'(s)|,$$

$$R(t, s) = \ln |\gamma(s) - \gamma(t)|.$$

Observe that:

- The kernel K is not singular at $t = s$ (it is equal to half the curvature of the boundary).
- The kernel R has an integrable singularity at $t = s$.

To compute:

$$\int_{\Gamma} \ln |x - y| g(x) d\Gamma = \int_0^{2\pi} R(t, s) g(\gamma(s)) |\gamma'(s)| ds,$$

we use **Nyström's method**.

Nyström's method (1/2)

Nyström's method allows to compute integrals having the form:

$$\int_0^{2\pi} R(t, s)\varphi(s)ds,$$

where φ is a smooth function. The kernel is first decomposed into:

$$\ln |\gamma(t) - \gamma(s)| = \ln \left| 2e^{-1/2} \sin \left(\frac{t-s}{2} \right) \right| + B(t, s),$$

where we have set:

$$B(t, s) = \begin{cases} \ln \frac{|e^{1/2}(\gamma(t) - \gamma(s))|}{|2 \sin((t-s)/2)|}, & \text{si } t \neq s, \\ \ln |e^{1/2}\gamma'(t)| & \text{si } t = s, \end{cases}$$

which is **not singular** at $t = s$: the integration of this kernel on Γ is easy.

Nyström's method (2/2)

The integration of the singular kernel is done by means of the formula:

$$-\frac{1}{\pi} \int_0^{2\pi} \varphi(s) \ln \left| 2e^{-1/2} \sin \left(\frac{t-s}{2} \right) \right| ds = \frac{1}{\sqrt{2\pi}} \left[\widehat{\varphi}(0) + \sum_{|m|>0} \frac{\widehat{\varphi}(m)}{|m|} e^{imt} \right],$$

where $\widehat{\varphi}(m)$ are the Fourier coefficients of φ . This quantity is computed by FFT.

Numerical simulation: scheme

Advantages of the method (Lagrangian formalism+Integral equations)

- Potential functions are computed on the bodies' boundaries only.
- No fluid meshing.
- The pressure is never computed.
- Computations are very fast.

