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Representation theory of $\mathscr{W}$-algebras

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Abstract. We study the representation theory of the $\mathscr{W}$-algebra $\mathscr{W}_k(\bar{\mathfrak{g}})$ associated with a simple Lie algebra $\bar{\mathfrak{g}}$ at level k. We show that the "$-$" reduction functor is exact and sends an irreducible module to zero or an irreducible module at any level $k \in \mathbb{C}$. Moreover, we show that the character of each irreducible highest weight representation of $\mathscr{W}_k(\bar{\mathfrak{g}})$ is completely determined by that of the corresponding irreducible highest weight representation of affine Lie algebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$. As a consequence we complete (for the "$-$" reduction) the proof of the conjecture of E. Frenkel, V. Kac and M. Wakimoto on the existence and the construction of the modular invariant representations of $\mathscr{W}$-algebras.

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1. Introduction

This paper is a continuation of the author's earlier work [2] in which we proved (completely for the "$-$" reduction and partially for the "$+$" reduction) the vanishing conjecture of E. Frenkel, V. Kac and M. Wakimoto [27].

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Let $\bar{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra, $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ its triangular decomposition, $\mathfrak{g}$ the Kac–Moody affinization of $\bar{\mathfrak{g}}$, k the level of $\mathfrak{g}$. Then the corresponding *$\mathscr{W}$-algebra* $\mathscr{W}_k(\bar{\mathfrak{g}})$ ([16,17,21, 50,51,59]), which is a Möbius conformal vertex algebra, can be constructed by the method of B. Feigin and E. Frenkel [21] (see Sect. 4). It is known [21] that for $k \neq -h^\vee$ the vertex algebra $\mathscr{W}_k(\bar{\mathfrak{g}})$ is conformal (or a vertex operator algebra) of central charge $c(k)$, where

$$c(k) = \operatorname{rank} \bar{\mathfrak{g}} - 12\left(\left(k+h^\vee\right)|\bar{\rho}^\vee|^2 - 2\langle\bar{\rho}, \bar{\rho}^\vee\rangle + \frac{1}{k+h^\vee}|\bar{\rho}|^2\right),$$

while $\mathscr{W}_{-h^\vee}(\bar{\mathfrak{g}})$ is the center of the universal affine vertex algebra $V_{-h^\vee}(\bar{\mathfrak{g}})$ (see Sect. 4.3) associated with $\bar{\mathfrak{g}}$ at level $-h^\vee$. Here $\bar{\rho}$ is the half sum of positive roots of $\bar{\mathfrak{g}}$, $\bar{\rho}^\vee$ is the half sum of positive coroots of $\bar{\mathfrak{g}}$ and $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$.

The simplest $\mathscr{W}$-algebra $\mathscr{W}_k(\mathfrak{sl}_2)$ is the Virasoro vertex algebra, whose representation theory is well-studied ([22,23,31,35]). However, $\mathscr{W}$-algebras for other simple Lie algebras are not generated by Lie algebras, and hence one cannot apply the Lie algebra theory for $\mathscr{W}$-algebras directly. As a consequence, surprisingly little is known about their representation theory (see e.g. [9]).

Let $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ be the *Zhu algebra* [29,60] of $\mathscr{W}_k(\bar{\mathfrak{g}})$ (see Sect. 3.12). Then, by construction [21], $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ is isomorphic to the center $\mathcal{Z}(\bar{\mathfrak{g}})$ of the universal enveloping algebra $U(\bar{\mathfrak{g}})$ of $\bar{\mathfrak{g}}$ (Theorem 4.16.3 (ii)). Therefore, by Zhu's theorem [60], the irreducible highest weight representations of $\mathscr{W}_k(\bar{\mathfrak{g}})$ are parameterized[1] by the central characters of $\mathcal{Z}(\bar{\mathfrak{g}}) \cong S(\bar{\mathfrak{h}})^{\bar{W}}$. Let $\gamma_{\bar{\lambda}} : \mathcal{Z}(\bar{\mathfrak{g}}) \to \mathbb{C}$ be the evaluation at the Verma module $\bar{M}(\bar{\lambda})$ of $\bar{\mathfrak{g}}$ with highest weight $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Denote by $\mathbf{L}(\gamma_{\bar{\lambda}})$, with $\bar{\lambda} \in -\bar{\rho} + \bar{W}\backslash\bar{\mathfrak{h}}^*$, be the corresponding irreducible highest weight representation of $\mathscr{W}_k(\bar{\mathfrak{g}})$. Here $\bar{W}$ is the Weyl group of $\bar{\mathfrak{g}}$. Then $\mathbf{L}(\gamma_{\bar{\lambda}})$ is the simple quotient of the Verma module $\mathbf{M}(\gamma_{\bar{\lambda}})$ with highest weight $\gamma_{\bar{\lambda}}$ (see Sect. 5). As in the case of Lie algebras, one has a simple character formula for $\mathbf{M}(\gamma_{\bar{\lambda}})$ (see Proposition 5.6.6).

One of the aim of this paper is to find the character formula of each irreducible highest weight representation $\mathbf{L}(\gamma_{\bar{\lambda}})$, that is, to determine the integer $m_{\bar{\lambda},\bar{\mu}}$ in the expression $\operatorname{ch}\mathbf{L}(\gamma_{\bar{\lambda}}) = \sum_{\bar{\mu}} m_{\bar{\lambda}\bar{\mu}} \operatorname{ch}\mathbf{M}(\gamma_{\bar{\mu}})$.

To this end we study the method of quantum reduction [21,27], in particular the "$-$" reduction functor [27], which is the modified version of the reduction functor originally introduced in [21] (the original reduction functor is referred to as the "$+$" reduction functor): Fix the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ of the affine Lie algebra $\mathfrak{g}$ in the standard manner. Then, one has the corresponding Bernstein–Gelfand–Gelfand category $\mathcal{O}_k$ of $\mathfrak{g}$ at level k (see Sect. 6.1). Let $\mathfrak{h}_k^* \subset \mathfrak{h}^*$ be the set of weights of level k,

[1] To be precise, in the case that the level k is critical, "irreducible highest weight representations" should be replaced by "graded irreducible highest weight representations", see Sect. 4.17.

$M(\lambda) \in \mathcal{O}_k$ the Verma module of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}_k^*$, $L(\lambda) \in \mathcal{O}_k$ the unique simple quotient of $M(\lambda)$. Denote by $H_-^\bullet(M)$ the cohomology with coefficient in $M \in \mathcal{O}_k$ associated with the quantized Drinfeld–Sokolov "−" reduction [21,27] (see Sect. 6.5). Then the correspondence

$$M \longmapsto H_-^0(M)$$

defines a functor from $\mathcal{O}_k$ to the category of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules[2].

Now recall that if $\lambda \in \mathfrak{h}^*$ is admissible [38] then $L(\lambda)$ is called an *Kac–Wakimoto admissible representation*. Conjecturally, Kac–Wakimoto admissible representations exhaust all modular invariant representations of $\mathfrak{g}$. In [27], E. Frenkel, V. Kac and M. Wakimoto conjectured that, for an admissible weight $\lambda \in \mathfrak{h}^*$, one has $H_-^i(L(\lambda)) = 0$ with $i \neq 0$ and $H_-^0(L(\lambda))$ is an irreducible $\mathscr{W}_k(\bar{\mathfrak{g}})$-module if λ is non-degenerate [39]. (They showed that $H_-^0(L(\lambda))$ is zero if λ is a degenerate admissible weight.[3]) The remarkable consequence of this conjecture is that the existence and the construction of modular invariant representations of $\mathscr{W}$-algebras. Namely, one can obtain (non-trivial) modular invariant representations of $\mathscr{W}$-algebras as the image of non-degenerate Kac–Wakimoto (principal) admissible representations of $\mathfrak{g}$ by the functor $H_-^0(?)$. In the case that $\bar{\mathfrak{g}} = \mathfrak{sl}_2$, this conjecture is a known to be true ([27]), and the corresponding modular invariant representations are the minimal series representations [5] of the Virasoro (vertex) algebra.

Of this conjecture the part concerning the vanishing of cohomology is proved in [2]. Therefore, it is remains to prove the irreducibility. It turns out that the method of quantum reduction produces not only modular invariant representations but also *all* the highest weight irreducible representations of $\mathscr{W}$-algebras, determining their characters.

Let $\mathfrak{h}^* \ni \lambda \mapsto \bar{\lambda} \in \bar{\mathfrak{h}}^*$ be the restriction. We refer to $\bar{\lambda}$ as the *classical part* of λ. The weight $\bar{\lambda} \in \bar{\mathfrak{h}}^*$ is called *anti-dominant* if $\bar{M}(\bar{\lambda})$ is irreducible over $\bar{\mathfrak{g}}$. One knows that this condition is equivalent to

(1)
$$\langle \bar{\lambda} + \bar{\rho}, \bar{\alpha}^\vee \rangle \notin \{1, 2, \dots\} \text{ for each element } \bar{\alpha} \text{ of positive roots } \bar{\Delta}_+ \text{ of } \bar{\mathfrak{g}}.$$

It is clear that any central character of $\mathcal{Z}(\bar{\mathfrak{g}})$ is of the form $\gamma_{\bar{\lambda}}$ for some anti-dominant $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Note that if $\lambda \in \mathfrak{h}^*$ an admissible weight then by definition it is non-degenerate if and only if its classical part $\bar{\lambda}$ is anti-dominant.[4]

Let $D(\mathbf{L}(\gamma_{\bar{\lambda}}))$ be the restricted dual of $\mathbf{L}(\gamma_{\bar{\lambda}})$ in the sense of [26]. Then

$$D(\mathbf{L}(\gamma_{\bar{\lambda}})) \cong \mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$$

as $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules, where w_0 is the longest element of $\bar{W}$ (Theorem 5.5.4 (i)).

[2] The action of $\mathscr{W}_k(\bar{\mathfrak{g}})$ on $H_-^\bullet(M)$ is slightly changed, see Sect. 6.4.

[3] A dominant integral weights of $\mathfrak{g}$ is a degenerate admissible weight. Therefore, an integrable representation of $\mathfrak{g}$ goes to zero by the functor $H_-^0(?)$.

[4] But a non-degenerate admissible weight λ is by no means "anti-dominant" as a weight of affine Lie algebra $\mathfrak{g}$, see [27,39]. (Indeed it is "regular dominant" so that the Weyl–Kac type character formula holds for $L(\lambda)$ ([37]).)

Main Theorem 1. *Let k be an arbitrary complex number.*

(i) (Theorem 7.6.1) *For any object M of $\mathcal{O}_k$, the cohomology $H^i_-(M)$ is zero unless $i = 0$.*
(ii) (Theorem 7.6.3) *For $\lambda \in \mathfrak{h}^*_k$, there is an isomorphism*

$$H^0_-(L(\lambda)) \cong \begin{cases} D(\mathbf{L}(\gamma_{\bar{\lambda}})) & \text{if the classical part } \bar{\lambda} \text{ is anti-dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

By Main Theorem 1 (1), which generalizes the result obtained in [2], it follows that the correspondence $M \mapsto H^0_-(M)$ defines an exact functor from $\mathcal{O}_k$ to the category of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules at any $k \in \mathbb{C}$. Also, it holds that $H^0_-(M(\lambda)) \cong \mathbf{M}(\gamma_{-w_0(\bar{\lambda})})$ (Theorem 7.5.1). Therefore, if we choose λ so that its classical part $\bar{\lambda}$ is anti-dominant, then we have

$$\operatorname{ch} \mathbf{L}(\gamma_{\bar{\lambda}}) = \sum_{\mu} [L(\lambda) : M(\mu)] \operatorname{ch} \mathbf{M}(\gamma_{\bar{\mu}})$$

(Theorem 7.7.1). Here $[L(\lambda) : M(\mu)] \in \mathbb{Z}$ is determined by the formula $\operatorname{ch} L(\lambda) = \sum_{\mu} [L(\lambda) : M(\mu)] \operatorname{ch} M(\mu)$. This integer $[L(\lambda) : M(\mu)]$ is known[5] and written in terms of Kazhdan–Lusztig polynomials [13,41–44]. Therefore the above formula gives the character of each irreducible highest weight representation $\mathbf{L}(\gamma_{\bar{\lambda}})$.

It is clear that the above explained conjecture of E. Frenkel, V. Kac and M. Wakimoto [27] follows immediately from Main Theorem 1 (Corollary 7.6.4).

Next we describe our results for the "+" reduction, through which $\mathscr{W}$-algebras themselves are defined. In [27], E. Frenkel, V. Kac and M. Wakimoto gave the similar conjecture also for the "+" reduction. But the "+" reduction is in general not as simple as the "−" reduction (if we consider the whole category $\mathcal{O}_k$). This is basically because the "+" reduction is not compatible with the degree operator $\mathbf{D}$ of $\mathfrak{g}$ as opposed to "−" reduction (however see [3] for the case that $\bar{\mathfrak{g}} = \mathfrak{sl}_2$). Nevertheless we have the following result for this case: Let $H^i_+(V)$ be the cohomology associated to the quantized Drinfeld–Sokolov "+" reduction ([21,25], see Sect. 6.4). Denote by $\mathcal{O}^{[\lambda]}_k$ the block of $\mathcal{O}_k$ corresponding to a weight $\lambda \in \mathfrak{h}^*_k$.

Main Theorem 2. *Suppose that $\lambda \in \mathfrak{h}^*$ is non-critical and satisfies the following condition:*

$$\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z} \quad \text{for all } \alpha \in \{-\bar{\alpha} + n\delta; \bar{\alpha} \in \bar{\Delta}_+, 1 \leq n \leq \operatorname{ht} \bar{\alpha}\},$$

where $\operatorname{ht} \bar{\alpha}$ *is the height of* $\bar{\alpha}$.

[5] The character formula of $L(\lambda)$ is not known if the level of λ is critical. However, since $\mathscr{W}_{-h^\vee}(\bar{\mathfrak{g}})$ is a commutative vertex algebra [21], in this case one knows that each $\mathbf{L}(\gamma_{\bar{\lambda}})$ is one-dimensional, see Sect. 4.17.

(i) ([2]) *Let M be any object of $\mathcal{O}_k^{[\lambda]}$. Then the cohomology $H_+^i(M)$ is zero for all $i \neq 0$.*
(ii) (Theorem 9.1.4) *There is an isomorphism $H_+^0(L(\lambda)) \cong \mathbf{L}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$.*

Let k be a rational number such that there exists a non-degenerated principal admissible weight [38] of level k. Then the condition of Main Theorem 2 is satisfied for $\lambda = k\Lambda_0$ and all the minimal series representations of $\mathscr{W}_k(\bar{\mathfrak{g}})$ at level k can be obtained as the image of the functor $H_+^0(?)$ (see Remark 9.1.8). This proves the conjecture of E. Frenkel, V. Kac and M. Wakimoto for the "+" reduction partially (Corollary 9.1.7).

This paper is organized as follows. In Sect. 2, we collect some fundamental results concerning the quantum reduction for finite-dimensional Lie algebras [45,46,55]. We remark that the finite-dimensional version of the functor $H_-^0(?)$ is identical to Soergel's functor [58] ([4], see Theorem 2.6.1 and Remark 6.5.1). In Sect. 3, we prepare some general theory about filtration of vertex superalgebras and BRST cohomology. In particular the notion of strict filtration of vertex algebras is introduced (Sect. 3.8). The definition of the current algebra [54] (= the universal enveloping algebra in the sense of I. Frenkel and Y. Zhu [29]) and the Zhu algebra of a vertex algebra is also recalled. Some technical notion needed for this is summarized in Appendix A. In Sect. 4, we recall the definition of $\mathscr{W}$-algebras and collect necessary information about its structure, following [25]. In particular we define an important strict filtration of $\mathscr{W}_k(\bar{\mathfrak{g}})$ arising from the argument of [25] (Sect. 4.11). We also describe the current algebra and the Zhu algebra of $\mathscr{W}_k(\bar{\mathfrak{g}})$ (Theorems 4.14.1, 4.14.2 and 4.16.3). Theorem 4.16.3 should be compared with Kostant's Theorem [45] (Theorem 2.2.1) through Kostant–Sternberg's Theorem [46] (Theorem 2.4.2). In Sect. 5, we define the Verma module $\mathbf{M}(\gamma_{\bar{\lambda}})$ of $\mathscr{W}_k(\bar{\mathfrak{g}})$ with highest weight $\gamma_{\bar{\lambda}}$ and its simple quotient $\mathbf{L}(\gamma_{\bar{\lambda}})$. The duality structure of modules over $\mathscr{W}$-algebras is also discussed (Theorem 5.5.4). In Sect. 6, we recall the definition of the category $\mathcal{O}_k$ and two reduction functors $H_+^\bullet(?)$ and $H_-^\bullet(?)$. We also recall some of the structure of the category $\mathcal{O}_k$ for the later purpose. Sect. 7 is the main part of this paper: we study the representation theory of $\mathscr{W}$-algebras through the "−" reduction functor $H_-^0(?)$. Sect. 8 is devoted to the proof of Theorem 7.5.3, which is the most crucial part in the proof of Main Theorem 1. In Sect. 9 we prove Main Theorem 2.

In view of [14,36,40], the $\mathscr{W}$-algebras $\mathscr{W}_k(\bar{\mathfrak{g}})$ considered in this article are the $\mathscr{W}$-algebras associated with principal nilpotent orbits of $\bar{\mathfrak{g}}$. There has been rapidly growing interest in $\mathscr{W}$-algebras associated with other nilpotent orbits (see e.g. [12,30,57]).[6] The method developed in the present paper works for general $\mathscr{W}$-algebras as well (cf. [3]). The detail will appear in our forthcoming papers.

[6] While the present article was being refereed, the paper [De Sole, A., Kac, V., *Finite vs. affine W-algebras*, Japanese J. Math. 1, No. 1, April, 2006, 137–261] appeared, which studies the relationship between affine $\mathscr{W}$-algebras and finite $\mathscr{W}$-algebras in full generality.

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Notation. Throughout this paper the ground field is the complex number $\mathbb{C}$ and tensor products and dimensions are always meant to be as vector spaces over $\mathbb{C}$ if not otherwise stated.

2. Quantum reduction for finite-dimensional simple Lie algebras

2.1. The setting. Let $\bar{\mathfrak{g}}$ be a finite dimensional complex simple Lie algebra, $l = \operatorname{rank} \bar{\mathfrak{g}}$. Let $(\cdot|\cdot)$ the normalized invariant inner product of $\bar{\mathfrak{g}}$, that is, $(\cdot|\cdot) = \frac{1}{2h^\vee} \times$Killing form where $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$.

For $x \in \bar{\mathfrak{g}}$ we write $\bar{\mathfrak{g}}^x$ for the centralizer of x in $\bar{\mathfrak{g}}$.

Let e be a principal nilpotent element of $\bar{\mathfrak{g}}$, so that $\dim \bar{\mathfrak{g}}^e = l$. One knows that $\bar{\mathfrak{g}}^e$ is commutative. By the Jacobson–Morozov theorem, there exists an $\mathfrak{sl}_2$-triple $\{e, h_0, f\}$ associated e, that is,

$$[h_0, e] = 2e, \quad [h_0, f] = -2f, \quad [e, f] = h_0.$$

Set

$$\bar{\mathfrak{g}}_j := \{x \in \bar{\mathfrak{g}}; [h_0, x] = 2jx\} \quad \text{for } \in \mathbb{Z}. \tag{2}$$

This gives a triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$, where

$$\bar{\mathfrak{h}} = \bar{\mathfrak{g}}_0, \quad \bar{\mathfrak{n}}_+ = \bigoplus_{j>0} \bar{\mathfrak{g}}_j, \quad \bar{\mathfrak{n}}_- = \bigoplus_{j<0} \bar{\mathfrak{g}}_j. \tag{3}$$

We often identify the Cartan subalgebra $\bar{\mathfrak{h}}$ with its dual $\bar{\mathfrak{h}}^*$ using the form $(\cdot|\cdot)$.

Let $\bar{\mathfrak{b}}_- = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}}$. Then there is an exact sequence

$$0 \longrightarrow \bar{\mathfrak{g}}^f \hookrightarrow \bar{\mathfrak{b}}_- \xrightarrow{\operatorname{ad} f} \bar{\mathfrak{n}}_- \longrightarrow 0. \tag{4}$$

Let $\bar{\Delta}$ be the set of roots of $\bar{\mathfrak{g}}$. We have the decomposition $\bar{\Delta} = \bigsqcup_{j \in \mathbb{Z}} \bar{\Delta}_j$, where $\bar{\Delta}_j = \{\alpha \in \bar{\Delta}; \langle \alpha, h_0 \rangle = 2j\}$. The set

$$\bar{\Pi} := \bar{\Delta}_1 \tag{5}$$

is a basis of $\bar{\Delta}$, $\bar{\Delta}_+ := \sqcup_{j>0}\bar{\Delta}_j$ is the set of positive roots and $\bar{\Delta}_+ := \sqcup_{j<0}\bar{\Delta}_j$ is the set of negative roots. Let $\bar{Q}$ be the roots lattice, $\bar{P}$ the weight lattice, $\bar{Q}^\vee$ the coroot lattice, $\bar{P}^\vee$ the coweight lattice. We write $\bar{W}$ for the Weyl groups of $\bar{\mathfrak{g}}$ and w_0 for the longest element of $\bar{W}$.

For $\alpha \in \bar{\Delta}_+$, the number $\langle \alpha, \bar{\rho}^\vee \rangle$ is called the *height* of α and denote by $\operatorname{ht}\alpha$. We have

$$h_0 = 2\bar{\rho}^\vee, \tag{6}$$

and so $\bar{\Delta}_j = \{\alpha \in \bar{\Delta}_+; \operatorname{ht}\alpha = j\}$ for $j > 0$.

Fix an anti-Lie algebra involution

$$\bar{\mathfrak{g}} \ni X \mapsto X^t \in \bar{\mathfrak{g}} \tag{7}$$

such that $e^t = f$, $f^t = e$ and $h^t = h$ for all $h \in \bar{\mathfrak{h}}$. Choose root vectors $\{J_\alpha\}$ such that $J_\alpha^t = J_{-\alpha}$ and $(J_\alpha, J_{-\alpha}) = 1$. Let $\{J_i; i \in \bar{I}\}$, where

$$\bar{I} = \{1, 2, \dots, l\}, \tag{8}$$

be a basis of $\bar{\mathfrak{h}}$. Then the set $\{J_a; a \in \bar{I} \sqcup \bar{\Delta}\}$ forms a basis of $\bar{\mathfrak{g}}$. Let $c_{a,b}^d$ be the structure constant with respect to this basis: $[J_a, J_b] = \sum_d c_{a,b}^d J_d$. We have $c_{\alpha,\beta}^\gamma = -c_{-\alpha,-\beta}^{-\gamma}$ for $\alpha, \beta, \gamma \in \bar{\Delta}_+$.

Let $d_1, d_2, \dots, d_l$ be the exponents of $\bar{\mathfrak{g}}$. There exists a basis $\{P_i; i \in \bar{I}\}$ of $\bar{\mathfrak{g}}^f$ such that that is,

$$P_i \in \bar{\mathfrak{g}}_{-d_i}, \quad \text{or equivalently,} \quad [\bar{\rho}^\vee, P_i] = -d_i P_i. \tag{9}$$

By (4) for $\alpha \in \bar{\Delta}_+$ there exists an element $I_{-\alpha} \in \bar{\mathfrak{b}}_-$ such that

$$[f, I_{-\alpha}] = J_{-\alpha}. \tag{10}$$

The set $\{P_i,\ I_{-\alpha}; i \in \bar{I}, \alpha \in \bar{\Delta}_+\}$ form a basis of $\bar{\mathfrak{b}}_-$.

2.2. Kostant's theorems. Define an element $\bar{\chi}_+ \in \bar{\mathfrak{n}}_+^*$ by

$$\bar{\chi}_+(x) = (f|x) \quad \text{for } x \in \bar{\mathfrak{n}}_+. \tag{11}$$

Then $\bar{\chi}_+$ is a character of $\bar{\mathfrak{n}}_+$, that is, $\bar{\chi}_+([\bar{\mathfrak{n}}_+, \bar{\mathfrak{n}}_+]) = 0$. The character χ_+ is non-degenerate in the sense of [45]: $\bar{\chi}_+(J_\alpha) \neq 0$ for all $\alpha \in \bar{\Pi}$. Set

$$\bar{\chi}_+^* = -\bar{\chi}_+.$$

Then $\bar{\chi}_+^*$ also defines a non-degenerate character of $\bar{\mathfrak{n}}_+$. Let $\ker \bar{\chi}_+^* = \ker(\bar{\chi}_+^* : U(\bar{\mathfrak{n}}_+) \to \mathbb{C})$ and set

$$\mathbb{C}_{\bar{\chi}_+^*} = U(\bar{\mathfrak{n}}_+)/\ker \bar{\chi}_+^*. \tag{12}$$

This defines a one-dimensional representation of $\bar{\mathfrak{n}}_+$.

Let M be a $\bar{\mathfrak{g}}$-module. Define

$$\mathrm{Wh}(M) := \{m \in M;\, xm = \bar{\chi}_+^*(x)m \ \text{ for all } x \in \bar{\mathfrak{n}}_+\}, \tag{13}$$

$$\mathrm{Wh}^{\mathrm{gen}}(M) := \left\{m \in M;\, (\ker \bar{\chi}_+^*)^r m = 0 \ \text{ for } r \gg 0\right\}. \tag{14}$$

Then $\mathrm{Wh}(M) \subset \mathrm{Wh}^{\mathrm{gen}}(M)$ and $\mathrm{Wh}^{\mathrm{gen}}(M)$ is a $\bar{\mathfrak{g}}$-submodule of M. An element of $\mathrm{Wh}(M)$ is called a *Whittaker vector*.

Let $H^\bullet(\bar{\mathfrak{n}}_+, M)$ be the Lie algebra cohomology of $\bar{\mathfrak{n}}_+$ with coefficient in a $\bar{\mathfrak{n}}_+$-module M. By definition we have

$$\mathrm{Wh}(M) = H^0(\bar{\mathfrak{n}}_+, M \otimes \mathbb{C}_{\bar{\chi}_+}), \tag{15}$$

where $\bar{\mathfrak{n}}_+$ acts on $V \otimes \mathbb{C}_{\bar{\chi}_+}$ by the tensor product action.

Define a $\bar{\mathfrak{g}}$-module Y by

$$\mathsf{Y} := U(\bar{\mathfrak{g}}) \otimes_{U(\bar{\mathfrak{n}}_+)} \mathbb{C}_{\bar{\chi}_+^*}. \tag{16}$$

Then $\mathsf{Y} = \mathrm{Wh}^{\mathrm{gen}}(\mathsf{Y})$. By the Frobenius reciprocity, we have an isomorphism

$$\mathrm{Wh}(M) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathsf{Y}, M) \quad \text{ for any } \bar{\mathfrak{g}}\text{-module } M.$$

In particular $\mathrm{Wh}(\mathsf{Y}) \cong \mathrm{End}_{\mathfrak{g}}(\mathsf{Y})$. The space $\mathrm{Wh}(\mathsf{Y})$ is considered as a $\mathbb{C}$-algebra through the identification

$$\mathrm{Wh}(\mathsf{Y}) = \mathrm{End}_{\mathfrak{g}}(\mathsf{Y})^{\mathrm{op}}. \tag{17}$$

Let $\mathcal{Z}(\bar{\mathfrak{g}})$ be the center of the universal enveloping algebra $U(\bar{\mathfrak{g}})$ of $\bar{\mathfrak{g}}$. The following theorem is well-known.

Theorem 2.2.1 (B. Kostant [45]). *The correspondence*

$$\begin{array}{ccc} \mathcal{Z}(\bar{\mathfrak{g}}) & \longrightarrow & \mathrm{Wh}(\mathsf{Y}) = H^0(\bar{\mathfrak{n}}_+, \mathsf{Y} \otimes \mathbb{C}_{\bar{\chi}_+}) \\ z & \longmapsto & z \otimes 1 \end{array}$$

gives an algebra isomorphism.

Let $\widetilde{\mathcal{C}}$ be the Serre full subcategory of the category of $\bar{\mathfrak{g}}$-modules consisting of objects M such that $M = \mathrm{Wh}^{\mathrm{gen}}(M)$.

Theorem 2.2.2 (B. Kostant [45], see also [57, Appendix by S. Skryabin]).

(i) *Let M be any object of $\widetilde{\mathcal{C}}$. Then $H^i(\bar{\mathfrak{n}}_+, V \otimes \mathbb{C}_{\bar{\chi}_+}) = 0$ for all $i \neq 0$.*
(ii) *The functor $M \mapsto \mathrm{Wh}(M)$ gives an equivalence of the category $\widetilde{\mathcal{C}}$ and the category of left $\mathcal{Z}(\bar{\mathfrak{g}})$-modules. The inverse functor is given by the functor $E \mapsto \mathsf{Y} \otimes_{\mathcal{Z}(\bar{\mathfrak{g}})} E$.*

Let $\mathcal{C}$ be the full subcategory of $\widetilde{\mathcal{C}}$ consisting of objects V satisfying (1) M is finitely generated over $U(\bar{\mathfrak{g}})$, and (2) M is locally finite over $\mathcal{Z}(\bar{\mathfrak{g}})$. Let $\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$ be the category of finite-dimensional $\mathcal{Z}(\bar{\mathfrak{g}})$-modules. Then, $\mathsf{Y} \otimes_{\mathcal{Z}(\bar{\mathfrak{g}})} E \in \mathcal{C}$ for $E \in \mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$. Therefore, by Theorem 2.2.2, we have an equivalence of categories

$$\mathcal{C} \cong \mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}}). \tag{18}$$

2.3. Superspaces and superalgebras. A *superspace* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We set $p(v) = \bar{j}$ if $v \in V_{\bar{j}}$. The $p(v)$ is called the *parity* of v. We use the convention that when we write $p(v)$ we are assuming the vector v is homogeneous. A vector v is called *even* if $p(v) = \bar{0}$, and *odd* if $p(v) = \bar{1}$. In this paper we write V^{even} for $V_{\bar{0}}$ and V^{odd} for $V_{\bar{1}}$. A supersubspace W of V is a $\mathbb{Z}_2$-graded subspace of V: $W = W^{\text{even}} \oplus W^{\text{odd}}$, $W^{\text{even}} = W \cap V^{\text{even}}$, $W^{\text{odd}} = W \cap V^{\text{odd}}$.

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra $A = A^{\text{even}} \oplus A^{\text{odd}}$. For $a, b \in A$ we set

$$[a, b] = ab - (-1)^{p(a)p(b)} ba.$$

Any superalgebra can be viewed as a *Lie superalgebra* with respect to this bracket.

The tensor product $A \otimes B$ of two superalgebras A and B is considered as a superalgebra by $p(a \otimes 1) = p(a)$, $p(1 \otimes b) = p(b)$,

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{p(b_1)p(a_2)} (a_1 a_2) \otimes (b_1 b_2).$$

If no confusion can arise we often omit the tensor symbol for elements, e.g. we write ab for $a \otimes b$.

2.4. The BRST construction of the center $\mathcal{Z}(\bar{\mathfrak{g}})$. The form $(\cdot|\cdot)$ of $\bar{\mathfrak{g}}$ restricts to a non-degenerate symmetric bilinear form on $\bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{n}}_+$. Let $\overline{\mathcal{C}l}$ be the associated Clifford algebra. The $\overline{\mathcal{C}l}$ may be defined as the superalgebra with

$$\text{odd generators: } \psi_\alpha \quad \text{with } \alpha \in \bar{\Delta},$$
$$\text{relations: } [\psi_\alpha, \psi_\beta] = \delta_{\alpha+\beta,0} \quad \text{with } \alpha, \beta \in \bar{\Delta}.$$

Here ψ_α is regarded as the element of $\overline{\mathcal{C}l}$ corresponding to the root vector $J_\alpha \in \bar{\mathfrak{g}}$. The algebra $\overline{\mathcal{C}l}$ contains the Grassmann algebra $\Lambda(\bar{\mathfrak{n}}_\pm)$ of $\bar{\mathfrak{n}}_\pm$ as its subalgebra:

$$\Lambda(\bar{\mathfrak{n}}_\pm) = \langle \psi_\alpha; \alpha \in \bar{\Delta}_\pm \rangle \subset \overline{\mathcal{C}l}. \tag{19}$$

There is an isomorphism of linear spaces

$$\overline{\mathcal{C}l} \cong \Lambda(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_-).$$

The space $\Lambda(\bar{\mathfrak{n}}_+)$ and $\Lambda(\bar{\mathfrak{n}}_-)$ can be also considered as $\overline{\mathcal{C}l}$-modules through the identification

$$\Lambda(\bar{\mathfrak{n}}_+) = \overline{\mathcal{C}l}/(\overline{\mathcal{C}l} \cdot \bar{\mathfrak{n}}_-), \quad \Lambda(\bar{\mathfrak{n}}_-) = \overline{\mathcal{C}l}/(\overline{\mathcal{C}l} \cdot \bar{\mathfrak{n}}_+).$$

These are irreducible $\overline{\mathcal{C}l}$-modules.

Define a superalgebra $\bar{C}(\bar{\mathfrak{g}})$ by

$$\bar{C}(\bar{\mathfrak{g}}) := U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}, \tag{20}$$

where $U(\bar{\mathfrak{g}})$ is considered as a purely even superalgebra. Let $\bar{Q}_+ \in \bar{C}(\bar{\mathfrak{g}})$ be an odd element defined by

$$\bar{Q}_+ = \bar{Q}_+^{\mathrm{st}} + \bar{\chi}_+, \tag{21}$$

$$\text{where} \quad \bar{Q}_+^{\mathrm{st}} := \sum_{\alpha \in \bar{\Delta}_+} J_\alpha \psi_{-\alpha} - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \bar{\Delta}_+} c_{\alpha,\beta}^{\gamma} \psi_{-\alpha} \psi_{-\beta} \psi_\gamma . \tag{22}$$

Here the character $\bar{\chi} \in \bar{\mathfrak{n}}_+^*$ is considered as an element of $\overline{\mathcal{C}l}$, that is,

$$\bar{\chi}_+ = \sum_{\alpha \in \bar{\Pi}} \bar{\chi}_+(J_\alpha) \psi_{-\alpha}.$$

The following assertion can be checked by a direct calculation.

Proposition 2.4.1. *We have*

$$\left[\bar{Q}_+^{\mathrm{st}}, \bar{Q}_+^{\mathrm{st}}\right] = [\bar{\chi}_+, \bar{\chi}_+] = \left[\bar{Q}_+^{\mathrm{st}}, \bar{\chi}_+\right] = 0 \quad \textit{in } \bar{C}(\bar{\mathfrak{g}}).$$

In particular, $(\bar{Q}_+^{\mathrm{st}})^2 = \bar{\chi}_+^2 = 0$ *and* $\bar{Q}_+^2 = 0$.

By Proposition 2.4.1 we have

$$(\operatorname{ad} \bar{Q}_+)^2 = 0 \quad \text{in } \operatorname{End} \bar{C}(\bar{\mathfrak{g}}), \tag{23}$$

where $\operatorname{ad} \bar{Q}_+(c) = [\bar{Q}_+, c]$. This follows from the relation

$$[\bar{Q}_+, [\bar{Q}_+, c]] = [[\bar{Q}_+, \bar{Q}_+], c] - [\bar{Q}_+, [\bar{Q}_+, c]] \quad \text{for } c \in \bar{C}(\bar{\mathfrak{g}}). \tag{24}$$

Define a $\mathbb{Z}$-grading

$$\bar{C}(\bar{\mathfrak{g}}) = \bigoplus_{n \in \mathbb{Z}} \bar{C}^n(\bar{\mathfrak{g}}), \quad \bar{C}^m(\bar{\mathfrak{g}}) \cdot \bar{C}^n(\bar{\mathfrak{g}}) \subset \bar{C}^{m+n}(\bar{\mathfrak{g}}), \tag{25}$$

by setting

$$\deg \psi_\alpha = -1, \deg \psi_{-\alpha} = 1 \quad \text{for } \alpha \in \bar{\Delta}_+ \quad \text{and} \quad \deg u = 0 \quad \text{for } u \in U(\bar{\mathfrak{g}}). \tag{26}$$

Then $\deg \bar{Q}_+ = 1$. Thus by (23) $(\bar{C}(\bar{\mathfrak{g}}), \operatorname{ad} \bar{Q}_+)$ is a cochain complex. Let

$$H^\bullet(\bar{C}(\bar{\mathfrak{g}})) = \bigoplus_{i \in \mathbb{Z}} H^i(\bar{C}(\bar{\mathfrak{g}})) \tag{27}$$

be the corresponding cohomology space. The space $H^\bullet(\bar{C}(\bar{\mathfrak{g}}))$ inherits the $\mathbb{Z}$-graded superalgebra structure from $\bar{C}(\bar{\mathfrak{g}})$.

Theorem 2.4.2 (B. Kostant and S. Sternberg [46]).

(i) *The cohomology* $H^i(\bar{C}(\bar{\mathfrak{g}}))$ *is zero for all* $i \neq 0$.
(ii) *The correspondence*

$$\begin{array}{ccc} \mathcal{Z}(\bar{\mathfrak{g}}) & \to & H^0(\bar{C}(\bar{\mathfrak{g}})) \\ z & \mapsto & z \otimes 1 \end{array}$$

gives an algebra isomorphism.

Proof. We identify $\Lambda(\bar{\mathfrak{n}}_-)$ with $\Lambda(\bar{\mathfrak{n}}_+^*)$ via $(\cdot|\cdot)$ and set $\psi_\alpha{}^* = \psi_{-\alpha} \in \Lambda(\bar{\mathfrak{n}}_+^*)$ for $\alpha \in \bar{\Delta}_+$. For convenience put $\bar{C} = \bar{C}(\bar{\mathfrak{g}}) = U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$, $\bar{C}^n = \bar{C}^n(\bar{\mathfrak{g}})$. Then

$$\bar{C}^n = \sum_{i-j=n} U(\bar{\mathfrak{g}}) \otimes \Lambda^i(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^j(\bar{\mathfrak{n}}_+). \tag{28}$$

The following assertion can be checked by a direct calculation.

Claim 1. We have

$$\operatorname{ad} \bar{Q}_+ = \bar{d}_+ + \bar{d}_- \tag{29}$$

on $\bar{C}$, where $\bar{d}_\pm \in \operatorname{End} \bar{C}(\bar{\mathfrak{g}})$ is defined by

$$\begin{aligned} \bar{d}_+(u \otimes \omega_1 \otimes \omega_2) = &\sum_{\alpha \in \bar{\Delta}_+} ((\operatorname{ad} J_\alpha(u)) \otimes \psi_\alpha^* \omega_1 \otimes \omega_2 \\ &+ u \otimes \psi_\alpha^* \omega_1 \otimes \operatorname{ad} J_\alpha(\omega_2)) \\ &- \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \bar{\Delta}_+} c_{\alpha,\beta}^\gamma u \otimes \psi_\alpha^* \psi_\beta^* (\operatorname{ad} \psi_\gamma(\omega_1)) \otimes \omega_2, \end{aligned} \tag{30}$$

(31)

$$\begin{aligned} (-1)^i \bar{d}_-(u \otimes \omega_1 \otimes \omega_2) = &\sum_{\alpha \in \bar{\Delta}_+} u(J_\alpha + \bar{\chi}(J_\alpha)) \otimes \omega_1 \otimes \operatorname{ad} \psi_\alpha^*(\omega_2) \\ &- \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \bar{\Delta}_+} c_{\alpha,\beta}^\gamma u \otimes \omega_1 \otimes \psi_\gamma \operatorname{ad} \psi_\beta^* (\operatorname{ad} \psi_\alpha^*(\omega_2)) \end{aligned}$$

for $u \in U(\bar{\mathfrak{g}})$, $\omega_1 \in \Lambda^i(\bar{\mathfrak{n}}_+^*)$, $\omega_2 \in \Lambda(\bar{\mathfrak{n}}_+)$.

Note that

$$\begin{aligned} \bar{d}_+(U(\bar{\mathfrak{g}}) \otimes \Lambda^i(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^j(\bar{\mathfrak{n}}_+)) &\subset U(\bar{\mathfrak{g}}) \otimes \Lambda^{i+1}(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^j(\bar{\mathfrak{n}}_+), \\ \bar{d}_-(U(\bar{\mathfrak{g}}) \otimes \Lambda^i(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^j(\bar{\mathfrak{n}}_+)) &\subset U(\bar{\mathfrak{g}}) \otimes \Lambda^i(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^{j-1}(\bar{\mathfrak{n}}_+). \end{aligned} \tag{32}$$

Thus we have

$$\bar{d}_+^2 = \bar{d}_-^2 = [\bar{d}_+, \bar{d}_-] = 0. \tag{33}$$

Define

$$F^p\bar{C} := \sum_{i\geq p} U(\bar{\mathfrak{g}}) \otimes \Lambda^i(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^\bullet(\bar{\mathfrak{n}}_+) \subset \bar{C}. \tag{34}$$

Then

$$\bar{C} = F^0\bar{C} \supset F^1\bar{C} \supset \cdots \supset F^{\dim \bar{\mathfrak{n}}_+ +1}\bar{C} = 0, \tag{35}$$

$$\bar{d}_+ F^p\bar{C} \subset F^{p+1}\bar{C}, \quad \bar{d}_- F^p\bar{C} \subset F^p\bar{C}. \tag{36}$$

Hence there is a corresponding converging spectral sequence $E_r \Rightarrow H^\bullet(\bar{C})$. By definition and (31), we have

$$E_1^{p,q} = H^{p+q}(F^p\bar{C}/F^{p+1}\bar{C}, \operatorname{ad} \bar{Q}) = H^q(U(\bar{\mathfrak{g}}) \otimes \Lambda^p(\bar{\mathfrak{n}}_+^*) \otimes \Lambda^\bullet(\bar{\mathfrak{n}}_+), \bar{d}_-).$$

The formula (31) shows that the cochain complex $(\bar{C}, \bar{d}_-)$ is identical to the Chevalley complex for calculating the $\bar{\mathfrak{n}}_+$-homology

$$H_\bullet(\bar{\mathfrak{n}}_+, (U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}+}) \otimes \Lambda(\bar{\mathfrak{n}}_+^*)) = H_\bullet(\bar{\mathfrak{n}}_+, (U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}+})) \otimes \Lambda(\bar{\mathfrak{n}}_+^*)$$

(equipped with the opposite homological gradation). Here $\big(U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}_+}\big) \otimes \Lambda(\bar{\mathfrak{n}}_+^*)$ is regarded as a right $U(\bar{\mathfrak{n}}_+)$-module on which $U(\bar{\mathfrak{n}}_+)$ acts only on the first factor $U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}}$.

Obviously the right $\bar{\mathfrak{n}}_+$-module $U(\bar{\mathfrak{g}})$ is free over $\bar{\mathfrak{n}}_+$, and thus, so is $U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}+}$. Therefore

(37)

$$\begin{aligned} E_1^{p,q} &= \begin{cases} ((U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}+})/(U(\bar{\mathfrak{g}}) \otimes \mathbb{C}_{\bar{\chi}+})\bar{\mathfrak{n}}_+) \otimes \Lambda^p(\bar{\mathfrak{n}}_+^*) & \text{for } q = 0, \\ 0 & \text{for } q \neq 0 \end{cases} \\ &= \begin{cases} (U(\bar{\mathfrak{g}}) \otimes_{U(\bar{\mathfrak{n}}_+)} \mathbb{C}_{\bar{\chi}_+^*}) \otimes \Lambda^p(\bar{\mathfrak{n}}_+^*) & \text{for } q = 0, \\ 0 & \text{for } q \neq 0. \end{cases} \end{aligned}$$

Next we calculate the term E_2. From (16), (30) and (37) it follows that $E_2^{p,q} = 0$ for $q \neq 0$ and that

$$E_2^{p,0} = H^p(\bar{\mathfrak{n}}_+, \mathsf{Y} \otimes \mathbb{C}_{\bar{\chi}+}).$$

But then by Theorem 2.2.2 (i) we have $E_2^{p,q} = 0$ unless $(p,q) = (0,0)$ and our spectral sequence collapses at $E_2 = E_\infty$. This proves (i). Because the term $E_\infty^{p,q}$ lies entirely in $E_2^{0,0}$ the corresponding filtration of $H^0(\bar{C}(\bar{\mathfrak{g}}))$ is trivial: $H^0(\bar{C}(\bar{\mathfrak{g}})) \cong \mathrm{Wh}(\mathsf{Y})$. Thus Theorem 2.2.1 proves (ii). □

Let

$$\overline{\mathcal{C}l} \ni a \mapsto a^t \in \overline{\mathcal{C}l} \tag{38}$$

be the anti-superalgebra automorphism defined by $\psi_\alpha^t = \psi_{-\alpha}$ for $\alpha \in \bar{\Delta}$.

This induces the anti-superalgebra automorphism

$$\bar{C}(\bar{\mathfrak{g}}) \ni c \mapsto c^t \in \bar{C}(\bar{\mathfrak{g}}) \tag{39}$$

defined by $(u \otimes a)^t = u^t \otimes a^t$ for $u \in U(\bar{\mathfrak{g}})$ and $a \in \overline{\mathcal{C}l}$.

Set

$$\bar{Q}_-^{\mathrm{st}} := \left(\bar{Q}_+^{\mathrm{st}}\right)^t = \sum_{\alpha \in \bar{\Delta}_-} J_\alpha \psi_{-\alpha} - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \bar{\Delta}_-} c_{\alpha,\beta}^{\gamma} \psi_{-\alpha}\psi_{-\beta}\psi_\gamma, \tag{40}$$

$$\bar{\chi}_- := (\bar{\chi}_+)^t = \sum_{\alpha \in \bar{\Delta}_-} \bar{\chi}_-(J_\alpha)\psi_{-\alpha}, \tag{41}$$

$$\bar{Q}_- := (\bar{Q}_+)^t = \bar{Q}_-^{\mathrm{st}} + \bar{\chi}_-. \tag{42}$$

The element $\bar{\chi}_-$ can be considered as a character of $\bar{\mathfrak{n}}_-$ such that

$$\bar{\chi}_-(J_\alpha) = (e|J_\alpha) \quad \text{for } \alpha \in \bar{\Delta}_-. \tag{43}$$

We have $(\bar{Q}_-^{\mathrm{st}})^2 = (\bar{\chi}_-)^2 = \bar{Q}_-^2 = 0$, $(\mathrm{ad}\, \bar{Q}_-)^2 = 0$ and $\mathrm{ad}\, \bar{Q}_- \cdot \bar{C}^i(\bar{\mathfrak{g}}) \subset \bar{C}^{i-1}(\bar{\mathfrak{g}})$. Thus $(\bar{C}(\bar{\mathfrak{g}}), Q_-)$ is a chain complex. Put

$$H_\bullet(\bar{C}(\bar{\mathfrak{g}})) := H_\bullet(\bar{C}(\bar{\mathfrak{g}}), \bar{Q}_-).$$

The anti-automorphism (39) induces an isomorphism $H^i(\bar{C}(\bar{\mathfrak{g}})) \cong H_{-i}(\bar{C}(\bar{\mathfrak{g}}))$ for all $i \in \mathbb{Z}$. Therefore, by Theorem 2.4.2 we have $H_i(\bar{C}(\bar{\mathfrak{g}})) = 0$ for all $i \neq 0$ and the algebra isomorphism

$$\begin{array}{ccc} \mathcal{Z}(\bar{\mathfrak{g}}) & \xrightarrow{\sim} & H_0(\bar{C}(\bar{\mathfrak{g}})) \\ z & \mapsto & z \otimes 1 \end{array} \tag{44}$$

2.5. Whittaker functor. Let $\mathcal{O}(\bar{\mathfrak{g}})$ be the Bernstein–Gelfand–Gelfand category [6] of $\bar{\mathfrak{g}}$, that is, a full subcategory of the category of left $\bar{\mathfrak{g}}$-modules consisting of objects M such that the following hold:

- M is finitely generated over $\bar{\mathfrak{g}}$;
- $\bar{\mathfrak{h}}$ acts semisimply on M;
- $\bar{\mathfrak{n}}_+$ acts locally nilpotently on M.

Let $\bar{M}(\bar{\lambda}) \in \mathcal{O}(\bar{\mathfrak{g}})$ be the Verma module of $\bar{\mathfrak{g}}$ with highest weight $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, $\bar{L}(\bar{\lambda}) \in \mathcal{O}(\bar{\mathfrak{g}})$ its unique simple quotient. Then every simple object of $\mathcal{O}(\bar{\mathfrak{g}})$ is isomorphic to exactly one of the $\bar{L}(\bar{\lambda})$ with $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Every object of $\mathcal{O}(\bar{\mathfrak{g}})$ has finite length. It is known that $\bar{M}(\bar{\lambda}) = \bar{L}(\bar{\lambda})$ if and only if $\bar{\lambda}$ is *anti-dominant*, that is,

$$\langle \bar{\lambda} + \bar{\rho}, \bar{\alpha}^\vee \rangle \notin \{1, 2, 3, \dots\} \quad \text{for all } \bar{\alpha} \in \bar{\Delta}_+. \tag{45}$$

Let $\mathcal{O}^{\triangle}(\bar{\mathfrak{g}})$ be the full subcategory of $\mathcal{O}(\bar{\mathfrak{g}})$ consisting of objects M that admit a *Verma flag*, that is, a finite filtration $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ such that each successive subquotient M_i/M_{i+1} is isomorphic to some Verma module $\bar{M}(\bar{\lambda}_i)$ with $\bar{\lambda}_i \in \bar{\mathfrak{h}}^*$. Let $\bar{P}(\bar{\lambda})$ be the projective cover of $\bar{L}(\bar{\lambda})$. It is known that $\bar{P}(\bar{\lambda}) \in \mathcal{O}^{\triangle}(\bar{\mathfrak{g}})$ for any $\bar{\lambda} \in \bar{\mathfrak{h}}^*$.

For a $\bar{\mathfrak{g}}$-module M set

$$\bar{C}(M) := M \otimes \Lambda(\bar{\mathfrak{n}}_-). \tag{46}$$

The space $\bar{C}(M)$ can be viewed naturally as a module over the superalgebra $\bar{C}(\bar{\mathfrak{g}}) = U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$. Let $\bar{C}_i(M) = M \otimes \Lambda^i(\bar{\mathfrak{n}}_-)$, so that $\bar{C}(M) = \bigoplus_{i \geq 0} \bar{C}_i(M)$. Then,

$$\bar{Q}_- \cdot \bar{C}_i(M) \subset \bar{C}_{i-1}(M).$$

One sees that the chain complex $(\bar{C}(M), \bar{Q}_-)$ is identically the Chevalley complex for the Lie algebra homology $H_\bullet(\bar{\mathfrak{n}}_-, M \otimes \mathbb{C}_{\bar{\chi}_-})$. Put

$$\bar{H}_i(M) := H_i(\bar{\mathfrak{n}}_-, M \otimes \mathbb{C}_{\bar{\chi}_-}) = H_i(\bar{C}(M), \bar{Q}_-). \tag{47}$$

Note that the center $\mathcal{Z}(\bar{\mathfrak{g}}) = H_0(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}, \operatorname{ad} \bar{Q}_-)$ acts on $\bar{H}_i(M)$ naturally:

$$[c] \cdot [m] = [c \cdot m] \quad \text{for } c \in U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l},\ v \in \bar{C}_i(M)$$
$$\text{such that } [\bar{Q}_+, c] = \bar{Q}_+ m = 0.$$

By definition we have

$$\bar{H}_0(M) = M/(\ker \bar{\chi}_-^*)M. \tag{48}$$

Here, $\bar{\chi}_-^* = -\bar{\chi}_-$ and $\ker \bar{\chi}_-^* = \ker(\bar{\chi}_-^* : U(\bar{\mathfrak{n}}_-) \to \mathbb{C})$.

Lemma 2.5.1. *Let M be any object of $\mathcal{O}^{\triangle}(\bar{\mathfrak{g}})$.*

(i) *The homology $\bar{H}_i(M)$ is zero for all $i > 0$.*
(ii) *The space $M/(\ker \bar{\chi}_-^*)^r M$ is finite-dimensional for all $r \in \mathbb{Z}_{\geq 1}$. In particular $\bar{H}_0(M)$ is finite-dimensional.*

Proof. (i) Since an object M of $\mathcal{O}^{\triangle}(\bar{\mathfrak{g}})$ is free of finite rank over $U(\bar{\mathfrak{n}}_-)$, we have

$$M \otimes \mathbb{C}_{\bar{\chi}_-} \cong M \quad \text{as } \bar{\mathfrak{n}}_-\text{-modules}. \tag{49}$$

Hence, $\bar{H}_i(M) \cong H_i(\bar{\mathfrak{n}}_-, M)$. This gives $\bar{H}_i(M) = 0$ for $i > 0$ because M is free over $\bar{\mathfrak{n}}_-$. (ii) By (49) we have

$$M/(\ker \bar{\chi}_-^*)^r M \cong M/(\bar{\mathfrak{n}}_-)^r M. \tag{50}$$

Hence the assertion follows. □

Lemma 2.5.2. *Let M be any object of $\mathcal{O}(\bar{\mathfrak{g}})$. Then the space $M/(\ker \bar{\chi}_-^*)^r M$ is finite-dimensional for all $r \in \mathbb{Z}_{\geq 1}$. In particular $\bar{H}_0(M)$ is finite-dimensional.*

Proof. The assertion follows from Lemma 2.5.1 because $\mathcal{O}(\bar{\mathfrak{g}})$ has enough projectives. □

Let $M^\vee$ the full dual space of M:

$$M^\vee := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}).$$

We regard $M^\vee$ as a $\bar{\mathfrak{g}}$-module on which $\bar{\mathfrak{g}}$ acts by $(xf)(v) = f(x^t v)$ for $f \in M^\vee,\ v \in M$. Then by definition we have

$$\mathrm{Wh}(M^\vee) = \mathrm{Hom}_{\mathbb{C}}(M/(\ker \bar{\chi}_-^*)M, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\bar{H}_0(M), \mathbb{C}). \tag{51}$$

Hence, by Lemma 2.5.2, $\mathrm{Wh}(M^\vee)$ is finite-dimensional for any object M of $\mathcal{O}(\bar{\mathfrak{g}})$.

Remark 2.5.3. By Theorem 2.2.2 (ii) we have

$$\mathrm{Wh}^{\mathrm{gen}}(M^\vee) = \mathsf{Y} \otimes_{\mathcal{Z}(\bar{\mathfrak{g}})} \mathrm{Wh}(M^\vee)$$

and $\mathrm{Wh}^{\mathrm{gen}}(M^\vee)$ belongs to $\mathcal{C}$.

Theorem 2.5.4 (B. Kostant [45]). *The cofunctor*

$$\begin{array}{rcl} \mathcal{O}(\bar{\mathfrak{g}}) & \to & \mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}}) \\ M & \mapsto & \mathrm{Wh}(M^\vee) \end{array}$$

is exact.

Theorem 2.5.5. *The functor*

$$\begin{array}{rcl} \mathcal{O}(\bar{\mathfrak{g}}) & \to & \mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}}) \\ M & \mapsto & \bar{H}_0(M) \end{array}$$

is exact.

Proof. By (51), Lemma 2.5.2 and Theorem 2.5.4 the functor $\bar{H}_0(?)$ is exact since it is the composition of two exact functors $M \mapsto \mathrm{Wh}(M^\vee)$ and $M \mapsto \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. □

Theorem 2.5.6. *Let M be any object of $\mathcal{O}(\bar{\mathfrak{g}})$. Then the homology $\bar{H}_i(M)$ is zero for all $i \neq 0$.*

Proof. We prove the assertion by induction on $i \geq 1$. Because the category $\mathcal{O}(\bar{\mathfrak{g}})$ has enough projectives, there exists a projective object $P \in \mathcal{O}(\bar{\mathfrak{g}})$ and an exact sequence

$$0 \to N \to P \to M \to 0 \tag{52}$$

in $\mathcal{O}(\bar{\mathfrak{g}})$. Consider the corresponding long exact sequence

$$\begin{aligned} &\cdots \to \bar{H}_i(P) \to \bar{H}_i(M) \to \bar{H}_{i-1}(N) \to \cdots \\ &\cdots \to \bar{H}_1(P) \to \bar{H}_1(M) \to \bar{H}_0(N) \to \bar{H}_0(P) \to \bar{H}_0(M) \to 0. \end{aligned} \tag{53}$$

By Lemma 2.5.1, we have

$$\bar{H}_i(P) = 0 \text{ for } i > 0. \tag{54}$$

Hence, from Theorem 2.5.5 it follows that $\bar{H}_1(M) = 0$. Let $i \geq 2$. Then from (53) and (54) we see that the vanishing of $\bar{H}_{i-1}(N)$ for any object N of $\mathcal{O}(\bar{\mathfrak{g}})$ implies the vanishing of $\bar{H}_i(M)$. This completes the proof. □

Let

$$\gamma : \mathcal{Z}(\bar{\mathfrak{g}}) \xrightarrow{\sim} S(\bar{\mathfrak{h}})^{\bar{W}} \tag{55}$$

be the Harish–Chandra isomorphism with respect to the triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$. For $\bar{\lambda} \in \bar{\mathfrak{h}}^*$ let

$$\gamma_{\bar{\lambda}} := (\text{evaluation at } \bar{\lambda} + \bar{\rho}) \circ \gamma : \mathcal{Z}(\bar{\mathfrak{g}}) \to \mathbb{C}. \tag{56}$$

Then $\gamma_{\bar{\lambda}}$ is the infinitesimal character of $\bar{M}(\bar{\lambda})$, that is, $zm = \gamma_{\bar{\lambda}}(z)m$ for $z \in \mathcal{Z}(\bar{\mathfrak{g}})$ and $m \in \bar{M}(\bar{\lambda})$. We have

$$\gamma_{\bar{\lambda}} = \gamma_{\bar{w} \circ \bar{\lambda}} \quad \text{for all } \bar{w} \in \bar{W}. \tag{57}$$

Here,

$$\bar{w} \circ \bar{\lambda} = \bar{w}(\bar{\lambda} + \bar{\rho}) - \bar{\rho}. \tag{58}$$

Let

$$\mathbb{C}_{\gamma_{\bar{\lambda}}} := \mathcal{Z}(\bar{\mathfrak{g}}) / \ker \gamma_{\bar{\lambda}}. \tag{59}$$

Theorem 2.5.7 **(**[55, Corollary 3.4.6, Theorem 3.4.7]**).** *Let $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. There are the following isomorphisms of $\mathcal{Z}(\bar{\mathfrak{g}})$-modules:*

(i) $\bar{H}_0(\bar{M}(\lambda)) \cong \mathbb{C}_{\gamma_{\bar{\lambda}}}$;

(ii) $\bar{H}_0(\bar{L}(\bar{\lambda})) = \begin{cases} \mathbb{C}_{\gamma_{\bar{\lambda}}} & \textit{if } \bar{\lambda} \textit{ is anti-dominant}, \\ 0 & \textit{otherwise}; \end{cases}$

(iii) $\bar{H}_0(\bar{M}(\lambda)^*) \cong \mathbb{C}_{\gamma_{\bar{\lambda}}}$.

Proof. (i) The assertion follows from the fact that $\bar{M}(\bar{\lambda}) \otimes \mathbb{C}_{\chi_-} \cong \bar{M}(\bar{\lambda})$ as $\bar{\mathfrak{n}}_-$-modules. (See the proof of Lemma 2.5.1 (i).) (ii) Suppose that $\bar{\lambda}$ is anti-dominant. Then $\bar{L}(\bar{\lambda}) = \bar{M}(\bar{\lambda})$, and hence the assertion follows from the first assertion. Next suppose that $\bar{\lambda}$ is not anti-dominant. Then there exist exact sequences

$$0 \to \bar{M}(\bar{\mu}) \to \bar{M}(\bar{\lambda}) \to \bar{M}(\bar{\lambda})/\bar{M}(\bar{\mu}) \to 0$$
$$\text{and} \quad \bar{M}(\bar{\lambda})/\bar{M}(\bar{\mu}) \to \bar{L}(\bar{\lambda}) \to 0$$

with some $\bar{\mu} \in \bar{W} \circ \bar{\lambda}$. By applying the exact functor $\bar{H}_0(?)$ we get the exact sequences

$$0 \to \bar{H}_0(\bar{M}(\bar{\mu})) \to \bar{H}_0(\bar{M}(\bar{\lambda})) \to \bar{H}_0(\bar{M}(\bar{\lambda})/\bar{M}(\bar{\mu})) \to 0$$
$$\text{and} \quad \bar{H}_0(\bar{M}(\bar{\lambda})/\bar{M}(\bar{\mu})) \to \bar{H}_0(\bar{L}(\bar{\lambda})) \to 0.$$

But $\bar{H}_0(\bar{M}(\bar{\mu})) \cong \bar{H}_0(\bar{M}(\bar{\lambda}))$ by the first assertion. Therefore, $\bar{H}_0(\bar{M}(\bar{\lambda})/\bar{M}(\bar{\mu})) = 0$, and hence, $\bar{H}_0(\bar{L}(\bar{\lambda})) = 0$. (iii) Let $K_0(\mathcal{O}(\bar{\mathfrak{g}}))$ and $K_0(\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}}))$ denote the Grothendieck groups of $\mathcal{O}(\bar{\mathfrak{g}})$ and $\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$, respectively. Then, by Theorem 2.5.5, $\bar{H}_0(?)$ defines a well-defined map from $K_0(\mathcal{O}(\bar{\mathfrak{g}}))$ to $K_0(\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}}))$. Because $[M(\lambda)^*] = [M(\lambda)]$ in $K_0(\mathcal{O}(\bar{\mathfrak{g}}))$, we have $[\bar{H}_0(M(\lambda)^*)] = [\bar{H}_0(M(\lambda))] = [\mathbb{C}_{\gamma_{\bar{\lambda}}}]$ in $K_0(\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}}))$. Because $\mathbb{C}_{\gamma_{\bar{\lambda}}}$ is simple, this completes the proof. □

By Theorem 2.5.7, we see in particular that any simple object of $\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$ is isomorphic to $\bar{H}_0(\bar{L}(\bar{\lambda}))$ for some anti-dominant weight $\bar{\lambda}$.

2.6. Identification with Soergel's functor. For $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, let $\bar{W}(\bar{\lambda}) := \langle s_{\bar{\alpha}}; \langle \lambda + \bar{\rho}, \bar{\alpha}^\vee \rangle \in \mathbb{Z} \rangle \subset \bar{W}$. It is known that $\bar{W}(\bar{\lambda})$ is a Coxeter group and it is called the *integral Weyl group* of $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Let $\mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]}$ be the block of $\mathcal{O}(\bar{\mathfrak{g}})$ corresponding to $\bar{\lambda}$, that is, $\mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]}$ is the Serre full subcategory of $\mathcal{O}(\bar{\mathfrak{g}})$ whose objects have all their local composition factors isomorphic to $\bar{L}(w \circ \bar{\lambda})$ with $w \in \bar{W}(\bar{\lambda})$. Then we have

$$\mathcal{O}(\bar{\mathfrak{g}}) = \bigoplus_{\substack{\bar{\lambda} \in \bar{\mathfrak{h}}^* \\ \bar{\lambda} \text{ is anti-dominant}}} \mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]}.$$

Let $\bar{\lambda}$ be anti-dominant. For an object M of $\mathcal{O}(\bar{\mathfrak{g}})^{[\lambda]}$, we regard $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), M)$ as a $\mathcal{Z}(\bar{\mathfrak{g}})$-module by $(zf)(v) = zf(v)$ for $z \in \mathcal{Z}(\bar{\mathfrak{g}})$, $f \in \mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), M)$ and $v \in M$. Then, both $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), ?)$ and $\bar{H}_0(?)$ define functors from $\mathcal{O}(\bar{\mathfrak{g}})^{[\lambda]}$ to $\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$. They are both exact functors. Indeed, $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), ?)$ is exact because $\bar{P}(\bar{\lambda})$ is projective and $\bar{H}_0(?)$ is exact by Theorem 2.5.5.

The functor $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), ?) : \mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]} \to \mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$ was studied by W. Soergel [58]. By a result of E. Backelin [4] it follows that the functor $\bar{H}_0(?)$ coincides the Soergel's functor $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), ?)$. Indeed one can describe the corresponding natural transformation in the following manner:

Let $\tilde{v}_\lambda \in \bar{P}(\bar{\lambda})$ be any inverse image of the highest weight vector of $\bar{L}(\lambda)$ by the canonical homomorphism $\bar{P}(\bar{\lambda}) \twoheadrightarrow \bar{L}(\bar{\lambda})$. Define a natural transformation

$$\Phi : \mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), ?) \to \bar{H}_0(?) \tag{60}$$

by

$$\begin{array}{rcccl} \Phi_M : \mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), M) & \to & \bar{H}_0(M) = M/(\ker \bar{\chi}_-^*) \\ f & \mapsto & [f(\tilde{v}_\lambda)] \end{array} \tag{61}$$

with $M \in Obj\ \mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]}$. (It is clear that Φ_M is a homomorphism of $\mathcal{Z}(\bar{\mathfrak{g}})$-module.)

Theorem 2.6.1 (E. Backelin [4])**.** *For each anti-dominant* $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, Φ *defines a natural isomorphism* $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), ?) \cong \bar{H}_0(?)$ *of functors from* $\mathcal{O}(\bar{\mathfrak{g}})^{[\lambda]}$ *to* $\mathsf{Fin}\mathcal{Z}(\bar{\mathfrak{g}})$.

Proof. We have to show that Φ_M is isomorphism for each object M of $\mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]}$. We prove this by induction on the length $l(M)$ of the composition series of M. Let $l(M) = 1$, that is, $M = L(\bar{\mu})$ for some $\bar{\mu} \in \bar{W}(\lambda) \circ \bar{\lambda}$. If $\bar{\mu} \neq \bar{\lambda}$, then both $\mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), \bar{L}(\bar{\mu}))$ and $\bar{H}_0(\bar{L}(\bar{\mu}))$ are zero (see Theorem 2.5.7). On the other hand, if $\bar{\mu} = \bar{\lambda}$, then the map $\Phi_{\bar{L}(\bar{\lambda})}$ is clearly an isomorphism by Theorem 2.5.7. Next let $l(M) > 1$. Then we have an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathcal{O}(\bar{\mathfrak{g}})^{[\bar{\lambda}]}$ such that $l(M_1), l(M_2) < l(M)$. This induces a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), M_1) & \to & \mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), M) & \to & \mathrm{Hom}_{\bar{\mathfrak{g}}}(\bar{P}(\bar{\lambda}), M_2) & \to 0 \\ & \downarrow \Phi_{M_1} & & \downarrow \Phi_M & & \downarrow \Phi_{M_2} & \\ 0 \to & \bar{H}_0(M_1) & \to & \bar{H}_0(M) & \to & \bar{H}_0(M_2) & \to 0. \end{array}$$

The upper row is obviously exact and the lower row is exact by Theorem 2.5.5. Hence we are done by the induction hypothesis and the five-lemma. □

3. Filtration of vertex (super)algebras and BRST cohomology

3.1. Some notation. Let V be any superspace. We set

$$L(V) := V \otimes \mathbb{C}[t, t^{-1}]. \tag{62}$$

This is considered as a superspace such that $L(V)^{\mathrm{even}} = L(V^{\mathrm{even}})$ and $L(V)^{\mathrm{odd}} = L(V^{\mathrm{odd}})$. For any subring R of $\mathbb{C}[t, t^{-1}]$ we consider $V \otimes R$ as a supersubspace of $L(V)$.

The *symmetric algebra* $S(V)$ of a superspace V is the superalgebra

$$S(V) = S\left(V^{\mathrm{even}}\right) \otimes \Lambda\left(V^{\mathrm{odd}}\right), \tag{63}$$

where $S(V^{\mathrm{even}})$ is the (usual) symmetric algebra of U and $\Lambda(V^{\mathrm{odd}})$ is the Grassmann algebra of V^{odd}.

A filtration $F = \{F_p V;\, p \in \mathbb{Z}\}$ of a superspace V is a $\mathbb{Z}_2$-graded filtration of the vector space V: $F_p V = (F_p V)^{\mathrm{even}} \oplus (F_p V)^{\mathrm{odd}}$, where $(F_p V)^{\mathrm{even}} = F_p V \cap V^{\mathrm{even}}$, $(F_p V)^{\mathrm{odd}} = F_p V \cap V^{\mathrm{odd}}$.

A filtration F of V is called *exhaustive* if $V = \bigcup_p F_p V$ and *separated* if $\bigcap_p F_p V = 0$.

For an increasing filtration $\{F_p V\}$ of a superspace V, we set $\mathrm{gr}^F V = \bigoplus_p \mathrm{gr}_p^F V$, where $\mathrm{gr}_p^F V = F_p V / F_{p-1} V$. For any filtration F we write σ_p for the symbol map $F_p V \to \mathrm{gr}_p^F V = F_p V / F_{p-1}$.

Below up until the end of Sect. 3 we treat superspaces, supersubspaces, Lie superalgebras, vertex superalgebras etc., unless otherwise stated. However for convention we shall often drop the prefix "super".

3.2. Vertex algebras (see [25,33]). Let $\mathbb{V}$ be a vertex algebra,

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1} \in (\mathrm{End}\, \mathbb{V})[[z, z^{-1}]] \tag{64}$$

the field associated with $v \in \mathbb{V}$, $|0\rangle$ the vacuum so that $Y(|0\rangle, z) = \mathrm{id}_{\mathbb{V}}$, T the translation operator:

$$[T, Y(v, z)] = Y(Tv, z) = \frac{d}{dz} Y(v, z) \quad \forall v \in \mathbb{V}. \tag{65}$$

We have

$$[u_{(m)}, v_{(n)}] = \sum_{i \geq 0} \binom{m}{i} (u_{(i)} v)_{(m+n-i)}, \tag{66}$$

$$\begin{aligned} (u_{(m)} v)_{(n)} = \sum_{i \geq 0} \binom{m}{i} (-1)^i \Big(& u_{(m-i)} v_{(n+i)} \\ & -(-1)^{p(u)p(v)} (-1)^m v_{(m+n-i)} u_{(i)} \Big) \end{aligned} \tag{67}$$

for $u, v \in \mathbb{V}$. The sum in the right hand side of (66) is finite because the axiom of vertex algebras requires that $u_{(n)} v = 0$ for sufficiently large n. The relations (67) is called the *Borchards identity*.

A subspace U of $\mathbb{V}$ is said to *strongly generate* $\mathbb{V}$ if $\mathbb{V}$ is spanned by the vectors

$$a^{j_1}_{(-n_1)} \dots a^{j_r}_{(-n_r)}|0\rangle$$

with $a^{j_s} \in U$, $r \geq 0$, $j_s \in J$, $n_s \geq 1$.

An even element H of End $\mathbb{V}$ is called a *Hamiltonian* if H acts semisimply on H satisfying

$$[H, Y(v, z)] = Y(Hv, z) + zY(Tv, z) \quad \forall v \in \mathbb{V}. \tag{68}$$

If $v \in \mathbb{V}$ is a eigenvector of H then its eigenvalue Δ is called the *conformal weight* of v and denoted by Δ_v. We use the convention that when we write Δ_v we are assuming that v is an eigenvector of H. By (68) one has

$$\Delta_{|0\rangle} = 0, \tag{69}$$

$$\Delta_{u_{(n)}v} = \Delta_u + \Delta_v - n - 1. \tag{70}$$

Set

$$\mathbb{V}_{-\Delta} = \{v \in \mathbb{V};\, H \cdot v = \Delta v\}, \tag{71}$$

so that $\mathbb{V} = \bigoplus_{\Delta \in \mathbb{C}} \mathbb{V}_{-\Delta}$. The $\mathbb{V}$ is called $\mathbb{Z}$-*graded* (by H) if $\mathbb{V}_{-\Delta} = 0$ unless $\Delta \in \mathbb{Z}$; $\mathbb{V}$ is called $\mathbb{Z}_{\geq 0}$-*graded* if $\mathbb{V}_{-\Delta} = 0$ unless $\Delta \in \mathbb{Z}_{\geq 0}$.

Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra, H the Hamiltonian. We say that $\mathbb{V}$ is *compatibly* $\mathbb{Z}_{\geq 0}$-*gradable* if $\mathbb{V}$ admits a Hamiltonian H' which gives a $\mathbb{Z}_{\geq 0}$-grading of $\mathbb{V}$ satisfying $[H, H'] = 0$.

The vertex algebra $\mathbb{V}$ is called *commutative* if $[u_{(m)}, v_{(n)}] = 0$ for all $u, v \in \mathbb{V}$, $m, n \in \mathbb{Z}$. It is well-known that $\mathbb{V}$ is commutative if and only if $u_{(m)}v = 0$ for all $u, v \in \mathbb{V}$, $m \geq 0$.

3.3. Möbius conformal vertex algebras and conformal vertex algebras. Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra, H the Hamiltonian. Suppose there exists an even operator T^* on $\mathbb{V}$ satisfying the following:

$$[T^*, H] = T^*, \tag{72}$$

$$[T^*, Y(v, z)] = Y(T^*v, z) + 2zY(Hv, z) + z^2Y(Tv, z) \quad \forall v \in \mathbb{V}. \tag{73}$$

Then $\mathbb{V}$ is called a *Möbius conformal* vertex algebra [33, Section 4.9]. In this case the triplet $\{T^*, H, T\}$ forms the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ in End $\mathbb{V}$:

$$[H, T^*] = -T^*, \quad [T^*, T] = 2H, \quad [H, T] = T.$$

A vector which is annihilated by T^* is called a *quasi-primary vector*.

A *conformal vector* of $\mathbb{V}$ is an even vector $\omega \in \mathbb{V}_{-2}$ such that the corresponding field $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ has the following properties:

(i) $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c_{\mathbb{V}}\,\mathrm{id}_{\mathbb{V}}$ for $m, n \in \mathbb{Z}$, where $c_{\mathbb{V}} \in \mathbb{C}$ (*the central charge);*
(ii) $T = L(-1)$*;*
(iii) *the action of* $L(0)$ *is semisimple and a Hamiltonian (i.e. satisfies* (68)*)*

In this case $\mathbb{V}$ has the Möbius conformal vertex algebra structure with $T^* = L(1)$ and $H = L(0)$. A vertex algebra with a conformal vector is called *conformal* (or *vertex operator algebra*).

3.4. Filtration of vertex algebras. A vertex algebra $\mathbb{V}$ is called *filtered* if there exists an increasing filtration $F = \{F_p\mathbb{V}\}$ of $\mathbb{V}$ as a superspace which is compatible with the vertex algebra structure in the following sense:

(74) $|0\rangle \in F_0\mathbb{V}\backslash F_{-1}\mathbb{V},$

(75) $T \cdot F_p\mathbb{V} \subset F_p\mathbb{V};$

(76) $v_{(n)} \cdot F_q\mathbb{V} \subset F_{p+q}\mathbb{V}$ for all $p, q \in \mathbb{Z}$, $v \in F_p\mathbb{V}$ and $n \in \mathbb{Z}$.

Also, unless otherwise stated, we require a filtration of a vertex algebra $\mathbb{V}$ to be separated and exhaustive.

A filtration of a $\mathbb{Z}$-graded vertex algebra $\mathbb{V}$ is a filtration F of $\mathbb{V}$ which is compatible with the action of the Hamiltonian: $H \cdot F_p\mathbb{V} \subset F_p\mathbb{V}$.

If $(\mathbb{V}, F)$ is a filtered vertex algebra (resp. a filtered $\mathbb{Z}$-graded vertex algebra) then $\mathrm{gr}^F\mathbb{V}$ is naturally a vertex algebra (resp. a $\mathbb{Z}$-graded vertex algebra), see e.g. [49]. The vertex algebra $\mathrm{gr}^F\mathbb{V}$ is called the *graded vertex algebra associated with the filtered vertex algebra* $\mathbb{V}$.

A filtered vertex (super)algebra $(\mathbb{V}, F)$ is called *quasi-commutative* if $\mathrm{gr}^F\mathbb{V}$ is commutative. If $(\mathbb{V}, F)$ is quasi-commutative then $\mathrm{gr}^F\mathbb{V}$ is naturally a *vertex Poisson algebra* ([49, Proposition 4.2]). However we do not use this fact in this article.

3.5. Standard filtration

Theorem 3.5.1 ([49]). *Let* $\mathbb{V}$ *be a vertex algebra, which is* $\mathbb{Z}$*-graded by* H*, and compatibly* $\mathbb{Z}_{\geq 0}$*-gradable by* H'*. Take a strongly generating* H*,* H'*-invariant subspace* U *of* $\mathbb{V}$*. Let* $\{a^j; j \in J\}$ *be a basis of* U*,* $\Delta_i = \Delta_{a_i}$*. Let* $G_p\mathbb{V}$*, with* $p \in \mathbb{Z}$*, be the subspace of* $\mathbb{V}$ *spanned by all the vectors*

$$a^{j_1}_{(-n_1)}a^{j_2}_{(-n_2)}\dots a^{j_r}_{(-n_r)}|0\rangle$$

with $r \geq 0$*,* $j_s \in J$*,* $n_s \geq 1$ *satisfying the relation*

$$\Delta_{j_1} + \Delta_{j_2} + \dots + \Delta_{j_r} \leq p.$$

Here by convention $|0\rangle \in G_0\mathbb{V}\backslash G_{-1}\mathbb{V}$. *Then*

(i) $G = \{G_p\mathbb{V}\}$ *gives a filtration of the* $\mathbb{Z}$*-graded vertex algebra* $\mathbb{V}$*;*
(ii) $(\mathbb{V}, G)$ *is quasi-commutative;*
(iii) G *is the finest filtration of* $\mathbb{V}$ *such that* $V_{-\Delta} \subset G_\Delta\mathbb{V}$ *for all* Δ*. In particular the filtration* G *is independent of the choice of a strongly generating* H*,* H'*-invariant subspace* U *of* $\mathbb{V}$*.*

Remark 3.5.2. In [49] it was assume that $\mathbb{V}$ is $\mathbb{Z}_{\geq 0}$-graded and $\mathbb{V}_0 = \mathbb{C}|0\rangle$. However it is easy to see that Theorem 3.5.1 follows from the the assertion in the case that $\mathbb{V}$ is $\mathbb{Z}_{\geq 0}$-graded. Also, in the case that $\mathbb{V}$ is $\mathbb{Z}_{\geq 0}$-graded, it is not difficult to remove the condition that $\mathbb{V}_0 = \mathbb{C}|0\rangle$.

The filtration G given in Theorem 3.5.1 is called the *standard filtration* of a $\mathbb{Z}$-graded, compatibly $\mathbb{Z}_{\geq 0}$-gradable vertex algebra $\mathbb{V}$.

3.6. PBW basis of vertex algebras. Let $\mathbb{V}$ be a vertex algebra, F a filtration of $\mathbb{V}$ such that $(\mathbb{V}, F)$ is quasi-commutative. Let U be a subspace of $\mathbb{V}$ such that the its image $\bar{U}$ in $\mathrm{gr}^F\mathbb{V}$ strongly generates $\mathrm{gr}^F\mathbb{V}$. Then U strongly generates $\mathbb{V}$. We have the surjective linear map

$$\begin{array}{ccc} S(\bar{U}\otimes\mathbb{C}[t^{-1}]t^{-1}) & \longrightarrow & \mathrm{gr}^F\mathbb{V} \\ (\bar{a}^1\otimes t^{-n_1})\dots(\bar{a}^r\otimes t^{-n_r}) & \longmapsto & \bar{a}^1_{(-n_1)}\dots\bar{a}^r_{(-n_r)}|0\rangle \end{array} \tag{77}$$

with $\bar{a}^i \in \bar{U}$, $a_i \geq 1$.

We say that a subspace $U \subset \mathbb{V}$ *generates a PBW basis* of $\mathbb{V}$ if there exists a filtration F of $\mathbb{V}$ such that $(\mathbb{V}, F)$ is quasi-commutative and the map (77) is a linear isomorphism; In this case we say that $\mathbb{V}$ *admits a PBW basis.*

3.7. Vacuum subalgebras. For a vertex algebra $\mathbb{V}$, let

$$\mathrm{Vac}\,\mathbb{V} := \{v \in \mathbb{V};\, T\cdot v = 0\}. \tag{78}$$

Then $|0\rangle \in \mathrm{Vac}\,\mathbb{V}$. The $\mathrm{Vac}\,\mathbb{V}$ is a vertex subalgebra of $\mathbb{V}$ and called the *vacuum subalgebra of* $\mathbb{V}$ [33, Remark 4.4b].

Suppose that $\mathbb{V}$ is filtered, F its filtration. Then F induces a filtration of $\mathrm{Vac}\,\mathbb{V}$: $F_p\,\mathrm{Vac}\,\mathbb{V} = \mathrm{Vac}\,\mathbb{V}\cap F_p\mathbb{V}$. There is a natural embedding of vertex algebras

$$\mathrm{gr}^F\mathrm{Vac}\,\mathbb{V} \hookrightarrow \mathrm{Vac}(\mathrm{gr}^F\mathbb{V}) \tag{79}$$

which is given by the correspondence $\sigma_p(v) \mapsto \sigma_p(v)$ with $v \in F_p\,\mathrm{Vac}\,\mathbb{V}$.

Proposition 3.7.1. *Let* $\mathbb{V}$ *be a vertex algebra that admits a PBW basis. Then* $\mathrm{Vac}\,\mathbb{V} = \mathbb{C}|0\rangle$.

Proof. By assumption there exist a subspace U of $\mathbb{V}$ and a filtration F of $\mathbb{V}$ such that $\mathrm{gr}^F \mathbb{V}$ commutative and the map (77) is a linear isomorphism. By (79) it is sufficient to show that $\mathrm{Vac}(\mathrm{gr}^F \mathbb{V}) = \mathbb{C}|0\rangle$.

By assumption $\mathrm{gr}^F \mathbb{V}$ is isomorphic to the space of the following form:

$$R = \mathbb{C}\big[x^{(j)}_{(-n)};\, j \in J_{\bar{0}}, n \geq 1\big] \otimes \Lambda\big(y^{(j)}_{(-n)};\, j \in J_{\bar{1}}, n \geq 1\big),$$

where $\Lambda(y^{(j)}_{(-n)};\, j \in J_{\bar{1}}, n \geq 1)$ is the Grassmann algebra with generators $y^{(j)}_{(-n)}$, $j \in J_{\bar{1}}$, $n \geq 1$. Under this identification T acts as the following derivation:

$$\big[T, x^{(j)}_{(-n)}\big] = n x^{(j)}_{(-n-1)}, \quad \big[T, y^{(j)}_{(-n)}\big] = n y^{(j)}_{(-n-1)}$$

Define the even derivations $[H, ?]$, $[T^*, ?]$ on R by the following:

$$\big[H, x^{(j)}_{(-n)}\big] = n x^{(j)}_{(-n)}, \quad \big[H, y^{(j)}_{(-n)}\big] = n y^{(j)}_{(-n)}$$
$$\big[T^*, x^{(j)}_{(-n)}\big] = n x^{(j)}_{(-n+1)}, \quad \big[T^*, y^{(j)}_{(-n)}\big] = n y^{(j)}_{(-n+1)}.$$

Here by convention $x^{(j)}_{(0)} = y^{(j')}_{(0)} = 0$ for all j, j'. This gives the well-defined action of $\mathfrak{sl}_2(\mathbb{C})$ on R. Therefore by [33, Proposition 4.9(a)] we have $\mathrm{Vac}(\mathrm{gr}^F \mathbb{V}) = \{v \in R;\, H \cdot v = 0\} = \mathbb{C}|0\rangle$. □

Remark 3.7.2. In general if $\mathbb{V}$ is a $\mathbb{Z}_{\geq 0}$-graded Möbius conformal vertex algebra then $\mathrm{Vac}\, \mathbb{V} \subset \mathbb{V}_0$, see [33, Proposition 4.9].

3.8. Strict filtration. Let $\mathbb{V}$ be a filtered vertex algebra, F its filtration. Then $TF_p \mathbb{V} \subset F_p \mathbb{V} \cap T\mathbb{V}$. We call F *strict* if

(80) $$F_p \mathbb{V} \cap T\mathbb{V} = TF_p \mathbb{V} \quad \text{for all } p.$$

Lemma 3.8.1.

(i) *The filtration F is strict if and only if the natural embedding* $\mathrm{gr}^F \mathrm{Vac}\, \mathbb{V} \hookrightarrow \mathrm{Vac}(\mathrm{gr}^F \mathbb{V})$ *is an isomorphism of vertex algebras.*
(ii) *If* $\mathrm{Vac}(\mathrm{gr}^F \mathbb{V}) = \mathbb{C}|0\rangle$, *then* $\mathrm{Vac}\, \mathbb{V} = \mathbb{C}|0\rangle$ *and F is strict.*

Proof. (i) Let $\mathrm{Vac}(\mathrm{gr}^F_p \mathbb{V}) = \mathrm{Vac}(\mathrm{gr}^F \mathbb{V}) \cap \mathrm{gr}^F_p \mathbb{V} = \{v \in F_p \mathbb{V};\, Tv \in F_{p-1} \mathbb{V}\}/F_{p-1} \mathbb{V}$. Then

$$\mathrm{Vac}(\mathrm{gr}^F_p \mathbb{V})/\, \mathrm{gr}^F_p \mathrm{Vac}\, \mathbb{V} = \{v \in F_p \mathbb{V};\, Tv \in F_{p-1} \mathbb{V}\}/(F_{p-1} \mathbb{V} + F_p \mathrm{Vac}\, \mathbb{V}).$$

One sees that the correspondence $v \mapsto Tv$ induces an isomorphism

(81) $$\mathrm{Vac}(\mathrm{gr}^F_p \mathbb{V})/\, \mathrm{gr}^F_p \mathrm{Vac}(V) \xrightarrow{\sim} (F_{p-1} \mathbb{V} \cap TF_p \mathbb{V})/TF_{p-1} \mathbb{V}.$$

But as easily seen F is strict if and only if $F_{p-1} \mathbb{V} \cap TF_p \mathbb{V} = TF_{p-1} \mathbb{V}$ for all p. Therefore, the assertion follows. (ii) The first assertion is obvious. The second assertion follows from (i). □

Proposition 3.8.2. *Let $\mathbb{V}$ be a vertex algebra, F its filtration. Suppose that $\mathrm{gr}^F \mathbb{V}$ admits a PBW basis. Then $\mathrm{Vac}\,\mathbb{V} = \mathbb{C}|0\rangle$ and F is strict.*

Proof. By assumption and Proposition 3.7.1, $\mathrm{Vac}\,(\mathrm{gr}^F \mathbb{V}) = \mathbb{C}|0\rangle$. Therefore Lemma 3.8.1 (ii) gives the assertion. □

Proposition 3.8.3. *Let $\mathbb{V}$ be a $\mathbb{Z}$-graded, compatibly $\mathbb{Z}_{\geq 0}$-gradable vertex algebra that admits a PBW basis. Then the standard filtration G is strict.*

Proof. Under the assumption of Proposition, $\mathrm{gr}^G \mathbb{V}$ admits a PBW basis. Thus the assertion follows from Proposition 3.8.2. □

3.9. Lie algebras attached to vertex algebras. Let $\mathbb{V}$ be a vertex algebra. Define

(82) $$\mathfrak{L}(\mathbb{V}) := L(\mathbb{V})/\operatorname{Im}\partial_{-1},$$

where

(83) $$\partial_{-1} := T\otimes \mathrm{id} + \mathrm{id}\otimes \frac{d}{dt}.$$

Then $\mathfrak{L}(\mathbb{V})$ has the Lie algebra structure [8], whose the commutation relation is given by

(84) $$[u_{\{m\}}, v_{\{n\}}] = \sum_{r\geq 0} \binom{m}{r} (u_{(r)}v)_{\{m+n-r\}},$$

where $v_{\{n\}}$ denotes the image of $v\otimes t^n$ with $v\in\mathbb{V}$ and $n\in\mathbb{Z}$ in $\mathfrak{L}(\mathbb{V})$. By (66) the correspondence $v_{\{n\}}\mapsto v_{(n)}$ defines a representation of $\mathfrak{L}(\mathbb{V})$ on $\mathbb{V}$.

Define the adjoint action $\mathrm{ad}\,T$ of T on $\mathfrak{L}(\mathbb{V})$ by the following.

(85) $$\mathrm{ad}\,T\cdot v_{\{n\}} = [T, v_{\{n\}}] := (Tv_{\{n\}}) = -nv_{\{n-1\}}.$$

If $\mathbb{V}$ is graded by a Hamiltonian H, then there is a adjoint action $\mathrm{ad}\,H$ of the Hamiltonian H on $\mathfrak{L}(\mathbb{V})$ in view of (68):

(86) $$\mathrm{ad}\,H\cdot u_{\{n\}} = [H, u_{\{n\}}] := (Hu)_{\{n\}} + (Tu)_{\{n+1\}}.$$

This gives a $\mathbb{Z}$-grading of $\mathfrak{L}(\mathbb{V})$: $\mathfrak{L}(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}} \mathfrak{L}(\mathbb{V})_d$, $[\mathfrak{L}(\mathbb{V})_d, \mathfrak{L}(\mathbb{V})_{d'}]\subset \mathfrak{L}(\mathbb{V})_{d+d'}$, where

(87) $$\mathfrak{L}(\mathbb{V})_d = \{u\in\mathfrak{L}(\mathbb{V});\ \mathrm{ad}\,H\cdot u = -du\}.$$

Set

(88) $$v_n := v_{\{n+\Delta_v-1\}}.$$

Then $\operatorname{ad} H \cdot v_n = -nv_n$ and $\mathfrak{L}(\mathbb{V})_d$ is spanned by v_d with homogeneous vectors $v \in \mathbb{V}$. One has

$$[u_m, v_n] = \sum_{r\geq 0} \binom{m+\Delta_u - 1}{r} (u_{(r)}v)_{m+n}. \tag{89}$$

Next suppose that $\mathbb{V}$ is Möbius conformal. Then we also have the adjoint action of T^*:

$$\operatorname{ad} T^* \cdot v_{\{n\}} = [T^*, v_{\{n\}}] := (T^*v)_{\{n\}} + 2(Hv)_{\{n+1\}} + (Tv)_{\{n+2\}}. \tag{90}$$

Thus $\mathfrak{sl}_2(\mathbb{C}) = \operatorname{span}\{T^*, H, T\}$ acts on $\mathfrak{L}(\mathbb{V})$ by derivation:

$$\operatorname{ad} X \cdot [u_m, v_n] = [\operatorname{ad} X \cdot u_m, \operatorname{ad} X \cdot v_n] \quad \text{for } X \in \{T^*, H, T\}, \tag{91}$$

$u, v \in \mathbb{V}, m, n \in \mathbb{Z}$.

The same proof of [52, Proposition 4.1.1] applies for the following assertion because only the Möbius conformal structure of $\mathbb{V}$ is used in the argument (cf. [26, Section 2.8]).

Proposition 3.9.1 ([52, Proposition 4.1.1], cf. [26, Section 5]). *Let $\mathbb{V}$ be a $\mathbb{Z}$-graded Möbius conformal vertex algebra on which T^* acts locally nilpotently. Set*

$$\theta(v_n) = \begin{cases} (-1)^{\Delta_v}(e^{T^*}v)_{-n} & \text{if } p(v) = \bar{0} \\ \sqrt{-1}(-1)^{\Delta_v}(e^{T^*}v)_{-n} & \text{if } p(v) = \bar{1} \end{cases}$$

for $v \in \mathbb{V}$, $n \in \mathbb{Z}$. Then θ defines a well-defined anti-Lie algebra isomorphism of $\mathfrak{L}(\mathbb{V})$ such that $\theta^2(v) = (-1)^{p(v)}v$.

If $\mathbb{V}$ is purely even then θ is an anti-Lie algebra involution of $\mathfrak{L}(\mathbb{V})$.

3.10. Filtration of $\mathfrak{L}(\mathbb{V})$.

Let V be a filtered vertex algebra, F its filtration. Define an increasing filtration on $L(\mathbb{V})$ by

$$F_p L(\mathbb{V}) = L(F_p\mathbb{V})$$

(Notation (62)). This gives the quotient filtration $\{F_p\, \mathfrak{L}(\mathbb{V})\}$ of the Lie algebra $\mathfrak{L}(\mathbb{V})$.

Proposition 3.10.1. *Let $\mathbb{V}$ be a filtered vertex algebra, $F = \{F_p\mathbb{V}\}$ its filtration. Then there is a natural surjective Lie algebra homomorphism $\mathfrak{L}(\operatorname{gr}^F \mathbb{V}) \to \operatorname{gr}^F \mathfrak{L}(\mathbb{V})$ given by the correspondence*

$$(\sigma_p(v))_{\{n\}} \mapsto \sigma_p(v_{\{n\}}) \quad \textit{for } p \in \mathbb{Z}, v \in \mathbb{V}, n \in \mathbb{Z}. \tag{92}$$

Moreover if F is strict then this is an isomorphism.

Proof. By definition $\mathfrak{L}(\mathrm{gr}^F \mathbb{V}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{L}(\mathrm{gr}^F_p \mathbb{V})$, where

$$\begin{aligned}\mathfrak{L}\big(\mathrm{gr}^F_p \mathbb{V}\big) &= \mathrm{gr}^F_p L(\mathbb{V})/\partial_{-1}\big(\mathrm{gr}^F_p L(\mathbb{V})\big)\\ &= F_p L(\mathbb{V})/\big(F_{p-1} L(\mathbb{V}) + \partial_{-1} F_p L(\mathbb{V})\big).\end{aligned}$$

On the other hand

$$\begin{aligned}\mathrm{gr}^F_p \mathfrak{L}(\mathbb{V}) &= F_p \mathfrak{L}(\mathbb{V})/F_{p-1}\mathfrak{L}(\mathbb{V})\\ &= F_p L(\mathbb{V})/\big(F_{p-1} L(\mathbb{V}) + \partial_{-1} L(\mathbb{V}) \cap F_p L(\mathbb{V})\big).\end{aligned}$$

Clearly $\partial_{-1} F_p L(\mathbb{V}) \subset \partial_{-1} L(\mathbb{V}) \cap F_p L(\mathbb{V})$. Thus (92) is well-defined and surjective. It is straightforward to see that it is a Lie algebra homomorphism.

Next assume that F is strict. It is sufficient to show that the opposite correspondence

$$\sigma_p(v_{\{n\}}) \mapsto (\sigma_p(v))_{\{n\}} \quad \text{for } p \in \mathbb{Z}, v \in \mathbb{V}, n \in \mathbb{Z}. \tag{93}$$

is well-defined. For this we need to show that

$$\partial_{-1} L(\mathbb{V}) \cap F_p L(\mathbb{V}) \subset \partial_{-1} F_p L(\mathbb{V}). \tag{94}$$

Let $w \in \partial_{-1} L(\mathbb{V}) \cap F_p L(\mathbb{V})$. Then w has the form

$$w = \sum_{i=1}^{r} \partial_{-1}(v_i \otimes f_i) = \sum_{i=1}^{r} ((Tv_i) \otimes f_i + v_i \otimes f_i')$$

with $v_i \in \mathbb{V}$, $f_i \in \mathbb{C}[t, t^{-1}]$. We have to show that each v_i belongs to $F_p\mathbb{V}$. We may assume that $\{f_i\}$ are linearly independent homogeneous elements of $\mathbb{C}[t, t^{-1}]$, and $\deg f_i \geq \deg f_{i+1}$ for all i. Because f_1 has the largest degree, it follows that Tv_1 must belong to $F_p\mathbb{V}$. Therefore v_1 must belong to $F_p\mathbb{V}$ since F is strict. Repeating this argument, we get $v_i \in F_p\mathbb{V}$ for all i. □

3.11. Current algebras of vertex algebras and $\mathbb{V}$-modules. Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra, H its Hamiltonian. Following [54], we associate with $\mathbb{V}$ the *current algebra* $\mathfrak{U}(\mathbb{V})$, which is essentially the universal enveloping algebra of $\mathbb{V}$ in the sense of I. Frenkel and Y. Zhu [29]: Let $\mathbf{U}(\mathbb{V})$ be the quotient of the universal enveloping algebra $U(\mathfrak{L}(\mathbb{V}))$ of $\mathfrak{L}(\mathbb{V})$ by the two-sided ideal generated by $(|0\rangle)_0 - 1$. The action ad H naturally extends to $\mathbf{U}(\mathbb{V})$. Let $\mathbf{U}(\mathbb{V})_d = \{u \in \mathbf{U}(\mathbb{V}); \operatorname{ad} H \cdot u = -du\}$. This makes $\mathbf{U}(\mathbb{V})$ a graded algebra:

$$\mathbf{U}(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}} \mathbf{U}(\mathbb{V})_d, \quad \mathbf{U}(\mathbb{V})_d \cdot \mathbf{U}(\mathbb{V})_{d'} \subset \mathbf{U}(\mathbb{V})_{d+d'}.$$

Denote by $\widetilde{\mathbf{U}}(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}} \widetilde{\mathbf{U}}(\mathbb{V})_d$ the *standard degreewise completion* (see Sect. A.2) of $\mathbf{U}(\mathbb{V})$. The $\widetilde{\mathbf{U}}(\mathbb{V})$ is equipped with the left linear topology defined by the decreasing sequence of the left ideals

$$\mathscr{I}_N(\widetilde{\mathbf{U}}(\mathbb{V})) = \ker\Big(\widetilde{\mathbf{U}}(\mathbb{V}) \twoheadrightarrow \mathbf{U}(\mathbb{V})/\Big(\bigoplus_{d\in\mathbb{Z}}\sum_{r>N}\mathbf{U}(\mathbb{V})_{d-r}\mathbf{U}(\mathbb{V})_r\Big)\Big)$$

with $N \geq 0$. Let $\mathbf{B}(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}} \mathbf{B}(\mathbb{V})_d$ be the ideal of $\widetilde{\mathbf{U}}(\mathbb{V})$ corresponding to the Borchards identity, i.e. the $\mathbb{Z}$-graded two-sided ideal generated by

(95)
$$(u_{(m)}v)_{\{n\}} - \sum_{i\geq 0}\binom{m}{i}(-1)^i\left(u_{\{m-i\}}v_{\{n+i\}} - (-1)^{p(u)p(v)}(-1)^m v_{\{m+n-i\}}u_{\{i\}}\right)$$

with $u, v \in \mathbb{V}$, $m, n \in \mathbb{Z}$. Let $\widetilde{\mathbf{B}}(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}} \widetilde{\mathbf{B}}(\mathbb{V})_d$ be the *degreewise closure* (see Sect. A.2) of $\mathbf{B}(\mathbb{V})$ in $\widetilde{\mathbf{U}}(\mathbb{V})$. The current algebra of $\mathbb{V}$ is the quotient algebra

(96) $$\mathfrak{U}(\mathbb{V}) := \widetilde{\mathbf{U}}(\mathbb{V})/\widetilde{\mathbf{B}}(\mathbb{V}).$$

By definition the $\mathfrak{U}(\mathbb{V})$ is graded by ad H:

$$\mathfrak{U}(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}}\mathfrak{U}(\mathbb{V})_d, \quad \mathfrak{U}(\mathbb{V})_d = \{u \in \mathfrak{U}(\mathbb{V}); \operatorname{ad} H \cdot u = -du\}.$$

The $\mathfrak{U}(\mathbb{V})$ is equipped with the left linear topology induced by that of $\widetilde{\mathbf{U}}(\mathbb{V})$. Its topology is defined by the decreasing sequence of left ideals $\mathscr{I}_N(\mathfrak{U}(\mathbb{V}))$, where $\mathscr{I}_N(\mathfrak{U}(\mathbb{V}))$ is the image of $\mathscr{I}_N(\widetilde{\mathbf{U}}(\mathbb{V}))$ in $\mathfrak{U}(\mathbb{V})$. The space $\mathscr{I}_N(\mathfrak{U}(\mathbb{V}))$ coincides with the degreewise closure of $\mathfrak{U}(\mathbb{V}) \cdot \sum_{r>N}\mathfrak{U}(\mathbb{V})_r$ in $\mathfrak{U}(\mathbb{V})$.

The $\mathfrak{U}(\mathbb{V})$ is a *compatible degreewise complete algebra* [54]: Each $\mathfrak{U}(\mathbb{V})_d$ is complete with respect to the relative topology:

(97) $$\mathfrak{U}(\mathbb{V})_d = \lim_{\substack{\leftarrow \\ N}}\mathfrak{U}_N(\mathbb{V})_d, \qquad \text{where } \mathfrak{U}_N(\mathbb{V})_d := \mathfrak{U}(\mathbb{V})_d/\mathscr{I}_N(\mathfrak{U}(\mathbb{V}))_d,$$

and the multiplication map $\mathfrak{U}(\mathbb{V})_d \times \mathfrak{U}(\mathbb{V})_{d'} \to \mathfrak{U}(\mathbb{V})_{d+d'}$ is continuous, see [54] for details. Here $\mathscr{I}_N(\mathfrak{U}(\mathbb{V}))_d = \{u \in \mathscr{I}_N(\mathfrak{U}(\mathbb{V})); \operatorname{ad} H \cdot u = -du\}$. We set $\mathfrak{U}_N(\mathbb{V}) = \bigoplus_{d\in\mathbb{Z}}\mathfrak{U}_N(\mathbb{V})_d$.

The image of v_n, with $v \in \mathbb{V}$, $n \in \mathbb{Z}$, by the natural map $\mathfrak{L}(\mathbb{V}) \to \mathfrak{U}(\mathbb{V})$ is denoted also by v_n, and so is its image in $\mathfrak{U}_N(\mathbb{V})$.

Proposition 3.11.1 **(**[54, Proposition 6.4.1]**).** *The image of $\mathfrak{L}(\mathbb{V})$ in $\mathfrak{U}(\mathbb{V})$ is dense, that is, the natural map $\mathfrak{L}(\mathbb{V}) \ni v_n \mapsto v_n \in \mathfrak{U}_N(\mathbb{V})$ is surjective for all $N \geq 0$.*

Set

$$\mathfrak{U}(\mathbb{V})_{\geq 0} := \bigoplus_{d \geq 0} \mathfrak{U}(\mathbb{V})_d, \quad \mathfrak{U}(\mathbb{V})_{>0} := \bigoplus_{d>0} \mathfrak{U}(\mathbb{V})_d. \tag{98}$$

These are subalgebras of $\mathfrak{U}(\mathbb{V})$.

Suppose that $\mathbb{V}$ is Möbius conformal. Then, by [28, Section 5], the anti-Lie algebra isomorphism $\theta : \mathfrak{L}(\mathbb{V}) \to \mathfrak{L}(\mathbb{V})$ induces a anti-algebra isomorphism $\theta : \mathfrak{U}(\mathbb{V}) \to \mathfrak{U}(\mathbb{V})$ such that $\theta(\mathfrak{U}(\mathbb{V})_d) = \mathfrak{U}(\mathbb{V})_{-d}$.

3.12. $\mathbb{V}$-modules and Zhu algebras. Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra, H its Hamiltonian. A *$\mathbb{V}$-module* is by definition a $\mathfrak{U}(\mathbb{V})$-module. A *graded $\mathbb{V}$-module* is a $\mathfrak{U}(\mathbb{V})$-module M which carries a $\mathbb{C}$-grading $M = \bigoplus_{d \in \mathbb{C}} M_d$ such that $\mathfrak{U}(\mathbb{V})_n \cdot M_d \subset M_{d+n}$ for all n, d. Let $\mathbb{V}$-grMod be the category of graded $\mathbb{V}$-modules: the objects of $\mathbb{V}$-grMod are graded $\mathbb{V}$-modules and the morphisms of $\mathbb{V}$-grMod are all the graded $\mathfrak{U}(\mathbb{V})$-module homomorphisms. Here a $\mathfrak{U}(\mathbb{V})$-module homomorphism $\phi : M \to N$ is called graded if for each $d \in \mathbb{C}$ there exists $d' \in \mathbb{C}$ satisfying $\phi(M_{d+n}) \subset N_{d'+n}$ for all $n \in \mathbb{Z}$.

An *admissible $\mathbb{V}$-module* [1] is a graded $\mathbb{V}$-module such that $M_{d+n} = 0$ for $n \gg 0$ with a fixed d. If M is a admissible $\mathbb{V}$-module then the action $\mathfrak{U}(\mathbb{V}) \times M \to M$ is continuous with respect to the topology on $\mathfrak{U}(\mathbb{V})$ and the discrete topology on M. Let $\mathbb{V}$-adMod be the full subcategory of the category of graded $\mathbb{V}$-modules consisting of admissible $\mathbb{V}$-modules. If $\mathbb{V}$ is $\mathbb{Z}_{\geq 0}$-graded then $\mathbb{V}$ belongs to $\mathbb{V}$-adMod when considered as a $\mathbb{V}$-module.

Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra. With it one associated the *Zhu Algebra* $\mathfrak{Zh}(\mathbb{V})$ [60] of $\mathbb{V}$, which is the unital associative algebra in the usual sense. As remarked in [29] one may define $\mathfrak{Zh}(\mathbb{V})$ as

$$\mathfrak{Zh}(\mathbb{V}) = \mathfrak{U}_0(\mathbb{V})_0 (= \{a \in \mathfrak{U}_0(\mathbb{V}); \operatorname{ad} H \cdot a = 0\}), \tag{99}$$

or equivalently,

$$\mathfrak{Zh}(\mathbb{V}) = \mathfrak{U}(\mathbb{V})_0 / \overline{\sum_{r>0} \mathfrak{U}(\mathbb{V})_{-r} \mathfrak{U}(\mathbb{V})_r} \tag{100}$$

(see [52, Theorem A.2.11] for the proof of the equivalence with the usual definition).

For a graded $\mathbb{V}$-module M, let M_{top} be the sum of nonzero homogeneous subspace M_d with $M_{d+n} = 0$ for all $n > 0$. Then M_{top} is naturally a module over $\mathfrak{Zh}(\mathbb{V})$. If M is a simple object of $\mathbb{V}$-adMod then $M_{\mathrm{top}} = M_d$ for some d.

Theorem 3.12.1 (Zhu [60])**.** *Let $\mathbb{V}$ be a $\mathbb{Z}_{\geq 0}$-graded vertex algebra. The correspondence $M \mapsto M_{\mathrm{top}}$ gives a bijection between equivalence classes of simple objects of $\mathbb{V}$-adMod and the equivalence classes of irreducible representations of $\mathfrak{Zh}(\mathbb{V})$.*

3.13. Filtration of $\mathfrak{U}(\mathbb{V})$. Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra, F its filtration. Then F induces a filtration $\{F_p\mathbf{U}(\mathbb{V})\}$ of $\mathbf{U}(\mathbb{V})$:

$$F_p\mathbf{U}(\mathbb{V}) = \sum_{p_1+p_2+\cdots+p_r\le p} F_{p_1}\mathfrak{L}(\mathbb{V})\cdot F_{p_2}\mathfrak{L}(\mathbb{V})\cdot\cdots\cdot F_{p_r}\mathfrak{L}(\mathbb{V}).$$

The following assertion follows from Proposition 3.10.1.

Lemma 3.13.1. *There is a natural surjective homomorphism* $\mathbf{U}(\operatorname{gr}^F\mathbb{V})\to\operatorname{gr}^F\mathbf{U}(\mathbb{V})$ *of graded algebra given by the correspondence*

$$\left(\sigma_{p_1}(v^1)\right)_{n_1}\cdot\cdots\cdot\left(\sigma_{p_r}(v^r)\right)_{n_r}\mapsto\sigma_{p_1+\cdots+p_r}\left(v^1_{n_1}\cdot\cdots\cdot v^r_{n_r}\right)$$

with $v^i\in\mathbb{V}$, $n_i\in\mathbb{Z}$. *This is an isomorphism if* F *is strict.*

Let $F_p\widetilde{\mathbf{U}}(\mathbb{V})$ be the degreewise closure of the image of $F_p\mathbf{U}(\mathbb{V})$ in $\widetilde{\mathbf{U}}(\mathbb{V})$. This gives a filtration of $\widetilde{\mathbf{U}}(\mathbb{V})$. Let $\{F_p\mathfrak{U}(\mathbb{V})\}$ be the quotient filtration: $F_p\mathfrak{U}(\mathbb{V})=F_p\widetilde{\mathbf{U}}(\mathbb{V})/F_p\widetilde{\mathbf{B}}(\mathbb{V})$, where $F_p\widetilde{\mathbf{B}}(\mathbb{V})=\widetilde{\mathbf{B}}(\mathbb{V})\cap F_p\widetilde{\mathbf{U}}(\mathbb{V})$. The union $\bigcup_p F_p\mathfrak{U}(\mathbb{V})$ is dense in $\mathfrak{U}(\mathbb{V})$. Let $\{F_p\mathfrak{U}_N(\mathbb{V})\}$ with $N\ge 0$ be the quotient filtration of $\mathfrak{U}_N(\mathbb{V})$. This is an exhaustive filtration.

Remark 3.13.2. If $F_{-1}\mathbb{V}=0$ then $\{F_p\mathfrak{U}_N(\mathbb{V})\}$ is obviously separated. Let $\mathbb{V}$ be a $\mathbb{Z}$-graded, compatibly $\mathbb{Z}_{\ge 0}$-gradable vertex algebra, G the standard filtration. Then from Proposition 3.11.1 one sees that the filtration $\{G_p\mathfrak{U}_N(\mathbb{V})\}$ is separated.

The following assertion follows from Proposition 3.10.1.

Theorem 3.13.3. *Let* $\mathbb{V}$ *be a* $\mathbb{Z}$*-graded vertex algebra,* F *its filtration. Then there is a natural surjective linear map* $\mathfrak{U}_N(\operatorname{gr}^F\mathbb{V})\to\operatorname{gr}^F\mathfrak{U}_N(\mathbb{V})$ *given by the correspondence*

$$\left(\sigma_{p_1}(v^1)\right)_{n_1}\cdot\cdots\cdot\left(\sigma_{p_r}(v^r)\right)_{n_r}\mapsto\sigma_{p_1+\cdots+p_r}\left(v^1_{n_1}\cdot\cdots\cdot v^r_{n_r}\right)$$

with $v^i\in\mathbb{V}$, $n_i\in\mathbb{Z}$ *for each* N. *Moreover if the filtration* F *is strict then this is an isomorphism.*

Let $\widetilde{\operatorname{gr}}^F\mathfrak{U}(\mathbb{V})$ denote the degreewise completion of $\operatorname{gr}^F\mathfrak{U}(\mathbb{V})$. Then the surjection in Theorem 3.13.3 extends to the surjective homomorphism $\mathfrak{U}(\operatorname{gr}^F\mathbb{V})\to\widetilde{\operatorname{gr}}^F\mathfrak{U}(\mathbb{V})$ of compatible degreewise complete algebras, which is an isomorphism if F is strict.

Let $F_p\,\mathfrak{Zh}(\mathbb{V})=\mathfrak{Zh}(\mathbb{V})\cap F_p\mathfrak{U}_0(\mathbb{V})$. This gives the filtration $\mathfrak{Zh}(\mathbb{V})$. The following assertion is obvious from Theorem 3.13.3 and the Definition (99).

Theorem 3.13.4. *Let* $\mathbb{V}$ *be a* $\mathbb{Z}$*-graded vertex algebra,* F *its filtration. Then there is a natural surjective algebra homomorphism* $\mathfrak{Zh}(\operatorname{gr}^F\mathbb{V})\to\operatorname{gr}^F\mathfrak{Zh}(\mathbb{V})$ *given by the correspondence*

$$\left(\sigma_{p_1}(v^1)\right)_0\cdot\cdots\cdot\left(\sigma_{p_r}(v^r)\right)_0\mapsto\sigma_{p_1+\cdots+p_r}\left(v^1_0\cdot\cdots\cdot v^r_0\right)$$

with $v^i\in\mathbb{V}$. *Moreover if the filtration* F *is strict then this is an isomorphism.*

3.14. The PBW theorem for current algebras and Zhu algebras. Let $\mathbb{V}$ be a $\mathbb{Z}$-graded vertex algebra, H its Hamiltonian, F its filtration such that $(\mathbb{V}, F)$ is quasi-commutative. Take a H-invariant subspace U of $\mathbb{V}$ such that its image $\bar{U}$ in $\mathrm{gr}^F U$ strongly generates $\mathrm{gr}^F \mathbb{V}$.

We define the adjoint action on $L(\bar{U})$ (Notation 3.9) by $\mathrm{ad}\, H \cdot u \otimes t^n = (Hu) \otimes t^n - (n+1) u \otimes t^n$. This action extends naturally to the symmetric algebra $S(L(\bar{U}))$:

$$S(L(\bar{U})) = \bigoplus_{d \in \mathbb{Z}} S(L(\bar{U}))_d,$$
$$S(L(\bar{U}))_d = \{a \in S(L(\bar{U}));\, \mathrm{ad}\, H \cdot a = -da\}.$$

Let $\mathbb{S}(\bar{U})$ denote the standard degreewise completion of $S(L(\bar{U}))$:

(101) $$\mathbb{S}(\bar{U}) = \lim_{\overleftarrow{N}} \mathbb{S}_N(\bar{U}),$$

(102) $$\text{where } \mathbb{S}_N(\bar{U}) = S(L(\bar{U}))/\Big(\bigoplus_{d \in \mathbb{Z}} \sum_{r > N} S(L(\bar{U}))_{d-r} S(L(\bar{U}))_r\Big).$$

The space $\mathbb{S}_N(\bar{U})$ can be identified with the subalgebra of $S(L(\bar{U}))$ spanned by the elements of the form

(103) $$\sigma_{p_1}(u^1) \otimes t^{n_1 + \Delta_{u^1} - 1} \cdot \cdots \cdot \sigma_{p_r}(u^r) \otimes t^{n_r + \Delta_{u^r} - 1}$$

with $p_i \in \mathbb{Z}$, $u^i \in U$, $n_i \in \mathbb{Z}$ satisfying

(104) $$n_1 + \cdots + n_r \leq N.$$

Theorem 3.14.1. *Let $\mathbb{V}$, F, U, $\bar{U}$ be as above. Then the map*

$$\mathbb{S}_N(\bar{U}) \to \mathrm{gr}^F \mathfrak{U}_N(\mathbb{V})$$
$$\sigma_{p_1}(u^1) \otimes t^{n_1 + \Delta_{u^1} - 1} \ldots \sigma_{p_r}(u^r) \otimes t^{n_r + \Delta_{u^r} - 1} \mapsto \sigma_{p_1 + \cdots + p_r}\big(u^1_{n_1} \ldots u^r_{n_r}\big)$$

is surjective for each $N \geq 0$. Further, if $\bar{U}$ generates a PBW basis of $\mathrm{gr}^F \mathbb{V}$ *then this is an isomorphism.*

Proof. First, the map

$$\mathbb{S}_N(\bar{U}) \to \mathfrak{U}_N(\mathrm{gr}^F \mathbb{V})$$
$$\sigma_{p_1}(u^1) \otimes t^{n_1 + \Delta_{u^1} - 1} \ldots \sigma_{p_r}(u^r) \otimes t^{n_r + \Delta_{u^r} - 1} \mapsto \sigma_{p_1}\big(u^1_{n_1}\big) \ldots \sigma_{p_r}\big(u^r_{n_r}\big)$$

is surjective, and bijective if $\bar{U}$ generates a PBW basis of $\mathrm{gr}^F \mathbb{V}$. To show this, the proof of [25, Lemma 4.3.2] applies. (In fact the proof is easier because $\mathrm{gr}^F \mathbb{V}$ is supercommutative.) Second, if $\bar{U}$ generates a PBW basis of $\mathrm{gr}^F \mathbb{V}$ then F is strict by Proposition 3.8.2. Therefore the assertion follows from Theorem 3.13.3. □

The map in Theorem 3.14.1 extends to the surjective homomorphism $\mathbb{S}(\bar{U}) \to \widetilde{\mathrm{gr}}^F \mathfrak{U}(\mathbb{V})$ of compatible degreewise complete algebra, which is an isomorphism if F is strict.

If $\mathbb{V}$ is compatibly $\mathbb{Z}_{\geq 0}$-gradable then one can take the standard filtration as the filtration in Theorem 3.14.1.

The following immediately follows from Theorem 3.14.1.

Theorem 3.14.2. *Let $\mathbb{V}$, F, U, $\bar{U}$ be as above. Then the map*

$$S(\bar{U}) \to \mathrm{gr}^F \mathfrak{Zh}(\mathbb{V})$$
$$\sigma_{p_1}(u^1) \cdot \cdots \cdot \sigma_{p_r}(u^r) \mapsto \sigma_{p_1+\cdots+p_r}\left(u^1_0 \cdot \cdots \cdot u^r_0\right)$$

is a surjective algebra homomorphism. Further, if $\bar{U}$ generates a PBW basis then this is an isomorphism.

3.15. BRST construction of vertex algebras. Let $\mathbb{V} = \bigoplus_{i \in \mathbb{Z}} \mathbb{V}^i$ be a vertex algebra with an additional $\mathbb{Z}$-gradation, which is shown by the upper index: $v_{(n)} \cdot \mathbb{V}^j \subset \mathbb{V}^{i+j}$ for $v \in \mathbb{V}^i$, $n \in \mathbb{Z}$, $\mathbb{V}^i = \bigoplus_{\Delta \in \mathbb{Z}} \mathbb{V}^i_{-\Delta}$, where $\mathbb{V}^i_{-\Delta} = \mathbb{V}^i \cap \mathbb{V}_{-\Delta}$.

Suppose there exists an odd operator $Q \in \mathrm{End}(\mathbb{V})$ satisfying the following:

(105) $$Q^2 = 0,$$

(106) $$Q \cdot \mathbb{V}^i \subset \mathbb{V}^{i+1}, \quad Q|0\rangle = 0, \quad [Q, T] = 0,$$

(107) $$[Q, Y(v, z)] = Y(Qv, z), \quad \forall v \in \mathbb{V}.$$

Then $(\mathbb{V}, Q)$ is a cochain complex. Let

$$H^\bullet(\mathbb{V}) := H^\bullet(\mathbb{V}, Q) = \bigoplus_{i \in \mathbb{Z}} H^i(\mathbb{V}, Q).$$

From (107) it follows that the action of the field $Y([v], z)$ with $[v] \in H^\bullet(\mathbb{V})$ is well-defined on $H^\bullet(\mathbb{V})$. Therefore $H^\bullet(\mathbb{V})$ is a vertex algebra and $H^0(\mathbb{V})$ is a vertex subalgebra of $H^\bullet(\mathbb{V})$. We call $(\mathbb{V}, Q)$ a *weak BRST complex* of vertex algebras, and Q the corresponding *BRST operator*. If Q is compatible with a Hamiltonian H of $\mathbb{V}$, that is, if

(108) $$[Q, H] = 0,$$

then $H^\bullet(\mathbb{V})$ is graded by the Hamiltonian H. In this case $(\mathbb{V}, Q)$ is called *weak BRST complex of $\mathbb{Z}$-graded vertex algebras*. Further, if $\mathbb{V}$ is Möbius conformal and the BRST operator Q commutes with T^* then $H^\bullet(\mathbb{V})$ is also Möbius conformal.

A *BRST complex of vertex algebras* is a weak BRST complex $(\mathbb{V}, Q)$ of vertex algebras such that Q coincides with the residue $A_{(0)}$ of the filed $Y(A, z)$ associated with some odd element $A \in \mathbb{V}^1$. In this case the conditions (106) and (107) are automatically satisfied.

3.16. Filtration of vertex algebras and spectral sequences. Let $(\mathbb{V}, Q)$ be a weak BRST complex of vertex algebras. Let F be a filtration of $\mathbb{V}$ such that

$$F_{-1}\mathbb{V} = 0 \quad \text{and} \quad Q \cdot F_p\mathbb{V} \subset F_p\mathbb{V} \quad \text{for all } p. \tag{109}$$

Then the action of Q on $\operatorname{gr}^F \mathbb{V}$ is well-defined and $(\operatorname{gr}^F \mathbb{V}, Q)$ is a weak BRST complex of vertex algebras. Also, F induces a filtration of $H^\bullet(V)$:

$$F_p H^\bullet(\mathbb{V}) = \operatorname{Im} : (H^\bullet(F_p\mathbb{V}) \to H^\bullet(\mathbb{V})). \tag{110}$$

This is an increasing, separated, exhaustive filtration compatible with the vertex algebra structure of $H^\bullet(\mathbb{V})$. Further, there is a natural vertex algebra homomorphism

$$\operatorname{gr}^F H^\bullet(\mathbb{V}) \to H^\bullet(\operatorname{gr}^F \mathbb{V}) \tag{111}$$

given by the correspondence $\sigma_p([v]) \mapsto [\sigma_p(v)]$.

An increasing filtration $\{F_p\mathbb{V}\}$ can be transformed into a decreasing filtration by setting $F^p\mathbb{V} = F_{-p}\mathbb{V}$. Thus we have the following.

Proposition 3.16.1. *Let $\mathbb{V}$, Q, F be as above. Then there exists a converging spectral sequence $E_1^{p,q} \Rightarrow H^\bullet(\mathbb{V})$ such that*

$$E_1^{p,q} = H^{p+q}\left(\operatorname{gr}^F_{-p} \mathbb{V}\right), \quad E_\infty^{p,q} = \operatorname{gr}^F_{-p} H^{p+q}(\mathbb{V}).$$

If $H^i(\operatorname{gr}^F \mathbb{V}) = 0$ for all $i \neq 0$ then the spectral sequence collapses at $E_1 = E_\infty$, and consequently $H^i(\mathbb{V}) = 0$ for $i \neq 0$, and the natural map $\operatorname{gr}^F H^0(\mathbb{V}) \to H^0(\operatorname{gr}^F \mathbb{V})$ is an isomorphism of vertex algebras.

3.17. BRST cohomology of attached Lie algebras. Let $(\mathbb{V}, Q)$ be a weak BRST complex of vertex algebras. The action of Q on $\mathbb{V}$ induces the adjoint action $\operatorname{ad} Q$ on $\mathfrak{L}(\mathbb{V})$:

$$\operatorname{ad} Q \cdot v_{\{n\}} = (Qv)_{\{n\}} \quad \text{for } v \in \mathbb{V}, n \in \mathbb{Z}.$$

We have $(\operatorname{ad} Q)^2 = 0$ and $(\mathfrak{L}(\mathbb{V}), \operatorname{ad} Q)$ can be viewed as a cochain complex. The cohomology $H^\bullet(\mathfrak{L}(\mathbb{V})) = \bigoplus_{i\in\mathbb{Z}} H^i(\mathfrak{L}(\mathbb{V}))$ is naturally a Lie algebra. If $\mathbb{V}$ is graded by H and Q is compatible with H then $H^\bullet(\mathfrak{L}(\mathbb{V}))$ is $\mathbb{Z}$-graded by $\operatorname{ad} H$.

There is a natural Lie algebra homomorphism

$$\mathfrak{L}(H^\bullet(\mathbb{V})) \to H^\bullet(\mathfrak{L}(\mathbb{V})) \tag{112}$$

given by the correspondence $[v]_{\{n\}} \mapsto [v_{\{n\}}]$ with $v \in \mathbb{V}$, $n \in \mathbb{Z}$. Here $[v]_{\{n\}}$ denotes the image of $[v]\otimes t^n$ in $\mathfrak{L}(H^\bullet(\mathbb{V}))$, where $[v]$ is the cohomology class of a cocycle $v \in \mathbb{V}$, and $[v_{\{n\}}]$ denotes the cohomology class of a cocycle $v_{\{n\}} \in \mathfrak{L}(\mathbb{V})$. (We shall use similar convention throughout the paper).

Theorem 3.17.1. *Suppose that* $\operatorname{Vac}\mathbb{V} = \mathbb{C}|0\rangle$, $\operatorname{Vac}H^\bullet(\mathbb{V}) = \mathbb{C}|0\rangle$, $H^0(\mathbb{V}) \neq 0$ *and* $H^i(\mathbb{V}) = 0$ *for all* $i \neq 0$. *Then* $H^i(\mathfrak{L}(\mathbb{V})) = 0$ *for* $i \neq 0$ *the natural map* $\mathfrak{L}(H^0(\mathbb{V})) \to H^0(\mathfrak{L}(\mathbb{V}))$ *is an isomorphism of Lie algebras.*

The following assertion is easily seen.

Lemma 3.17.2. *Let* $\mathbb{V}$ *be a vertex algebra such that* $\operatorname{Vac}\mathbb{V} = \mathbb{C}|0\rangle$. *Then* $\ker(\partial_{-1} : L(\mathbb{V}) \to L(\mathbb{V})) = \mathbb{C}(|0\rangle \otimes 1)$.

Proof of Theorem 3.17.1. First, considering $L(\mathbb{V})$ as a cochain complex with the differential $Q \otimes \mathrm{id}$, we have

$$H^i(L(\mathbb{V})) = \begin{cases} L(H^0(\mathbb{V})) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases} \tag{113}$$

Second, consider the long exact sequence associated with the short exact sequence of cochain complexes

$$0 \to \mathbb{C}(|0\rangle \otimes 1) \to L(\mathbb{V}) \to L(\mathbb{V})/\mathbb{C}(|0\rangle \otimes 1) \to 0.$$

Then by (113) we obtain the exact sequences

(114)
$$0 \to H^i(L(\mathbb{V})/\mathbb{C}(|0\rangle \otimes 1)) \to H^{i+1}(\mathbb{C}(|0\rangle \otimes 1)) \to 0 \quad \text{for } i \neq -1, 0,$$
(115)
$$0 \to H^{-1}(L(\mathbb{V})/\mathbb{C}(|0\rangle \otimes 1)) \to H^0(\mathbb{C}(|0\rangle \otimes 1)) \to L(H^0(\mathbb{V}))$$
$$\to H^0(L(\mathbb{V})/\mathbb{C}(|0\rangle \otimes 1)) \to 0.$$

We obviously have that $H^i(\mathbb{C}(0\rangle \otimes 1)) = 0$ for $i \neq 0$ and

$$H^0(\mathbb{C}(0\rangle \otimes 1)) = \mathbb{C}(|0\rangle \otimes 1). \tag{116}$$

By (116) it follows that the middle map $H^0(\mathbb{C}(|0\rangle \otimes 1)) \to L(H^0(\mathbb{V}))$ in (115) is injective. Therefore (114) and (115) give

$$H^i(L(\mathbb{V})/\mathbb{C}(|0\rangle \otimes 1)) = \begin{cases} L(H^0(\mathbb{V}))/\mathbb{C}(|0\rangle \otimes 1) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases} \tag{117}$$

Third, by assumption and Lemma 3.17.2, we have the following exact sequences:

$$0 \to L(\mathbb{V})/\mathbb{C}(|0\rangle \otimes 1) \xrightarrow{\partial_{-1}} L(\mathbb{V}) \to \mathfrak{L}(\mathbb{V}) \to 0, \tag{118}$$

$$0 \to L(H^0(\mathbb{V}))/\mathbb{C}(|0\rangle \otimes 1) \xrightarrow{\partial_{-1}} L(H^0(\mathbb{V})) \to \mathfrak{L}(H^0(\mathbb{V})) \to 0. \tag{119}$$

The (118) is an exact sequence of cochain complexes. Thus we have the corresponding long exact sequence. This together with (113),(117) gives $H^i(\mathfrak{L}(\mathbb{V})) = 0$ for $i \neq 0, -1$ and the exact sequence

$$0 \to H^{-1}(\mathfrak{L}(\mathbb{V})) \to L(H^0(\mathbb{V}))/\mathbb{C}(|0\rangle \otimes \mathbf{1}) \to L(H^0(\mathbb{V}))$$
$$\to H^0(\mathfrak{L}(\mathbb{V})) \to 0.$$

However the middle map $L(H^0(\mathbb{V}))/\mathbb{C}(|0\rangle \otimes \mathbf{1}) \to L(H^0(\mathbb{V}))$ is the derivation ∂_{-1}. Therefore by (119) this is exact. Hence $H^{-1}(\mathfrak{L}(\mathbb{V}))$ is zero and that $\mathfrak{L}(H^0(\mathbb{V}))$ is isomorphic to $H^0(\mathfrak{L}(\mathbb{V}))$. This completes the proof. □

3.18. BRST cohomology of current algebras. Let $(\mathbb{V}, Q)$ be a weak BRST complex of $\mathbb{Z}$-graded vertex algebras. The action of ad Q extends to a degree-preserving derivation of $\mathfrak{U}(\mathbb{V})$ such that $(\operatorname{ad} Q)^2 = 0$ because Q is odd (cf. (24)). Thus each $(\mathfrak{U}_N(\mathbb{V}), \operatorname{ad} Q)$ is a cochain complex, which is $\mathbb{Z}$-graded by ad H:

$$H^\bullet(\mathfrak{U}_N(\mathbb{V})) = \bigoplus_{d\in\mathbb{Z}} H^\bullet(\mathfrak{U}_N(\mathbb{V}))_d,$$
$$H^\bullet(\mathfrak{U}_N(\mathbb{V}))_d = \{[v] \in H^\bullet(\mathfrak{U}_N(\mathbb{V})); \operatorname{ad} H \cdot [v] = -d[v]\}.$$

This makes a projective system $\{H^\bullet(\mathfrak{U}_N(\mathbb{V}))_d\}$ of linear spaces for each d. Set

(120)
$$H^\bullet(\mathfrak{U}(\mathbb{V})) = \bigoplus_{d\in\mathbb{Z}} H^\bullet(\mathfrak{U}(\mathbb{V}))_d, \quad \text{where } H^\bullet(\mathfrak{U}(\mathbb{V}))_d = \lim_{\substack{\leftarrow \\ N}} H^\bullet(\mathfrak{U}_N(\mathbb{V}))_d.$$

There is a natural linear map

(121)
$$\mathfrak{U}_N(H^\bullet(\mathbb{V})) \to H^\bullet(\mathfrak{U}_N(\mathbb{V}))$$

for each N given by the correspondence

$$[v^1]_{n_1} \dots [v^r]_{n_r} \mapsto \left[v^1_{n_1} \dots v^r_{n_r}\right].$$

Let F a filtration of $\mathbb{V}$ which is compatible with H such that $F_{-1}\mathbb{V} = 0$ and $Q \cdot F_p\mathbb{V} \subset F_p\mathbb{V}$ for all p. Then the corresponding filtration $\{F_p\mathfrak{U}_N(\mathbb{V})\}$ satisfies

(122)
$$F_{-1}\mathfrak{U}_N(\mathbb{V}) = 0, \quad \operatorname{ad} Q \cdot F_p\mathfrak{U}_N(\mathbb{V}) \subset F_p\mathfrak{U}_N(\mathbb{V}).$$

Thus we have the following.

Proposition 3.18.1. *Let $\mathbb{V}$, Q, F be as above. There exists a converging spectral sequence $E_1^{p,q} \Rightarrow H^\bullet(\mathfrak{U}_N(\mathbb{V}))$ such that*

$$E_1^{p,q} = H^{p+q}\left(\operatorname{gr}^F_{-p}\mathfrak{U}_N(\mathbb{V})\right), \quad E_\infty^{p,q} = \operatorname{gr}^F_{-p} H^{p+q}(\mathfrak{U}_N(\mathbb{V})).$$

If $H^i(\mathrm{gr}^F \mathfrak{U}_N(\mathbb{V})) = 0$ for all $i \neq 0$ then the spectral sequence collapses at $E_1 = E_\infty$, and consequently $H^i(\mathfrak{U}_N(\mathbb{V})) = 0$ for $i \neq 0$, and the natural map

$$\mathrm{gr}^F H^0(\mathfrak{U}_N(\mathbb{V})) \to H^0\left(\mathrm{gr}^F \mathfrak{U}_N(\mathbb{V})\right)$$

given by the correspondence $\sigma_p([a]) \mapsto [\sigma_p(a)]$ is a linear isomorphism.

Proposition 3.18.2. *Let $\mathbb{V}$, Q, F be as above. Assume that*

- *the filtration F of $\mathbb{V}$ is strict;*
- *the filtration of $H^\bullet(\mathbb{V})$ induced by F is also strict;*
- *the natural map $\mathrm{gr}^F H^\bullet(\mathbb{V}) \to H^\bullet(\mathrm{gr}^F \mathbb{V})$ is an isomorphism of vertex algebras;*
- *the natural map $\mathrm{gr}^F H^\bullet(\mathfrak{U}_N(\mathbb{V})) \to H^\bullet(\mathrm{gr}^F \mathfrak{U}_N(\mathbb{V}))$ is an linear isomorphism;*
- *the natural map $\mathfrak{U}_N(H^\bullet(\mathrm{gr}^F \mathbb{V})) \to H^\bullet(\mathfrak{U}_N(\mathrm{gr}^F \mathbb{V}))$ is a linear isomorphism.*

Then the natural map $\mathfrak{U}_N(H^\bullet(\mathbb{V})) \to H^\bullet(\mathfrak{U}_N(\mathbb{V}))$ is also a linear isomorphism.

Proof. The filtration $\{F_p H^\bullet(\mathbb{V})\}$ of $H^\bullet(\mathbb{V})$ induces a filtration $\{F_p \mathfrak{U}_N(H^\bullet(\mathbb{V}))\}$ of $\mathfrak{U}_N(H^\bullet(\mathbb{V}))$. The natural map $\mathfrak{U}_N(H^\bullet(\mathbb{V})) \to H^\bullet(\mathfrak{U}_N(\mathbb{V}))$ preserves the filtration. Therefore it is sufficient to show that the induced homomorphism $\mathrm{gr}^F \mathfrak{U}_N(H^\bullet(\mathbb{V})) \to \mathrm{gr}^F H^\bullet(\mathfrak{U}_N(\mathbb{V}))$ is an isomorphism.

We have

$$\mathrm{gr}^F \mathfrak{U}_N(H^\bullet(\mathbb{V})) \cong \mathfrak{U}_N(\mathrm{gr}^F H^\bullet(\mathbb{V})) \cong \mathfrak{U}_N(H^\bullet(\mathrm{gr}^F \mathbb{V})), \tag{123}$$

$$\mathrm{gr}^F H^\bullet(\mathfrak{U}_N(\mathbb{V})) \cong H^\bullet\left(\mathrm{gr}^F \mathfrak{U}_N(\mathbb{V})\right) \cong H^\bullet\left(\mathfrak{U}_N(\mathrm{gr}^F \mathbb{V})\right) \tag{124}$$

by assumption and Theorem 3.13.3. By chasing the isomorphisms one finds that it is enough to show that $\mathfrak{U}_N(H^\bullet(\mathrm{gr}^F \mathbb{V})) \cong H^\bullet(\mathfrak{U}_N(\mathrm{gr}^F \mathbb{V}))$. But this is contained in the assumption. □

3.19. Our favorite cases

Theorem 3.19.1. *Let $(\mathbb{V}, Q)$ be a weak BRST complex of $\mathbb{Z}$-graded vertex algebras, H the Hamiltonian. Assume that there exists an H-invariant subspace U and a filtration F of $\mathbb{V}$ (compatible with H) satisfying the following:*

(a) $F_{-1}\mathbb{V} = 0$;
(b) $(\mathbb{V}, F)$ *is quasi-commutative;*
(c) $Q \cdot F_p \mathbb{V} \subset F_p \mathbb{V}$ *for all p;*
(d) *The image $\bar{U}$ of U in $\mathrm{gr}^F \mathbb{V}$ is preserved by the action of Q. Further, $H^i(\bar{U}) = 0$ for $i \neq 0$;*
(e) *$\bar{U}$ generates a PBW basis of the commutative vertex algebra $\mathrm{gr}^F \mathbb{V}$, so that U generates a PBW basis of $\mathbb{V}$.*

Then we have the following:

(i) $H^i(\mathbb{V}) = 0$ *and* $H^i(\operatorname{gr}^F \mathbb{V}) = 0$ *for* $i \neq 0$.
(ii) *F induces a strict filtration* $\{F_p H^0(\mathbb{V})\}$ *of the* $\mathbb{Z}$*-graded vertex algebra* $H^0(\mathbb{V})$ *such that* $(H^0(\mathbb{V}), F)$ *is quasi-commutative.*
(iii) *The natural map* $\operatorname{gr}^F H^0(\mathbb{V}) \to H^0(\operatorname{gr}^F \mathbb{V})$ *is an isomorphism of vertex algebras.*
(iv) *The natural map* $H^0(\bar{U}) \to H^0(\operatorname{gr}^F \mathbb{V})$ *is injective and its image generates a PBW basis of* $H^0(\operatorname{gr}^F \mathbb{V}) = \operatorname{gr}^F H^0(\mathbb{V})$.
(v) *For each N,* $H^i(\mathfrak{U}_N(\mathbb{V})) = 0$ *with* $i \neq 0$.
(vi) *For each N, the natural map* $\operatorname{gr}^F H^0(\mathfrak{U}_N(\mathbb{V})) \to H^0(\mathfrak{U}_N(\operatorname{gr}^F \mathbb{V}))$ *is an isomorphism.*
(vii) *For each N, the natural map* $\mathfrak{U}_N(H^0(\mathbb{V})) \to H^0(\mathfrak{U}_N(\mathbb{V}))$ *is an isomorphism.*

Proof. First, we calculate $H^\bullet(\operatorname{gr}^F \mathbb{V})$. By assumption, there is the isomorphism

$$S(\bar{U} \otimes \mathbb{C}[t^{-1}]t^{-1}) \xrightarrow{\sim} \operatorname{gr}^F \mathbb{V} \tag{125}$$

given by (77). Consider $\bar{U} \otimes \mathbb{C}[t^{-1}]t^{-1}$ as a cochain complex with the differential $Q \otimes 1$. Extend the action of Q to an odd derivation of $S(\bar{U} \otimes \mathbb{C}[t^{-1}]t^{-1})$. Then (125) is a cochain map. Because

$$H^i(\bar{U} \otimes \mathbb{C}[t^{-1}]t^{-1}) = \begin{cases} H^0(\bar{U}) \otimes \mathbb{C}[t^{-1}]t^{-1} & (i = 0), \\ 0 & (i \neq 0) \end{cases} \tag{126}$$

by assumption, it follows that

$$\begin{aligned} H^i(\operatorname{gr}^F \mathbb{V}) &= H^i(S(\bar{U} \otimes \mathbb{C}[t^{-1}]t^{-1})) \\ &= \begin{cases} S(H^0(\bar{U}) \otimes \mathbb{C}[t^{-1}]t^{-1}) & (i = 0), \\ 0 & (i \neq 0), \end{cases} \end{aligned} \tag{127}$$

see [40, Lemma 3.2]. Applying Proposition 3.16.1, the assertion (127) for $i \neq 0$ proves (i) and (iii), while the assertion (127) for $i = 0$ proves (iv). We have also proved (ii) because the only non-trivial assertion is the strictness of the filtration and this follows from (iv) and Proposition 3.8.2.

The assertions (v), (vi) and (vii) remain to be proved. For this we compute the cohomology $H^\bullet(\operatorname{gr}^F \mathfrak{U}_N(\mathbb{V}))$. By Theorem 3.14.1 and (ii), we have

$$H^\bullet\big(\operatorname{gr}^F \mathfrak{U}_N(\mathbb{V})\big) = H^\bullet(\mathbb{S}_N(\bar{U})). \tag{128}$$

Here, $L(\bar{U})$ is considered as a complex with the differential $Q \otimes \mathbf{1}$, and its action is extended to $\mathbb{S}_N(\bar{U})$ as an odd derivation. By definition, we have the exact sequence of cochain maps

$$0 \to S(L(\bar{U})) \sum_{r>N} S(L(\bar{U})_r) \to S(L(\bar{U})) \to \mathbb{S}_N(\bar{U}) \to 0. \tag{129}$$

But the differential is degree-preserving, therefore by identifying $\mathbb{S}_N(\bar{U})$ with the subalgebra of $S(L(\bar{U}))$ as in Sect. 3.14, $\mathbb{S}_N(\bar{U})$ can be identified with a subcomplex of $S(L(\bar{U}))$. Namely, the exact sequence (129) splits;

$$H^\bullet(S(L(\bar{U}))) = H^\bullet\Big(S(L(\bar{U}))\sum_{r>N} S(L(\bar{U})_r)\Big) \oplus H^\bullet(\mathbb{S}_N(\bar{U})). \tag{130}$$

On the other hand we have

$$H^\bullet(S(L(\bar{U}))) = \begin{cases} S(L(H^0(\bar{U}))) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases} \tag{131}$$

in the same manner as (127). Together with (130), this gives

$$H^i(\mathbb{S}_N(\bar{U})) = \begin{cases} \mathbb{S}_N(H^0(\bar{U})) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases} \tag{132}$$

Thus we can apply Proposition 3.18.1 (ii) to obtain $H^i(\mathfrak{U}_N(\mathbb{V})) = 0$ for $i \neq 0$ and the isomorphism

$$\begin{aligned} \operatorname{gr}^F H^0(\mathfrak{U}_N(\mathbb{V})) &\xrightarrow{\sim} H^0\left(\operatorname{gr}^F \mathfrak{U}_N(\mathbb{V})\right) \\ &\xrightarrow{\sim} H^0\left(\mathfrak{U}_N(\operatorname{gr}^F \mathbb{V})\right) \xrightarrow{\sim} \mathbb{S}_N(H^0(\bar{U})). \end{aligned} \tag{133}$$

Here the second isomorphism follows from (ii) and Theorem 3.13.3 and the last isomorphism follows from (132) (iv) and Theorem 3.14.1. We have proved (v) and (vi). Finally, because $H^0(\bar{U})$ generates a PBW basis of $H^0(\operatorname{gr}^F \mathbb{V})$,

$$\mathfrak{U}_N(H^0(\operatorname{gr}^F \mathbb{V})) \cong \mathbb{S}_N(H^0(\bar{U})),$$

by Theorem 3.14.1. This, together with (133), implies that the natural map $\mathfrak{U}_N(H^0(\operatorname{gr}^F \mathbb{V})) \to H^0(\mathfrak{U}_N(\operatorname{gr}^F \mathbb{V}))$ is an isomorphism. Therefore we conclude that all the conditions in Proposition 3.18.2 are satisfied. Hence (vii) is proved. This completes the proof. □

If the assumption of Theorem 3.19.1 is satisfied then the isomorphism in (viii) extends to the isomorphism

$$\mathfrak{U}(H^0(\mathbb{V})) \xrightarrow{\sim} H^0(\mathfrak{U}(\mathbb{V})). \tag{134}$$

In particular, in this case $H^0(\mathfrak{U}(\mathbb{V}))$ has the compatible degreewise complete algebra structure.

Theorem 3.19.2. *Let* $(\mathbb{V}, Q)$ *be a weak BRST complex of* $\mathbb{Z}_{\geq 0}$*-graded vertex algebra, H the Hamiltonian. Assume that there exists an H-invariant subspace U satisfying the following:*

(a) *U generates a PBW basis of* $\mathbb{V}$*;*
(b) $Q \cdot U \subset U$*, so that U is a subcomplex of V. Further,* $H^i(U) = 0$ *for* $i \neq 0$.

Let G be the standard filtration of $\mathbb{V}$*. Then we have the following:*

(i) $H^i(\mathbb{V}) = 0$ *and* $H^i(\mathrm{gr}^G \mathbb{V}) = 0$ *for* $i \neq 0$.
(ii) *The filtration* $\{G_p H^0(\mathbb{V})\}$ *induced by G is the standard filtration of* $H^0(\mathbb{V})$.
(iii) *The natural map*

$$\mathrm{gr}^G H^0(\mathbb{V}) \to H^0(\mathrm{gr}^G \mathbb{V})$$

is an isomorphism of vertex algebras.
(iv) *The natural map* $H^0(U) \to H^0(\mathbb{V})$ *is injective and its image generates a PBW basis of* $H^0(\mathbb{V})$.
(v) *For each N,* $H^i(\mathfrak{U}_N(\mathbb{V})) = 0$ *with* $i \neq 0$.
(vi) *For each N, the natural map* $\mathrm{gr}^G H^0(\mathfrak{U}_N(\mathbb{V})) \to H^0(\mathfrak{U}_N(\mathrm{gr}^G \mathbb{V}))$ *is an linear isomorphism.*
(vii) *For each N, the natural map* $\mathfrak{U}_N(H^0(\mathbb{V})) \xrightarrow{\sim} H^0(\mathfrak{U}_N(\mathbb{V}))$ *is a linear isomorphism.*

Proof. If we take G as the filtration F in Theorem 3.19.1 then (U, G) satisfies the conditions (a)–(e) in Theorem 3.19.1. Indeed, because $Q \cdot U \subset U$, it follows that $Q \cdot G_p \mathbb{V} \subset G_p \mathbb{V}$, from the definition of the standard filtration (see Theorem 3.5.1). So we only need to show that the condition (d) is satisfied, that is,

$$H^\bullet(U) \cong H^\bullet(\bar{U}). \tag{135}$$

Because U generate a PBW basis of $\mathbb{V}$, $\sigma_{\Delta_u}(u) \neq 0$ for a nonzero element of $u \in U$. Because $\Delta_{Qu} = \Delta_u$, this implies that $\sigma_{\Delta_u}(Qu) = 0$ means $Qu = 0$. Similarly, if $\sigma_{\Delta_u}(u) = Q\sigma_{\Delta_u}(u') = \sigma_{\Delta_u}(Qu')$ for $u, u' \in U$, then $u = Qu'$. Thus (135) is proved. Therefore all the assertions except for (ii) follow directly from Theorem 3.19.1. But (ii) also, follows from (iii), (iv) and the definition of the standard filtration. □

4. $\mathscr{W}$-algebras

Throughout this paper, k represents a complex number (= the level of the affine Lie algebra $\mathfrak{g}$ associated with $\bar{\mathfrak{g}}$). There is no restriction on k unless otherwise stated.

4.1. Affine Lie algebras (see [32] for details). We freely use the notation of Sect. 2. Let $\mathfrak{g}$ be the (non-twisted) affine Lie algebra associated with $(\bar{\mathfrak{g}}, (\ ,\ ))$. This is the Lie algebra given by

$$\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$$

with the commutation relations

$$[X(m), Y(n)] = [X, Y](m+n) + m\delta_{m+n,0}(X, Y)K,$$
$$[\mathbf{D}, X(m)] = mX(m), \quad [K, \mathfrak{g}] = 0$$

for $X, Y \in \bar{\mathfrak{g}}$, $m, n \in \mathbb{Z}$. Here

(136) $$X(n) = X \otimes t^n \quad \text{for } X \in \bar{\mathfrak{g}}, n \in \mathbb{Z}.$$

The subalgebra $\bar{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g}$ is naturally identified with $\bar{\mathfrak{g}}$. The invariant symmetric bilinear form $(\mid)$ is extended from $\bar{\mathfrak{g}}$ to $\mathfrak{g}$ as follows:

$$(X(m)|Y(n)) = (X, Y)\delta_{m+n,0}, \quad (\mathbf{D}|K) = 1,$$
$$(X(m)|\mathbf{D}) = (X(m)|K) = (\mathbf{D}|\mathbf{D}) = (K|K) = 0.$$

Fix the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ in the standard way. That is,

$$\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D},$$
$$\mathfrak{g}_- = \bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}] \oplus \bar{\mathfrak{h}} \otimes \mathbb{C}[t^{-1}]t^{-1} \oplus \bar{\mathfrak{n}}_+ \otimes \mathbb{C}[t^{-1}]t^{-1},$$
$$\mathfrak{g}_+ = \bar{\mathfrak{n}}_- \otimes \mathbb{C}[t]t \oplus \bar{\mathfrak{h}} \otimes \mathbb{C}[t]t \oplus \bar{\mathfrak{n}}_+ \otimes \mathbb{C}[t].$$

Let

(137) $$\mathfrak{h}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$$

be the dual of $\mathfrak{h}$. Here, Λ_0 and δ are dual elements of K and $\mathbf{D}$, respectively. For $\lambda \in \mathfrak{h}^*$, the number $\langle \lambda, K \rangle$ is called the *level of* λ. Let $\mathfrak{h}^*_k$ denote the set of the weights of level k:

(138) $$\mathfrak{h}^*_k := \{\lambda \in \mathfrak{h}^*; \langle \lambda, K \rangle = k\}.$$

Let $\bar{\lambda}$ be the restriction of $\lambda \in \mathfrak{h}^*$ to $\bar{\mathfrak{h}}^*$. We refer to $\bar{\lambda}$ as the *classical part* of λ.

Let Δ be the set of roots of $\mathfrak{g}$, Δ_+ the set of positive roots, $\Delta_- = -\Delta_+$. Then, $\Delta = \Delta^{\mathrm{re}} \sqcup \Delta^{\mathrm{im}}$, where Δ^{re} is the set of real roots and Δ^{im} is the set of imaginary roots. Let Π be the standard basis of Δ^{re}, $\Delta^{\mathrm{re}}_\pm = \Delta^{\mathrm{re}} \cap \Delta_\pm$, $\Delta^{\mathrm{im}}_\pm = \Delta^{\mathrm{im}} \cap \Delta_\pm$. Then

$$\Delta^{\mathrm{re}}_+ = \{\alpha + n\delta; \alpha \in \bar{\Delta}_+,\ n \geq 0\} \sqcup \{-\alpha + n\delta; \alpha \in \bar{\Delta}_+,\ n \geq 1\}.$$

Let Q be the root lattice, $Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0}\alpha \subset Q$. We define a partial ordering $\mu \leq \lambda$ on $\mathfrak{h}^*$ by $\lambda - \mu \in Q_+$.

Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$ generated by the reflections s_α with $\alpha \in \Delta^{\mathrm{re}}$, where $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\lambda \in \mathfrak{h}^*$. We have $W = \bar{W} \ltimes \bar{Q}^\vee$. Let $\widetilde{W}$ be the extended Weyl group of $\mathfrak{g}$: $\widetilde{W} = \bar{W} \ltimes \bar{P}^\vee$. We write t_μ for the element of $\widetilde{W}$ corresponding to $\mu \in \bar{P}^\vee$:

$$t_\mu(\lambda) = \lambda + \langle \lambda, K \rangle \mu - \left(\langle \lambda, \mu \rangle + \frac{1}{2}|\mu|^2 \langle \lambda, K \rangle \right) \delta \quad \text{for } \lambda \in \mathfrak{h}^*.$$

Let

(139) $$\widetilde{W}_+ := \left\{ w \in \widetilde{W};\, \Delta^{\mathrm{re}}_+ \cap w^{-1}\left(\Delta^{\mathrm{re}}_-\right) = \emptyset \right\}.$$

Then $\widetilde{W} = \widetilde{W}_+ \ltimes W$.

The dot action of $\widetilde{W}$ on $\mathfrak{h}^*$ is defined by $w \circ \lambda = w(\lambda + \rho) - \rho$, where $\rho = \bar{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$.

For $\lambda \in \mathfrak{h}^*$, let

(140) $$\Delta(\lambda) := \{\alpha \in \Delta^{\mathrm{re}};\, \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\},$$

(141) $$\Delta_+(\lambda) := \Delta(\lambda) \cap \Delta_+,$$

(142) $$W(\lambda) := \langle s_\alpha;\, \alpha \in \Delta(\lambda) \rangle \subset W.$$

One knows that $W(\lambda)$ is a Coxeter subgroup of W and it is called the *integral Weyl group* of $\lambda \in \mathfrak{h}^*$.

4.2. Graded duals. Let M be any semisimple $\mathfrak{h}$-module. We write M^λ for the weight space of M of weight λ:

(143) $$M^\lambda := \{m \in M;\, hm = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}.$$

Let $P(M) = \{\lambda \in \mathfrak{h}^*;\, M^\lambda \neq 0\}$, the set of weights of M. If $\dim M^\lambda < \infty$ for all $\lambda \in P(M)$, then we define the graded dual M^* of M by

(144) $$M^* = \bigoplus_\lambda \mathrm{Hom}_{\mathbb{C}}(M^\lambda, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}).$$

It is clear that $((M)^*)^* = M$.

4.3. Universal affine vertex algebras associated with Lie algebras. Define a $\mathfrak{g}$-module $V_k(\bar{\mathfrak{g}})$ by

(145) $$V_k(\bar{\mathfrak{g}}) := U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D})} \mathbb{C}.$$

Here, $\mathbb{C}$ is considered as a $\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$-module on which $\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{D}$ acts trivially and K acts as the multiplication by k.

It is well-known that the space $V_k(\bar{\mathfrak{g}})$ has the natural vertex algebra structure: The vacuum vector is given by $|0\rangle = 1 \otimes 1$; The translation operator is defined by the relations

(146) $$T|0\rangle = 0, \quad [T, J(n)] = -nJ(n-1) \quad \text{for } J \in \bar{\mathfrak{g}} \text{ and } n \in \mathbb{Z};$$

The filed corresponding to $J(-1)|0\rangle$ with $J \in \bar{\mathfrak{g}}$ is

$$J(z) := \sum_{n\in\mathbb{Z}} J(n)z^{-n-1} \tag{147}$$

and the fields corresponding to other vectors determined by (147) in the sense of the reconstruction theorem [33, Theorem 4.5], [25, Theorem 4.4.1]: Explicitly, they are given by

$$\begin{aligned} &Y(J_{a_1}(-n_1-1)\dots J_{a_r}(-n_r-1)|0\rangle, z) \\ &\qquad = \frac{1}{n_1!\dots n_r!} : \partial_z^{n_1} J_{a_1}(z)\dots\partial_z^{n_r} J_{a_r}(z) : \end{aligned}$$

for $a_i \in \bar{I} \sqcup \bar{\Delta}, n_i \geq 0$. Here $\partial_z = \frac{d}{dz}$ and $: a(z)b(z) :$ is the normally ordered product [33, Section 3.1].

The vertex algebra $V_k(\bar{\mathfrak{g}})$ is Möbius conformal: The Hamiltonian is $-\mathbf{D}$; The operator T^* is defined by the relations

$$T^*|0\rangle = 0, \quad [T^*, J(n)] = -nJ(n+1) \quad \text{for } J \in \bar{\mathfrak{g}} \text{ and } n \in \mathbb{Z}, \tag{148}$$

see [33, Remark 5.7.d].

The Möbious conformal vertex algebra $V_k(\bar{\mathfrak{g}})$ is called the *universal affine vertex algebra at level k* associated with $\bar{\mathfrak{g}}$.

It is clear that $V_k(\bar{\mathfrak{g}})$ is $\mathbb{Z}_{\geq 0}$-graded:

$$V_k(\bar{\mathfrak{g}}) = \bigoplus_{\Delta\in\mathbb{Z}_{\geq 0}} V_k(\bar{\mathfrak{g}})_{-\Delta} \tag{149}$$

$$V_k(\bar{\mathfrak{g}})_0 = \mathbb{C}|0\rangle, \quad \dim V_k(\bar{\mathfrak{g}})_{-\Delta} < \infty \text{ for all } \Delta. \tag{150}$$

The standard filtration of $V_k(\bar{\mathfrak{g}})$ coincides with the one induced by the standard filtration of $U(\bar{\mathfrak{g}} \otimes \mathbb{C}[t^{-1}]t^{-1})$ (in the usual sense). The subspace $V_k(\bar{\mathfrak{g}})_{-1}$ generates a PBW basis of $V_k(\bar{\mathfrak{g}})$ (see Sect. 3.6).

For $k \neq -h^\vee$ $V_k(\bar{\mathfrak{g}})$ is conformal by the Sugawara construction: The conformal vector is $\omega_{\bar{\mathfrak{g}}} = 1/2(k+h^\vee)\sum_{a\in\bar{I}\sqcup\bar{\Delta}} J_a(-1)J^a(-1)|0\rangle$, where $\{J^a; a \in \bar{I} \sqcup \bar{\Delta}\}$ is a basis dual to $\{J_a\}$. We write

$$Y(\omega_{\bar{\mathfrak{g}}}, z) = L^{\mathfrak{g}}(z) = \sum_{n\in\mathbb{Z}} L^{\mathfrak{g}}(n)z^{-n-2}. \tag{151}$$

4.4. Affine clifford algebras and the corresponding vertex algebras. Define the nilpotent subalgebra $L\bar{\mathfrak{n}}_\pm$ of $\mathfrak{g}$ by

$$L\bar{\mathfrak{n}}_\pm = \bar{\mathfrak{n}}_\pm \otimes \mathbb{C}[t, t^{-1}]. \tag{152}$$

Let $\mathcal{C}l$ be the Clifford algebra associated with $L\bar{\mathfrak{n}}_+ \oplus L\bar{\mathfrak{n}}_-$ and its symmetric bilinear form defined by the identification $(L\bar{\mathfrak{n}}_\pm)^* = L\bar{\mathfrak{n}}_\mp$ through $(\cdot|\cdot)$.

As $\overline{\mathcal{C}l}$ (see Sect. 2.4), $\mathcal{C}l$ may be defined as the superalgebra with

odd generators: $\psi_\alpha(n)$ with $\alpha \in \bar{\Delta}, n \in \mathbb{Z}$,

relations: $[\psi_\alpha(m), \psi_\beta(n)] = \delta_{\alpha+\beta,0}\delta_{m+n,0}$ with $\alpha, \beta \in \bar{\Delta}, m, n \in \mathbb{Z}$.

Here $\psi_\alpha(n)$ is regarded as the element of $\mathcal{C}l$ corresponding to the vector $J_\alpha(n) \in \mathfrak{g}$.

The algebra $\mathcal{C}l$ contains the Grassmann algebra $\Lambda(L\bar{\mathfrak{n}}_\pm)$ of $L\bar{\mathfrak{n}}_\pm$ as its subalgebra: $\Lambda(L\bar{\mathfrak{n}}_\pm) = \langle \psi_\alpha(n); \alpha \in \bar{\Delta}_\pm, n \in \mathbb{Z}\rangle$. One has

(153) $$\mathcal{C}l = \Lambda(L\bar{\mathfrak{n}}_+) \otimes \Lambda(L\bar{\mathfrak{n}}_-)$$

as linear spaces.

In view of (153), the adjoint action of $\mathfrak{h}$ on $L\bar{\mathfrak{n}}_\pm$ induces an action of $\mathfrak{h}$ on $\mathcal{C}l$. In particular there is the adjoint action of $\mathbf{D}$ on $\mathcal{C}l$.

Let $\mathcal{F}$ be the irreducible representation of $\mathcal{C}l$ generated by a vector $\mathbf{1}$ such that

(154) $$\psi_\alpha(n)\mathbf{1} = 0 \quad \text{if } \alpha + n\delta \in \Delta_+^{\mathrm{re}}.$$

Then

(155) $$\mathcal{F} = \Lambda(L\bar{\mathfrak{n}}_- \cap \mathfrak{g}_-) \otimes \Lambda(L\bar{\mathfrak{n}}_+ \cap \mathfrak{g}_-)$$

as linear spaces. We regard $\mathcal{F}$ as an $\mathfrak{h}$-module under this identification:

(156) $$\begin{aligned}&\mathcal{F} = \textstyle\bigoplus_{\lambda \in -Q_+} \mathcal{F}^\lambda,\\ &\mathbf{1} \in \mathcal{F}^0, \quad \psi_\alpha(n)\mathcal{F}^\lambda \subset \mathcal{F}^{\lambda+\alpha+n\delta} \quad \text{for } \alpha \in \bar{\Delta},\ n \in \mathbb{Z}.\end{aligned}$$

Then $\mathcal{F}^\lambda$ is finite-dimensional for all λ.

The space $\mathcal{F}$ is naturally a conformal vertex superalgebra: The vacuum vector is $\mathbf{1}$; The translation operator T is defined by the relations

$$\begin{aligned}&T\mathbf{1} = 0,\\ &[T, \psi_\alpha(n)] = -n\psi_\alpha(n-1) \quad \text{for } \alpha \in \bar{\Delta}_+, n \in \mathbb{Z},\\ &[T, \psi_\alpha(n)] = -(n-1)\psi_\alpha(n-1) \quad \text{for } \alpha \in \bar{\Delta}_-, n \in \mathbb{Z};\end{aligned}$$

The Hamiltonian is $-\mathbf{D}$; The fields are determined by the reconstruction theorem and the following:

(157) $$Y(\psi_\alpha(-1)\mathbf{1}, z) = \psi_\alpha(z) := \sum_{n\in\mathbb{Z}} \psi_\alpha(n) z^{-n-1} \quad \text{for } \alpha \in \bar{\Delta}_+,$$

(158) $$Y(\psi_\alpha(0)\mathbf{1}, z) = \psi_\alpha(z) := \sum_{n\in\mathbb{Z}} \psi_\alpha(n) z^{-n} \text{ for } \alpha \in \bar{\Delta}_-;$$

The conformal vector is chosen as $\omega_{\mathcal{F}} = \sum_{\alpha\in\bar{\Delta}_+} \psi_{-\alpha}(-1)\psi_{\alpha}(-1)|0\rangle$. We write

$$(159)\qquad Y(\omega_{\mathcal{F}}, z) = L^f(z) = \sum_{n\in\mathbb{Z}} L^f(z)z^{-n-2}.$$

The vertex superalgebra $\mathcal{F}$ is $\mathbb{Z}_{\geq 0}$-graded:

$$(160)\qquad \mathcal{F} = \bigoplus_{\Delta\mathfrak{n}\mathbb{Z}_{\geq 0}} \mathcal{F}_{-\Delta}, \quad \dim \mathcal{F}_{-\Delta} < \infty \text{ for all } \Delta.$$

Denote by $U_{\mathcal{F}}$ the subspace of $\mathcal{F}$ spanned by $\psi_\alpha(-1)\mathbf{1}$, $\psi_{-\alpha}(0)\mathbf{1}$ with $\alpha\in\bar{\Delta}_+$. Then $U_{\mathcal{F}}$ generates a PBW basis of $\mathcal{F}$.

4.5. The BRST complex of quantized Drinfeld–Sokolov reduction. Recall that a tensor product of vertex superalgebras is naturally a vertex superalgebra ([25, Lemma 1.3.6]).

Define a vertex algebra $C_k(\bar{\mathfrak{g}})$ by

$$(161)\qquad C_k(\bar{\mathfrak{g}}) := V_k(\bar{\mathfrak{g}}) \otimes \mathcal{F}.$$

The vacuum vector $|0\rangle\otimes\mathbf{1}$ is also denoted by $|0\rangle$. The vertex algebra $C_k(\bar{\mathfrak{g}})$ is naturally Möbious conformal with the diagonal action of $\mathfrak{sl}_2(\mathbb{C})$.

We consider $C_k(\bar{\mathfrak{g}})$ as an $\mathfrak{h}$-module by the tensor product action. Then $-\mathbf{D}$ is the Hamiltonian. The vertex superalgebra $C_k(\bar{\mathfrak{g}})$ is clearly $\mathbb{Z}_{\geq 0}$-graded; The subspace

$$(162)\qquad U = V_k(\bar{\mathfrak{g}})_{-1} \oplus U_{\mathcal{F}}$$

generates a PBW basis of $C_k(\bar{\mathfrak{g}})$. As before, we often omit the tensor product symbol: $V_k(\bar{\mathfrak{g}})_{-1} = V_k(\bar{\mathfrak{g}})_{-1}\otimes\mathbb{C}\mathbf{1}$, $U_{\mathcal{F}} = |0\rangle\otimes U_{\mathcal{F}}$. We have

$$(163)\qquad \operatorname{Vac} C_k(\bar{\mathfrak{g}}) = \mathbb{C}|0\rangle$$

by Proposition 3.8.2.

Define

$$(164)\qquad Q_+^{\mathrm{st}}(z) = Q_+^{\mathrm{st}}(z) + \chi_+(z) \in (\operatorname{End} C_k(\bar{\mathfrak{g}}))[[z, z^{-1}]],$$

by

$$(165)\qquad Q_+^{\mathrm{st}}(z) = \sum_{n\in\mathbb{Z}} Q_+^{\mathrm{st}}(n)z^{-n-1}$$
$$:= \sum_{\alpha\in\bar{\Delta}_+} J_\alpha(z)\psi_{-\alpha}(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\bar{\Delta}_+} c_{\alpha,\beta}^{\gamma}\psi_{-\alpha}(z)\psi_{-\beta}(z)\psi_\gamma(z),$$

$$(166)\qquad \chi_+(z) = \sum_{n\in\mathbb{Z}} \chi_+(n)z^{-n} := \sum_{\alpha\in\bar{\Delta}_+} \bar{\chi}_+(J_\alpha)\psi_{-\alpha}(z),$$

where $\bar{\chi}_+$ is the character of $\bar{\mathfrak{n}}_+$ defined by (11), and $c^{\gamma}_{\alpha,\beta}$ is the structure constant of $\bar{\mathfrak{g}}$ as in Sect. 2.1. The $Q^{\mathrm{st}}_+(z)$ and $\chi_+(z)$ are fields corresponding to the vectors $Q^{\mathrm{st}}_+(-1)|0\rangle$ and $\chi_+(0)|0\rangle$, respectively.

By abuse of notation, we set

$$(167)\quad Q^{\mathrm{st}}_+ := \left(Q^{\mathrm{st}}_+(-1)|0\rangle\right)_{(0)} = Q^{\mathrm{st}}_+(0) = \sum_{\alpha\in\bar{\Delta}_+, n\in\mathbb{Z}} J_\alpha(-n)\psi_{-\alpha}(n) - \frac{1}{2}\sum_{\substack{\alpha,\beta,\gamma\in\bar{\Delta}_+\\ k+l+m=0}} c^{\gamma}_{\alpha,\beta}\psi_{-\alpha}(k)\psi_\beta(l)\psi_\gamma(m),$$

$$(168)\quad \chi_+ := (\chi_+(0)|0\rangle)_{(0)} = \chi_+(1) = \sum_{\alpha\in\bar{\Delta}_+}\bar{\chi}_+(J_\alpha)\psi_{-\alpha}(1),$$

$$(169)\quad Q_+ := \left(Q^{\mathrm{st}}_+(-1) + \chi_+(0)|0\rangle\right)_{(0)} = Q^{\mathrm{st}}_+ + \chi_+.$$

Lemma 4.5.1. *We have* $(Q^{\mathrm{st}}_+)^2 = \chi_+^2 = [Q^{\mathrm{st}}_+, \chi_+] = 0$. *In particular we have* $Q_+^2 = 0$.

Proof. Direct calculation. □

Let $\mathcal{F} = \bigoplus_{i\in\mathbb{Z}}\mathcal{F}^i$ be an additional $\mathbb{Z}$-gradation of the vertex algebra $\mathcal{F}$ defined by

$$\deg \mathbf{1} = 0, \quad \deg\psi_\alpha(n) = \begin{cases} 1 & \text{for } \alpha\in\bar{\Delta}_- \\ -1 & \text{for } \alpha\in\bar{\Delta}_+. \end{cases}$$

Set

$$C^i_k(\mathfrak{g}) = V_k(\mathfrak{g})\otimes\mathcal{F}^i \quad \text{with } i\in\mathbb{Z}.$$

This gives a $\mathbb{Z}$-gradation of $C_k(\bar{\mathfrak{g}})$: $C_k(\bar{\mathfrak{g}}) = \bigoplus_{i\in\mathbb{Z}} C^i_k(\mathfrak{g})$.

By definition,

$$Q_+\cdot C^i_k(\bar{\mathfrak{g}}) \subset C^{i+1}_k(\bar{\mathfrak{g}}).$$

Therefore, by Lemma 4.5.1, $(C_k(\bar{\mathfrak{g}}), Q_+)$ is a BRST complex of vertex algebras in the sense of Sect. 3.15. This complex is called *the BRST complex of the quantized Drinfeld–Sokolov ("+") reduction* [21,25].

Remark 4.5.2. As in Sect. 2.4, χ_+ may be identified with the character of the Lie algebra $L\bar{\mathfrak{n}}_+$ defined by

$$\chi_+(J(n)) = \delta_{n,-1}\bar{\chi}_+(J) \quad \text{for } J\in\bar{\mathfrak{n}}_+ \text{ and } n\in\mathbb{Z}.$$

4.6. Change of hamiltonian. The BRST operator Q_+ above is not compatible with the Hamiltonian $-\mathbf{D}$ of $C_k(\bar{\mathfrak{g}})$. Consider the weight space decomposition $C_k(\bar{\mathfrak{g}}) = \bigoplus_{\lambda\in\mathfrak{h}^*} C_k(\bar{\mathfrak{g}})^\lambda$ with respect to the action of $\mathfrak{h}$. Then one has

$$Q_+^{\mathrm{st}} \cdot C_k(\bar{\mathfrak{g}})^\lambda \subset C_k(\bar{\mathfrak{g}})^\lambda, \quad \chi_+ \cdot C_k(\bar{\mathfrak{g}})^\lambda \subset \sum_{\alpha\in\bar{\Pi}} C_k(\bar{\mathfrak{g}})^{\lambda-\alpha+\delta}. \tag{170}$$

Set

$$\mathbf{D}_{\mathrm{new}} := \mathbf{D} + \bar{\rho}^\vee \in \mathfrak{h} \tag{171}$$

where $\bar{\rho}^\vee$ is as in Sect. 2.1.

Lemma 4.6.1. *The operator $-\mathbf{D}_{\mathrm{new}}$ defines a Hamiltonian on $C_k(\bar{\mathfrak{g}})$. The action of $\mathbf{D}_{\mathrm{new}}$ on $C_k(\bar{\mathfrak{g}})$ commutes with that of Q_+.*

Proof. The first assertion is easy to check. The second assertion follows from (170). □

We denote by $C_k(\bar{\mathfrak{g}})_{\mathrm{new}}$ the vertex algebra $C_k(\bar{\mathfrak{g}})$ equipped with the new Hamiltonian $-\mathbf{D}_{\mathrm{new}}$. We write

$$C_k(\bar{\mathfrak{g}})_{\mathrm{new}} = \bigoplus_{\Delta\in\mathbb{Z}} C_k(\bar{\mathfrak{g}})_{-\Delta,\mathrm{new}}, \tag{172}$$

where

$$C_k(\bar{\mathfrak{g}})_{-\Delta,\mathrm{new}} = \{c \in C_k(\bar{\mathfrak{g}}); \mathbf{D}_{\mathrm{new}} \cdot c = -\Delta c\}.$$

Note that $C_k(\bar{\mathfrak{g}})_{\mathrm{new}}$ is no more $\mathbb{Z}_{\geq 0}$-graded. It is compatibly $\mathbb{Z}_{\geq 0}$-gradable by $-\mathbf{D}$ (see Sect. 3.2).

The Möbious conformal structure is changed accordingly: Let $T^* = T^* \otimes 1 + 1 \otimes L^f(1)$. Set

$$T^*_{\mathrm{new}} := T^* - 2\widehat{\bar{\rho}^\vee}(1), \tag{173}$$

where $\widehat{\bar{\rho}^\vee}(1)$ is the operator on $C_k(\bar{\mathfrak{g}})''$ such that

$$\begin{aligned}
\widehat{\bar{\rho}^\vee}(1)|0\rangle &= 0,\\
\left[\widehat{\bar{\rho}^\vee}(1), J(n)\right] &= [\bar{\rho}^\vee, J](n+1) + k\delta_{n,-1}(\bar{\rho}^\vee|J)\,\mathrm{id} \quad \text{for } J \in \bar{\mathfrak{g}}, n \in \mathbb{Z},\\
\left[\widehat{\bar{\rho}^\vee}(1), \psi_\alpha(n)\right] &= \alpha(\bar{\rho}^\vee)\psi_\alpha(n+1) \quad \text{for } \alpha \in \bar{\Delta}, n \in \mathbb{Z}.
\end{aligned}$$

The triplet $\{T^*_{\mathrm{new}}, -\mathbf{D}_{\mathrm{new}}, T\}$ gives $C_k(\bar{\mathfrak{g}})$ a Möbious conformal structure (cf. Sect. 4.17). We have

$$\left[Q_+, T^*_{\mathrm{new}}\right] = 0. \tag{174}$$

Also, the action of T^*_{new} on $C_k(\bar{\mathfrak{g}})$ is locally nilpotent.

Put

$$(175) \qquad \mathfrak{r} := \mathbb{C}T^*_{\mathrm{new}} + \mathbb{C}\mathbf{D}_{\mathrm{new}} + \mathbb{C}T.$$

We consider $\mathfrak{r}$ as a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Below if no confusion can arise we write just $C_k(\bar{\mathfrak{g}})$ for $C_k(\bar{\mathfrak{g}})_{\mathrm{new}}$. We write $C_k(\bar{\mathfrak{g}})_{\mathrm{old}}$ for the vertex algebra $C_k(\bar{\mathfrak{g}})$ with the (old) Hamiltonian $-\mathbf{D}$.

4.7. Definition of $\mathscr{W}$-algebras. The following assertion was proved by B. Feigin and E. Frenkel [21] for generic k, by J. de Boer and T. Tjin [14] for all k in the case that $\bar{\mathfrak{g}} = \mathfrak{sl}_n$ and by E. Frenkel [25] for the general case.

Theorem 4.7.1. *The cohomology $H^i(C_k(\bar{\mathfrak{g}}))$ is zero for all $i \neq 0$.*

Define

$$(176) \qquad \mathscr{W}_k(\bar{\mathfrak{g}}) := H^0(C_k(\bar{\mathfrak{g}})).$$

This is a Möbious conformal vertex algebra for all $k \in \mathbb{C}$, because the action of $\mathfrak{r}$ on $C_k(\bar{\mathfrak{g}})$ commutes with the BRST operator Q_+. The vertex algebra $\mathscr{W}_k(\bar{\mathfrak{g}})$ is called the *$\mathscr{W}$-algebra associated with $\bar{\mathfrak{g}}$ at level k.*

For the later purpose we recall the proof of Theorem 4.7.1 given by [25] in next sections.

4.8. The tensor product decomposition of the complex $C_k(\bar{\mathfrak{g}})$. Following [25, Chapter 15], we set

$$\widehat{J}_a(z) = \sum_{n\in\mathbb{Z}} \widehat{J}_a(n) z^{-n-1} := J_a(z) - \sum_{\beta,\gamma\in\bar{\Delta}_+} c^{\gamma}_{a,\beta} : \psi_{-\beta}(z)\psi_\gamma(z) :$$

for $a \in \bar{I} \sqcup \bar{\Delta}$.

Lemma 4.8.1. *We have the following relations for $m, n \in \mathbb{Z}$:*

(i) $[\widehat{J}_a(m), \widehat{J}_b(n)] = \sum_d c^d_{a,b} \widehat{J}_d(m+n) + (k+h^\vee)m(J_a, J_b)\delta_{m+n,0}$ *if $a, b \in \bar{I} \sqcup \bar{\Delta}_+$ or $a, b \in \bar{I} \sqcup \bar{\Delta}_-$.*

(ii) $[\widehat{J}_a(m), \psi_\beta(n)] = \sum_\gamma c^\gamma_{a,\beta} \psi_\gamma(m+n)$ *if $a \in \bar{I} \sqcup \bar{\Delta}_+, \beta \in \bar{\Delta}_+$ or $a \in \bar{I} \sqcup \bar{\Delta}_-, \beta \in \bar{\Delta}_-$.*

Proof. Direct caculation. □

Let $C_k(\bar{\mathfrak{g}})'$ be the subspace of $C_k(\bar{\mathfrak{g}})$ spanned by the elements

$$\widehat{J}_{\alpha_{i_1}}(-n_1) \dots \widehat{J}_{\alpha_{i_p}}(-n_p)\psi_{\alpha_{j_1}}(-m_1) \dots \psi_{\alpha_{j_q}}(-m_q)|0\rangle$$

with $\alpha_{i_s}, \alpha_{j_s} \in \bar{\Delta}_+$, $n_i, m_i \in \mathbb{Z}$. Then, by Lemma 4.8.1, $C_k(\bar{\mathfrak{g}})'$ is a vertex subalgebra of $C_k(\bar{\mathfrak{g}})$. Similarly, let $C_k(\bar{\mathfrak{g}})''$ be the vertex subalgebra of $C_k(\bar{\mathfrak{g}})$

spanned by the elements

$$\widehat{J}_{a_1}(-n_1)\dots\widehat{J}_{a_p}(-n_p)\psi_{\alpha_{j_1}}(-m_1)\dots\psi_{\alpha_{j_q}}(-m_q)|0\rangle$$

with $a_i \in \bar{I} \sqcup \bar{\Delta}_-$, $\alpha_{j_i} \in \bar{\Delta}_-$, $n_i, m_i \in \mathbb{Z}$. Then $C_k(\bar{\mathfrak{g}})''$ is a vertex subalgebra of $C_k(\bar{\mathfrak{g}})$ by Lemma 4.8.1.

Lemma 4.8.2. *We have the following relations for $n \in \mathbb{Z}$:*

$$\left[Q_+^{\mathrm{st}}, \psi_\alpha(n)\right] = \widehat{J}_\alpha(n) \text{ and } \left[Q_+^{\mathrm{st}}, \widehat{J}_\alpha(n)\right] = 0 \quad \text{for } \alpha \in \bar{\Delta}_+;$$

$$[\chi_+, \psi_\alpha(n)] = \chi_+(J_\alpha(n)) \text{ and } [\chi_+, \widehat{J}_\alpha(n)] = 0 \quad \text{for } \alpha \in \bar{\Delta}_+;$$

$$\left[Q_+^{\mathrm{st}}, \psi_{-\alpha}(n)\right] = -\frac{1}{2}\sum_{\substack{\beta,\gamma\in\bar{\Delta}_+\\ k+l=n}} c^{\alpha}_{\beta,\gamma}\psi_{-\beta}(k)\psi_{-\gamma}(l)$$

$$\text{and } [\chi_+, \psi_{-\alpha}(n)] = 0 \quad \text{for } \alpha \in \bar{\Delta}_+;$$

$$\left[Q_+^{\mathrm{st}}, \widehat{J}_a(n)\right] = \sum_{\substack{\alpha\in\bar{\Delta}_+, b\in\bar{\Delta}_-\sqcup\bar{I}\\ k+l=n}} c^{b}_{\alpha,a} :\psi_{-\alpha}(l)\widehat{J}_b(k): n - k_a\psi_a(n)$$

$$\text{for } a \in \bar{\Delta}_- \sqcup \bar{I};$$

$$[\chi_+, \widehat{J}_a(n)] = \sum_{\beta\in\bar{\Delta}_+} ([f, J_a], J_\beta)\psi_{-\beta}(n+1) \quad \text{for } a \in \bar{\Delta}_- \sqcup \bar{I},$$

where

$$k_a = \begin{cases} k - \sum_{\beta,\gamma\in\bar{\Delta}_-} c^{\gamma}_{a,\beta}c^{-\gamma}_{-a,-\beta} & \text{for } a \in \bar{\Delta}_-,\\ 0 & \text{for } a \in \bar{I}.\end{cases}$$

Proof. Direct calculation. □

By Lemma 4.8.2 and the fact that $Q_+|0\rangle = 0$, it follows that

(177) $$Q_+^{\mathrm{st}} \cdot C_k(\bar{\mathfrak{g}})' \subset C_k(\bar{\mathfrak{g}})', \chi_+ \cdot C_k(\bar{\mathfrak{g}})' \subset C_k(\bar{\mathfrak{g}})' \\ \text{so } Q_+ \cdot C_k(\bar{\mathfrak{g}})' \subset C_k(\bar{\mathfrak{g}})',$$

(178) $$Q_+^{\mathrm{st}} \cdot C_k(\bar{\mathfrak{g}})'' \subset C_k(\bar{\mathfrak{g}})'', \chi_+ \cdot C_k(\bar{\mathfrak{g}})'' \subset C_k(\bar{\mathfrak{g}})'' \\ \text{so } Q_+ \cdot C_k(\bar{\mathfrak{g}})'' \subset C_k(\bar{\mathfrak{g}})''.$$

Thus both $C_k(\bar{\mathfrak{g}})'$ and $C_k(\bar{\mathfrak{g}})''$ are subcomplexes of $C_k(\bar{\mathfrak{g}})$. Hence both $(C_k(\bar{\mathfrak{g}})', Q_+)$ and $(C_k(\bar{\mathfrak{g}})'', Q_+)$ are weak BRST complexes of vertex algebras (see Sect. 3.15). The cohomological gradation takes only non-positive values on $C_k(\bar{\mathfrak{g}})'$ and only non-negative values on $C_k(\bar{\mathfrak{g}})''$:

(179) $$C_k(\bar{\mathfrak{g}})' = \bigoplus_{i\le 0} C_k^i(\bar{\mathfrak{g}})', \quad C_k(\bar{\mathfrak{g}})'' = \bigoplus_{i\ge 0} C_k^i(\bar{\mathfrak{g}})'',$$

where $C_k^i(\bar{\mathfrak{g}})' = C_k^i(\bar{\mathfrak{g}}) \cap C_k(\bar{\mathfrak{g}})'$ and $C_k^i(\bar{\mathfrak{g}})'' = C_k^i(\bar{\mathfrak{g}}) \cap C_k(\bar{\mathfrak{g}})''$.

Proposition 4.8.3.

(i) *Let U' be the subspace of $C_k(\bar{\mathfrak{g}})'$ spanned by the vectors*

$$\widehat{J}_\alpha(-1)|0\rangle, \quad \psi_\alpha(-1)|0\rangle \quad \textit{with } \alpha \in \bar{\Delta}_+.$$

Then U' generates a PBW basis of $C_k(\bar{\mathfrak{g}})'$.

(ii) *Let U'' be the subspace of $C_k(\bar{\mathfrak{g}})''$ spanned by the vectors*

$$\widehat{J}_a(-1)|0\rangle, \quad \psi_{-\alpha}(0)|0\rangle \quad \textit{with } a \in \bar{I} \sqcup \bar{\Delta}_-, \alpha \in \bar{\Delta}_+.$$

Then U'' generates a PBW basis of $C_k(\bar{\mathfrak{g}})''$.

(iii) *The subspace $U' + U'' = U' \oplus U''$ generates a PBW basis of $C_k(\bar{\mathfrak{g}})$, and the multiplication map*

$$\begin{aligned}&\widehat{J}_{\alpha_{i_1}}(-n_1)\dots\psi_{\alpha_{j_1}}(-m_1)\dots|0\rangle \otimes \widehat{J}_{a_1}(-n_1)\dots\psi_{\alpha_{k_1}}(-m_1)\dots|0\rangle \\ &\mapsto \widehat{J}_{\alpha_{i_1}}(-n_1)\dots\psi_{\alpha_{j_1}}(-m_1)\dots\widehat{J}_{a_1}(-n_1)\dots\psi_{\alpha_{k_1}}(-m_1)\dots|0\rangle\end{aligned}$$

gives the isomorphism of cochain complexes $C_k(\bar{\mathfrak{g}})' \otimes C_k(\bar{\mathfrak{g}})'' \xrightarrow{\sim} C_k(\bar{\mathfrak{g}})$. In particular by the Künneth theorem one has

$$H^n(C_k(\bar{\mathfrak{g}})) = \bigoplus_{p+q=n} H^p(C_k(\bar{\mathfrak{g}})') \otimes H^q(C_k(\bar{\mathfrak{g}})'')$$

for all $n \in \mathbb{Z}$.

It is clear that $C_k(\bar{\mathfrak{g}})'$ and $C_k(\bar{\mathfrak{g}})''$ is preserved by the action of $\mathfrak{r}$. Therefore both $C_k(\bar{\mathfrak{g}})'$ and $C_k(\bar{\mathfrak{g}})''$ are Möbious conformal, and thus so are $H^\bullet(C_k(\bar{\mathfrak{g}})')$ and $H^\bullet(C_k(\bar{\mathfrak{g}})'')$.

Proposition 4.8.4 **(**[25, 15.2.6]**).** *The cohomology $H^i(C_k(\bar{\mathfrak{g}})')$ is zero for all $i \neq 0$ and one has $H^0(C_k(\bar{\mathfrak{g}})') = \mathbb{C}$.*

The following assertion follows directly from Propositions 4.8.3 and 4.8.4.

Theorem 4.8.5 **(**[25, Lemma 15.2.7]**).** *The natural embedding $C_k(\bar{\mathfrak{g}})'' \hookrightarrow C_k(\bar{\mathfrak{g}})$ induces the isomorphism $H^\bullet(C_k(\bar{\mathfrak{g}})'') \xrightarrow{\sim} H^\bullet(C_k(\bar{\mathfrak{g}}))$ of Möbius conformal vertex algebras. In particular,*

$$H^0(C_k(\bar{\mathfrak{g}})'') \cong \mathscr{W}_k(\bar{\mathfrak{g}})\big(= H^0(C_k(\bar{\mathfrak{g}}))\big)$$

as Möbius conformal vertex algebras.

Remark 4.8.6. By (179) we have $H^0(C_k(\bar{\mathfrak{g}})'') = \{c \in C_k^0(\bar{\mathfrak{g}})''; Q_+c = 0\} \subset C_k(\bar{\mathfrak{g}})''$. Therefore, by Theorem 4.8.5 $\mathscr{W}_k(\bar{\mathfrak{g}})$ can be regarded as a vertex subalgebra of $C_k(\bar{\mathfrak{g}})''$, and thus of $C_k(\bar{\mathfrak{g}})$.

4.9. The filtration F of $C_k(\bar{\mathfrak{g}})''$. We keep the notation of previous sections. The vertex algebra $C_k(\bar{\mathfrak{g}})''$ is considered to be equipped with the Hamiltonian $-\mathbf{D}_{\text{new}}$. We have

$$\Delta_{\widehat{J}_{-\alpha}(-1)|0\rangle} = \operatorname{ht}\alpha + 1, \quad \Delta_{\psi_{-\alpha}(0)|0\rangle} = \operatorname{ht}\alpha \quad \text{for } \alpha \in \bar{\Delta}_+, \tag{180}$$

$$\Delta_{\widehat{J}_i(-1)|0\rangle} = 1 \quad \text{for } i \in \bar{I}. \tag{181}$$

Thus $C_k(\bar{\mathfrak{g}})''$ is $\mathbb{Z}_{\geq 0}$-graded:

(182)

$$C_k(\bar{\mathfrak{g}})'' = \bigoplus_{\Delta\in\mathbb{Z}_{\geq 0}} C_k(\bar{\mathfrak{g}})''_{-\Delta}, \quad C_k(\bar{\mathfrak{g}})''_0 = \mathbb{C}|0\rangle, \quad \dim C_k(\bar{\mathfrak{g}})''_{-\Delta} < \infty \quad \forall\Delta.$$

It is clear that $C_k(\bar{\mathfrak{g}})''$ is an $\mathfrak{h}$-submodule of $C_k(\bar{\mathfrak{g}})$ and we have

$$C_k^n(\bar{\mathfrak{g}})'' = \bigoplus_{\langle\lambda,\bar{\rho}^\vee\rangle\leq -n} \left(C_k^n(\bar{\mathfrak{g}})''\right)^\lambda \tag{183}$$

for all n.

Proposition 4.9.1. *Define an increasing filtration $F = \{F_p C_k(\bar{\mathfrak{g}})''\}$ of $C_k(\bar{\mathfrak{g}})''$ by*

$$\begin{aligned} F_p C_k(\bar{\mathfrak{g}})'' &= \sum_n F_p C_k^n(\bar{\mathfrak{g}})'', \\ F_p C_k^n(\bar{\mathfrak{g}})'' &:= \bigoplus_{\substack{\lambda \\ \langle\lambda,\bar{\rho}^\vee\rangle\geq -p-n}} \left(C_k^n(\bar{\mathfrak{g}})''\right)^\lambda. \end{aligned} \tag{184}$$

Then we have the following:

(i) *F is a (separated, exhaustive,) strict filtration of the vertex algebra $C_k(\bar{\mathfrak{g}})''$ such that*

$$\begin{gathered} \mathfrak{r}\cdot F_p C_k(\bar{\mathfrak{g}})'' \subset F_p C_k(\bar{\mathfrak{g}})'' \quad \forall p, \\ F_{-1} C_k(\bar{\mathfrak{g}})'' = 0. \end{gathered}$$

(ii) $\operatorname{gr}^F C_k(\bar{\mathfrak{g}})'' \cong C_k(\bar{\mathfrak{g}})''$ *as Möbius conformal vertex algebras.*

(iii) *For all p we have $Q_+^{\text{st}} \cdot F_p C_k(\bar{\mathfrak{g}})'' \subset F_{p-1} C_k(\bar{\mathfrak{g}})''$ and $\chi_+ \cdot F_p C_k(\bar{\mathfrak{g}})'' \subset F_p C_k(\bar{\mathfrak{g}})''$, and thus $Q_+ \cdot F_p C_k(\bar{\mathfrak{g}})'' \subset F_p C_k(\bar{\mathfrak{g}})''$.*

Proof. For (i), the strictness of F follows from (ii), together with Proposition 4.8.3 (ii) and Proposition 3.8.2. Other assertions are easily seen. (ii) is easily seen. (iii) follows from (170). □

The $(C_k(\bar{\mathfrak{g}})'', \chi_+)$ is also a weak BRST complex of vertex algebras. By Proposition 4.9.1,

$$H^\bullet\left(\operatorname{gr}^F C_k(\bar{\mathfrak{g}})''\right) \cong H^\bullet(C_k(\bar{\mathfrak{g}})'', \chi_+), \tag{185}$$

as Möbius conformal vertex algebras.

4.10. The vertex algebra $H^{\bullet}(C_k(\bar{\mathfrak{g}})'', \chi_+)$. According to [25, 15.2.9], the cohomology $H^{\bullet}(C_k(\bar{\mathfrak{g}}), \chi_+)$ is easy to calculate, in view of Theorem 3.19.2: Let

$$\widehat{P}_i(z) = \sum_{n\in\mathbb{Z}} \widehat{P}_i(n) z^{-n-1},$$

with $i \in \bar{I}$, be the linear combination of $\widehat{J}_a(z)$ corresponding to $P_i \in \bar{\mathfrak{g}}^f$ (see Sect. 2.1). Similarly, let $\widehat{I}_{-\alpha}(z) = \sum_{n\in\mathbb{Z}} \widehat{I}_{-\alpha}(n) z^{-n-1}$ with $\alpha \in \bar{\Delta}_+$ be the linear combination of $J_a(z)$ corresponding to $I_{-\alpha}$. Then from the fifth formula of Lemma 4.8.2 it follows that

$$[\chi_+, \widehat{P}_i(n)] = 0 \quad \forall i \in \bar{I},\ n \in \mathbb{Z}, \tag{186}$$

$$[\chi_+, \widehat{I}_{-\alpha}(n)] = \psi_{-\alpha}(n+1) \quad \forall \alpha \in \bar{\Delta}_+,\ n \in \mathbb{Z}. \tag{187}$$

Let $L\bar{\mathfrak{g}}^f = \bar{\mathfrak{g}}^f \otimes \mathbb{C}[t, t^{-1}]$ (notation Sect. 2.1). This is a commutative subalgebra of $\mathfrak{g}$. Set $V(\bar{\mathfrak{g}}^f) = U(L\bar{\mathfrak{g}}^f) \cdot |0\rangle \subset V_k(\bar{\mathfrak{g}})$. This is a commutative Möbius conformal vertex subalgebra of $V_k(\bar{\mathfrak{g}})$. Here, the Möbius conformal structure of $V(\bar{\mathfrak{g}}^f)$ is considered to be given by $\mathfrak{r}$.

By (186) there is an embedding of vertex algebras of the following form:

(188)

$$\begin{array}{ccc} V(\bar{\mathfrak{g}}^f) & \to & H^0(C_k(\bar{\mathfrak{g}})'', \chi_+) \subset C_k(\bar{\mathfrak{g}})'' \\ P_{i_1}(-n_1)P_{i_2}(-n_2)\dots P_{i_r}(-n_r)|0\rangle & \mapsto & \widehat{P}_{i_1}(-n_1)\widehat{P}_{i_2}(-n_2)\dots \widehat{P}_{i_r}(-n_r)|0\rangle. \end{array}$$

One sees that (188) is $\mathfrak{r}$-equivalent.

Proposition 4.10.1.

(i) [25, Lemma 15.2.10] *The cohomology* $H^i(C_k(\bar{\mathfrak{g}})'', \chi_+)$ *is zero for all* $i \neq 0$ *and the map* (188) *gives an isomorphism*

$$V(\bar{\mathfrak{g}}^f) \xrightarrow{\sim} H^0(C_k(\bar{\mathfrak{g}})'', \chi_+)$$

of Möbius conformal vertex algebras.

(ii) *Let* $N \geq 0$. *The cohomology* $H^i(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})''), \mathrm{ad}\,\chi_+)$ *is zero for all* $i \neq 0$ *and the isomorphism* $V(\bar{\mathfrak{g}}^f) \xrightarrow{\sim} H^0(C_k(\bar{\mathfrak{g}})'', \chi_+)$ *in (i) induces an isomorphism*

$$\mathfrak{U}_N(V(\bar{\mathfrak{g}}^f)) \xrightarrow{\sim} H^0(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})''), \mathrm{ad}\,\chi_+)$$

(notation Section 3.11*).*

Proof. The set $\{\widehat{P}_i(-1)|0\rangle,\ \widehat{I}_{-\alpha}(-1)|0\rangle,\ \psi_{-\alpha}(0)|0\rangle;\, i \in \bar{I}, \alpha \in \bar{\Delta}_+\}$ forms a basis of U'' (notation Proposition 4.8.3), which generates a PBW basis of $C_k(\bar{\mathfrak{g}})''$.

By (186) and (187), we have

$$\chi_+ \cdot \widehat{P}_i(-1)|0\rangle = 0 \quad \forall i \in \bar{I}, \tag{189}$$

$$\chi_+ \cdot \widehat{I}_{-\alpha}(-1)|0\rangle = \psi_{-\alpha}(0)|0\rangle, \tag{190}$$
$$\chi_+ \cdot \psi_{-\alpha}(0)|0\rangle = 0 \quad \forall \alpha \in \bar{\Delta}_+.$$

Hence

$$\chi_+ \cdot U'' \subset U'', \tag{191}$$

$$H^i(U'', \chi_+) \cong \begin{cases} \bigoplus_{i\in\bar{I}} \mathbb{C}\widehat{P}_i(-1)|0\rangle & (i=0) \\ 0 & (i \neq 0). \end{cases} \tag{192}$$

Therefore the assertion follows from Proposition 4.8.3 (ii) and Theorem 3.19.2. □

4.11. The filtration F of $\mathscr{W}_k(\bar{\mathfrak{g}})$. Let $\{F_p\mathscr{W}_k(\bar{\mathfrak{g}})\}$ be the filtration of $\mathscr{W}_k(\bar{\mathfrak{g}}) = H^0(C_k(\bar{\mathfrak{g}})'')$ induced by F:

$$F_p\mathscr{W}_k(\bar{\mathfrak{g}}) := \operatorname{Im}\left(H^0(F_pC_k(\bar{\mathfrak{g}})'') \to H^0(C_k(\bar{\mathfrak{g}})'')\right). \tag{193}$$

Then F is compatible with the Möbius conformal structure of $\mathscr{W}_k(\bar{\mathfrak{g}})$: $\mathfrak{r} \cdot F_p\mathscr{W}_k(\bar{\mathfrak{g}}) \subset F_p\mathscr{W}_k(\bar{\mathfrak{g}})$.

The following assertion proves Theorem 4.7.1:

Theorem 4.11.1 **(**[25, Theorem 15.1.9]**).** *The cohomology $H^i(C_k(\bar{\mathfrak{g}})'')$ is zero for $i \neq 0$ and there is an isomorphism $\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}}) \cong V(\bar{\mathfrak{g}}^f)$ of Möbius conformal vertex algebras.*

Proof. By Propositions 3.16.1, the assertion follows from (185) and Proposition 4.10.1 (i). □

By Theorem 4.11.1, $(\mathscr{W}_k(\bar{\mathfrak{g}}), F)$ is quasi-commutative.

Lemma 4.11.2.

(i) *The filtration $F = \{F_p\mathscr{W}_k(\bar{\mathfrak{g}})\}$ of $\mathscr{W}_k(\bar{\mathfrak{g}})$ is strict.*
(ii) *We have* $\operatorname{Vac}\mathscr{W}_k(\bar{\mathfrak{g}}) = \mathbb{C}|0\rangle$.

Proof. Because

$$U_{V(\bar{\mathfrak{g}}^f)} := \bigoplus_{i\in\bar{I}} P_i(-1)|0\rangle \tag{194}$$

generates a PBW basis of $V(\bar{\mathfrak{g}}^f) = \operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})$, we have that $\operatorname{Vac}\mathscr{W}_k(\bar{\mathfrak{g}}) = \mathbb{C}|0\rangle$ and F is strict, by Proposition 3.8.2. □

Remark 4.11.3. The filtration F differs from the standard filtration of $\mathscr{W}_k(\bar{\mathfrak{g}})$. One can characterize F as the finest filtration of $\mathscr{W}_k(\bar{\mathfrak{g}})$ such that $\mathscr{W}_k(\bar{\mathfrak{g}})_{-\Delta} \subset F_{\Delta-1}\mathscr{W}_k(\bar{\mathfrak{g}})$ for all $\Delta > 0$ (compare Theorem 3.5.1 (iii), cf. Remark 4.12.2). The existence of such a filtration F can be proved for any $\mathbb{Z}_{\geq 0}$-graded vertex algebra $\mathbb{V}$ with $\mathbb{V}_0 = \mathbb{C}|0\rangle$.

4.12. Quasi-primary generators of $\mathscr{W}_k(\bar{\mathfrak{g}})$. Let $\{F_p C_k(\bar{\mathfrak{g}})''\}$ be the filtration of $C_k(\bar{\mathfrak{g}})''$ defined in Proposition 4.9.1, $\{F_p \mathscr{W}_k(\bar{\mathfrak{g}})\}$ the induced filtration (193) of $\mathscr{W}_k(\bar{\mathfrak{g}})$.

Recall the exponent d_i of $\bar{\mathfrak{g}}$ with $i \in \bar{I}$, see Sect. 2.1. By definition $\widehat{P}_i(-1)|0\rangle \in F_{d_i} C_k(\bar{\mathfrak{g}})''$.

Proposition 4.12.1.

(i) *The action of* $\mathfrak{r} = \mathbb{C}T^*_{\mathrm{new}} + \mathbb{C}\mathbf{D}_{\mathrm{new}} + \mathbb{C}T$ *on* $\mathscr{W}_k(\bar{\mathfrak{g}})$ *is completely reducible.*

(ii) *For each* $i \in \bar{I}$, *there exists a quasi-primary vector* $\mathsf{W}_i \in F_{d_i}\mathscr{W}_k(\bar{\mathfrak{g}}) \subset C_k(\bar{\mathfrak{g}})''$ *of conformal weight* $d_i + 1$ *such that* $\sigma_{d_i}(\mathsf{W}_i) = P_i(-1)|0\rangle$ (*notation Section* 3.1) *under the identification* $\mathrm{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}}) \cong V(\bar{\mathfrak{g}}^f)$.

(iii) *Let* $\mathsf{W}_1, \dots, \mathsf{W}_l$ *be as in* (ii). Then the subspace $U_{\mathscr{W}} := \bigoplus_{i\in\bar{I}} \mathbb{C}\mathsf{W}_i$ generates a PBW basis of $\mathscr{W}_k(\bar{\mathfrak{g}})$.

Proof. (i) Because the isomorphism in Theorem 4.11.1 is $\mathbf{D}_{\mathrm{new}}$-equivalent, it follows that there is no vector of conformal weight 1 in $\mathscr{W}_k(\bar{\mathfrak{g}})$. This together Lemma 4.11.2 (ii) shows that $\mathrm{Vac}\,\mathscr{W}_k(\bar{\mathfrak{g}}) \cap \mathrm{Im}\, T^*_{\mathrm{new}} = 0$. Thanks to [33, Proposition 4.9 (b)], this proves the assertion. (ii) Because F is compatible with the action of $\mathfrak{r}$,

$$0 \longrightarrow F_{d_i-1}\mathscr{W}_k(\bar{\mathfrak{g}}) \longrightarrow F_{d_i}\mathscr{W}_k(\bar{\mathfrak{g}}) \xrightarrow{\sigma_{d_i}} \mathrm{gr}^F_{d_i}\mathscr{W}_k(\bar{\mathfrak{g}}) \longrightarrow 0$$

is an exact sequence of $\mathfrak{r}$-modules. Since $\mathscr{W}_k(\bar{\mathfrak{g}})$ is completely reducible over $\mathfrak{r}$ by (i), there exist a $\mathfrak{r}$-equivalent linear map $s_{d_i} : \mathrm{gr}^F_{d_i}\mathscr{W}_k(\bar{\mathfrak{g}}) \to F_{d_i}\mathscr{W}_k(\bar{\mathfrak{g}})$ such that $\sigma_{d_i} \circ s_{d_i} = \mathrm{id}$. The vector $\mathsf{W}_i = s_{d_i}(P_i(-1)|0\rangle)$ satisfies the desired property, because $P_i(-1)|0\rangle$ is quasi-primary. (iii) is obvious from the fact that $U_{V(\bar{\mathfrak{g}}^f)}$ (see (194)) generates a PBW basis of $V(\bar{\mathfrak{g}}^f)$. □

We fix quasi-primary generates $\mathsf{W}_1, \dots, \mathsf{W}_l$ of $\mathscr{W}_k(\bar{\mathfrak{g}})$ which appeared in Proposition 4.12.1. We write $\bar{\mathsf{W}}_i$ for the image of W_i in $\mathrm{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})$.

Remark 4.12.2. From Proposition 4.12.1 it follows that

$$F_p\mathscr{W}_k(\bar{\mathfrak{g}}) = \mathrm{span}\left\{ (\mathsf{W}_{i_1})_{(-n_1)}(\mathsf{W}_{i_2})_{(-n_2)}\dots(\mathsf{W}_{i_r})_{(-n_r)}|0\rangle;\ \begin{array}{l} n_i \geq 1,\ r \in \mathbb{Z}_{\geq 0}, \\ d_{i_1} + d_{i_2} + \dots + d_{i_r} \leq p \end{array} \right\}.$$

One can prove that $F_p\mathscr{W}_k(\bar{\mathfrak{g}})$ is spanned by vectors

$$w^1_{(-n_1)} \dots w^r_{(-n_r)}|0\rangle$$

with $r \geq 0, n_i \geq 1$ and homogeneous vectors $w^1, \dots, w^r \in \mathscr{W}_k(\bar{\mathfrak{g}})$ satisfying

$$\Delta_{w^1} + \dots + \Delta_{w^r} \leq p - r.$$

4.13. The Lie algebra $\mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}}))$

Theorem 4.13.1.

(i) *The cohomology* $H^i(C_k(\bar{\mathfrak{g}})'')$ *is zero for all* $i \neq 0$ *and there is a natural Lie algebra isomorphism* $\mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{L}(C_k(\bar{\mathfrak{g}})''))$
(ii) *The cohomology* $H^i(C_k(\bar{\mathfrak{g}}))$ *is zero for all* $i \neq 0$ *and there is a natural Lie algebra isomorphism* $\mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{L}(C_k(\bar{\mathfrak{g}})))$.

Proof. By applying Theorem 3.17.1, the assertion follows from (163), Lemma 4.11.2 and Theorems 4.8.5 and 4.7.1. □

We write $\mathsf{W}_i(n)$ for $(\mathsf{W}_i)_n \in \mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. By Theorem 4.13.1, this can be considered also as an element of $H^0(\mathfrak{L}(C_k(\bar{\mathfrak{g}})'')) = H^0(\mathfrak{L}(C_k(\bar{\mathfrak{g}})))$.

Let $\{F_p\,\mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}}))\}$ be the filtration of $\mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ induced by (193). Then by Lemma 4.11.2 (i) and Proposition 3.10.1 we have the isomorphism of Lie algebras

$$\mathfrak{L}\big(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})\big) \xrightarrow{\sim} \operatorname{gr}^F \mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}})). \tag{195}$$

We write $\bar{\mathsf{W}}_i(n)$ for $(\bar{\mathsf{W}}_i)_n \in \mathfrak{L}(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})) = \operatorname{gr}^F \mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}}))$.

4.14. The current algebra of $\mathscr{W}_k(\bar{\mathfrak{g}})$

Theorem 4.14.1.

(i) *For each* N*, the cohomology* $H^i(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})''))$ *is zero for all* $i \neq 0$ *and the natural map* $\mathfrak{U}_N(\mathscr{W}_k(\bar{\mathfrak{g}})) \to H^0(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})''))$ *is a linear isomorphism.*
(ii) *The cohomology* $H^i(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})''))$ *is zero for all* $i \neq 0$ *and the natural map* $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}})) \to H^0(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})''))$ *is an algebra isomorphism.*

Proof. (i) By Proposition 4.10.1 (ii) and (185), one can apply Proposition 3.18.1 (ii) to get:

$$H^i(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')) = 0 \quad \text{for } i \neq 0, \tag{196}$$

$$\operatorname{gr}^F H^0(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')) \xrightarrow{\sim} H^0\big(\mathfrak{U}_N\big(\operatorname{gr}^F C_k(\bar{\mathfrak{g}})''\big)\big). \tag{197}$$

Because $\{F_p\,C_k(\bar{\mathfrak{g}})''\}$ is strict, we have $\mathfrak{U}_N(\operatorname{gr}^F C_k(\bar{\mathfrak{g}})'') \xrightarrow{\sim} \operatorname{gr}^F \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')$ by Theorem 3.13.3. Thus by (197)

$$\operatorname{gr}^F H^0(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')) \xrightarrow{\sim} H^0\big(\operatorname{gr}^F \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')\big). \tag{198}$$

Also, by Theorem 4.11.1 and Proposition 4.10.1 (ii),

$$\mathfrak{U}_N\big(H^0\big(\operatorname{gr}^F C_k(\bar{\mathfrak{g}})''\big)\big) \cong \mathfrak{U}_N(V(\bar{\mathfrak{g}}^f)) \cong H^0\big(\mathfrak{U}_N\big(\operatorname{gr}^F C_k(\bar{\mathfrak{g}})''\big)\big). \tag{199}$$

Therefore, by Proposition 4.9.1 (i), Lemma 4.11.2 (i), Theorem 4.11.1, (198) and (199), all the assumption of Proposition 3.18.2 are satisfied. Therefore one has $\mathfrak{U}_N(H^0(C_k(\bar{\mathfrak{g}})'')) \cong H^0(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})''))$. (ii) The assertion follows directly from (i). Indeed by definition $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ is a direct summand of the complex $\mathfrak{U}_0(\mathscr{W}_k(\bar{\mathfrak{g}}))$:

$$\begin{aligned}\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}})) &= \{a \in \mathfrak{U}_0(\mathscr{W}_k(\bar{\mathfrak{g}})); \operatorname{ad} H \cdot a = 0\}\\ &= \{a \in H^0(\mathfrak{U}_0(C_k(\bar{\mathfrak{g}})'')); \operatorname{ad} H \cdot a = 0\} = H^0(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})'')).\end{aligned}$$

□

By Theorem 4.14.1 (i) it follows that

$$\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')) \tag{200}$$

(see (120)). In particular $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ can be considered as s subalgebra of $\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')$, and thus of $\mathfrak{U}(C_k(\bar{\mathfrak{g}}))$ by Theorem 3.14.1.

Theorem 4.14.2.

(i) *For each N, the cohomology $H^i(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})))$ is zero for all $i \neq 0$ and the natural map $\mathfrak{U}_N(\mathscr{W}_k(\bar{\mathfrak{g}})) \to H^0(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})))$ is a linear isomorphism.*
(ii) *The cohomology $H^i(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})))$ is zero for all $i \neq 0$ and the natural map $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}})) \to H^0(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})))$ is an algebra isomorphism.*

Proof. Because (ii) follows from (i) (see above), it is sufficient to show (i).
Let $N \geq 0$. The embedding $C_k(\bar{\mathfrak{g}})'' \hookrightarrow C_k(\bar{\mathfrak{g}})$ induces an embedding

$$\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'') \hookrightarrow \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})), \tag{201}$$

by Theorem 3.14.1. By Theorem 4.14.1 (i), it is sufficient to show that the map (201) induces an isomorphism

$$H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')) \xrightarrow{\sim} H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))). \tag{202}$$

To prove this, consider the standard filtration $G = \{G_p C_k(\bar{\mathfrak{g}})\}$ of $C_k(\bar{\mathfrak{g}})$ with respect to the (old) grading defined by the Hamiltonian $-\mathbf{D}$. Then

$$G_{-1} C_k(\bar{\mathfrak{g}}) = 0. \tag{203}$$

Also, $G_p C_k(\bar{\mathfrak{g}})'' = G_p C_k(\bar{\mathfrak{g}}) \cap C_k(\bar{\mathfrak{g}})''$ is the standard filtration of $C_k(\bar{\mathfrak{g}})''$ (with respect to the old grading). Because

$$Q_+^{\mathrm{st}}(-1)|0\rangle \in G_1 C_k(\bar{\mathfrak{g}}), \quad \chi_+(0)|0\rangle \in G_0 C_k(\bar{\mathfrak{g}}) \tag{204}$$

and $(C_k(\bar{\mathfrak{g}}), G)$ is quasi-commutative, we have

$$Q_+^{\mathrm{st}} \cdot G_p C_k(\bar{\mathfrak{g}}) \subset G_p C_k(\bar{\mathfrak{g}}), \quad \chi_+ \cdot G_p C_k(\bar{\mathfrak{g}}) \subset G_{p-1} C_k(\bar{\mathfrak{g}}). \tag{205}$$

Therefore there are converging spectral sequences

$$(206) \quad E_r^{p,q} \Rightarrow H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))), \quad (E'')_r^{p,q} \Rightarrow H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'')),$$

see Proposition 3.18.1. Because the restriction of (201) gives a cochain map

$$(207) \qquad G_p \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})'') \to G_p \mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))$$

for each p, (201) induces a map $(E'')_r \to E_r$ of spectral sequences. If this is an isomorphism for $r = 1$, then this is an isomorphism for all $r \geq 1$, inducing the desired isomorphism (202). So it is suffice to show the following assertion:

Proposition 4.14.3. *The natural embedding* $\mathrm{gr}^G C_k(\bar{\mathfrak{g}})'' \hookrightarrow \mathrm{gr}^G C_k(\bar{\mathfrak{g}})$ *induces an isomorphism* $H^\bullet(\mathfrak{U}_N(\mathrm{gr}^G C_k(\bar{\mathfrak{g}})'')) \xrightarrow{\sim} H^\bullet(\mathfrak{U}_N(\mathrm{gr}^G C_k(\bar{\mathfrak{g}})))$ *for all* N.

Proof. By Proposition 4.8.3 we have

$$\mathrm{gr}^G C_k(\bar{\mathfrak{g}}) \cong \mathrm{gr}^G C_k(\bar{\mathfrak{g}})' \otimes \mathrm{gr}^G C_k(\bar{\mathfrak{g}})''$$

as complexes and as (supercommutative) vertex algebras, where $\mathrm{gr}^G C_k(\bar{\mathfrak{g}})'$ is the graded vertex algebra associated with the standard filtration of $C_k(\bar{\mathfrak{g}})'$ (with respect to the old grading). For convenience we put $\mathbb{V} = \mathrm{gr}^G C_k(\bar{\mathfrak{g}})$, $\mathbb{V}_1 = \mathrm{gr}^G C_k(\bar{\mathfrak{g}})'$, and $\mathbb{V}_2 = \mathrm{gr}^G C_k(\bar{\mathfrak{g}})''$, so that $\mathbb{V} = \mathbb{V}_1 \otimes \mathbb{V}_2$. Let U' and U'' be as in Proposition 4.8.3. Denote by U_1 (resp. U_2) the image of U' in $\mathbb{V}_1$ (resp. the image of U'' in $\mathbb{V}_2$). Then U_1, U_2 and $U_1 \oplus U_2 \subset \mathbb{V}$ generates PBW basis of $\mathbb{V}_1$, $\mathbb{V}_2$ and $\mathbb{V}$, respectively. By (205) the differential Q_+ acts as Q_+^{st} on $\mathbb{V}$, $\mathbb{V}_1$ and $\mathbb{V}_2$.

Let F be the filtration defined in Proposition 4.9.1. Let $F_p \mathbb{V}_2$ be the image of $F_p C_k(\bar{\mathfrak{g}})''$ in $\mathbb{V}_2$. Then $\{F_2 \mathbb{V}_2\}$ gives a filtration of the supercommutative vertex algebra $\mathbb{V}_2$ such that

$$(208) \qquad Q_+ \cdot F_p \mathbb{V}_2 \subset F_{p-1} \mathbb{V}_2$$

(see Proposition 4.9.1 (iii)). Set

$$F_p \mathbb{V} = \mathbb{V}_1 \otimes F_p \mathbb{V}_2.$$

Then this also gives a filtration of the supercommutative vertex algebra $\mathbb{V}$ which is compatible with Q_+. By (208) one has

$$(209) \qquad \left(\mathrm{gr}^F \mathbb{V}, Q_+\right) = \left(\mathbb{V}_1 \otimes \mathbb{V}_2, Q_+^{\mathrm{st}} \otimes \mathrm{id}\right)$$

as weak BRST complexes of supercommutative vertex algebras.

Denote by $\bar{U}_1$ (resp. $\bar{U}_2$) the image of U_1 in $\mathrm{gr}^F \mathbb{V}_1$ (resp. the image of U_2 in $\mathrm{gr}^F \mathbb{V}_2$). Then $\bar{U}_2$ generates a PBW basis of $\mathrm{gr}^F \mathbb{V}_2$ and $\bar{U} := \bar{U}_1 + \bar{U}_2 =$

$\bar{U}_1 \oplus \bar{U}_2$ generates a PBW basis of the supercommutative vertex algebra $\mathrm{gr}^F \mathbb{V}$. By (208) and the first formula in Lemma 4.8.2 it follows that

$$Q_+ \cdot \bar{U}_2 = 0, \quad H^i(\bar{U}_2) = \begin{cases} \bar{U}_2 & (i=0) \\ 0 & (i \neq 0), \end{cases}$$

$$Q_+ \cdot \bar{U} \subset \bar{U}, \quad H^i(\bar{U}) = \begin{cases} \bar{U}_2 & (i=0) \\ 0 & (i \neq 0). \end{cases}$$

Therefore, by Theorem 3.19.1 (i), (iv), it follows that the $H^i(\mathbb{V}_2) = H^i(\mathbb{V}) = 0$ for $i \neq 0$ and the natural embedding $\mathbb{V}_2 \hookrightarrow \mathbb{V}$ induces an isomorphism

$$H^0(\mathbb{V}_2) \xrightarrow{\sim} H^0(\mathbb{V}). \tag{210}$$

Further, by Theorem 3.19.1 (v), (vii) we have

$$H^i(\mathfrak{U}_N(\mathbb{V}_2)) = H^i(\mathfrak{U}_N(\mathbb{V})) = 0 \quad \text{for } i \neq 0, \tag{211}$$

$$\mathfrak{U}_N\big(H^0(\mathbb{V}_2)\big) \cong H^0(\mathfrak{U}_N(\mathbb{V}_2)), \tag{212}$$

$$\mathfrak{U}_N(H^0(\mathbb{V})) \cong H^0(\mathfrak{U}_N(\mathbb{V})). \tag{213}$$

Therefore, by (210), (211), (212), and (213), it follows that $\mathbb{V}_2 \hookrightarrow \mathbb{V}$ induces an isomorphism

$$H^\bullet(\mathfrak{U}_N(\mathbb{V}_2)) \xrightarrow{\sim} H^\bullet(\mathfrak{U}_N(\mathbb{V})). \tag{214}$$

This completes the proof. □

4.15. A realization of $\mathfrak{U}(C_k(\bar{\mathfrak{g}}))$ and $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. Let

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g},$$

and denote by $U_k(\mathfrak{g}')$ the quotient of $U(\mathfrak{g}')$ by the two-sided ideal generated by $K - k\,\mathrm{id}$. There is a diagonal action of $\mathrm{ad}\,\mathbf{D}_{\mathrm{new}}$ on the superalgebra $U(\mathfrak{g}') \otimes \mathcal{C}l$. Let

$$(U_k(\mathfrak{g}') \otimes \mathcal{C}l)_{d,\mathrm{new}} := \{u \in U_k(\mathfrak{g}) \otimes \mathcal{C}l;\, [\mathbf{D}_{\mathrm{new}}, u] = du\}.$$

This gives $U_k(\mathfrak{g}') \otimes \mathcal{C}l$ a graded algebra structure:

$$U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\mathrm{new}} = \bigoplus_{d \in \mathbb{Z}} (U_k(\mathfrak{g}') \otimes \mathcal{C}l)_{d,\mathrm{new}}. \tag{215}$$

There is a natural homomorphism $\Psi : U_k(\mathfrak{g}') \otimes \mathcal{C}l \to \mathfrak{U}(C_k(\bar{\mathfrak{g}}))$ of graded algebras induced by the correspondence

$$\mathfrak{g} \ni J(n) \mapsto J(n)(= (J(-1)|0\rangle)_{\{n\}}) \in \mathfrak{L}(C_k(\bar{\mathfrak{g}})) \tag{216}$$

$$\mathcal{C}l \ni \psi_\alpha(n) \mapsto \psi_\alpha(n)(= (\psi_\alpha(-1)|0\rangle))_{\{n\}} \in \mathfrak{L}(C_k(\bar{\mathfrak{g}})), \tag{217}$$

$$\mathcal{C}l \ni \psi_{-\alpha}(n) \mapsto \psi_{-\alpha}(n)(= (\psi_{-\alpha}(0)|0\rangle))_{\{n-1\}} \in \mathfrak{L}(C_k(\bar{\mathfrak{g}})) \tag{218}$$

for $J \in \bar{\mathfrak{g}}$, $\alpha \in \bar{\Delta}_+$, $n \in \mathbb{Z}$. Denote by Ψ_N the composite of Ψ with the natural surjection $\mathfrak{U}(C_k(\bar{\mathfrak{g}})) \to \mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))$.

Set

$$\mathsf{I}_N := U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}} \cdot \sum_{d>N} (U_k(\mathfrak{g}') \otimes \mathcal{C}l)_{d,\text{new}} \subset U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}. \tag{219}$$

Proposition 4.15.1. *For each $N \geq 0$, there is an exact sequence*

$$0 \longrightarrow \mathsf{I}_N \longrightarrow U_k(\mathfrak{g}') \otimes \mathcal{C}l \xrightarrow{\Psi_N} \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})) \longrightarrow 0.$$

Proof. Let $\{G_p \mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))\}$ be the filtration of $\mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))$ induced by the standard filtration G of $C_k(\bar{\mathfrak{g}})$ (with respect to the old grading). Let U be the space as in (162), which generates a PBW basis of $C_k(\bar{\mathfrak{g}})$. Then by Theorem 3.14.1 we have

$$\operatorname{gr}^G \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})) \cong \mathbb{S}_N(U). \tag{220}$$

(Here U is identified with its image in $\operatorname{gr}^G C_k(\bar{\mathfrak{g}})$.)

Let $\{G_p(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}})\}$ be the filtration of $U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}$ defined by

$$\begin{aligned} G_{-1}(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) &= 0, \\ G_0(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) &= \Lambda(L\bar{\mathfrak{n}}_-) = \langle \psi_{-\alpha}(n); \alpha \in \bar{\Delta}_+,\ n \in \mathbb{Z} \rangle, \\ G_p(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) &= \sum_{n \in \mathbb{Z},\ J \in \bar{\mathfrak{g}}} J(n) \cdot G_{p-1}(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) \\ &\quad + \sum_{n \in \mathbb{Z},\ \alpha \in \bar{\Delta}_+} \psi_\alpha(n) \cdot G_{p-1}(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) \end{aligned}$$

for $p \geq 1$. This defines a filtration of $U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}$ such that

$$\operatorname{gr}^G(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) \cong S(\bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]) \otimes \Lambda(L\bar{\mathfrak{n}}_+) \otimes \Lambda(L\bar{\mathfrak{n}}_-). \tag{221}$$

This can be regarded as an isomorphism

$$\operatorname{gr}^G(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) \cong S(L(U)). \tag{222}$$

By construction, Ψ preserves the filtration. Therefore, Ψ_N induces a map

$$\operatorname{gr}^G(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}}) \to \operatorname{gr}^G \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})). \tag{223}$$

But under the identification (220) and (222), the map (223) is identical to the natural surjection $S(L(U)) \to \mathbb{S}_N(U)$. Therefore (223) is surjective, and its kernel is $S(L(U)) \sum_{r>N} S(L(U))_r$, which is exactly the image of I_N in $\operatorname{gr}^G(U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{new}})$. This completes the proof. □

Let $U_k\widetilde{(\mathfrak{g}') \otimes \mathcal{C}}l_{\text{new}}$ be the standard degreewise completion (Sect. A.2) of $U_k(\mathfrak{g}') \otimes \mathcal{C}l$ with respect to the grading (215).

The following assertion immediately follows from Proposition 4.15.1 and (200).

Proposition 4.15.2. *There is an isomorphism* $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\text{new}}) \cong U_k\widetilde{(\mathfrak{g}') \otimes \mathcal{C}l}_{\text{new}}$ *as compatible degreewise complete algebras. Thus there is an isomorphism* $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \cong H^0(U_k\widetilde{(\mathfrak{g}') \otimes \mathcal{C}l}_{\text{new}}, \operatorname{ad} Q_+)$.

4.16. New grading vs. old grading, and a realization of $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ **(cf. [27, Section 2.2]).** Let

$$(U_k(\mathfrak{g}') \otimes \mathcal{C}l)_{d,\text{old}} := \{u \in U_k(\mathfrak{g}) \otimes C_k(\bar{\mathfrak{g}}); [\mathbf{D}, u] = du\}.$$

This also gives $U_k(\mathfrak{g}') \otimes \mathcal{C}l$ a graded algebra structure:

$$U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{old}} = \bigoplus_{d \in \mathbb{Z}} (U_k(\mathfrak{g}') \otimes \mathcal{C}l)_{d,\text{old}}. \tag{224}$$

Let $U_k\widetilde{(\mathfrak{g}') \otimes \mathcal{C}l}_{\text{old}}$ be the standard degreewise completion of $U_k(\mathfrak{g}') \otimes \mathcal{C}l$ with respect to the grading (224).

Proposition 4.16.1. *We have the following:*

(i) $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\text{old}}) \cong U_k\widetilde{(\mathfrak{g}') \otimes \mathcal{C}l}_{\text{old}}$ *as compatible degreewise complete algebras.*
(ii) *The correspondence* $\bar{\mathfrak{g}} \ni J \mapsto J(0) \in \mathfrak{U}_0(C_k(\bar{\mathfrak{g}})_{\text{old}})$, $\overline{\mathcal{C}l} \ni \psi_\alpha \mapsto \psi_\alpha(0) \in \mathfrak{U}_0(C_k(\bar{\mathfrak{g}})_{\text{old}})$ *gives the algebra isomorphism* $U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l} \xrightarrow{\sim} \mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})_{\text{old}})$.

Proof. The proof of (1) is exactly in the same manner as Proposition 4.15.2. (ii) follows from (i). □

For each $N \geq 0$, $\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\text{old}})$ and $\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\text{new}})$ admit weight space decompositions with respect to the adjoint action of $\mathfrak{h}$:

$$\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\text{old}}) = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\text{old}})^\lambda, \tag{225}$$

$$\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\text{new}}) = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\text{new}})^\lambda. \tag{226}$$

Let $t_{\bar{\lambda}} \in \widetilde{W}$ with $\bar{\lambda} \in \bar{P}^\vee$. Then its action on $\mathfrak{h}$ extends to an automorphism of $U_k(\mathfrak{g}') \otimes \mathcal{C}l$ by the following:

$$\begin{aligned}
&t_{\bar{\lambda}}(J_\alpha(n)) = J_\alpha(n - \langle \alpha, \bar{\lambda} \rangle) \quad \text{for } \alpha \in \bar{\Delta}, n \in \mathbb{Z};\\
&t_{\bar{\lambda}}(h) = h - k(\bar{\lambda}, h) \quad \text{for } h \in \bar{\mathfrak{h}};\\
&t_{\bar{\lambda}}(J_i(n)) = J_i(n) \quad \text{for } i \in \bar{I}, n \neq 0;\\
&t_{\bar{\lambda}}(\psi_\alpha(n)) = \psi_\alpha(n - \langle \alpha, \bar{\lambda} \rangle) \quad \text{for } \bar{\lambda} \in \bar{\Delta}, n \in \mathbb{Z}.
\end{aligned}$$

One has:

$$t_{\bar{\rho}^\vee}\left((U_k(\mathfrak{g}')\otimes \mathcal{C}l)^{\mu}_{d,\mathrm{old}}\right)\subset (U_k(\mathfrak{g}')\otimes \mathcal{C}l)^{t_{\bar{\rho}^\vee}(\mu)}_{d,\mathrm{new}},$$
$$t_{-\bar{\rho}^\vee}\left((U_k(\mathfrak{g}')\otimes \mathcal{C}l)^{\mu}_{d,\mathrm{new}}\right)\subset (U_k(\mathfrak{g}')\otimes \mathcal{C}l)^{t_{-\bar{\rho}^\vee}(\mu)}_{d,\mathrm{old}}.$$

This shows that $t_{\bar{\rho}^\vee}$ and $t_{-\bar{\rho}^\vee}$ extends to the mutually inverse isomorphisms

$$\widehat{t_{\bar{\rho}}}:\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})\xrightarrow{\sim}\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\mathrm{new}}),\tag{227}$$
$$\widehat{t_{-\bar{\rho}}}:\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\mathrm{new}})\xrightarrow{\sim}\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})\tag{228}$$

for all N, inducing isomorphisms

$$\widehat{t_{\bar{\rho}}}:\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})\xrightarrow{\sim}\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{new}}),\tag{229}$$
$$\widehat{t_{-\bar{\rho}}}:\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{new}})\xrightarrow{\sim}\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})\tag{230}$$

of compatible degreewise complete algebras. In particular, we have

$$\widehat{t_{\bar{\rho}}}:\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})\xrightarrow{\sim}\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\mathrm{new}}),\tag{231}$$
$$\widehat{t_{-\bar{\rho}}}:\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\mathrm{new}})\xrightarrow{\sim}\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}).\tag{232}$$

The following assertion follows immediately from Proposition 4.16.1.

Proposition 4.16.2. *The map $\widehat{t_{-\bar{\rho}^\vee}}$ induces the algebra isomorphism $\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\mathrm{new}})\xrightarrow{\sim}U(\bar{\mathfrak{g}})\otimes\overline{\mathcal{C}l}$.*

We have

$$\widehat{t_{-\bar{\rho}^\vee}}\left(Q^{\mathrm{st}}_+\right)=Q^{\mathrm{st}}_+,\quad \widehat{t_{-\bar{\rho}^\vee}}(\chi_+)=\chi'_+,\tag{233}$$
$$\widehat{t_{-\bar{\rho}^\vee}}(Q_+)=Q'_+:=Q^{\mathrm{st}}_++\chi'_+,\tag{234}$$

where

$$\chi'_+:=\sum_{\alpha\in\bar{\Pi}}\psi_{-\alpha}(0).\tag{235}$$

Of course we have

$$\left[\chi'_+,Q^{\mathrm{st}}_+\right]=0,\quad(\chi'_+)^2=0,\quad(Q'_+)^2=0,\tag{236}$$

which can be checked directly. Hence each $(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})''),\mathrm{ad}\,Q'_+)$ is a cochain complex and one can define the corresponding cohomology

$$H^\bullet(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}))=\bigoplus_{d\in\mathbb{Z}}H^\bullet(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}))_d,$$
$$H^\bullet(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}))_d:=\lim_{\substack{\leftarrow\\ N}}H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})).$$

In view of Proposition 4.16.1 (i),

$$H^\bullet(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}))=H^\bullet\left(U_k\widetilde{(\mathfrak{g}')\otimes}\,\mathcal{C}l_{\mathrm{old}},\mathrm{ad}\,Q'_+\right).$$

By Proposition 4.16.1 (ii), one finds that the subcomplex $(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}), \mathrm{ad}\, Q'_+)$ of $\mathfrak{U}_0(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}})$ is identical to cochain complex $(\bar{C}(\bar{\mathfrak{g}}), \mathrm{ad}\, \bar{Q}_+)$ considered in Sect. 2.4. Therefore by Theorem 2.4.2 we have

$$H^0(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}}))_{\mathrm{old}}) \cong \mathcal{Z}(\bar{\mathfrak{g}}). \tag{237}$$

By (228) and (234), it is clear that

$$H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}}))_{\mathrm{old}}, \mathrm{ad}\, Q'_+) = H^\bullet(\mathfrak{U}_N(C_k(\bar{\mathfrak{g}})_{\mathrm{new}}), \mathrm{ad}\, Q_+) \quad \forall N. \tag{238}$$

Therefore, Theorem 4.14.2 and (237) give the following assertion.

Theorem 4.16.3. *The* $H^i(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}))$ *is zero for all* $i \neq 0$ *and* $\widehat{t_{-\bar{\rho}^\vee}}$ *induces the following isomorphisms:*

(i) $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})) = H^0(U_k\widetilde{(\mathfrak{g}') \otimes \mathcal{C}l}_{\mathrm{old}}, \mathrm{ad}\, Q'_+)$ *as compatible degreewise topological algebras;*
(ii) (cf. [27, Proposition 3.3 (1)]) $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})) = \mathcal{Z}(\bar{\mathfrak{g}})$ *as algebras.*

Below we often identify $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ with $\mathcal{Z}(\bar{\mathfrak{g}})$ through the isomorphism $\widehat{t_{-\bar{\rho}^\vee}}$.

Similarly as above, let $C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}$ be the vertex algebra $C_k(\bar{\mathfrak{g}})''$ equipped with the Hamiltonian $-\mathbf{D}$. To avoid confusion we may write $C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}$ for the same vertex algebra with the Hamiltonian $-\mathbf{D}_{\mathrm{new}}$. By Theorem 3.14.1 and Proposition 4.8.3, $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}})$ and $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}})$ are subalgebras of $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}})$ and $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{new}})$, respectively.

The restriction of (229) and (230) give the mutually inverse isomorphisms

$$\widehat{t_{\bar{\rho}}} : \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}) \xrightarrow{\sim} \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}), \tag{239}$$

$$\widehat{t_{-\bar{\rho}^\vee}} : \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}) \xrightarrow{\sim} \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}). \tag{240}$$

The $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}})$ is a subcomplex of $(\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\mathrm{old}}), \mathrm{ad}\, Q'_+)$. The following assertion follows from Theorem 4.14.1.

Theorem 4.16.4. *The* $H^i(\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}))$ *is zero for all* $i \neq 0$ *and* $\widehat{t_{-\bar{\rho}^\vee}}$ *induces the following isomorphisms:*

(i) $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}))$ *as compatible degreewise topological algebras;*
(ii) $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} H^0(\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}))$ *as algebras.*

Remark 4.16.5. Let $\bar{\mathfrak{a}}$ be the Lie superalgebra such that $\bar{\mathfrak{a}}^{\mathrm{even}}$ is the Lie algebra $\bar{\mathfrak{b}}_-$, $\bar{\mathfrak{a}}^{\mathrm{odd}}$ is the supercommutative Lie superalgebra $\bar{\mathfrak{n}}_-$, and $[x, y] = \mathrm{ad}(x)(y)$ for $x \in \bar{\mathfrak{b}}_- = \bar{\mathfrak{a}}^{\mathrm{even}}$, $y \in \bar{\mathfrak{n}}_- = \bar{\mathfrak{a}}^{\mathrm{odd}}$. Then similarly as Proposition 4.16.1 (ii) one sees that the correspondence

$$\bar{\mathfrak{a}}^{\mathrm{even}} \ni J_a \mapsto \widehat{J_a}(0) \in \mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}),$$
$$\bar{\mathfrak{a}}^{\mathrm{odd}} \ni J_\alpha \mapsto \psi_\alpha(0) \in \mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}})$$

with $a \in \bar{I} \sqcup \bar{\Delta}_-$, $\alpha \in \bar{\Delta}_-$, give the isomorphism

$$U(\bar{\mathfrak{a}}) \xrightarrow{\sim} \mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\text{old}}) \tag{241}$$

By Theorem 4.16.4 $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}})) = \mathcal{Z}(\bar{\mathfrak{g}})$ can be identified with the commutative subalgebra of $U(\bar{\mathfrak{a}})$ consisting of elements that commutes with the adjoint action of Q'_+ under the identification (241).

4.17. The conformal vector at non-critical level. Assume that $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$. Define

$$L(z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2} := L^{\mathfrak{g}}(z) + L^f(z) + \frac{d}{dz}\widehat{\bar{\rho}^\vee}(z),$$

where

$$\widehat{\bar{\rho}^\vee}(z) = \sum_{n\in\mathbb{Z}} \widehat{\bar{\rho}^\vee}(n) z^{-n-1} = \bar{\rho}^\vee(z) + \sum_{\alpha\in\bar{\Delta}_+} \operatorname{ht}\alpha : \psi_\alpha(z)\psi_{-\alpha}(z) : .$$

A short calculation shows that

$$[Q_+, L(z)] = 0. \tag{242}$$

Thus $L(-2)|0\rangle$ defines an element of $\mathscr{W}_k(\bar{\mathfrak{g}})$. Further, one sees that

$$T^*_{\text{new}} = L(1), \quad -\mathbf{D}_{\text{new}} = L(0), \quad T = L(-1) \tag{243}$$

as elements of $\operatorname{End} C_k(\bar{\mathfrak{g}})$, and that

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c(k)\,\mathrm{id}, \tag{244}$$

where

$$c(k) = l - 12\left((k+h^\vee)|\bar{\rho}^\vee|^2 - 2\langle\bar{\rho}, \bar{\rho}^\vee\rangle + \frac{1}{k+h^\vee}|\bar{\rho}|^2\right), \tag{245}$$

see [21,25,27] (Recall that l is the rank of $\bar{\mathfrak{g}}$). Therefore $L(-2)|0\rangle$ is a conformal vector of central charge $c(k)$. Thus, the vertex algebra $\mathscr{W}_k(\bar{\mathfrak{g}})$ is conformal for $k \neq -h^\vee$.

Remark 4.17.1. Write k as $k = -h^\vee + p/q$. Then, as explained in [27], the formula (245) can be written as

$$c(k) = l - 12|q\bar{\rho} - p\bar{\rho}^\vee|^2/pq,$$

which in the simply laced case becomes

$$c(k) = l(1 - h^\vee(h^\vee+1)(p-q)^2/pq).$$

Proposition 4.17.2. *Assume that $k \neq -h^\vee$. Then under the isomorphism $\widehat{t_{-\bar\rho^\vee}} : \mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}})) \xrightarrow{\sim} \mathcal{Z}(\bar{\mathfrak{g}})$ in Theorem* 4.16.3 (ii), the image of the Hamiltonian $L(0)$ in $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ is mapped to the element

$$\frac{1}{2(k+h^\vee)}\Omega - \frac{1}{2}((k+h^\vee)|\bar\rho^\vee|^2 - 2\langle\bar\rho, \bar\rho^\vee\rangle)1 \in \mathcal{Z}(\bar{\mathfrak{g}})$$

Here Ω is the Casimir element of $U(\bar{\mathfrak{g}})$.

Proof. Direct calculation. □

5. Irreducible highest weight representations of $\mathscr{W}$-algebras

5.1. Verma modules of $\mathscr{W}$-algebras. We identify $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ with the center $\mathcal{Z}(\bar{\mathfrak{g}})$ of $U(\bar{\mathfrak{g}})$ through Theorem 4.16.3 (ii).

Recall that $\gamma_{\bar\lambda} : \mathcal{Z}(\bar{\mathfrak{g}}) \to \mathbb{C}$ with $\bar\lambda \in \bar{\mathfrak{h}}^*$ denotes the Harish–Chandra homomorphism (with respect to the triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$), see (56). The one-dimensional $\mathcal{Z}(\bar{\mathfrak{g}})$-module $\mathbb{C}_{\gamma_{\bar\lambda}}$ can be naturally regarded as a $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))_{\geq 0}$-module on which $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))_{>0}$ acts trivially.

Define

(246) $$\mathbf{M}(\gamma_{\bar\lambda}) := \mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \otimes_{\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))_{\geq 0}} \mathbb{C}_{\gamma_{\bar\lambda}}.$$

The $\mathscr{W}_k(\bar{\mathfrak{g}})$-module $\mathbf{M}(\gamma_{\bar\lambda})$ is called the *Verma module of $\mathscr{W}_k(\bar{\mathfrak{g}})$ with highest weight $\gamma_{\bar\lambda}$*. The canonical vector $1 \otimes 1 \in \mathbf{M}(\gamma_{\bar\lambda})$ is called the *highest weight vector* of $\mathbf{M}(\gamma_{\bar\lambda})$ and denoted by $|\gamma_{\bar\lambda}\rangle$.

Define a subspace $\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))$ of $\mathfrak{U}_0(\mathscr{W}_k(\bar{\mathfrak{g}}))$ (notation Sect. 3.11) by

(247)
$$\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) := \mathrm{span}\{\mathsf{W}_{i_1}(-n_1)\cdot\dots\cdot\mathsf{W}_{i_r}(-n_r);\, r \geq 0,\ i_s \in \bar{I},\ n_s \geq 1\}.$$

Then by Theorem 3.14.1 and Proposition 4.12.1 (iii) there is a linear isomorphism

(248) $$\begin{array}{ccc} \mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) & \xrightarrow{\sim} & \mathbf{M}(\gamma_{\bar\lambda}) \\ u & \mapsto & u|\gamma_{\bar\lambda}\rangle. \end{array}$$

The Verma module $\mathbf{M}(\gamma_{\bar\lambda})$ is an admissible $\mathscr{W}_k(\bar{\mathfrak{g}})$-module (see Sect. 3.12) by the natural grading such that

(249) $$\mathbf{M}(\gamma_{\bar\lambda})_{\mathrm{top}} = \mathbb{C}|\gamma_{\bar\lambda}\rangle,$$

which is unique up to a constant shift. By (248) each subspace $\mathbf{M}(\gamma_{\bar\lambda})_d$ is finite-dimensional.

Let M be any graded $\mathscr{W}_k(\bar{\mathfrak{g}})$-module. Denote by $S(M)$ the subspace of M spanned by the homogeneous vectors m such that $\mathsf{W}_i(n)m = 0$ for all

$i \in \bar{I}$, $n > 0$. Then $S(M)$ is naturally a $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}}))$-module containing M_{top} (notation Sect. 3.12). By definition we have the isomorphism

$$\operatorname{Hom}_{\mathscr{W}_k\text{-grMod}}(\mathbf{M}(\gamma_{\bar{\lambda}}), M) \cong \bigoplus_{d\in\mathbb{C}} \operatorname{Hom}_{\mathcal{Z}(\bar{\mathfrak{g}})}(\mathbb{C}_{\gamma_{\bar{\lambda}}}, S(M)_d), \tag{250}$$

where $S(M)_d = S(M) \cap M_d$.

Let $F = \{F_p\mathscr{W}_k(\bar{\mathfrak{g}})\}$ be the filtration of $\mathscr{W}_k(\bar{\mathfrak{g}})$ defined by (193), $\{F_p\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))\}$, the induced filtration of $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. Set

$$F_p\mathbf{M}(\gamma_{\bar{\lambda}}) := F_p\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \cdot |\gamma_{\bar{\lambda}}\rangle. \tag{251}$$

This defines an increasing, separated, exhaustive filtration of $\mathbf{M}(\gamma_{\bar{\lambda}})$. By definition, $F_p\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \cdot F_q\mathbf{M}(\gamma_{\bar{\lambda}}) \subset F_{p+q}\mathbf{M}(\gamma_{\bar{\lambda}})$ for all p, q. Thus the graded space $\operatorname{gr}^F \mathbf{M}(\gamma_{\bar{\lambda}})$ is naturally a module over the commutative vertex algebra $\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})$.

Proposition 5.1.1. *We have*

$$\operatorname{gr}^F \mathbf{M}(\gamma_{\bar{\lambda}}) \cong \mathfrak{U}\big(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})\big) \otimes_{\mathfrak{U}(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}}))_{\geq 0}} \mathbb{C}$$

as $\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})$*-modules, where* $\mathbb{C}$ *is the trivial* $\mathfrak{U}(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}}))_{\geq 0}$*-module. Namely the* $\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})$*-module* $\operatorname{gr}^F \mathbf{M}(\gamma_{\bar{\lambda}})$ *is isomorphic to the polynomial ring*

$$\mathbb{C}[\bar{\mathsf{W}}_i(-n_i); i \in \bar{I}, n_i \geq 1]$$

on which $\bar{\mathsf{W}}_i(n)$*, with* $n < 0$*, acts by the multiplication, and* $\bar{\mathsf{W}}_i(n)$*, with* $n \geq 0$*, acts trivially.*

Proof. We have the surjective homomorphism $\mathfrak{U}(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})) \otimes_{\mathfrak{U}(\operatorname{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}}))_{\geq 0}} \mathbb{C} \to \operatorname{gr}^F \mathbf{M}(\gamma_{\bar{\lambda}})$ that sends $1 \otimes 1$ to the image of $|\gamma_{\bar{\lambda}}\rangle$ in $\operatorname{gr}^F \mathbf{M}(\gamma_{\bar{\lambda}})$. From (248) it follows that this is a bijection. □

Remark 5.1.2. Let $\bar{M}^{\dagger}(\bar{\lambda})$ be the lowest weight Verma module of $\bar{\mathfrak{g}}$ with lowest weight $\bar{\lambda}$. Denote by $\gamma'_{\bar{\lambda}} : \mathcal{Z}(\bar{\mathfrak{g}}) \to \mathbb{C}$ be the evaluation at $\bar{M}^{\dagger}(\bar{\lambda})$. Then we have

$$\gamma'_{\bar{\lambda}} = \gamma_{w_0(\bar{\lambda})}, \tag{252}$$

and hence $\mathbf{M}(\gamma'_{\bar{\lambda}}) \cong \mathbf{M}(\gamma_{w_0(\bar{\lambda})})$, where w_0 is the longest element of $\bar{W}$. Indeed let

$$\bar{P}_{++} = \{\bar{\lambda} \in \bar{P}; \langle\lambda, \bar{\alpha}^{\vee}\rangle \in \mathbb{Z}_{\geq 0}\ \forall \bar{\alpha} \in \bar{\Delta}_+\}. \tag{253}$$

Then for $\bar{\lambda} \in \bar{P}_{++}$ the irreducible finite-dimensional $\bar{\mathfrak{g}}$-module $\bar{L}(\bar{\lambda})$ has the lowest weight $w_0(\bar{\lambda})$. Thus (252) holds for all $\lambda \in w_0(\bar{P}_{++})$. Hence it must hold for all $\lambda \in \bar{\mathfrak{h}}^*$.

5.2. A realization of $\mathbf{M}(\gamma_{\bar{\lambda}})$. Let $\mathbb{C}_\lambda = \mathbb{C}1_\lambda$ with $\lambda \in \mathfrak{h}^*$ be the one-dimensional representation of $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\text{old}})_{\geq 0}$ such that

$$\widehat{J}_{-\alpha}(n)1_\lambda = \psi_{-\alpha}(n)1_\lambda = 0 \quad \text{for } \alpha \in \bar{\Delta}_+, n \geq 0$$
$$\widehat{J}_i(n)1_\lambda = 0 \quad \text{for } i \in \bar{I}, n > 0,$$
$$\widehat{J}_i(0)1_\lambda = \lambda(J_i)1_\lambda \quad \text{for } i \in \bar{I}.$$

We consider $\mathbb{C}_\lambda$ as a cochain complex concentrated in degree 0 on which the differential Q'_+ acts trivially. Let $K(\lambda)$ be a $C_k(\bar{\mathfrak{g}})''_{\text{old}}$-module defined by

$$K(\lambda) = \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\text{old}}) \otimes_{\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\text{old}})_{\geq 0}} \mathbb{C}_\lambda. \tag{254}$$

This is an admissible $\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')$-module with the natural grading such that $K(\lambda)_{\text{top}} = \mathbb{C} \otimes \mathbb{C}_\lambda$. We regard $K(\lambda)$ as a cochain complex with the differential Q'_+, whose action is defined by the rule $Q'_+(u \otimes 1) = [Q'_+, u] \otimes 1$ for $u \in \mathfrak{U}(C_k(\bar{\mathfrak{g}})'')$.

By (200) and (240), $H^\bullet(K(\lambda))$ can be viewed as a $\mathscr{W}_k(\bar{\mathfrak{g}})$-module by the following action:

$$u \cdot [v] = [\widehat{t}_{-\bar{\rho}^\vee}(u)v] \quad \text{for } u \in \mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})) \text{ and } [v] \in H^\bullet(K(\lambda)). \tag{255}$$

Then $H^\bullet(K(\lambda))$ is an admissible $\mathscr{W}_k(\bar{\mathfrak{g}})$-module. By construction,

$$H^i(K(\lambda))_{\text{top}} = H^i(K(\lambda)_{\text{top}}) = \begin{cases} \mathbb{C} & i = 0, \\ 0 & i \neq 0. \end{cases}$$

This is a $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$-module.

Proposition 5.2.1. *We have $H^0(K(\lambda))_{\text{top}} \cong \mathbb{C}_{\gamma_\lambda}$ as $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$-modules.*

Proof. We shall use the identification

$$\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}})) = H^0(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})''_{\text{old}}), \text{ad}\, Q'_+) \tag{256}$$
$$= H^0(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\text{old}}), \text{ad}\, Q'_+) \tag{257}$$
$$= H^0(U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}, \text{ad}\, \bar{Q}_+) = \mathcal{Z}(\bar{\mathfrak{g}})$$

through Theorems 4.16.3 and 4.16.4.

Observe that $\bar{M}^\dagger(\bar{\lambda} + 2\bar{\rho})$ has the central character

$$\gamma'_{\bar{\lambda}+2\bar{\rho}} = \gamma_{w_0(\bar{\lambda}+2\bar{\rho})} = \gamma_{w_0 \circ \bar{\lambda}} = \gamma_{\bar{\lambda}} \tag{258}$$

(notation Remark 5.1.2).

The superalgebra $\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})_{\text{old}}) = U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$ acts on $\bar{M}^\dagger(\bar{\lambda}+2\rho) \otimes \Lambda(\bar{\mathfrak{n}}_+)$ as in Sect. 2.5, and by (257) this induces the action of $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ on the space

$$H_0\big(\bar{\mathfrak{n}}_+, \bar{M}^\dagger(\bar{\lambda} + 2\bar{\rho}) \otimes \mathbb{C}_{\bar{\chi}_+}\big) = H_0\big(\bar{M}^\dagger(\bar{\lambda} + 2\bar{\rho}) \otimes \Lambda(\bar{\mathfrak{n}}_+), \bar{Q}_+\big).$$

Let $|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger} = v^{\dagger}_{\bar{\lambda}+2\bar{\rho}} \otimes 1$, where $v^{\dagger}_{\bar{\lambda}+2\bar{\rho}}$ is the lowest weight vector of $\bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})$. Then $\mathbb{C}|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger}$ is a subcomplex of $(\bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\Lambda(\bar{\mathfrak{n}}_+), \bar{Q}_+)$. From (the proof of) Theorem 2.5.7 (i), it follows that the natural embedding $\mathbb{C}|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger} \hookrightarrow \bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\Lambda(\bar{\mathfrak{n}}_+)$ induces an isomorphism

$$(259)\quad H_0(\mathbb{C}|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger}) \cong H_0(\bar{\mathfrak{n}}_+, \bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\mathbb{C}_{\bar{\chi}_+}) = \mathbb{C}_{\gamma_{\bar{\lambda}}}$$

as $\mathcal{Z}(\bar{\mathfrak{g}})$-modules.

Consider $\bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\Lambda(\bar{\mathfrak{n}}_+)$ as a $\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\text{old}})$-module by the correspondence

$$(260)\qquad \widehat{J_a}(0) \mapsto J_a - \sum_{\beta,\gamma\in\bar{\Delta}_+} c^{\gamma}_{a,\beta}\psi_{-\beta}\psi_{\gamma} \quad \text{for } a\in\bar{\Delta}_-\sqcup\bar{I},$$

$$(261)\qquad \psi_{\alpha}(0)\mapsto\psi_{\alpha} \quad \text{for } \alpha\in\bar{\Delta}_-$$

(see Remark 4.16.5). Then a short calculation shows that

$$(262)\qquad \widehat{J_i}(0)|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger} = \lambda(J_i)|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger} \quad \text{for } i\in\bar{I},$$

$$(263)\qquad \widehat{J_{\alpha}}(0)|\bar{\lambda}+2\bar{\rho}\rangle^{\dagger} = 0 \quad \text{for } \alpha\in\bar{\Delta}_-.$$

Therefore the correspondence

$$(264)\qquad \bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\Lambda(\bar{\mathfrak{n}}_+) \ni |\bar{\lambda}+2\bar{\rho}\rangle^{\dagger} \mapsto 1_{\lambda}\in K(\lambda)_{\text{top}}$$

gives a $\mathfrak{Z}\mathfrak{h}(C_k(\bar{\mathfrak{g}})''_{\text{old}})$-equivalent cochain map. Here $\bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\Lambda(\bar{\mathfrak{n}}_+)$ is considered as a cochain complex by reversing the homological gradation. Therefore, it follows that (264) induces an isomorphism

$$H_0\big(\bar{M}^{\dagger}(\bar{\lambda}+2\bar{\rho})\otimes\Lambda(\bar{\mathfrak{n}}_+), \bar{Q}_+\big) \xrightarrow{\sim} H^0(K(\lambda))_{\text{top}}.$$

By (259) this completes the proof. □

Proposition 5.2.2. *We have* $H^i(K(\lambda)) = 0$ *for* $i\neq 0$ *and* $H^0(K(\lambda)) \cong \mathbf{M}(\gamma_{\bar{\lambda}})$ *as* $\mathscr{W}_k(\mathfrak{g})$*-modules.*

Proof. By (250) and Proposition 5.2.1 there is a unique $\mathscr{W}_k(\bar{\mathfrak{g}})$-module homomorphism

$$\phi : \mathbf{M}(\gamma_{\bar{\lambda}}) \to H^0(K(\lambda))$$

that sends $|\gamma_{\bar{\lambda}}\rangle$ to $[1\otimes 1_{\lambda}]\in H^0(K(\lambda))$.

Let F be the filtration of $C_k(\bar{\mathfrak{g}})''$ defined in Sect. 4.9, $\{F_p\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\text{old}})\}$ the induced filtration. Define

$$(265)\qquad F_p K(\lambda) := F_p\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\text{old}})\cdot K(\lambda)_{\text{top}}.$$

Then this defines an increasing, separated, exhaustive filtration of $K(\lambda)$ such that

(266)
$$Q_+^{\mathrm{st}} \cdot F_p K(\lambda) \subset F_{p-1} K(\lambda), \quad \chi'_+ \cdot F_p K(\lambda) \subset F_p K(\lambda), \quad F_{-1} K(\lambda) = 0.$$

Thus there is a converging spectral sequence $E_r^{p,q} \Rightarrow H^\bullet(K(\lambda))$ such that

(267)
$$E_1^{p,q} = H^{p+q}\big(\mathrm{gr}^F_{-p} K(\lambda)\big).$$

By construction, we have

(268)
$$H^\bullet(\mathrm{gr}^F K(\lambda)) = H^\bullet(K(\lambda), \chi'_+).$$

From the commutation relations

(269)
$$[\chi'_+, \widehat{J}_a(-n)] = \sum_{\beta \in \bar{\Delta}_+} ([f, J_a], J_\beta)\psi_{-\beta}(-n) \quad \text{for } a \in \bar{\Delta}_- \sqcup \bar{I},$$

(270)
$$[\chi'_+, \psi_{-\alpha}(-n)] = 0 \quad \text{for } \alpha \in \bar{\Delta}_+,$$

one can show exactly in the same manner as Proposition 4.10.1 that

(271)
$$H^i(K(\lambda), \chi_+) = 0 \quad \text{for } i \neq 0$$

and that there is a linear isomorphism

(272)
$$S(\bar{\mathfrak{g}}^f \otimes \mathbb{C}[t^{-1}]t^{-1}) \xrightarrow{\sim} H^0(K(\lambda), \chi'_+)$$

given by the correspondence

$$\big(P_{i_1} \otimes t^{-n_1}\big) \dots \big(P_{i_r} \otimes t^{-n_r}\big) \mapsto \widehat{P}_{i_1}(-n_1) \dots \widehat{P}_{i_r}(-n_r) \otimes 1.$$

By (268) and (271) it follows that our spectral sequence collapses at $E_1 = E_\infty$, and consequently we have $H^i(K(\lambda)) = 0$ for $i \neq 0$ and the isomorphism

(273)
$$\mathrm{gr}^F H^0(K(\lambda)) \cong S(\bar{\mathfrak{g}}^f \otimes \mathbb{C}[t^{-1}]t^{-1}).$$

Finally, $\phi(F_p \mathbf{M}(\gamma_{\bar{\lambda}}) \subset F_p H^0(K(\lambda))$ by construction. Thus we have a $\mathrm{gr}^F \mathscr{W}_k(\bar{\mathfrak{g}})$-module homomorphism

(274)
$$\mathrm{gr}^F \mathbf{M}(\gamma_{\bar{\lambda}}) \to \mathrm{gr}^F H^0(K(\lambda)).$$

But from (273) and Proposition 5.1.1 it follows that (274) is a bijection. Thus completes the proof. □

5.3. Irreducible highest weight representations of $\mathscr{W}$-algebras. Let $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Denote by $\mathbf{N}(\gamma_{\bar{\lambda}})$ the sum of all proper graded $\mathscr{W}_k(\bar{\mathfrak{g}})$-submodules of $\mathbf{M}(\gamma_{\bar{\lambda}})$. Define the graded $\mathscr{W}_k(\bar{\mathfrak{g}})$-module $\mathbf{L}(\gamma_{\bar{\lambda}})$ by

$$\mathbf{L}(\gamma_{\bar{\lambda}}) := \mathbf{M}(\gamma_{\bar{\lambda}})/\mathbf{N}(\gamma_{\bar{\lambda}}). \tag{275}$$

The image of $|\gamma_{\bar{\lambda}}\rangle \in \mathbf{M}(\gamma_{\bar{\lambda}})$ in $\mathbf{L}(\gamma_{\bar{\lambda}})$ is also denoted by $|\gamma_{\bar{\lambda}}\rangle$. We have $\mathbf{L}(\gamma_{\bar{\lambda}})_{\mathrm{top}} = \mathbb{C}|\gamma_{\bar{\lambda}}\rangle$.

Theorem 5.3.1. *The $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules $\mathbf{L}(\gamma_{\bar{\lambda}})$ is a unique simple graded quotient of $\mathbf{M}(\gamma_{\bar{\lambda}})$. The set*

$$\{\mathbf{L}(\gamma_{\bar{\lambda}}); \bar{\lambda} + \bar{\rho} \in \bar{W}\backslash\bar{\mathfrak{h}}\}$$

is the complete set of the isomorphism classes of simple objects of $\mathscr{W}_k(\bar{\mathfrak{g}})$-adMod.

Proof. It is clear that $\mathbf{N}(\gamma_{\bar{\lambda}}) \not\ni |\gamma_{\bar{\lambda}}\rangle$. Thus it follows that $\mathbf{N}(\gamma_{\bar{\lambda}})$ is the unique maximal graded submodule of $\mathbf{M}(\gamma_{\bar{\lambda}})$. The second assertion is Zhu's theorem Theorem 3.12.1, which is easily seen from (250). □

Note that we have

$$S(\mathbf{L}(\gamma_{\bar{\lambda}})) = \mathbb{C}|\gamma_{\bar{\lambda}}\rangle (\cong \mathbb{C}_{\gamma_{\bar{\lambda}}}), \tag{276}$$

because otherwise $\mathbf{L}(\gamma_{\bar{\lambda}})$ admits a non-trivial proper graded submodule.

5.4. The simple quotient of $\mathscr{W}_k(\bar{\mathfrak{g}})$ is $\mathbf{L}(\gamma_{\mathrm{vac_k}})$. It is clear that $\mathscr{W}_k(\bar{\mathfrak{g}})_{\mathrm{top}} = \mathbb{C}|0\rangle$ when $\mathscr{W}_k(\bar{\mathfrak{g}})$ is considered as a $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules. Set

$$\mathrm{vac_k} := \overline{t_{-\bar{\rho}^\vee} \circ k\Lambda_0} = -(k + h^\vee)\bar{\rho}^\vee \in \bar{\mathfrak{h}}^*. \tag{277}$$

A short calculation shows that

$$\widehat{t_{\bar{\rho}^\vee}}(\widehat{J_i}(0))|0\rangle = \langle \mathrm{vac_k}, J_i\rangle|0\rangle \quad \text{for } i \in \bar{I},$$
$$\widehat{t_{\bar{\rho}^\vee}}(\widehat{J_\alpha}(0))|0\rangle = 0, \quad \widehat{t_{\bar{\rho}^\vee}}(\psi_\alpha(0))|0\rangle = 0 \quad \text{for } \alpha \in \bar{\Delta}_-$$

(cf. [2, Proposition 4.2]). Hence by Proposition 5.2.1 it follows that

$$\mathscr{W}_k(\bar{\mathfrak{g}})_{\mathrm{top}} \cong \mathbb{C}_{\gamma_{\mathrm{vac_k}}} \tag{278}$$

as $\mathfrak{Z}\mathfrak{h}(\mathscr{W}_k(\bar{\mathfrak{g}}))$-modules. Thus there exist a unique surjection $\mathbf{M}(\gamma_{\mathrm{vac_k}}) \twoheadrightarrow \mathscr{W}_k(\bar{\mathfrak{g}})$ of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules that sends $|\gamma_{\mathrm{vac_k}}\rangle$ to $|0\rangle$. Therefore $\mathbf{L}(\gamma_{\mathrm{vac_k}})$ is a simple quotient of $\mathscr{W}_k(\bar{\mathfrak{g}})$. By [26, Remark 4.3.2], $\mathbf{L}(\gamma_{\mathrm{vac_k}})$ is a simple vertex algebra.

5.5. Duality functor D**.** Let $\theta : \mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}})) \to \mathfrak{L}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ be the anti-Lie algebra involution as in Proposition 3.9.1. We have

$$\theta(\mathsf{W}_i(n)) = (-1)^{d_i+1}\mathsf{W}_i(-n) \tag{279}$$

because W_i is quasi-primary.

The θ is induced by the anti-Lie superalgebra isomorphism $\mathfrak{L}(C_k(\bar{\mathfrak{g}})) \to \mathfrak{L}(C_k(\bar{\mathfrak{g}}))$ which is also denoted by θ. We have:

$$\theta(J_\alpha(-\langle\alpha,\bar{\rho}^\vee\rangle)) = -(-1)^{\langle\alpha,\bar{\rho}^\vee\rangle} J_\alpha(-\langle\alpha,\bar{\rho}^\vee\rangle) \quad \text{for } \alpha \in \bar{\Delta},$$
$$\theta(h(0)) = -h(0) \quad \text{for } h \in \bar{\mathfrak{h}},$$
$$\theta(\psi_\alpha(-\langle\alpha,\bar{\rho}^\vee\rangle)) = -\sqrt{-1}(-1)^{\langle\alpha,\bar{\rho}^\vee\rangle}\psi_\alpha(-\langle\alpha,\bar{\rho}^\vee\rangle) \quad \text{for } \alpha \in \bar{\Delta},$$
$$\theta(\psi_{-\alpha}(\langle\alpha,\bar{\rho}^\vee\rangle)) = \sqrt{-1}(-1)^{\langle\alpha,\bar{\rho}^\vee\rangle}\psi_\alpha(\langle\alpha,\bar{\rho}^\vee\rangle) \quad \text{for } \alpha \in \bar{\Delta}.$$

Therefore the following hold:

Lemma 5.5.1. *The* θ *induces an anti-algebra isomorphism* $\bar{\theta} : \mathfrak{Zh}(C_k(\bar{\mathfrak{g}})) \to \mathfrak{Zh}(C_k(\bar{\mathfrak{g}}))$ *with* $\bar{\theta}(a)^2 = (-1)^{p(a)}a$ *for* $a \in \mathfrak{Zh}(C_k(\bar{\mathfrak{g}}))$*, which is given by the following formula under the identification* $\mathfrak{Zh}(C_k(\bar{\mathfrak{g}})) = U(\bar{\mathfrak{g}}) \otimes \overline{\mathcal{C}l}$*:*

$$\bar{\theta}(J_{\pm\alpha}) = -(-1)^{\mathrm{ht}\,\alpha}J_{\pm\alpha} \quad \textit{for } \alpha \in \bar{\Delta}_+, \quad \bar{\theta}(h) = -h \quad \textit{for } h \in \bar{\mathfrak{h}},$$
$$\bar{\theta}(\psi_\alpha) = -\sqrt{-1}(-1)^{\mathrm{ht}\,\alpha}\psi_\alpha \quad \textit{for } \alpha \in \bar{\Delta}_+,$$
$$\bar{\theta}(\psi_{-\alpha}) = \sqrt{-1}(-1)^{\mathrm{ht}\,\alpha}\psi_{-\alpha} \quad \textit{for } \alpha \in \bar{\Delta}_+.$$

Because

$$\theta(Q_+) = -\sqrt{-1}Q_+, \tag{280}$$

$\bar{\theta}$ induces anti-algebra automorphism of $H^0(\mathfrak{Zh}(C_k(\bar{\mathfrak{g}}))) = \mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}})) = \mathcal{Z}(\bar{\mathfrak{g}})$, which is also denoted by $\bar{\theta}$.

Lemma 5.5.2. *For* $\bar{\lambda} \in \bar{\mathfrak{h}}^*$*, we have* $\gamma_{\bar{\lambda}} \circ \bar{\theta}_{|\mathcal{Z}(\bar{\mathfrak{g}})} = \gamma_{-w_0(\bar{\lambda})}$.

Proof. Let $\lambda \in \bar{P}_{++}$ (see (253)). Then $\bar{L}(\bar{\lambda})$ has the central character $\gamma_{\bar{\lambda}}$. Let $\bar{L}(\bar{\lambda})^{\bar{\theta}}$ denote $\bar{L}(\bar{\lambda})$ viewed as a $\bar{\mathfrak{g}}$-module on which $J \in \bar{\mathfrak{g}}$ acts by $J \cdot v = \bar{\theta}(J)v$. Then the formulas in Lemma 5.5.1 show that $\bar{L}(\bar{\lambda})$ has the central character $\gamma_{-w_0(\bar{\lambda})}$. This proves the assertion for all $\bar{\lambda} \in \bar{P}_{++}$. Hence it must hold for all $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. □

Denote by $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ the Serre full subcategory of $\mathscr{W}_k(\bar{\mathfrak{g}})$-$\mathsf{adMod}$ consisting of graded $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules $M = \bigoplus_{d\in\mathbb{C}} M_d$ such that $\dim M_d < \infty$ for all d. The Verma module $\mathbf{M}(\gamma_{\bar{\lambda}})$ belongs to $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. Thus by Theorem 5.3.1 any simple object of $\mathscr{W}_k(\bar{\mathfrak{g}})$-$\mathsf{grMod}$ belongs to $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$.

For an object M of $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$, define

$$D(M) := \bigoplus_{d \in \mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(M_d, \mathbb{C}). \tag{281}$$

Then the formula

$$\langle u \cdot f, m \rangle = \langle f, \theta(u) m \rangle \quad \text{for } u \in \mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})),\ f \in D(M),\ m \in M \tag{282}$$

gives a $\mathscr{W}_k(\bar{\mathfrak{g}})$-module structure on $D(M)$. By definition $D(M)$ is an object of $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$.

One has

$$D(D(M)) = M \tag{283}$$

for $M \in \mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. The correspondence $M \mapsto D(M)$ defines an exact cofunctor from $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ to itself (see Lemma 8.1.1).

The following assertion is clear.

Lemma 5.5.3. *Let $M \in \mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. The correspondence $N \mapsto D(N)$ gives a bijection between the set of graded submodules of M and the set graded quotients of $D(M)$*

Let $|\gamma_{\bar{\lambda}}\rangle^*$ denote the homogeneous vector of $D(\mathbf{M}(\gamma_{\bar{\lambda}}))$ dual to $|\gamma_{\bar{\lambda}}\rangle \in \mathbf{M}(\gamma_{\bar{\lambda}})$. Then $D(\mathbf{M}(\gamma_{\bar{\lambda}}))_{\mathrm{top}} = \mathbb{C}|\gamma_{\bar{\lambda}}\rangle^*$.

Theorem 5.5.4. *Let $\bar{\lambda}, \mu \in \bar{\mathfrak{h}}^*$.*

(i) *There is an isomorphism $D(\mathbf{L}(\gamma_{\bar{\lambda}})) \cong \mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$ of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules.*
(ii) *The $\mathscr{W}_k(\bar{\mathfrak{g}})$-submodule of $D(\mathbf{M}(\gamma_{\bar{\lambda}}))$ generated by $D(\mathbf{M}(\gamma_{\bar{\lambda}}))_{\mathrm{top}} = \mathbb{C}|\gamma_{\bar{\lambda}}\rangle^*$ is isomorphic to $\mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$. This is a unique simple graded submodule of $D(\mathbf{M}(\gamma_{\bar{\lambda}}))$.*
(iii) *Any nonzero (graded) homomorphic image of $\mathbf{M}(\gamma_{\bar{\lambda}})$ in $D(\mathbf{M}(\gamma_{\bar{\mu}}))$ is isomorphic to $\mathbf{L}(\gamma_{\bar{\lambda}})$. Hence*

$$\dim \operatorname{Hom}_{\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))}(\mathbf{M}(\gamma_{\bar{\lambda}}), D(\mathbf{M}(\gamma_{\bar{\mu}}))) = \begin{cases} 1 & \text{if } -w_0(\bar{\mu}) \in \bar{W} \circ \bar{\lambda}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) By Lemma 5.5.3 it is clear that $D(\mathbf{L}(\gamma_{\bar{\lambda}}))$ is a simple object of $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$. Thus it is sufficient to show that $D(\mathbf{L}(\gamma_{\bar{\lambda}}))$ is a quotient of $\mathbf{M}(\gamma_{-w_0(\bar{\lambda})})$. Because its simple, $S(D(\mathbf{L}(\gamma_{\bar{\lambda}}))) = D(\mathbf{L}(\gamma_{\bar{\lambda}}))_{\mathrm{top}} \cong \mathbb{C}_{\gamma_{-w_0(\bar{\lambda})}}$ by Lemma 5.5.2. By (250), this proves the assertion. (ii) By Lemma 5.5.3 and the first assertion of Theorem 5.3.1, it follows that $D(\mathbf{M}(\gamma_{\bar{\lambda}}))$ has the unique simple graded submodule $D(\mathbf{L}(\gamma_{\bar{\lambda}}))$, which is isomorphic to $\mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$, by (i). Further we have $|\gamma_{\bar{\lambda}}\rangle^* \in D(\mathbf{L}(\gamma_{\bar{\lambda}}))$, because it is the vector dual to $|\gamma_{\bar{\lambda}}\rangle \in \mathbf{M}(\gamma_{\bar{\lambda}})$. Therefore the assertion follows. (iii) Let $\phi : \mathbf{M}(\gamma_{\bar{\lambda}}) \to D(\mathbf{M}(\gamma_{\bar{\mu}}))$ be a homomorphism of graded $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules and suppose that $\phi(\mathbf{M}(\gamma_{\bar\lambda})) \neq 0$. Then $\phi(\mathbf{M}(\gamma_{\bar\lambda}))$ must contain $|\gamma_{\bar\mu}\rangle^*$, by (ii). From this it follows that

$$\phi(\mathbf{M}(\gamma_{\bar\lambda})_{\mathrm{top}}) = D(\mathbf{M}(\gamma_{\bar\mu}))_{\mathrm{top}} \tag{284}$$

because $\mathbf{M}(\gamma_{\bar\lambda})$ is generated by $\mathbf{M}(\gamma_{\bar\lambda})_{\mathrm{top}}$ and ϕ is graded. Therefore, by (250), $D(\mathbf{M}(\gamma_{\bar\mu}))_{\mathrm{top}} \cong \mathbb{C}_{\gamma_{\bar\lambda}}$ as $\mathcal{Z}(\bar{\mathfrak{g}})$-modules. But by (ii) we also have $D(\mathbf{M}(\gamma_{\bar\mu}))_{\mathrm{top}} = \mathbb{C}_{\gamma_{-w_0(\bar\mu)}}$, thus $-w_0(\bar\mu) \in \bar W \circ \bar\lambda$. Conversely, if $-w_0(\bar\mu) \in \bar W \circ \bar\lambda$ then (284) extends to the graded $\mathscr{W}_k(\bar{\mathfrak{g}})$-module homomorphism $\mathbf{M}(\gamma_{\bar\lambda}) \to D(\mathbf{M}(\gamma_{\bar\mu}))$, by (250). This completes the proof. □

Remark 5.5.5. For $k \neq -h^\vee$ one has $\theta(L(0)) = L(0)$. Also, it is easy to see that $\Delta_{\bar\lambda} = \Delta_{-w_0(\bar\lambda)}$.

Because $w_0(\bar\rho^\vee) = -\bar\rho^\vee$, we have the following assertion.

Proposition 5.5.6. *The simple vertex algebra $\mathbf{L}(\gamma_{\mathrm{vac_k}})$ is self-dual, that is, there is an isomorphism $D(\mathbf{L}(\gamma_{\mathrm{vac_k}})) \cong \mathbf{L}(\gamma_{\mathrm{vac_k}})$.*

5.6. Critical level cases vs. non-critical level cases. It is known [21] that the structure of $\mathscr{W}_k(\bar{\mathfrak{g}})$ drastically changes at the critical level $k = -h^\vee$. Thus, though our argument does not depend on the parameter k, it makes sense to discuss the difference between the critical level case and non-critical level cases.

First, let $k = -h^\vee$. The following remarkable fact is well-known.

Theorem 5.6.1 (Feigin–Frenkel[21]). *The vertex algebra $\mathscr{W}_{-h^\vee}(\bar{\mathfrak{g}})$ is commutative. In fact it coincides with the center of $V_{-h^\vee}(\bar{\mathfrak{g}})$ (in the sense of vertex algebras).*

Remark 5.6.2. If $k \neq -h^\vee$ then the center of $V_k(\bar{\mathfrak{g}})$ is trivial. This follows from the fact that $\mathrm{Vac}\, V_k(\bar{\mathfrak{g}}) = \mathbb{C}|0\rangle$ and[48, Proposition 3.11.2].

Since $\mathscr{W}_{-h^\vee}(\bar{\mathfrak{g}})$ is commutative it follows that $\mathbf{N}(\gamma_{\bar\lambda}) = \sum_{d<0} \mathbf{M}(\gamma_{\bar\lambda})_d$ and $\mathbf{L}(\gamma_{\bar\lambda})$ is one-dimensional. Therefore the following assertion immediately follows.

Proposition 5.6.3. *Let $k = -h^\vee$. Then $\mathbf{L}(\gamma_{\bar\lambda})$ is isomorphic to the one-dimensional $\mathscr{W}_{-h^\vee}(\bar{\mathfrak{g}})$-module on which $W_i(n)$, with $i \in \bar I$ and $n \neq 0$, acts trivially and $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}})) = \mathcal{Z}(\bar{\mathfrak{g}})$ acts by the central character $\gamma_{\bar\lambda}$.*

Next let $k \neq -h^\vee$. In this case $\mathscr{W}_k(\bar{\mathfrak{g}})$ has the conformal vector $L(-2)|0\rangle$. By (245) and Proposition 4.17.2, we have:

$$L(0)|\gamma_{\bar\lambda}\rangle = \Delta_{\bar\lambda}|\gamma_{\bar\lambda}\rangle, \tag{285}$$

where

$$\Delta_{\bar\lambda} = \frac{|\bar\lambda + \bar\rho|^2}{2(k + h^\vee)} - \frac{\mathrm{rank}\,\bar{\mathfrak{g}}}{24} + \frac{c(k)}{24}. \tag{286}$$

Because the grading of $\mathbf{M}(\gamma_{\bar{\lambda}})$ is determined by the action of $-L(0)$ (up to some constant shift), we have the following assertion.

Proposition 5.6.4. *Let $k \neq -h^\vee$. Then each $\mathbf{L}(\gamma_{\bar{\lambda}})$ is the unique irreducible quotient of the $\mathscr{W}_k(\bar{\mathfrak{g}})$-module $\mathbf{M}(\gamma_{\bar{\lambda}})$.*

Remark 5.6.5. In the case that $k = -h^\vee$, $\mathbf{M}(\gamma_{\bar{\lambda}})$ has (many) non-graded irreducible quotients.

For a $\mathscr{W}_k(\bar{\mathfrak{g}})$-module M, let $M_{[h]}$ be the generalized eigenspace of $L(0)$ of eigenvalue $-d \in \mathbb{C}$:

$$V_{[d]} := \{v \in V;\ (L(0) + d)^r v = 0 \quad \text{for } r \gg 0\}.$$

For an object M of $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$, define the normalized formal character $\operatorname{ch} M$ of M by

$$\operatorname{ch} M := q^{-\frac{c(k)}{24}} \sum_{d \in \mathbb{C}} q^{-d} \dim M_{[d]}. \tag{287}$$

One has $\operatorname{ch} M = q^{-\frac{c(k)}{24}} \operatorname{tr}_M q^{L(0)}$ if $L(0)$ acts on M semisimply.

The following assertion is easily seen from Propositions 4.17.2 and 5.1.1.

Proposition 5.6.6. *It holds that*

$$\operatorname{ch} \mathbf{M}(\gamma_{\bar{\lambda}}) = \frac{q^{\frac{|\bar{\lambda}+\bar{\rho}|^2}{2(k+h^\vee)}}}{\eta(q)^l}$$

for $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, where $\eta(q) = q^{\frac{1}{24}} \prod_{i \geq 1}(1 - q^i)$.

One of the main purpose of this paper is to determine each of $\operatorname{ch} \mathbf{L}(\gamma_{\bar{\lambda}})$.

6. Functors $H_+^\bullet(?)$ and $H_-^\bullet(?)$

6.1. Category $\mathcal{O}_k$. Let $\mathcal{O}_k$ be the Serre full subcategory of the category of left $\mathfrak{g}$-modules consisting of objects M satisfying the following:

- K acts as $k\,\mathrm{id}$ on M;
- $M = \bigoplus_{\lambda \in \mathfrak{h}_k^*} M^\lambda$ and $\dim M^\lambda < \infty$ for all λ;
- there exists a finite subset $\{\mu_1, \dots, \mu_n\} \subset \mathfrak{h}_k^*$ such that $P(M) \subset \bigcup_i \mu_i - Q_+$.

Let $M(\lambda) \in \mathit{Obj}\, \mathcal{O}_k$ be the Verma module with highest weight $\lambda \in \mathfrak{h}_k^*$, $v_\lambda \in M(\lambda)$ its highest weight vector. Let $L(\lambda)$ be the unique simple quotient of $M(\lambda)$. Then every irreducible object of $\mathcal{O}_k$ is isomorphic to exactly one of the $L(\lambda)$ with $\lambda \in \mathfrak{h}_k^*$.

The vertex algebra $V_k(\bar{\mathfrak{g}})$ is an object of $\mathcal{O}_k$ when considered as a $\mathfrak{g}$-module; indeed $V_k(\bar{\mathfrak{g}})$ is a quotient of $M(k\Lambda_0)$.

The correspondence $M \rightsquigarrow M^*$, where M^* is defined by (144), defines the duality functor in $\mathcal{O}_k$. Here, $\mathfrak{g}$ acts on M^* by

$$(Xf)(v) = f(X^t v) \tag{288}$$

where $X \mapsto X^t$ is the anti-automorphism of $\mathfrak{g}$ define by $K^t = K$, $\mathbf{D}^t = \mathbf{D}$ and $J(n)^t = J^t(-n)$ for $J \in \bar{\mathfrak{g}}$, $n \in \mathbb{Z}$ (Notation (7)).

6.2. Category $\mathcal{O}_k^{\triangle}$. Let $\mathcal{O}_k^{\triangle}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects M that admit a Verma flag, that is, a finite filtration $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ such that each successive subquotient M_i/M_{i+1} is isomorphic to some Verma module $M(\lambda_i)$ with $\lambda_i \in \mathfrak{h}_k^*$. The category $\mathcal{O}_k^{\triangle}$ is stable under taking direct summands. Dually, let $\mathcal{O}_k^{\nabla}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects M such that $M^* \in Obj\, \mathcal{O}_k^{\triangle}$.

6.3. Truncated category $\mathcal{O}_k^{\leq\lambda}$ (see [56, Section 2.10] for details). For $\lambda \in \mathfrak{h}_k^*$, let $\mathcal{O}_k^{\leq\lambda}$ be the Serre full subcategory of $\mathcal{O}_k$ consisting of objects M such that $M = \bigoplus_{\mu\leq\lambda} M^\mu$. Then $\mathcal{O}_k^{\leq\lambda}$ is stable under taking (graded) duals. It is known that every simple object $L(\mu) \in Obj\, \mathcal{O}_k^{\leq\lambda}$, $\mu \leq \lambda$, admits an indecomposable projective cover $P_{\leq\lambda}(\mu)$ in $\mathcal{O}_k^{\leq\lambda}$, and hence, every finitely generated object in $\mathcal{O}_k^{\leq\lambda}$ is an image of a projective object of the form $\bigoplus_{i=1}^r P_{\leq\lambda}(\mu_i)$. It is also known that $P_{\leq\lambda}(\mu) \in Obj\, \mathcal{O}_k^{\triangle}$. Dually, $I_{\leq\lambda}(\mu) = P_{\leq\lambda}(\mu)^*$ is the injective envelope of $L(\mu)$ in $\mathcal{O}_k^{\leq\lambda}$. In particular $M \in Obj\, \mathcal{O}_k^{\leq\lambda}$ is a submodule of an injective object of the form $\bigoplus_{i=1}^r I_{\leq\lambda}(\mu_i)$ if its dual M^* is finitely generated.

6.4. Functor $H_+^\bullet(?)$. Let M be an object of $\mathcal{O}_k$. Define

$$C(L\bar{\mathfrak{n}}_+, M) := M \otimes \mathcal{F} = \sum_{i\in\mathbb{Z}} C^i(L\bar{\mathfrak{n}}_+, M), \tag{289}$$

$$\text{where } C^i(L\bar{\mathfrak{n}}_+, M) := M \otimes \mathcal{F}^i.$$

Then $C(L\bar{\mathfrak{n}}_+, M)$ is naturally regarded as a module over the vertex superalgebra $C_k(\bar{\mathfrak{g}}) = V_k(\bar{\mathfrak{g}}) \otimes \mathcal{F} (= C(L\bar{\mathfrak{n}}_+, V_k(\bar{\mathfrak{g}})))$ (recall Sect. 4). In particular, Q_+ acts on $C(L\bar{\mathfrak{n}}_+, M)$ satisfying

$$Q_+^2 = 0, \quad Q_+ \cdot C^i(L\bar{\mathfrak{n}}_+, M) \subset C^{i+1}(L\bar{\mathfrak{n}}_+, M).$$

Thus $(C(L\bar{\mathfrak{n}}_+, M), Q_+)$ is a cochain complex. Set

$$H_+^i(M) := H^i(C(L\bar{\mathfrak{n}}_+, M), Q_+) \quad \text{for } i \in \mathbb{Z}. \tag{290}$$

The $H_+^\bullet(M)$ is called the *cohomology of the BRST complex of the quantized Drinfeld–Sokolov "+" reduction for $L\bar{\mathfrak{n}}_+$ with coefficients in M* ([21,25,27]).

Remark 6.4.1.

(i) We have $H^\bullet(C_k(\bar{\mathfrak{g}})) = H_+^\bullet(V_k(\bar{\mathfrak{g}}))$ and $\mathscr{W}_k(\bar{\mathfrak{g}}) = H_+^0(V_k(\bar{\mathfrak{g}}))$.

(ii) By definition

$$H_+^\bullet(M) = H^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_+, M \otimes \mathbb{C}_{\chi_+}),$$

where the right-hand-side is the semi-infinite cohomology [18] of the Lie algebra $L\bar{\mathfrak{n}}_+$ with coefficient in $M \otimes \mathbb{C}_{\chi_+}$.

The $C_k(\bar{\mathfrak{g}})$-module structure of $C(L\bar{\mathfrak{n}}_+, M)$ induces the $\mathscr{W}_k(\bar{\mathfrak{g}})$-module structure on $H_+^i(M)$ with $i \in \mathbb{Z}$, by Theorem 4.14.2. Further, there is a natural action of the operator $\mathbf{D}_{\text{new}}$ on $C(L\bar{\mathfrak{n}}_+, M)$ commuting with Q_+. Thus $H_+^\bullet(M)$ is graded by $\mathbf{D}_{\text{new}}$:

$$H_+^\bullet(M) = \bigoplus_{d \in \mathbb{C}} H_+^\bullet(M)_d, \;\; H_+^\bullet(M)_d = \{v \in H_+^\bullet(M); \mathbf{D}_{\text{new}} v = dv\}. \tag{291}$$

Therefore we have a functor form $\mathcal{O}_k$ to $\mathscr{W}_k(\bar{\mathfrak{g}})$-grMod defined by

$$M \mapsto H_+^i(M).$$

6.5. Functor $H_-^\bullet(?)$. Consider the automorphism of $U_k(\mathfrak{g}) \otimes \mathcal{C}l$ defined by

$$\begin{aligned} J(n) &\mapsto -J^t(n) \quad \text{for } J \in \bar{\mathfrak{g}}, n \in \mathbb{Z}, \\ \psi_\alpha(n) &\mapsto -\left(\psi_\alpha^t\right)(n) \quad \text{for } \alpha \in \bar{\Delta}, n \in \mathbb{Z} \end{aligned} \tag{292}$$

(Notation (7) and (38)). Because it is degree-preserving, (292) induces an algebra involution $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\text{old}}) = \widetilde{U_k(\mathfrak{g}') \otimes \mathcal{C}l_{\text{old}}}$, which we denote by $\sharp$. Then

$$\widehat{t}^{\sharp}_{-\bar{\rho}^\vee} := \sharp \circ \widehat{t}_{-\bar{\rho}^\vee} \tag{293}$$

defines an isomorphism $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\text{new}}) \xrightarrow{\sim} \mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\text{old}})$ of compatible degree-wise complete algebras. We have:

$$\widehat{t}^{\sharp}_{-\bar{\rho}^\vee}(Q_+) = Q_-, \quad Q_- = Q_-^{\text{st}} + \chi_-, \tag{294}$$

$$\widehat{t}^{\sharp}_{-\bar{\rho}^\vee}\left(Q_+^{\text{st}}\right) = Q_-^{\text{st}}, \quad \widehat{t}^{\sharp}_{-\bar{\rho}^\vee}(\chi_+) = \chi_-, \tag{295}$$

where Q_-^{st} and χ_- are the following elements of $\mathfrak{U}(C_k(\bar{\mathfrak{g}})_{\text{old}})_0$, respectively:

(296)

$$Q_-^{\text{st}} = \sum_{\alpha \in \bar{\Delta}_-, \; n \in \mathbb{Z}} J_\alpha(-n)\psi_{-\alpha}(n) - \frac{1}{2} \sum_{\substack{\alpha,\beta,\gamma \in \bar{\Delta}_- \\ k+l+m=0}} c_{\alpha,\beta}^\gamma \psi_{-\alpha}(k)\psi_{-\beta}(l)\psi_\gamma(m),$$

$$\chi_- = \sum_{\alpha \in \bar{\Delta}_-} \bar{\chi}_-(J_\alpha)\psi_{-\alpha}(0), \tag{297}$$

where $\bar{\chi}_-$ is defined by (43). Thus

(298) $$\left(Q_-^{\mathrm{st}}\right)^2 = \chi_-^2 = \left[Q_-^{\mathrm{st}}, \chi_-\right] = 0, \quad Q_-^2 = 0.$$

For $M \in \mathcal{O}_k$, set

(299) $$C(L\bar{\mathfrak{n}}_-, M) := M \otimes \mathcal{F} = \sum_{i \in \mathbb{Z}} C^i(L\bar{\mathfrak{n}}_-, M),$$

where

$$C^i(L\bar{\mathfrak{n}}_-, M) := M \otimes \mathcal{F}^{-i}$$

(compare (289)). Then

$$Q_- \cdot C^i(L\bar{\mathfrak{n}}_-, M) \subset C^{i+1}(L\bar{\mathfrak{n}}_-, M).$$

Thus $(C(L\bar{\mathfrak{n}}_-, M), Q_-)$ is a cochain complex. The corresponding cohomology is denoted by $H_-^{\bullet}(M)$:

(300) $$H_-^i(M) := H^i(C(L\bar{\mathfrak{n}}_-, M), Q_-) \quad \text{for } i \in \mathbb{Z}.$$

Remark 6.5.1. By definition we have

$$H_-^{\bullet}(M) = H^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_-, M \otimes \mathbb{C}_{\chi_-}),$$

where χ_- is considered as the character of $L\bar{\mathfrak{n}}_-$ defined by $\chi_-(J(n)) = \bar{\chi}_-(J)\delta_{n,0}$ with $J \in \bar{\mathfrak{n}}_-$ and $n \in \mathbb{Z}$ (compare (47)).

By (200) and (240), each $H_-^i(M)$ has a $\mathscr{W}_k(\bar{\mathfrak{g}})$-module structure:

$$u \cdot [v] = \left[\widehat{t}^{\sharp}_{-\bar{\rho}^\vee}(u)v\right] \quad \text{for } u \in \mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}})),\, v \in H_-^i(M).$$

There is a natural action of $\mathbf{D}$ on $C(L\bar{\mathfrak{n}}_-, M)$ commuting with Q_-. Thus $H_-^{\bullet}(M)$ is graded by $\mathbf{D}$:

(301) $$H_-^{\bullet}(M) = \bigoplus_{d \in \mathbb{C}} H^{\bullet}(M)_d, \quad H^{\bullet}(M)_d = \{v \in H^{\bullet}(M); \mathbf{D}v = dv\}.$$

Therefore the correspondence

$$M \mapsto H_-^i(M),$$

with $i \in \mathbb{Z}$, defines a functor from $\mathcal{O}_k$ to $\mathscr{W}_k(\bar{\mathfrak{g}})$-grMod, which is refereed as the "−" reduction functor. The functor $H_-^{\bullet}(?)$ is essentially the same functor studied by E. Frenkel, V. Kac and M. Wakimoto [27].

7. Representation theory of $\mathscr{W}$-algebras through the functor $H_-^0(?)$

In this section we study the representation theory of $\mathscr{W}$-algebras through the "−" reduction functor $H_-^0(?)$. As in the previous sections, the complex number k is arbitrary unless otherwise stated.

7.1. Category $\dot{\mathcal{O}}_k$. Let M be an object of $\mathcal{O}_k$. Regarded as a module over $V_k(\bar{\mathfrak{g}})$ with the Hamiltonian $\mathbf{D}$, its grading is given by the operator $\mathbf{D}$:

$$M = \bigoplus_{d \in \mathbb{Z}} M_d, \quad M_d := \{v \in M; \mathbf{D}v = dv\} = \sum_{\substack{\lambda \\ \langle \lambda, \mathbf{D} \rangle = d}} M^\lambda.$$

Each M_d is naturally regarded as a module over $\bar{\mathfrak{g}}$. Let $\dot{\mathcal{O}}_k$ be the full subcategory of $\mathcal{O}_k$ consisting of objects M such that each M_d, with $d \in \mathbb{C}$, belongs to the category $\mathcal{O}(\bar{\mathfrak{g}})$ (notation Sect. 2.5). It is easily seen that $\dot{\mathcal{O}}_k$ is a Serre full subcategory of $\mathcal{O}_k$.

Lemma 7.1.1.

(i) *Any Verma module $M(\lambda)$, with $\lambda \in \mathfrak{h}_k^*$, belongs to $\dot{\mathcal{O}}_k$.*
(ii) *Any simple module $L(\lambda)$, with $\lambda \in \mathfrak{h}_k^*$, belongs to $\dot{\mathcal{O}}_k$.*
(iii) *Any object of $\mathcal{O}_k^\triangle$ belongs to $\dot{\mathcal{O}}_k$.*
(iv) *Any object of $\mathcal{O}_k^\triangledown$ belongs to $\dot{\mathcal{O}}_k$.*

Proof. (i) Certainly $\bar{\mathfrak{n}}_+$ acts locally nilpotently and $\bar{\mathfrak{h}}$ acts semisimply on $M(\lambda)$. We have to show that each $M(\lambda)_d$, $d \in \mathbb{C}$, is finitely generated over $\bar{\mathfrak{g}}$. But this follows from the PBW theorem. (ii) and (iii) follow from (i). (iv) The category $\mathcal{O}(\bar{\mathfrak{g}})$ is closed under taking (graded) duals. Hence $\dot{\mathcal{O}}_k$ is closed under taking (graded) duals. Therefore (iv) follows from (ii). □

Let $\dot{\mathcal{O}}_k^{\leq\lambda}$ be the Serre full subcategory of $\mathcal{O}_k$ consisting of objects that belong to both $\dot{\mathcal{O}}_k$ and $\mathcal{O}_k^{\leq\lambda}$. Then by Lemma 7.1.1, $P_{\leq\lambda}(\mu)$ and $I_{\leq\lambda}(\mu)$, with $\mu \leq \lambda$, belong to $\dot{\mathcal{O}}_k^{\leq\lambda}$.

From the following assertion it follows that every object of $\mathcal{O}_k$ can be obtained as an injective limit of objects of $\dot{\mathcal{O}}_k$:

Proposition 7.1.2. *Let M be an object of $\mathcal{O}_k^{\leq\lambda}$ with $\lambda \in \mathfrak{h}_k^*$. Then there exists a sequence $M_1 \subset M_2 \subset M_3 \ldots$ of objects of $\dot{\mathcal{O}}_k^{\leq\lambda}$ such that $M = \bigcup_i M_i$.*

Proof. Finitely generated objects of $\mathcal{O}_k$ belong to $\dot{\mathcal{O}}_k$ since each projective module $P_{\leq\lambda}(\mu)$ does. Let $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \ldots$ be a highest weight filtration of M, so that $M = \bigcup_i M_i$ and each successive subquotient M_i/M_{i-1} is a highest weight module. In particular each M_i is finitely generated, and hence belongs to $\dot{\mathcal{O}}_k$. □

Lemma 7.1.3. *Fix $d \in \mathbb{C}$. Then for any object M of $\dot{\mathcal{O}}_k^{\leq\lambda}$, with $\lambda \in \mathfrak{h}_k^*$, there exists a finitely generated submodule M' of M such that $(M/M')_{d'} = 0$ for all $d' \geq d$.*

Proof. Let $\mathscr{P} = \{v_1, v_2, \ldots\}$ be a set of generators of M such that (1) each v_i is a weight vector of weight $\mu_i \in \mathfrak{h}^*$, (2) if we set $M_i = \sum_{r=1}^{i} U(\mathfrak{g})v_r$ (and $M_0 = 0$), then each M_i/M_{i-1} is a highest weight module with highest weight μ_i. (so $M_1 \subset M_2 \subset \ldots$ is a highest weight filtration of M). Then by definition $\sharp\{j \geq 1; \mu_j = \mu\} \leq [M : L(\mu)]$ for $\mu \in \mathfrak{h}^*$, where $[M : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor of M. Let

$$\mathcal{P}_{\geq d} = \{v_j \in \mathcal{P}; \langle \mu_j, \mathbf{D}\rangle \geq d\} \subset \mathcal{P}. \tag{302}$$

Then we see from the definition of $\dot{\mathcal{O}}_k$ that $\mathcal{P}_{\geq d}$ is a finite set. Thus $M' = \sum_{v\in\mathcal{P}_{\geq d}} U(\mathfrak{g})v \subset M$ satisfies the desired properties. □

7.2. Cohomology of top parts. Let M be an object of $\mathcal{O}_k$ such that $M = \bigoplus_{d\leq d_0} M_d$ for some $d_0 \in \mathbb{C}$. Then by (160) we have $C(L\bar{\mathfrak{n}}_-, M) = \bigoplus_{d\leq d_0} C(L\bar{\mathfrak{n}}_-, M)_d$, where $C(L\bar{\mathfrak{n}}_-, M)_d = \{c \in C(L\bar{\mathfrak{n}}_-, M); \mathbf{D}c = d\}$. Thus one has the following assertion.

Lemma 7.2.1. *Let M be an object of $\mathcal{O}_k$. Suppose that $M = \bigoplus_{d\leq d_0} M_d$ for some $d_0 \in \mathbb{C}$. Then $H^{\bullet}_-(M) = \bigoplus_{d\leq d_0} H^{\bullet}_-(M)_d$. In particular, $H^{\bullet}_-(M) = \bigoplus_{d\leq\langle\lambda,\mathbf{D}\rangle} H^{\bullet}_-(M)_d$ for any object M of $\mathcal{O}_k^{\leq\lambda}$.*

Recall that $\bar{H}_i(M) = H_i(\bar{\mathfrak{n}}_-, M\otimes\mathbb{C}_{\bar{\chi}_-}) = H_i(M\otimes\Lambda(\bar{\mathfrak{n}}_-), \bar{Q}_-)$ (notation Sect. 2.5).

Lemma 7.2.2. *Let M be an object of $\mathcal{O}_k^{\leq\lambda}$ with $\lambda \in \mathfrak{h}_k^*$. Then, we have*

$$H^i_-(M)_{\langle\lambda,\mathbf{D}\rangle} = \begin{cases} \bar{H}_{-i}(M_{\langle\lambda,\mathbf{D}\rangle}) & \text{for } i \leq 0,\\ 0 & \text{for } i > 0.\end{cases}$$

Proof. Because M is an object of $\mathcal{O}_k^{\leq\lambda}$, we have

$$C(L\bar{\mathfrak{n}}_-, M)_{\langle\lambda,\mathbf{D}\rangle} = M_{\langle\lambda,\mathbf{D}\rangle} \otimes \mathcal{F}_0.$$

Since $F_0 = \operatorname{span}\{\psi_{-\alpha_1}(0)\dots\psi_{-\alpha_r}(0)\mathbf{1}; \alpha_i \in \bar{\Delta}_+\} \subset \mathcal{F}$ and F_0 is naturally identified with $\Lambda(\bar{\mathfrak{n}}_-)$. Thus we have

$$C(L\bar{\mathfrak{n}}_-, M)_{\langle\lambda,\mathbf{D}\rangle} = M_{\langle\lambda,\mathbf{D}\rangle} \otimes \Lambda(\bar{\mathfrak{n}}_-) = \bar{C}(M_{\langle\lambda,\mathbf{D}\rangle}) \tag{303}$$

(see (46)). It is easily seen that the action of Q_- on $C(L\bar{\mathfrak{n}}_-, M)_{\langle\lambda,\mathbf{D}\rangle} = \bar{C}(M_{\langle\lambda,\mathbf{D}\rangle})$ coincides with that of $\bar{Q}_-$. This completes the proof. □

We identify $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ with $\mathcal{Z}(\bar{\mathfrak{g}})$ through Theorem 4.16.3(ii).

Proposition 7.2.3. *For each $\lambda \in \mathfrak{h}_k^*$ we have the following isomorphisms of $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$-modules*

(i) $H^i_-(M(\lambda))_{\text{top}} = H^i_-(M(\lambda))_{\langle\lambda,\mathbf{D}\rangle} \cong \begin{cases} \mathbb{C}_{\gamma_{w_0(\bar{\lambda})}} & \text{if } i = 0,\\ 0 & \text{if } i \neq 0;\end{cases}$

(ii) $H_{-}^{i}(L(\lambda))_{\langle\lambda,\mathbf{D}\rangle} \cong \begin{cases} \mathbb{C}_{\gamma_{-w_0(\bar{\lambda})}} & \textit{if } i=0 \textit{ and } \bar{\lambda}\in\bar{\mathfrak{h}}^* \textit{ is anti-dominant}, \\ 0 & \textit{otherwise;} \end{cases}$

(iii) $H_{-}^{i}(M(\lambda)^*)_{\mathrm{top}} = H_{-}^{i}(M(\lambda)^*)_{\langle\lambda,\mathbf{D}\rangle} \cong \begin{cases} \mathbb{C}_{\gamma_{-w_0(\bar{\lambda})}} & \textit{if } i=0, \\ 0 & \textit{if } i\neq 0. \end{cases}$

Proof. First, we have

$$\gamma_{\bar{\lambda}} \circ \natural_{|Z(\bar{\mathfrak{g}})} = \gamma_{-w_0(\bar{\lambda})}, \tag{304}$$

which can be shown in the same manner as Lemma 5.5.2.

Next we have $M(\lambda)_{\mathrm{top}} = M(\lambda)_{\langle\lambda,\mathbf{D}\rangle} = \bar{M}(\bar{\lambda})$, $L(\lambda)_{\mathrm{top}} = L(\lambda)_{\langle\lambda,\mathbf{D}\rangle} = \bar{L}(\bar{\lambda})$ and $M(\lambda)^*_{\mathrm{top}} = M(\lambda)^*_{\langle\lambda,\mathbf{D}\rangle} = \bar{M}(\bar{\lambda})^*$ as $\bar{\mathfrak{g}}$-modules. Therefore the assertion follows from Theorems 2.5.6, 2.5.7 and Lemma 7.2.2. □

7.3. A Technical Lemma. We need the following assertion for the later argument (because the category $\mathcal{O}_k^{\leq\lambda}$ is not Artinian in general).

Lemma 7.3.1. *Fix $\lambda \in \mathfrak{h}_k^*$ and $d \in \mathbb{C}$. For any object M of $\dot{\mathcal{O}}_k^{\leq\lambda}$ we have the following:*

(i) *There exists a finitely generated submodule M' of M such that $H_{-}^{\bullet}(M)_d = H_{-}^{\bullet}(M')_d$.*
(ii) *There exists a finitely generated submodule N of M^* such that $H_{-}^{\bullet}(M)_d = H_{-}^{\bullet}(N^*)_d$.*

Proof. (i) Let M' be a finitely generated submodule of M as in Lemma 7.1.3, so that

$$(M/M')_{d'} = 0 \quad \text{for all } d' \geq d.$$

From the exact sequence $0 \to M' \to M \to M/M' \to 0$, we get the following long exact sequence:

$$\cdots \to H_{-}^{i-1}(M/M') \to H_{-}^{i}(M') \to H_{-}^{i}(M) \to H_{-}^{i}(M/M') \to \cdots \tag{305}$$

Clearly, the restriction of (305) to a **D**-eigenspace remains exact. We are done by Lemma 7.2.1. (ii) Similarly as above, let N be a finitely generated submodule of M^* such that $(M^*/N)_{d'} = 0$ for all $d' \geq d$. Then $(M^*/N)^*_{d'} = 0$ for all $d' \geq d$. Therefore $H_{-}^{\bullet}((M^*/N)^*)_{d'} = 0$ for all $d' \geq d$ by Lemma 7.2.1. The assertion follows by considering the long exact sequence corresponding to the exact sequence $0 \to (M^*/N)^* \to M \to N^* \to 0$. □

7.4. Estimate of D-eigenvalues. For $M \in \mathcal{O}_k$, let $\bar{H}_{\bullet}(M) = \bigoplus_{d\in\mathbb{C}} \bar{H}_{\bullet}(M)_d$ where $\bar{H}_{\bullet}(M)_d = \bar{H}_{\bullet}(M_d)$.

Proposition 7.4.1. (i) *Let M be an object of $\dot{\mathcal{O}}_k$. Then $\bar{H}_i(M) = 0$ for $i \neq 0$ and $\bar{H}_0(M)_d$ is finite-dimensional for all $d \in \mathbb{C}$.*
(ii) *For each object M of $\mathcal{O}_k$ we have $\bar{H}_i(M) = 0$ with $i \neq 0$.*

Proof. (i) follows from Lemma 2.5.2 and Theorem 2.5.6. (ii) follows from (i) and Proposition 7.1.2 since a homology functor commutes with injective limits. □

Theorem 7.4.2. *Let $\lambda \in \mathfrak{h}_k^*$.*

(i) ([27]) *Let M be an object or $\dot{\mathcal{O}}_k^{\leq\lambda}$. Then each eigenspace $H^i_-(M)_d$, with $d \in \mathbb{C}$, is finite-dimensional.*
(ii) *For each object M of $\mathcal{O}_k^{\leq\lambda}$ we have $H^i_-(M)_d = 0$ unless $|i| \leq \langle\lambda, \mathbf{D}\rangle - d$.*

Proof. Because $L\bar{\mathfrak{n}}_- \cap \mathfrak{g}_- = (\bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}]t^{-1}) \oplus \bar{\mathfrak{n}}_-$, the space $\Lambda^n(L\bar{\mathfrak{n}}_- \cap \mathfrak{g}_-)$ decomposes as

$$\Lambda^n(L\bar{\mathfrak{n}}_- \cap \mathfrak{g}_-) \bigoplus_{i+j=n} \Lambda^i\left(\bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}]t^{-1}\right) \otimes \Lambda^j(\bar{\mathfrak{n}}_-) \quad \forall n.$$

Thus by (155) the space $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^n$ decomposes as

$$\mathcal{F} = \Lambda^{\frac{\infty}{2}}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) \otimes \Lambda(\bar{\mathfrak{n}}_-),$$
$$\mathcal{F}^n = \bigoplus_{\substack{p+q=n \\ p\in\mathbb{Z}, q\in\mathbb{Z}_{\leq 0}}} \Lambda^{\frac{\infty}{2}+p}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) \otimes \Lambda^{-q}(\bar{\mathfrak{n}}_-),$$

where

(306) $$\Lambda^{\frac{\infty}{2}}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) = \bigoplus_{p\in\mathbb{Z}} \Lambda^{\frac{\infty}{2}+p}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-),$$

(307) $$\Lambda^{\frac{\infty}{2}+p}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) := \bigoplus_{\substack{i-j=p \\ i,j\in\mathbb{Z}_{\geq 0}}} \Lambda^i\left(\bar{\mathfrak{n}}_+ \otimes \mathbb{C}[t^{-1}]t^{-1}\right) \otimes \Lambda^j\left(\bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}]t^{-1}\right).$$

Let $M \in \mathcal{O}_k^{\leq\lambda}$. Put $C^n = C^n(L\bar{\mathfrak{n}}_-, M)$ and set

$$F^pC^n := \bigoplus_{\substack{i\geq p \\ i-j=n}} M \otimes \Lambda^{\frac{\infty}{2}+i}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) \otimes \Lambda^j(\bar{\mathfrak{n}}_-).$$

Then

(308) $$\cdots \supset F^pC^n \supset F^{p+1}C^n \supset \cdots \supset F^{n+\dim\bar{\mathfrak{n}}_-+1}C^n = 0,$$

(309) $$C^n = \bigcup_p F^pC^n,$$

(310) $$Q_-F^pC^n \subset F^pC^{n+1}.$$

Thus there is a corresponding converging spectral sequence $E_r \Rightarrow H^{\bullet}_{-}(M)$. This is the semi-infinite analogue of Hochschild–Serre spectral sequence for the subalgebra $\bar{\mathfrak{n}}_- \subset L\bar{\mathfrak{n}}_-$.

By definition we have

$$E_1^{p,q} = \begin{cases} \bar{H}_{-q}(M) \otimes \Lambda^{\frac{\infty}{2}+p}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) & \text{for } q \leq 0, \\ 0 & \text{for } q > 0. \end{cases} \tag{311}$$

Hence by Proposition 7.4.1 it follows that

$$E_1^{p,q} = \begin{cases} \bar{H}_0(M) \otimes \Lambda^{\frac{\infty}{2}+p}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-) & \text{for } q = 0, \\ 0 & \text{for } q \neq 0. \end{cases} \tag{312}$$

Therefore we have

$$\begin{aligned} \dim H^i_-(M)_{\langle\lambda,\mathbf{D}\rangle-d} &\leq \sum_{p+q=i} \dim \left(E_1^{p,q}\right)_{\langle\lambda,\mathbf{D}\rangle-d} = \dim \left(E_1^{i,0}\right)_{\langle\lambda,\mathbf{D}\rangle-d} \\ &= \sum_{\substack{d'+d''=d \\ d',d''\geq 0}} \dim \bar{H}_0(M)_{\langle\lambda,\mathbf{D}\rangle-d'} \cdot \dim \Lambda^{\frac{\infty}{2}+i}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-)_{-d''}, \end{aligned} \tag{313}$$

where $\Lambda^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-)_d = \{\omega \in \Lambda^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-); \mathbf{D}\omega = d\omega\}$. But we have

$$\dim \Lambda^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-)_{-d} < \infty \quad \text{for all } d, \tag{314}$$

$$\Lambda^{\frac{\infty}{2}+i}(L\bar{\mathfrak{n}}_-/\bar{\mathfrak{n}}_-)_{-d} = 0 \quad \text{if } d < |i| \tag{315}$$

which follows immediately from the Definition (307).

By (313) and (315) we have $H^i(M)_{\langle\lambda,\mathbf{D}\rangle-d} = 0$ for $d < |i|$. This shows (ii). Further, if $M \in \dot{\mathcal{O}}_k^{\leq\lambda}$ then by Proposition 7.4.1 $\bar{H}_0(M)_d$ is finite-dimensional for all d. Therefore (313) and (314) give the assertion (i). □

Remark 7.4.3. Theorem 7.4.2 (ii) was partially proved in [2, Proposition 7.6]. (It also has a similar (weaker) statement for the "+" reduction.)

Recall the category $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ defines in Sect. 5.5.

Corollary 7.4.4. *The $\mathscr{W}_k(\bar{\mathfrak{g}})$-module $H^0_-(M)$, with $M \in \dot{\mathcal{O}}_k$, belongs to $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$.*

7.5. Images of verma modules and their duals by the "–" reduction. The following assertion is essentially due to [25], because its proof is the same as Theorem 4.11.1.

Theorem 7.5.1 ([2, Theorem 5.7])**.** *Let $\lambda \in \mathfrak{h}_k^*$. The cohomology $H^i_-(M(\lambda))$ is zero for all $i \neq 0$ and there is an isomorphism $H^0_-(M(\lambda)) \cong \mathbf{M}(\gamma_{-w_0(\bar{\lambda})})$.*

Remark 7.5.2.

(i) The fact that $H^0_-(M(\lambda)) = \mathbf{M}(\gamma_{-w_0(\bar{\lambda})})$ is not explicitly stated in [2], but it easily follows from the proof of [2, Theorem 5.7]. In fact by using exactly the same argument as Sect. 4.8, it can be proved in the same manned as Proposition 5.2.2.

(ii) We have $H^0_-(M(w \circ \lambda)) \cong \mathbf{M}(\gamma_{-w_0(\bar{\lambda})})$ for all $w \in \bar{W}$.

Theorem 7.5.3. *Let $\lambda \in \mathfrak{h}_k^*$. The cohomology $H^i_-(M(\lambda)^*)$ is zero for all $i \neq 0$ and there is an isomorphism $H^0_-(M(\lambda)^*) \cong D(\mathbf{M}(\gamma_{\bar{\lambda}}))$.*

The proof of Theorem 7.5.3 is given in Sect. 8.

Theorem 7.5.1 in particular implies the following assertion.

Theorem 7.5.4 **(**[2, Theorem 8.1]**).** *Let $\lambda, \mu \in \mathfrak{h}_k^*$ such that $\mu \leq \lambda$. The cohomology $H^i_-(P_{\leq\lambda}(\mu))$ is zero for all $i \neq 0$.*

Similarly, the following assertion follows from Theorem 7.5.3.

Theorem 7.5.5. *Let $\lambda, \mu \in \mathfrak{h}_k^*$ such that $\mu \leq \lambda$. The cohomology $H^i_-(I_{\leq\lambda}(\mu))$ is zero for all $i \neq 0$.*

7.6. Results for the "−" reduction. The following result is a generalization of the result established in [2].

Theorem 7.6.1. *Let M be any object of $\mathcal{O}_k$. The cohomology $H^i_-(M)$ is zero for all $i \neq 0$.*

Proof. We may assume that $M \in Obj\, \mathcal{O}_k^{\leq\lambda}$ for some $\lambda \in \mathfrak{h}_k^*$. Also, by Proposition 7.1.2, we may further assume that $M \in Obj\, \dot{\mathcal{O}}_k^{\leq\lambda}$, since the cohomology functor commutes with injective limits. Clearly, it is sufficient to show that $H^i_-(M)_d = 0$ for $i \neq 0$ with each $d \in \mathbb{C}$.

First, we show that $H^i_-(M)_d = 0$ for all $i > 0$. By Lemma 7.3.1 (1), for a given d, there exists a finitely generated submodule M' of M such that

$$H^i_-(M)_d = H^i_-(M')_d \quad \text{for all } i \in \mathbb{Z}. \tag{316}$$

Because M' is finitely generated, there exists a projective object P of the form $\bigoplus_{i=1}^r P_{\leq\lambda}(\mu_i)$ and an exact sequence

$$0 \to N \to P \to M' \to 0$$

in $\dot{\mathcal{O}}_k^{\leq\lambda}$. As the corresponding long exact sequence we obtain:

$$\cdots \to H^i_-(P) \to H^i_-(M') \to H^{i+1}_-(N) \to H^{i+1}_-(P) \to \ldots. \tag{317}$$

Hence, it follows that $H^i_-(M') \cong H^{i+1}_-(N)$ for all $i > 0$, by Theorem 7.5.4. Therefore, we find

$$H^i_-(M)_d \cong H^{i+1}_-(N)_d \quad \text{for all } i > 0, \tag{318}$$

by (316). But then, because $N \in Obj\, \dot{\mathcal{O}}_k^{\leq\lambda}$, we can repeat this argument to find, for each $r > 0$, some object N_r of $\dot{\mathcal{O}}_k^{\leq\lambda}$ such that

$$H_-^i(M)_d \cong H_-^{i+r}(N_r)_d \qquad \text{for all } i > 0. \tag{319}$$

By Theorem 7.4.2 (ii), this implies that $H_-^i(M)_d = 0$ for $i > 0$.

Next, we show that $H_-^i(M)_d = 0$ for all $i < 0$. (The argument is parallel to the above.) By Lemma 7.3.1 (ii) there exists a finitely generated submodule V of M^* such that $H_-^i(M)_d = H_-^i(V^*)_d$ for all $i \in \mathbb{Z}$. Because V is finitely generated, there exists an injective object I of the form $\bigoplus_{i=1}^r I_{\leq\lambda}(\mu_i)$ and an exact sequence

$$0 \to V^* \to I \to L \to 0$$

in $\dot{\mathcal{O}}_k^{\leq\lambda}$. Considering the corresponding long exact sequence, Theorem 7.5.5 gives $H_-^i(V^*) \cong H_-^{i-1}(L)$ for all $i < 0$. This shows that $H_-^i(M)_d \cong H_-^{i-1}(L)_d$ for all $i < 0$. By repeating this argument we find that $H_-^i(M)_d = 0$ for all $i < 0$. □

Corollary 7.6.2. *The correspondence $M \rightsquigarrow H_-^0(M)$ defines an exact functor from $\mathcal{O}_k$ to $\mathscr{W}_k(\bar{\mathfrak{g}})$*-adMod. *Its restriction gives the exact functor $H_-^0(?)$: $\dot{\mathcal{O}}_k \to \mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$.*

Theorem 7.6.3. *Let $\lambda \in \mathfrak{h}_k^*$. There is an isomorphism*

$$H_-^0(L(\lambda)) \cong \begin{cases} \mathbf{L}(\gamma_{-w_0(\bar{\lambda})}) & \text{if } \bar{\lambda} \in \bar{\mathfrak{h}}^* \text{is anti-dominant}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Because $L(\lambda)$ is a quotient of $M(\lambda)$, the exactness of the functor $H_-^0(?)$ (Corollary 7.6.2) imply that there is an exact sequence of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules

$$H_-^0(M(\lambda)) \to H_-^0(L(\lambda)) \to 0. \tag{320}$$

By Theorem 7.5.1 we have

$$H_-^0(M(\lambda)) \cong \mathbf{M}(\gamma_{-w_0(\bar{\lambda})}). \tag{321}$$

Thus $H_-^0(L(\lambda))$ is generated by $H_-^0(M(\lambda))_{\mathrm{top}}$ over $\mathscr{W}_k(\bar{\mathfrak{g}})$. By Proposition 7.2.3 (i) $H_-^0(M(\lambda))_{\mathrm{top}} = H_-^0(M(\lambda))_{\langle\lambda,\mathbf{D}\rangle}$. Because (320) is $\mathbf{D}$-equivalent, $H_-^0(L(\lambda)) \neq 0$ if and only if $H_-^0(L(\lambda))_{\langle\lambda,\mathbf{D}\rangle} \neq 0$. Hence, by Proposition 7.2.3 (ii), it follows that $H_-^0(L(\lambda))$ is nonzero if and only if its classical part $\bar{\lambda}$ is anti-dominant.

Next suppose that $\bar{\lambda}$ is anti-dominant, so that $H_-^0(L(\lambda)) \neq 0$. Because $L(\lambda)$ is also a submodule of $M(\lambda)^*$, by Theorem 7.5.3 and Corollary 7.6.2, there is an exact sequence of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules

$$0 \to H_-^0(L(\lambda)) \to D(\mathbf{M}(\gamma_{\bar{\lambda}})). \tag{322}$$

By Theorem 5.5.4 (iii), we immediately obtain from (320), (321) and (322) that $H_-^0(L(\lambda)) \cong \mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$. □

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called *principal admissible* [38] if (1) λ is regular dominant, that is, $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta_+^{\mathrm{re}}$, and (2) $\Delta(\lambda) \cong \Delta^{\mathrm{re}}$ as root systems.

Let Pr_k be the set of principal admissible weights of level k. A principal admissible weight λ is called *non-degenerate* [39] if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \bar{\Delta}$, or equivalently, if $\bar{\lambda}$ is anti-dominant. Let Pr_k^{nondeg} denote the set of non-degenerated principal admissible weighs of level k.

Corollary 7.6.4 ([27, Conjecture 3.4_]). *Let* $\lambda \in Pr_k^{\mathrm{nondeg}}$.

(i) [2] *The cohomology* $H_-^i(L(\lambda))$ *is zero for all* $i \neq 0$.
(ii) *There is an isomorphism* $H_-^0(L(\lambda)) \cong \mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$.

Remark 7.6.5.

(i) It is clear from Theorems 9.1.3 and 9.1.4 that $H^\bullet(L(\lambda)) = 0$ for $Pr_k \backslash Pr_k^{\mathrm{nondeg}}$, as stated in [27] (without a complete proof).
(ii) It is known [39] that $Pr_k^{\mathrm{nondeg}} \neq \emptyset$ if and only if

(323)
$$k + h^\vee = p/q \in \mathbb{Q}, \quad (p, q) = 1, \quad p \geq h^\vee, \quad q \geq h, \quad (q, r^\vee) = 1,$$

where h is the Coxeter number of $\bar{\mathfrak{g}}$ and $r^\vee$ is the lacing number of $\bar{\mathfrak{g}}$. Let k be as (323). Set

$$I_{p,q} = \left(P_{++}^{p-h^\vee} \times P_{++}^{\vee, q-h}\right)/\widetilde{W}_+$$

where $w(\lambda, \mu) = (w\lambda, w\mu)$ for $w \in \widetilde{W}_+$, $P_{++}^{p-h^\vee}$ is the set of integral weights of level $p - h^\vee$ and $P_{++}^{\vee, q-h}$ is the integral coweights of level $q - h$:

$$P_{++}^{p-h^\vee} = \left\{\lambda \in \mathfrak{h}_{p-h^\vee}^*; \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \quad \text{for all } \alpha \in \Delta_+^{\mathrm{re}}\right\},$$
$$P_{++}^{\vee q-h} = \left\{\mu \in \mathfrak{h}_{q-h}^*; (\alpha|\mu) \in \mathbb{Z}_{\geq 0} \quad \text{for all } \alpha \in \Delta_+^{\mathrm{re}}\right\}.$$

Then there exists a bijection

$$\begin{array}{rcl} \bar{W} \times I_{p,q} & \stackrel{\sim}{\rightarrow} & Pr_k^{\mathrm{nondeg}} \\ (w, (\lambda, \mu)) & \mapsto & w \circ \Lambda_{\lambda,\mu}, \end{array}$$

where

$$\Lambda_{\lambda,\mu} = \bar{\lambda} - (k + h^\vee)(\bar{\mu} + \bar{\rho}^\vee) + (k + h^\vee)\Lambda_0.$$

Note that $\mathrm{vac_k} = \Lambda_{(p-h^\vee)\Lambda_0, (q-h)\Lambda_0}$.

By Corollary 7.6.4 we have $H^0_-(L(w \circ \Lambda_{\lambda,\mu})) \cong H^0_-(L(\Lambda_{\lambda,\mu}))$ for all $w \in \bar{W}$, $(\lambda, \mu) \in I_{p,q}$. Moreover all the simple modules

$$\left\{H^0_-(L(\Lambda_{\lambda,\mu})); (\lambda, \mu) \in I_{p,q}\right\} \tag{324}$$

are non-isomorphic. This is all explained in [27].

It is natural to expect that $\mathbf{L}(\gamma_{\mathrm{vac_k}})$ is rational and C_2-cofinite, and the above set (324) are exactly the isomorphism classes of simple $\mathbf{L}(\gamma_{\mathrm{vac_k}})$-modules.

(iii) It is clear that our results apply for non-principal Kac–Wakimoto admissible representations also, see [27, Remark 3.4.(c)].

7.7. Characters. For an object M of $\mathcal{O}_k$, define the integer $[M : M(\mu)]$ by

$$\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}_k^*} [M : M(\mu)] \operatorname{ch} M(\mu), \tag{325}$$

where $\operatorname{ch} M$ is the formal character of M: $M = \sum_{\lambda \in \mathfrak{h}^*} e^{\lambda} \dim M^{\lambda}$. It is well-known [13,41–43] that the coefficient $[L(\lambda) : M(\mu)]$ is expressed in terms of Kazhdan–Lusztig polynomials provided that $k \neq -h^{\vee}$, see the formula [44, Theoreom 1.1].

The following assertion follows immediately from Proposition 5.6.6, Theorems 7.5.1, 7.6.3, Corollary 7.6.2 and Remark 5.5.5.

Theorem 7.7.1. *Let $k \neq -h^{\vee}$ and $\lambda \in \mathfrak{h}_k^*$. Suppose that the classical part $\bar{\lambda}$ of λ is anti-dominant. Then we have*

$$\operatorname{ch} \mathbf{L}(\gamma_{-w_0(\bar{\lambda})}) = \sum_{\mu} [L(\lambda) : M(\mu)] \operatorname{ch} \mathbf{M}(\gamma_{-w_0(\bar{\mu})}),$$

or equivalently,

$$\begin{aligned} \operatorname{ch} \mathbf{L}(\gamma_{\bar{\lambda}}) &= \sum_{\mu} [L(\lambda) : M(\mu)] \operatorname{ch} \mathbf{M}(\gamma_{\bar{\mu}}), \\ &= \sum_{\mu} [L(\lambda) : M(\mu)] q^{|\bar{\mu}+\bar{\rho}|^2/2(k+h^{\vee})} \eta(q)^{-l}. \end{aligned}$$

By Theorem 7.7.1 we have obtained the character formula of all $\mathbf{L}(\gamma_{\bar{\lambda}})$. In particular we have proved the character formula of the "minimal series representations" of $\mathscr{W}$-algebras conjectured by E. Frenkel, V. Kac and M. Wakimoto [27].

Remark 7.7.2.

(i) Using the vanishing of cohomology one may also apply the Euler–Poincare character method [27] to obtain Theorem 7.7.1.

(ii) By setting the formal character $\widetilde{\mathrm{ch}} H^0_-(M) := \sum_{d\in\mathbb{C}} q^d \dim H^0_-(M)_d$ for $M \in \dot{\mathcal{O}}_k$ we obtain the formula

$$\widetilde{\mathrm{ch}} H^0_-(L(\lambda)) = \sum_{\mu} [L(\lambda) : M(\mu)] q^{\langle \mu, \mathbf{D}\rangle} \prod_{i\geq 1} (1-q^{-i})^{-l}$$

which is valid even at $k = -h^\vee$. If $k = -h^\vee$ then by Proposition 5.6.3 we have[7]

(326) $$\sum_{\mu} [L(\lambda) : M(\mu)] q^{\langle \mu, \mathbf{D}\rangle} \prod_{i\geq 1} (1-q^{-i})^{-l} = q^{\langle \lambda, \mathbf{D}\rangle}$$

if $\bar{\lambda}$ is anti-dominant, and $\sum_{\mu} [L(\lambda) : M(\mu)] q^{\langle \mu, \mathbf{D}\rangle} \prod_{i\geq 1} (1-q^{-i})^{-l} = 0$ if $\bar{\lambda}$ is not anti-dominant.

8. Proof of Theorem 7.5.3.

In this section we prove Theorem 7.5.3. Fix $k \in \mathbb{C}$ and $\lambda \in \mathfrak{h}^*_k$.

8.1. Preliminaries. The following elementary fact has been already used in this article and will be frequently used in the sequel.

Lemma 8.1.1. *The correspondence* $M \mapsto \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ *defines an exact cofunctor from the category of finite-dimensional vector spaces to itself.*

The following assertion will be also often used.

Lemma 8.1.2 (the universal coefficient theorem). *Let* $C = \bigoplus_{i\in\mathbb{Z}} C^i$ *be a cochain complex of vector spaces,* Q *its differential. Let* $C^\vee = \bigoplus_{i\in\mathbb{Z}} (C^\vee)^i$ *with* $(C^\vee)^i = \mathrm{Hom}(C^{-i}, \mathbb{C})$*. Then* $C^\vee$ *can be regarded as a cochain complex with the differential* $Q^\vee$*, where* $(Q^\vee f)(c) = f(Qc)$ *with* $f \in C^\vee$, $c \in C$*. If* C *is finite-dimensional then*

(327) $$H^i(C^\vee) = \mathrm{Hom}(H^{-i}(C), \mathbb{C})$$

for all i.

For a semisimple $\mathbb{C}\mathbf{D}$-module M we set

(328) $$M_d = \{m \in M; \mathbf{D}m = dm\}.$$

If $\dim M_d < \infty$ for all d we write

(329) $$\mathsf{D}(M) := \bigoplus_{d\in\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(M_d, \mathbb{C}).$$

[7] For an application of the formula (326), see [T. Arakawa, *A new proof of the Kac–Kazhdan conjecture*, International Mathematics Research Notices Volume 2006 (2006), Article ID 27091, 5 pages], which appeared while the present article was being refereed.

Recall the cochain complex $(K(\lambda), Q'_+)$ with $\lambda \in \mathfrak{h}^*$ (see Sect. 5.2). In view of Lemma 8.1.2, we have

$$(330)\qquad H^i(\mathsf{D}(K(\lambda))) = \mathsf{D}(H^{-i}(K(\lambda))) = \begin{cases} \mathsf{D}(H^0(K(\lambda))) & \text{for } i = 0,\\ 0 & \text{for } i \neq 0 \end{cases}$$

by Proposition 5.2.2, where $\mathsf{D}(K(\lambda))$ is considered as a subcomplex of the dual complex $(K(\lambda)^\vee, (Q'_+)^\vee)$.

The space $\mathsf{D}(K(\lambda))$ is a $C_k(\bar{\mathfrak{g}})''$-module through the anti-superalgebra isomorphism $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}) \xrightarrow{\sim} \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{old}})$ induced by the anti-Lie superalgebra isomorphism in Proposition 3.9.1, with respect to the (old) Möbius conformal structure $\{T^*, -\mathbf{D}, T\}$. We write θ_{old} for this anti-superalgebra isomorphism. To avoid confusion we write θ_{new} for the anti-superalgebra isomorphism $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}) \xrightarrow{\sim} \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}})$ with respect to the Möbius conformal structure $\{T^*_{\mathrm{new}}, -\mathbf{D}_{\mathrm{new}}, T\}$.

By (200) the $C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}$-modules structure on $\mathsf{D}(K(\lambda))$ gives rise to the $\mathscr{W}_k(\bar{\mathfrak{g}})$-module structure on $H^0(\mathsf{D}(K(\lambda))) = \mathsf{D}(H^0(K(\lambda)))$ through the map (230). Because we have

$$(331)\qquad \theta_{\mathrm{old}} \circ \widehat{t}_{-\bar{\rho}^\vee} = \widehat{t}_{-\bar{\rho}^\vee} \circ \theta_{\mathrm{new}},$$

it follows that

$$(332)\qquad \mathsf{D}(H^0(K(\lambda))) \cong D(\mathbf{M}(\gamma_{\bar{\lambda}}))$$

as $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules.

8.2. Identification with duals. For any semisimple $\mathfrak{h}$-module M such that M^λ is finite-dimensional for all λ, we write M^* for the linear space $\bigoplus_\lambda \mathrm{Hom}_{\mathbb{C}}(M^\lambda, \mathbb{C})$, as in (144).

Let

$$(333)\qquad (\cdot|\cdot)_{\mathcal{F}} : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$$

be the symmetric bilinear form defined by

$$\begin{gathered}(\mathbf{1}|\mathbf{1})_{\mathcal{F}} = 1,\\ (\psi_\alpha(n)\omega_1|\omega_2)_{\mathcal{F}} = -\sqrt{-1}(\omega_1|\psi_{-\alpha}(-n)\omega_2)_{\mathcal{F}}\\ \text{for } \alpha \in \bar{\Delta}_+, \mathfrak{n} \in \mathbb{Z}, \omega_1 \in \mathcal{F}, \omega_2 \in \mathcal{F},\\ (\psi_{-\alpha}(n)\omega_1|\omega_2)_{\mathcal{F}} = \sqrt{-1}(\omega_1|\psi_\alpha(-n)\omega_2)_{\mathcal{F}}\\ \text{for } \alpha \in \bar{\Delta}_+, \mathfrak{n} \in \mathbb{Z}, \omega_1 \in \mathcal{F}, \omega_2 \in \mathcal{F}\end{gathered}$$

(recall Sect. 4.4). Then the form $(\cdot|\cdot)_{\mathcal{F}}$ is non-degenerate. Indeed its restriction to $(\mathcal{F}^i)^\mu \times (\mathcal{F}^i)^\mu$ is non-degenerate for each $i \in \mathbb{Z}$ and $\mu \in \mathfrak{h}^*$.

Let M be an object of $\mathcal{O}_k$. One may identify $C(L\bar{\mathfrak{n}}_-, M^*)$ with $C(L\bar{\mathfrak{n}}_+, M)^*$ via the linear isomorphism

$$\begin{array}{rcl} C(L\bar{\mathfrak{n}}_-, M^*) = M^* \otimes \mathcal{F} & \tilde{\rightarrow} & C(L\bar{\mathfrak{n}}_+, M)^* = (M \otimes \mathcal{F})^* \\ f \otimes \omega & \mapsto & (m \otimes \omega' \mapsto f(m)(\omega|\omega')_{\mathcal{F}}). \end{array} \tag{334}$$

Under the identification (334), we have

$$(\sqrt{-1}Q_-\phi)(c) = \phi(Q'_+c) \tag{335}$$

for $\phi \in C(L\bar{\mathfrak{n}}_+, M)^* = C(L\bar{\mathfrak{n}}_-, M^*)$ and $c \in C(L\bar{\mathfrak{n}}_+, M)$, where Q'_+ is defined in (234). Thus, in view of Lemma 8.1.2, the cochain complex $(C(L\bar{\mathfrak{n}}_-, M^*), Q_-)$ may be identified the subcomplex $C(L\bar{\mathfrak{n}}_+, M)^*$ of $(C(L\bar{\mathfrak{n}}_{+,M})^\vee, (Q'_+)^\vee)$.

We would like to apply the formula (327) to the case that $M = M(\lambda)$, but unfortunately the complex $(C(L\bar{\mathfrak{n}}_+, M(\lambda)), Q'_+)$ is not a direct sum of finite-dimensional subcomplexes.

Remark 8.2.1. We have $H^\bullet(C(L\bar{\mathfrak{n}}_+, M), Q'_+) \neq H^\bullet_+(M)$ in general. Indeed, the cohomology $H^i(C(L\bar{\mathfrak{n}}_+, M), Q'_+)$ is zero for all i and $M \in \mathcal{O}_k$, because $J_\alpha(0)$, with $\alpha \in \bar{\Pi}$, acts locally nilpotently on M ([27, Theorem 2.3]).

Under the identification $C(L\bar{\mathfrak{n}}_-, M^*) = C(L\bar{\mathfrak{n}}_+, M)^*$, the action of $\mathcal{W}_k(\bar{\mathfrak{g}})$ of $H^\bullet_-(M^*)$ induced by the following $C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}$-module structure of $C(L\bar{\mathfrak{n}}_+, M)^*$:

$$(u \cdot f)(c) = f(\theta_{\mathrm{old}}(\widehat{t_{-\bar{\rho}^\vee}}(u))c) = f(\widehat{t_{-\bar{\rho}^\vee}}(\theta_{\mathrm{new}}(u))c) \tag{336}$$

for $u \in \mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}})$, $f \in C(L\bar{\mathfrak{n}}_+, M)^*$ and $c \in C(L\bar{\mathfrak{n}}_+, M)$.

8.3. The first reduction of the computation of $H^\bullet_-(M(\lambda)^*)$. Below we mainly consider $C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}$, so if no confusion can arise we write just $C_k(\bar{\mathfrak{g}})''$ for $C_k(\bar{\mathfrak{g}})''_{\mathrm{old}}$.

The cochain complex $C(L\bar{\mathfrak{n}}_+, M(\lambda)) = M(\lambda) \otimes \mathcal{F}$ is endowed with the grading defined by $\mathbf{D}$: $C(L\bar{\mathfrak{n}}_+, M(\lambda)) = \bigoplus_d C(L\bar{\mathfrak{n}}_+, M(\lambda))_d$. As a $C_k(\bar{\mathfrak{g}})$-module

$$C(L\bar{\mathfrak{n}}_+, M(\lambda))_{\mathrm{top}} = C(L\bar{\mathfrak{n}}_+, M(\lambda))_{\langle\lambda,\mathbf{D}\rangle} \cong \bar{M}(\bar{\lambda}) \otimes \Lambda(\bar{\mathfrak{n}}_-). \tag{337}$$

Let $C_+(\lambda)$ be the $C_k(\bar{\mathfrak{g}})''$-submodule of $C(L\bar{\mathfrak{n}}_+, M(\lambda))$ generated by the vector $|\lambda\rangle$:

$$C_+(\lambda) = \mathfrak{U}(C_k(\bar{\mathfrak{g}})'') \cdot |\lambda\rangle, \tag{338}$$

where $|\lambda\rangle = v_\lambda \otimes \mathbf{1}$ and v_λ is the highest weight vector of $M(\lambda)$. By definition $C_+(\lambda)$ is linearly spanned by the vectors

$$\widehat{J}_{a_1}(m_1)\dots\widehat{J}_{a_r}(m_r)\psi_{\alpha_1}(n_1)\dots\psi_{\alpha_s}(n_s)|\lambda\rangle$$

with $a_i \in \bar{\Delta}_- \sqcup \bar{I}$, $\alpha_i \in \bar{\Delta}_-$, $m_i, n_i \in \mathbb{Z}$. Clearly,

$$C_+(\lambda) = \bigoplus_{d \leq \langle\lambda, \mathbf{D}\rangle} C_+(\lambda)_d, \tag{339}$$

$$C_+(\lambda)_{\mathrm{top}} = C(L\bar{\mathfrak{n}}_+, M(\lambda))_{\mathrm{top}} = \bar{M}(\bar{\lambda}) \otimes \Lambda(\bar{\mathfrak{n}}_-). \tag{340}$$

Remark 8.3.1. By considering the filtration induced by the standard filtration of $C_k(\bar{\mathfrak{g}})''$, one sees that

$$C_+(\lambda) \cong S\big(\bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}] \oplus \bar{\mathfrak{h}} \otimes \mathbb{C}[t^{-1}]t^{-1}\big) \otimes \Lambda\big(\bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}]\big). \tag{341}$$

Proposition 8.3.2. *We have $Q_+^{\mathrm{st}} \cdot C_+(\lambda) \subset C_+(\lambda)$ and $\chi'_+ \cdot C_+(\lambda) \subset C_+(\lambda)$. Hence $Q'_+ \cdot C_+(\lambda) \subset C_+(\lambda)$, that is, $(C_+(\lambda), Q'_+)$ is a subcomplex of $(C(L\bar{\mathfrak{n}}_+, M(\lambda)), Q'_+)$.*

Proof. The assertion that $Q_+^{\mathrm{st}} \cdot C_+(\lambda) \subset C_+(\lambda)$ follows from the third and the fourth formulas in Lemma 4.8.2 and the fact that $Q_+^{\mathrm{st}} \cdot |\lambda\rangle = 0$. The fact that $\chi'_+ \cdot C_+(\lambda) \subset C_+(\lambda)$ is obvious because this is an inner action of $C_k(\bar{\mathfrak{g}})''$. □

Remark 8.3.3. Note that $\chi'_+ \cdot |\lambda\rangle \neq 0$.

The space $C(L\bar{\mathfrak{n}}_+, M(\lambda))$ is a direct sum of finite-dimensional weight spaces $C(L\bar{\mathfrak{n}}_+, M(\lambda))^\mu$ with $\mu \in \mathfrak{h}^*$, and so is $C_+(\lambda)$. Thus the inclusion $C_+(\lambda) \hookrightarrow C(L\bar{\mathfrak{n}}_+, M(\lambda))$ induces the surjection

$$C(L\bar{\mathfrak{n}}_-, M(\lambda)^*) = C(L\bar{\mathfrak{n}}_+, M(\lambda))^* \twoheadrightarrow C_+(\lambda)^* \tag{342}$$

by Lemma 8.1.1. The map (342) is a cochain map if we consider $C_+(\lambda)^*$ as a cochain complex with the differential Q_-, where $(Q_-\phi)(c) = \phi(Q'_+c)$ for $\phi \in C_+(\lambda)^*$ and $c \in C_+(\lambda)$. The map (342) is also a $\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}})$-module homomorphism, whose action is given by (336). Thus the map (342) induces a homomorphism of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules

$$H^\bullet(M(\lambda)^*) \to H^\bullet(C_+(\lambda)^*). \tag{343}$$

Proposition 8.3.4 ([2, Proposition 6.3])**.** *The map* (343) *gives the following isomorphism of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules: $H_-^\bullet(M(\lambda)^*) \cong H^\bullet(C_+(\lambda)^*)$.*

Remark 8.3.5. One can use the argument of Sect. 4.8 to prove Proposition 8.3.4.

8.4. The assertion to be proved

Proposition 8.4.1. *There exists an exhaustive, decreasing filtration*

$$\mathsf{F}: \quad \cdots \supset \mathsf{F}^p C_+(\lambda)^* \supset \mathsf{F}^{p+1} C_+(\lambda)^* \cdots \supset \mathsf{F}^{-1} C_+(\lambda)^* \supset \mathsf{F}^0 C_+(\lambda)^* = 0$$

of $C_+(\lambda)^*$ *such that*

$$\mathfrak{U}(C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}) \cdot \mathsf{F}^p C_+(\lambda)^* \subset \mathsf{F}^p C_+(\lambda)^*,$$
$$Q_- \cdot \mathsf{F}^p C_+(\lambda)^* \subset \mathsf{F}^p C_+(\lambda), \quad \mathbf{D} \cdot \mathsf{F}^p C_+(\lambda)^* \subset \mathsf{F}^p C_+(\lambda)^*$$

for all p, *and the associated spectral sequence* $E_r \Rightarrow H^\bullet(C_+(\lambda)^*) = H^{\bullet}_{-}(M(\lambda)^*)$ *satisfies the following: there exists a* $\mathscr{W}_k(\bar{\mathfrak{g}})$*-equivalent isomorphism of cochain maps*

$$E_r^{p,q} \cong D(\mathbf{M}(\gamma_{\bar{\lambda}})) \otimes \left(E_r^{p,q}\right)_{\langle\lambda,\mathbf{D}\rangle} \quad \textit{for all } r \geq 1. \tag{344}$$

Here $D(\mathbf{M}(\gamma_{\bar{\lambda}})) \otimes (E_r^{p,q})_{\langle\lambda,\mathbf{D}\rangle}$ *is considered as a cochain complex with the differential* $1 \otimes d_r$ *and as a* $\mathscr{W}_k(\bar{\mathfrak{g}})$*-module on which* $\mathscr{W}_k(\bar{\mathfrak{g}})$ *acts on the first factor* $D(\mathbf{M}(\gamma_{\bar{\lambda}}))$.

Let us show that Proposition 8.4.1 implies Theorem 7.5.3.

Here and after, for a decreasing filtration $F = \{F^p M\}$ of a superspace M we write $\mathrm{gr}_F M = \bigoplus_{p\in\mathbb{Z}} \mathrm{gr}_F^p M$, $\mathrm{gr}_F^p M = F^p M / F^{p+1} M$.

Because F is compatible with the action of $\mathbf{D}$, each E_r is a direct sum of $\mathbf{D}$-eigenspaces:

$$E_r = \bigoplus_{d\in\mathbb{C}} (E_r)_d. \tag{345}$$

Thus $(E_r)_d$ converges to $H^{\bullet}_{-}(M(\lambda)^*)_d$:

$$(E_r)_d \Rightarrow H^\bullet(C_+(\lambda)^*)_d = H^{\bullet}_{-}(M(\lambda)^*)_d \quad \text{for each } d \in \mathbb{C}. \tag{346}$$

Also, because F is a filtration of $C_k(\bar{\mathfrak{g}})''$-modules, it follows that each $E_r^{p,q}$ is a $\mathscr{W}_k(\bar{\mathfrak{g}})$-module. In particular

$$E_\infty \cong \mathrm{gr}_{\mathsf{F}} H^\bullet(C_+(\lambda)^*) \quad \text{as } \mathscr{W}_k(\bar{\mathfrak{g}})\text{-modules}. \tag{347}$$

Here, $\mathsf{F}^p H^\bullet(C_+(\lambda)^*) = \mathrm{Im}(H^\bullet(\mathsf{F}^p C_+(\lambda)^*) \to H^\bullet(C_+(\lambda)^*))$.

By (346) and (347), we have the linear isomorphism $\bigoplus_{p+q=n} (E_\infty^{p,q})_d \cong H^n_-(M(\lambda)^*)_d$ for all d. In particular by Proposition 7.2.3 (iii) we have

$$\bigoplus_p \left(E_\infty^{p,n-p}\right)_{\langle\lambda,\mathbf{D}\rangle} \cong H^n(M(\lambda)^*)_{\mathrm{top}} = \begin{cases} \mathbb{C} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{348}$$

Because $\bigoplus_{p+q=0}(E_\infty^{p,n-p})_{\langle\lambda,\mathbf{D}\rangle}$ is one-dimensional, it follows that there exists $p_0 \in \mathbb{Z}$ such that

$$(349)\qquad \left(E_\infty^{p,q}\right)_{\langle\lambda,\mathbf{D}\rangle} = \begin{cases}\mathbb{C} & \text{if } p = p_0 \text{ and } q = -p_0\\ 0 & \text{otherwise.}\end{cases}$$

But by (344) we have

$$(350)\qquad E_\infty^{p,q} \cong D(\mathbf{M}(\gamma_{\bar{\lambda}})) \otimes \left(E_\infty^{p,q}\right)_{\langle\lambda,\mathbf{D}\rangle},$$

and therefore there is the following isomorphism of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules:

$$(351)\qquad E_\infty^{p,q} \cong \begin{cases}D(\mathbf{M}(\gamma_{\bar{\lambda}})) & \text{if } (p,q) = (p_0, -p_0),\\ 0 & \text{otherwise.}\end{cases}$$

This immediately gives $H_-^i(M(\lambda)^*) = 0$ for $i \neq 0$. Also, because the term E_∞ lies entirely in $E_\infty^{p_0,-p_0}$, the corresponding filtration of $H_-^0(M(\lambda)^*)$ is trivial. Therefore we conclude that

$$H_-^0(M(\lambda)^*) \cong D(\mathbf{M}(\gamma_{\bar{\lambda}}))$$

as $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules.

In next sections we shall construct the filtration F satisfying the conditions of Proposition 8.4.1.

8.5. The filtration F. The space $C_+(\lambda)_{\text{top}} = \bar{M}(\bar{\lambda}) \otimes \Lambda(\bar{\mathfrak{n}}_-)$ is spanned by the vectors

$$\widehat{J}_{\alpha_1}(0)\ldots\widehat{J}_{\alpha_r}(0)\psi_{\beta_1}(0)\ldots\psi_{\beta_s}(0)|\lambda\rangle$$

with $\alpha_i, \beta_i \in \bar{\Delta}_-$.

Note there is a $\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')$-module isomorphism

$$(352)\qquad C_+(\lambda) \cong \mathfrak{U}(C_k(\bar{\mathfrak{g}})'') \otimes_{\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')_{\geq 0}} C_+(\lambda)_{\text{top}}.$$

We shall define a filtration $\{\mathsf{F}_p C_+(\lambda)_{\text{top}}\}$ of $C_+(\lambda)_{\text{top}}$, and then extend it to a filtration $\{\mathsf{F}_p C_+(\lambda)\}$ of the whole space $C_+(\lambda)$. After that, we define the filtration $\{\mathsf{F}^p C_+(\lambda)^*\}$ of $C_+(\lambda)^*$.

Set

$$(353)\qquad \mathsf{F}_p C_+(\lambda)_{\text{top}} := \sum_{\substack{\mu\in\mathfrak{h}^*\\ \langle\mu-\lambda,\bar{\rho}^\vee\rangle\leq p}} C_+(\lambda)_{\text{top}}^{\mu} \subset C_+(\lambda)_{\text{top}} \quad \text{for } p \leq 0.$$

Then, $\mathsf{F}_p C_+(\lambda)_{\text{top}}$ is a subspace of $C_+(\lambda)_{\text{top}}$ linearly spanned by the vectors

$$\widehat{J}_{\alpha_1}(0)\ldots\widehat{J}_{\alpha_r}(0)\psi_{\beta_1}(0)\ldots\psi_{\beta_s}(0)|\lambda\rangle$$

with $\sum_i \langle\alpha_i, \bar{\rho}^\vee\rangle + \sum_j \langle\beta_j, \bar{\rho}^\vee\rangle \leq p(\leq 0)$.

The following assertion is a direct consequence of the definition.

Lemma 8.5.1. *Each $\mathsf{F}_p C_+(\lambda)_{\mathrm{top}}$ is preserved by the action of $\widehat{J}_a(n)$, $\psi_\alpha(n)$ with $a \in \bar{\Delta}_- \sqcup \bar{I}$, $\alpha \in \bar{\Delta}_-$, $n \geq 0$. Thus $\mathsf{F}_p C_+(\lambda)_{\mathrm{top}}$ is a $\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')_{\geq 0}$-submodule of $C_+(\lambda)_{\mathrm{top}}$.*

Proposition 8.5.2. *The following hold:*

(i) $\{\mathsf{F}_p C_+(\lambda)_{\mathrm{top}}\}$ *is an increasing filtration of $C_+(\lambda)_{\mathrm{top}}$:*

$$\cdots \subset \mathsf{F}_p C_+(\lambda)_{\mathrm{top}} \subset \mathsf{F}_{p+1} C_+(\lambda)_{\mathrm{top}} \subset \dots$$
$$\cdots \subset \mathcal{F} F_0 C_+(\lambda)_{\mathrm{top}} = C_+(\lambda)_{\mathrm{top}};$$

(ii) $\bigcap_p \mathsf{F}_p C_+(\lambda)_{\mathrm{top}} = 0$;

(iii) *For each p,*

$$Q_+^{\mathrm{st}} \cdot \mathsf{F}_p C_+(\lambda)_{\mathrm{top}} \subset \mathsf{F}_p C_+(\lambda)_{\mathrm{top}},$$
$$\chi'_+ \cdot \mathsf{F}_p C_+(\lambda)_{\mathrm{top}} \subset \mathsf{F}_{p-1} C_+(\lambda)_{\mathrm{top}}.$$

Hence $\mathsf{F}_p C_+(\lambda)_{\mathrm{top}}$ is a subcomplex of $(C_+(\lambda)_{\mathrm{top}}, Q'_-)$.

(iv) *The space $C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_p C_+(\lambda)_{\mathrm{top}}$ is finite-dimensional for each $p \leq 0$.*

Proof. (i)–(ii) are easily seen. (iii) The first inclusion follows from the fact Q_+^{st} preserves weight spaces with respect to the action of $\mathfrak{h}$. The second assertion has been already seen in Lemma 8.5.1. (iv) is obvious. □

Let $\mathsf{F}_p C_+(\lambda)$ with $p \leq 0$ be the $C_k(\bar{\mathfrak{g}})''$-submodule of $C_+(\lambda)$ generated by $\mathsf{F}_p C_+(\lambda)_{\mathrm{top}}$:

$$\mathsf{F}_p C_+(\lambda) := \mathfrak{U}(C_k(\bar{\mathfrak{g}})'') \cdot \mathsf{F}_p C_+(\lambda)_{\mathrm{top}}. \tag{354}$$

By Lemma 8.5.1, $\mathsf{F}_p C_+(\lambda)$ is spanned by the vectors

$$\widehat{J}_{a_1}(-m_1) \dots \widehat{J}_{a_r}(-m_r) \psi_{\beta_1}(-n_1) \dots\dots \psi_{\beta_s}(-n_s) v,$$

with $a \in \bar{\Delta}_- \sqcup \bar{I}$, $\beta_i \in \bar{\Delta}_-$ $m_i, n_i > 0$, $v \in \mathsf{F}_p C_+(\lambda)_{\mathrm{top}}$. Further, by (352), there is a $C_k(\bar{\mathfrak{g}})''$-module isomorphism

$$\mathsf{F}_p C_+(\lambda) \cong \mathfrak{U}(C_k(\bar{\mathfrak{g}})'') \otimes_{\mathfrak{U}(C_k(\bar{\mathfrak{g}})'')_{\geq 0}} \mathsf{F}_p C_+(\lambda)_{\mathrm{top}}. \tag{355}$$

The following assertion follows from Proposition 8.5.2 and (355).

Proposition 8.5.3. *The following hold:*

(i) $\cdots \subset \mathsf{F}_p C_+(\lambda) \subset \mathsf{F}_{p+1} C_+(\lambda) \subset \cdots \subset \mathcal{F} F_0 C_+(\lambda) = C_+(\lambda)$;

(ii) $\bigcap_p \mathsf{F}_p C_+(\lambda) = 0$.

(iii) *Each $C_p C_+(\lambda)$ is a subcomplex of $(C_+(\lambda), Q'_+)$:*

$$Q_+^{\mathrm{st}} \cdot \mathsf{F}_p C_+(\lambda) \subset \mathsf{F}_p C_+(\lambda), \quad \chi'_+ \cdot \mathsf{F}_p C_+(\lambda) \subset \mathsf{F}_p C_+(\lambda);$$

(iv) *The space $(C_+(\lambda)/\mathsf{F}_p C_+(\lambda))_d$ is finite-dimensional for all p and d.*

Define

$$\mathsf{F}^p C_+(\lambda)^* := (C_+(\lambda)/\mathsf{F}_p C_+(\lambda))^*. \tag{356}$$

This is a $C_k(\bar{\mathfrak{g}})''$-submodule of $C_+(\lambda)^*$. By Proposition 8.5.3 (iii), we have

$$Q_- \cdot \mathsf{F}^p C_+(\lambda)^* \subset \mathsf{F}^p C_+(\lambda)^* \quad \text{for all } p. \tag{357}$$

Also, it is clear that $\mathsf{F}^p C_+(\lambda)^*$ is preserved by the action of $\mathbf{D}$.

The following assertion is a consequence of Proposition 8.5.3 (iv).

Lemma 8.5.4. *The space $\mathsf{F}^p C_+(\lambda)^*_d$ is finite-dimensional for all p, d and we have $\mathsf{F}^p C_+(\lambda)^* = \mathsf{D}(C_+(\lambda)/\mathsf{F}_p C_+(\lambda))$ as subspaces of* $\mathrm{Hom}(C_+(\lambda), \mathbb{C})$.

By definition we have exact sequences

$$0 \to \mathsf{F}^p C_+(\lambda)^* \to C_+(\lambda)^* \to (\mathsf{F}_p C_+(\lambda))^* \to 0, \tag{358}$$

$$0 \to \mathsf{F}^p C_+(\lambda)^* \to \mathsf{F}^{p-1} C_+(\lambda)^* \to (\mathsf{F}_p C_+(\lambda)/\mathsf{F}_{p-1} C_+(\lambda))^* \to 0. \tag{359}$$

These are cochain maps by Proposition 8.5.3 (iii).

Proposition 8.5.5. *The $\mathsf{F}^p = \{\mathsf{F}^p C_+(\lambda)^*\}$ gives a decreasing, separated, exhaustive filtration of the $C_k(\bar{\mathfrak{g}})''$-module $C_+(\lambda)^*$:*

$$\cdots \supset \mathsf{F}^p C_+(\lambda)^* \supset \mathsf{F}^{p+1} C_+(\lambda)^* \supset \cdots \supset \mathsf{F}^0 C_+(\lambda)^* = 0,$$
$$C_+(\lambda)^* = \bigcup_p \mathsf{F}^p C_+(\lambda)^*.$$

Proof. We need only to show that that F^p is exhaustive. For this, it is enough to show that

$$(C_+(\lambda)^*)^\mu = \bigcup_p (\mathsf{F}^p C_+(\lambda)^*)^\mu \tag{360}$$

for each μ. By Proposition 8.5.3 (ii), $\bigcap_p \mathsf{F}_p C_+(\lambda)^\mu = 0$. Therefore, there exists p such that $\mathsf{F}_p C_+(\lambda)^\mu = 0$, because $C_+(\lambda)^\mu$ is finite-dimensional. Thus $(C_+(\lambda)^*)^\mu = (\mathsf{F}^p C_+(\lambda)^*)^\mu$, and this proves (360). □

By Proposition 8.5.5 and (357) we obtain the converging spectral sequence corresponding to the filtration $\{\mathsf{F}^p C_+(\lambda)^*\}$:

$$E_r \Rightarrow H^\bullet(C_+(\lambda)^*) = H^\bullet_-(M(\lambda)^*) \tag{361}$$

By definition,

$$E_1^{p,q} = H^{p+q}\big(\mathrm{gr}^{\mathsf{F}}_p\, C_+(\lambda)^*, Q_-\big). \tag{362}$$

We shall show our spectral sequence (361) satisfies the property (344) in Proposition 8.4.1 in next sections.

8.6. Computation of the term $(E_1)_{\mathrm{top}}$**.** First we compute $(E_1^{p,q})_{\langle\lambda,\mathbf{D}\rangle}$. By definition,

$$\big(E_1^{p,q}\big)_{\langle\lambda,\mathbf{D}\rangle} = H^{p+q}\big(\mathrm{gr}_p^{\mathsf{F}}\, C_+(\lambda)^*_{\mathrm{top}}\big). \tag{363}$$

We have:

$$\mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}} = (C_+(\lambda)/\mathsf{F}_p C_+(\lambda))^*_{\mathrm{top}} = (C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_p C_+(\lambda)_{\mathrm{top}})^*, \tag{364}$$

$$\mathrm{gr}^p_{\mathsf{F}}\, C_+(\lambda)^*_{\mathrm{top}} = \big(\mathrm{gr}^{\mathsf{F}}_{p+1}\, C_+(\lambda)_{\mathrm{top}}\big)^*, \tag{365}$$

by (359). By Proposition 8.5.2 (iii),

$$Q_-^{\mathrm{st}} \cdot \mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}} \subset \mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}} \qquad \chi_- \cdot \mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}} \subset \mathsf{F}^{p+1} C_+(\lambda)^*_{\mathrm{top}}. \tag{366}$$

Hence

$$\big(E_1^{p,q}\big)_{\langle\lambda,\mathbf{D}\rangle} = \bigoplus_{\substack{\mu \\ \langle\mu,\bar\rho^\vee\rangle=-p-1}} H^{p+q}\big(C_+(\lambda)^*_{\mathrm{top}}, Q_-^{\mathrm{st}}\big)^\mu, \tag{367}$$

by (353), because Q_-^{st} commutes with the action of $\mathfrak{h}$. Therefore, we can apply Lemma 8.1.2 to get:

$$\big(E_1^{p,q}\big)_{\langle\lambda,\mathbf{D}\rangle} = \bigoplus_{\substack{\mu \\ \langle\mu,\bar\rho^\vee\rangle=-p-1}} H^{-p-q}\left(C_+(\lambda)^\mu_{\mathrm{top}}, Q_+^{\mathrm{st}}\right)^*. \tag{368}$$

On the other hand we have the following assertion, which is easily seen from (340).

Lemma 8.6.1. *The complex $(C_+(\lambda)_{\mathrm{top}}, Q_+^{\mathrm{st}})$ is identical to the Chevalley complex for calculating the Lie algebra cohomology $H^\bullet(\bar{\mathfrak{n}}_+, \bar M(\bar\lambda))$:*

$$H^i\big(C_+(\lambda)_{\mathrm{top}}, Q_+^{\mathrm{st}}\big)^\mu = \begin{cases} H^i(\bar{\mathfrak{n}}_+, \bar M(\bar\lambda))^\mu & \text{for } i \geq 0, \\ 0 & \text{for } i < 0. \end{cases}$$

As is well-known. $H^\bullet(\bar{\mathfrak{n}}_+, \bar M(\bar\lambda))^\mu = 0$ unless $\mu \in \bar W \circ \bar\lambda$. Thus by Lemma 8.6.1 it follows that

$$H^\bullet\big(C_+(\lambda)_{\mathrm{top}}, Q_+^{\mathrm{st}}\big)^\mu = 0 \quad \text{unless } \mu \in \bar W \circ \bar\lambda. \tag{369}$$

8.7. Computation of the term E_1**.** By Lemma 8.5.4, each $\mathsf{F}^p C_+(\lambda)^*$ is a direct sum of finite-dimensional subcomplexes $\mathsf{F}^p C_+(\lambda)^*_d$, and thus so is the complex $\mathrm{gr}_{\mathsf{F}}^{p} C_{+}(\lambda)^{*}$. Hence we have

$$\mathrm{gr}_{\mathsf{F}}^{p} C_{+}(\lambda)^{*} = \left(\mathrm{gr}_{\mathsf{F}}^{p+1} C_{+}(\lambda)\right)^{*} = \mathsf{D}\left(\mathrm{gr}_{p+1}^{\mathsf{F}} C_{+}(\lambda)\right),$$

by (359). Therefore, applying Lemma 8.1.2, we obtain the following assertion.

Proposition 8.7.1. *There is an isomorphism*

$$E_{1}^{p,q} \cong \mathsf{D}\left(H^{-p-q}\left(\mathrm{gr}_{p+1}^{\mathsf{F}} C_{+}(\lambda)\right)\right)$$

of $\mathscr{W}_{k}(\bar{\mathfrak{g}})$-modules for each p and q.

By Proposition 8.7.1 the computation of the term E_1 is reduced to the computation of the cohomology of the cochain complex $(\mathrm{gr}^{\mathsf{F}} C_{+}(\lambda), Q'_{+})$.

By (355) we have

$$\mathrm{gr}_{p}^{\mathsf{F}} C_{+}(\lambda) \cong \mathfrak{U}(C_{k}(\bar{\mathfrak{g}})'') \otimes_{\mathfrak{U}(C_{k}(\bar{\mathfrak{g}})'')_{\geq 0}} \left(\mathrm{gr}_{p}^{\mathsf{F}} C_{+}(\lambda)_{\mathrm{top}}\right) \tag{370}$$

as $C_{k}(\bar{\mathfrak{g}})''$-modules. Under this identification the action of Q'_{+} is described by the following formula:

$$Q'_{+}(u \otimes v) = [Q'_{+}, u] \otimes v + (-1)^{p(u)} u \otimes Q_{+}^{\mathrm{st}} v \tag{371}$$

with $u \in \mathfrak{U}(C_{k}(\bar{\mathfrak{g}})'')$, $v \in \mathrm{gr}^{\mathsf{F}} C_{+}(\lambda)_{\mathrm{top}}$.

Observe that the $\mathfrak{U}(C_{k}(\bar{\mathfrak{g}})'')_{\geq 0}$-module $\mathrm{gr}_{p}^{\mathsf{F}} C_{+}(\lambda)_{\mathrm{top}}$ is a direct sum of one-dimensional representations. Indeed, for a weight vector $v \in \mathrm{gr}_{p}^{\mathsf{F}} C_{+}(\lambda)_{\mathrm{top}}^{\mu}$, we have

$$\widehat{J}_{\alpha}(n) v = \psi_{\alpha}(n) v = 0 \quad \text{for } \alpha \in \bar{\Delta}_{-}, n \geq 0; \tag{372}$$

$$\widehat{J}_{i}(n) v = 0 \quad \text{for } i \in \bar{I}, n > 0; \tag{373}$$

$$\widehat{J}_{i}(0) v = \langle \lambda, J_{i} \rangle v \quad \text{for } i \in \bar{I}. \tag{374}$$

Hence it follows that

$$\mathrm{gr}_{p}^{\mathsf{F}} C_{+}(\lambda) \cong \bigoplus_{\substack{\mu \\ \langle \mu, \bar{\rho}^{\vee} \rangle = -p}} K(\mu) \oplus C_{+}(\lambda)_{\mathrm{top}}^{\mu} \tag{375}$$

(notation Sect. 5.2) as cochain complexes and $C_{k}(\bar{\mathfrak{g}})''_{\mathrm{old}}$-modules, where the right-hand-side is considered as a direct sum of tensor products of cochain complexes $(K(\mu), Q'_{+})$ and $(C_{+}(\lambda)_{\mathrm{top}}^{\mu}, Q_{+}^{\mathrm{st}})$, and as a $C_{k}(\bar{\mathfrak{g}})''_{\mathrm{old}}$-module on which $C_{k}(\bar{\mathfrak{g}})''_{\mathrm{old}}$ acts on the first factor $K(\mu)$. Therefore the following assertion follows from Proposition 5.2.2 and (369).

Proposition 8.7.2. *There is an isomorphism*

$$H^{i}\left(\mathrm{gr}_{p}^{\mathsf{F}} C_{+}(\lambda)\right) \cong \mathbf{M}(\gamma_{\bar{\lambda}}) \otimes \bigoplus_{\substack{\mu \in \mathfrak{h}^{*} \\ \langle \mu, \bar{\rho}^{\vee} \rangle = -p}} H^{i}\left(C_{+}(\lambda)_{\mathrm{top}}^{\mu}, Q_{+}^{\mathrm{st}}\right)$$

of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules for all i. Here $H^i(\mathrm{gr}_p^{\mathsf{F}}\, C_+(\lambda))$ is considered as a $\mathscr{W}_k(\bar{\mathfrak{g}})$-module via the map $\widehat{t_{-\bar{\rho}^\vee}}$.

By (332), (336), (368), (369), Propositions 8.7.1 and 8.7.2 it follows that we have

$$E_1^{p,q} \xrightarrow{\sim} D(\mathbf{M}(\gamma_{\bar{\lambda}})) \otimes \left(E_1^{p,q}\right)_{\langle\lambda,\mathbf{D}\rangle}. \tag{376}$$

as $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules for any p, q.

8.8. Computation of the term E_r. The $\mathscr{W}_k(\bar{\mathfrak{g}})$ is a vertex subalgebra of $C_k(\bar{\mathfrak{g}})''_{\mathrm{new}}$ (see Remark 4.8.6). Therefore $\mathscr{W}_k(\bar{\mathfrak{g}})$ acts on $C_+(\lambda)$ itself, rather than its cohomology, and $C_+(\lambda)_{\mathrm{top}}$ is a $\mathfrak{U}(\mathscr{W}_k(\bar{\mathfrak{g}}))_{\geq 0}$-submodule of $C_+(\lambda)$. By (200) we have the cochain map

$$\begin{array}{ccc} \mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) \otimes (C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_p\, C_+(\lambda)_{\mathrm{top}}) & \to & C_+(\lambda)/\mathsf{F}_p\, C_+(\lambda) \\ u \otimes v & \mapsto & t_{-\bar{\rho}^\vee}(u) \cdot v \end{array} \tag{377}$$

(Notation (247)), where $\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) \otimes (C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_p\, C_+(\lambda)_{\mathrm{top}})$ is regarded as a cochain complex with the differential $1 \otimes Q'_+$. By Proposition 8.5.2 (iv),

$$\mathsf{D}\left(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) \otimes (C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_p\, C_+(\lambda)_{\mathrm{top}})\right) \cong \mathsf{D}(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))) \otimes \mathsf{F}^p C(\lambda)^*_{\mathrm{top}}.$$

Here $\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))$ is considered as a $\mathbf{D}$-module by $\mathbf{D}\cdot u = [\mathbf{D}_{\mathrm{new}}, u]$. Therefore (377) induces a cochain map

$$\mathsf{F}^p C_+(\lambda)^* \to \mathsf{D}(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))) \otimes \mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}}, \tag{378}$$

where $\mathsf{D}(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))) \otimes \mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}}$ is considered as a cochain complex with the differential $1 \otimes Q_-$.

It is clear the following diagram commutes:

(379)

$$\begin{array}{ccc} \mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) \otimes (C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_p\, C_+(\lambda)_{\mathrm{top}}) & \longrightarrow & C_+(\lambda)/\mathsf{F}_p\, C_+(\lambda) \\ \downarrow & & \downarrow \\ \mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}})) \otimes (C_+(\lambda)_{\mathrm{top}}/\mathsf{F}_{p+1}\, C_+(\lambda)_{\mathrm{top}}) & \longrightarrow & C_+(\lambda)/\mathsf{F}_{p+1}\, C_+(\lambda). \end{array}$$

Thus we have the following commutative diagram of cochain complexes:

$$\begin{array}{ccc} \mathsf{F}^p C_+(\lambda)^* & \longrightarrow & \mathsf{D}(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))) \otimes \mathsf{F}^p C_+(\lambda)^*_{\mathrm{top}} \\ \uparrow & & \uparrow \\ \mathsf{F}^{p+1} C_+(\lambda)^* & \longrightarrow & \mathsf{D}(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))) \otimes \mathsf{F}^{p+1} C_+(\lambda)^*_{\mathrm{top}}. \end{array} \tag{380}$$

Therefore the map (378) induces the homomorphism of spectral sequence

$$(E_r, d_r) \to (\mathsf{D}(\mathfrak{U}^-(\mathscr{W}_k(\bar{\mathfrak{g}}))) \otimes (E_r)_{\langle\lambda,\mathbf{D}\rangle}, 1 \otimes d_r) \tag{381}$$

We claim that the map (381) is an isomorphism for $r \geq 1$. To prove this it is sufficient to show the assertion for $r = 1$. But for $r = 1$ (381) is identical to the isomorphism (376), under the isomorphism (248). Moreover for $r \geq 1$ we have

$$E_r^{p,q} \cong D(\mathbf{M}(\gamma_{\bar{\lambda}})) \otimes \left(E_r^{p,q}\right)_{\langle\lambda,\mathbf{D}\rangle} \tag{382}$$

as $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules, where $(E_r^{p,q})_{\mathrm{top}}$ is considered as the space of multiplicity. Indeed this is true for $r = 1$ by (376), and thus must be true for all $r \geq 1$.

We have shown that the filtration $\{\mathsf{F}^p C_+(\lambda)^*\}$ satisfies the desired property. Proposition 8.4.1 is proved. Hence Theorem 7.5.3 is proved. □

9. Results for the functor $H_+^0(?)$

In this section we assume that $k \neq -h^\vee$.

9.1. Results for the "+" reduction. For $\lambda \in \mathfrak{h}_k^*$ let $\mathcal{O}_k^{[\lambda]}$ be the Serre full subcategory of $\mathcal{O}_k$ whose objects have all their local composition factors isomorphic to $L(w \circ \lambda)$ with $w \in W(\lambda)$. We have

$$\mathcal{O}_k = \bigoplus_{[\lambda]\in\mathfrak{h}_k^*/\sim} \mathcal{O}_k^{[\lambda]}. \tag{383}$$

where $\sim$ is the equivalent relation defined by $\lambda \sim \mu \iff \mu \in W(\lambda) \circ \lambda$ ([47]).

Consider the following condition on $\lambda \in \mathfrak{h}_k^*$:

$$\Delta_+(\lambda) \cap t_{\bar{\rho}^\vee}\left(\Delta_-^{\mathrm{re}}\right) = \emptyset. \tag{384}$$

Since

$$\Delta_+^{\mathrm{re}} \cap t_{\bar{\rho}^\vee}\left(\Delta_-^{\mathrm{re}}\right) = \{-\beta + n\delta;\, \beta \in \bar{\Delta}_+,\ n = 1, 2, \dots, \mathrm{ht}\,\beta\}, \tag{385}$$

this condition is equivalent to

(386)
$$\langle\lambda + \rho, \alpha^\vee\rangle \notin \mathbb{Z} \text{ for all } \alpha \in \{-\beta + n\delta;\, \beta \in \bar{\Delta}_+,\ n = 1, 2, \dots, \mathrm{ht}\,\beta\}.$$

Because $\Delta(w \circ \lambda) = \Delta(\lambda)$ for $w \in W(\lambda)$, the condition (384) can be regarded as a condition on the block $\mathcal{O}_k^{[\lambda]}$.

The following assertion is straightforward from definition.

Lemma 9.1.1. *Suppose that* $\lambda \in \mathfrak{h}_k^*$ *satisfies the condition* (384).

(i) *The correspondence* $\alpha \mapsto t_{-\bar{\rho}^\vee}(\alpha)$ *gives a bijective map from* $\Delta_+(\lambda)$ *to* $\Delta_+(t_{-\rho^\vee} \circ \lambda)$.

(ii) *We have* $\Delta(t_{-\rho^\vee} \circ \lambda) \cap \bar{\Delta}_+ = \emptyset$. *That is, the classical part*

$$\overline{t_{-\rho^\vee} \circ \lambda} = \bar{\lambda} - (k + h^\vee)\bar{\rho}^\vee \in \bar{\mathfrak{h}}^*$$

is anti-dominant.

Lemma 9.1.2. *Assume that* $\lambda \in \mathfrak{h}_k^*$ *satisfies the condition* (384), *and let* $M(w \circ \lambda) \hookrightarrow M(\lambda)$, *with* $w \in W(\lambda)$, *be a non-trivial homomorphism of* $\mathfrak{g}$-*modules. Then we have* $\langle w \circ \lambda, \mathbf{D}_{\text{new}} \rangle < \langle \lambda, \mathbf{D}_{\text{new}} \rangle$.

Proof. See [2, Lemma 7.5]. □

The following result was established in [2].

Theorem 9.1.3. *Assume that* $\lambda \in \mathfrak{h}_k^*$ *satisfies the condition* (384). *Let* M *be an object of* $\mathcal{O}_k^{[\lambda]}$. *Then the cohomology* $H_+^i(M)$ *is zero for all* $i \neq 0$. *The correspondence* $M \mapsto H_+^0(M)$ *defines an exact functor from* $\mathcal{O}_k^{[\lambda]}$ *to* $\mathscr{W}_k(\bar{\mathfrak{g}})$-adMod.

Theorem 9.1.4. *Assume that* $\lambda \in \mathfrak{h}_k^*$ *satisfies the condition* (384). *Then there is an isomorphism*

$$H_+^0(L(\lambda)) \cong \mathbf{L}(\gamma_{\overline{t_{-\bar{\rho}^\vee} \circ \lambda}}) = \mathbf{L}(\gamma_{\bar{\lambda} - (k+h^\vee)\bar{\rho}^\vee}).$$

Remark 9.1.5.

(i) If $\lambda \in \mathfrak{h}_k^*$ satisfies the condition (384), then any element of $W(\lambda) \circ \lambda$ also satisfies the condition (384). Thus $H_+^0(L(w \circ \lambda)) \cong \mathbf{L}(\gamma_{\overline{t_{-\bar{\rho}^\vee} w \circ \lambda}})$ for any $w \in W(\lambda)$ under the assumption of Theorem 9.1.4.
(ii) For $\bar{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$ we do not need any condition on the block, see [3].

Corollary 9.1.6.

(i) ([21]) *Let* $k \notin \mathbb{Q}$. *Then the cohomology* $H_+^i(M)$ *is zero for all* $i \neq 0$ *with any object* M *of* $\mathcal{O}_k$ *and there is an isomorphism* $H_+^0(L(\lambda)) \cong \mathbf{L}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$ *for all* $\lambda \in \mathfrak{h}_k^*$.
(ii) *Suppose that*

$$r^\vee jk \notin \mathbb{Z} \quad \textit{for } j = 1, 2, \ldots, h,$$

where h *is the Coxeter number of* $\bar{\mathfrak{g}}$. *Then the cohomology* $H_+^i(L(k\Lambda_0))$ *is zero for all* $i \neq 0$ *and* $H_+^0(L(k\Lambda_0))$ *is isomorphic to* $\mathbf{L}(\gamma_{\text{vac}_k})$ *as vertex algebras.*

Suppose that $Pr_k^{\text{nondeg}} \neq \emptyset$. Then $k = p/q - h^\vee$ with $p, q \in \mathbb{N}$ satisfying (323). Let

$$\dot{P}r_k := \left\{\bar{\lambda} - (k + h^\vee)\bar{\mu} + k\Lambda_0; \lambda \in P_{++}^{p-h^\vee},\ \mu \in P_{++}^{\vee, q-h}\right\} \tag{387}$$

(Notation Sect. 7.6). Then $\dot{P}r_k$ is a subset of Pr_k containing $k\Lambda_0$.

One checks that any element of $\dot{P}r_k$ satisfies the condition (384). Thus we have the following assertion, which proves [27, Conjecture 3.4$_+$(a)] partially.

Corollary 9.1.7. *Suppose that there exists a non-degenerate principal admissible weight of* $\mathfrak{g}$ *of level k. Let* $\lambda \in \dot{P}r_k$. *Then the cohomology* $H_+^i(L(\lambda))$ *is zero for all* $i \neq 0$ *and there is an isomorphism* $H_+^0(L(\lambda)) \cong \mathbf{L}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$.

Remark 9.1.8. From Remark 7.6.5 it follows that there is the surjective map

$$\begin{aligned} \dot{P}r_k &\twoheadrightarrow Pr_k^{\text{nondeg}} \\ \Lambda &\mapsto t_{-\bar{\rho}^\vee} \circ \Lambda. \end{aligned}$$

Hence $H_-^0(L(\lambda)) \cong H_+^0(L(t_{\bar{\rho}^\vee} \circ \lambda))$ for all $\lambda \in Pr_k^{\text{nondeg}}$ and all the isomorphism classes in the set (324) can be obtained as the image of the functor $H_+^0(?)$.

9.2. Proof of Theorem 9.1.4. For $M \in \mathcal{O}_k$, let

$$H_+^\bullet(M)_d := \{[m] \in H_+^\bullet(M); \mathbf{D}_{\text{new}}[m] = d[m]\}. \tag{388}$$

Then we have $H_+^\bullet(M) = \bigoplus_{d \in \mathbb{C}} H_+^\bullet(M)_d$.

First, we have the following assertion.

Proposition 9.2.1. *For each* $\lambda \in \mathfrak{h}_k^*$ *we have the following:*

$$\begin{aligned} &H_+^i(M(\lambda)) = 0 \textit{ for } i \neq 0; \\ &H_+^0(M(\lambda))_{\text{top}} = H_+^0(M(\lambda))_{\langle \lambda, \mathbf{D}_{\text{new}} \rangle} = \mathbb{C}|\lambda\rangle, \\ &\operatorname{ch} H_+^0(M(\lambda)) = \operatorname{ch} \mathbf{M}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee}), \end{aligned}$$

where $|\lambda\rangle = v_\lambda \otimes \mathbf{1}$.

Proof. See [2, Theorem 5.8] and its proof. □

Remark 9.2.2. It is not true that $H_+^0(M(\lambda)) \cong \mathbf{M}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$ in general (compare Theorem 7.5.1). However this is probably true for λ that satisfies the condition (384).

Proposition 9.2.3. *For any* $\lambda \in \mathfrak{h}_k^*$ *we have* $H_+^0(M(\lambda))_{\text{top}} \cong \mathbb{C}_{\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee}}$ *as* $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$*-modules.*

Proof. The assertion follows by observing that $\mathbb{C}|\lambda\rangle \cong K(t_{-\bar{\rho}^\vee} \circ \lambda)_{\text{top}}$ as $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$-modules, see Proposition 5.2.1, Sect. 5.4 and [2, Proposition 4.2]. □

Lemma 9.2.4. *Assume that* $\lambda \in \mathfrak{h}_k^*$ *satisfies the condition* (384). *Then* $H_+^0(L(\lambda))$ *belongs to* $\mathcal{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ *and there is the following isomorphism* *of* $\mathfrak{Zh}(\mathscr{W}_k(\bar{\mathfrak{g}}))$*-modules:*

$$H^0_+(L(\lambda))_{\mathrm{top}} = H^0_+(L(\lambda))_{\langle\lambda,\mathbf{D}_{\mathrm{new}}\rangle} \cong \mathbb{C}_{\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee}}.$$

Proof. By the exactness of the functor $H^0_+(?)$ (Theorem 9.1.3), we have the exact sequence of $\mathscr{W}_k(\bar{\mathfrak{g}})$-modules

$$0 \to H^0_+(N(\lambda)) \to H^0_+(M(\lambda)) \to H^0_+(L(\lambda)) \to 0,$$

where $N(\lambda)$ is the maximal proper submodule of $M(\lambda)$. Hence $H^0_+(L(\lambda))$ belongs to $\mathscr{O}(\mathscr{W}_k(\bar{\mathfrak{g}}))$ because $H^0_+(M(\lambda))$ does, by Proposition 9.2.1. To show the rest of the assertion, we need only to show that $H^0_+(N(\lambda))_{\langle\lambda,\mathbf{D}_{\mathrm{new}}\rangle} = 0$, by Proposition 9.2.3. But this follows from Lemma 9.1.2. □

Proposition 9.2.5. *Assume that* $\lambda \in \mathfrak{h}^*_k$ *satisfies the condition* (384). *Then we have* $\mathrm{ch}\, H^0_+(L(\lambda)) = \mathrm{ch}\, \mathbf{L}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$.

Proof. By the exactness of the functor $H^0_+(?)$ and Proposition 9.2.1 we have

$$\begin{aligned}\mathrm{ch}\, H^0_+(L(\lambda)) &= \sum_\mu [L(\lambda) : M(\mu)]\, \mathrm{ch}\, H^0_+(M(\mu)) \\ &= \sum_\mu [L(\lambda) : M(\mu)]\, \mathrm{ch}\, \mathbf{M}(\gamma_{\bar{\mu}-(k+h^\vee)\bar{\rho}^\vee}).\end{aligned}$$

On the other hand one knows from Theorem 7.7.1 and Lemma 9.1.1 that

$$\mathrm{ch}\, \mathbf{L}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee}) = \sum_{\mu'} [L(t_{-\bar{\rho}^\vee} \circ \lambda) : M(\mu')]\, \mathrm{ch}\, \mathbf{M}(\gamma_{\bar{\mu}'}).$$

Hence one needs only to show that

$$[L(\lambda) : M(\mu)] = [L(t_{-\bar{\rho}^\vee} \circ \lambda) : M(t_{-\bar{\rho}^\vee} \circ \mu)] \quad \text{for any } \mu \tag{389}$$

under the condition (384) on λ. But this follow from Lemma 9.1.1 and the character formula [44, Theorem 1.1] of $L(\lambda)$. □

Proof of Theorem 9.1.4. By Lemma 9.2.4 there is a non-trivial $\mathscr{W}_k(\bar{\mathfrak{g}})$-module homomorphism from $\mathbf{M}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$ to $H^0_+(L(\lambda))$ that sends $|\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee}\rangle$ to $|\lambda\rangle$. But then $H^0_+(L(\lambda))$ must be isomorphic to $\mathbf{L}(\gamma_{\bar{\lambda}-(k+h^\vee)\bar{\rho}^\vee})$, by Proposition 9.2.5. □

A. Compatible degreewise complete algebras

A.1. Linear topology. Let $\mathscr{I}_0 \supset \mathscr{I}_1 \supset \dots$ be a decreasing sequence of linear subspaces of a vector space V. A *linear topology define by* $\mathscr{I}_n$ is a topology on V such that the set $\{v + \mathscr{I}_N; N \in \mathbb{N}\}$ forms a fundamental system of open neighborhoods of $v \in V$. The closure of a subspace $U \subset V$ is given by $\bigcap_N (U + \mathscr{I}_N)$. A completion $\widetilde{V}$ of V with respect to the linear topology defined by $\mathscr{I}_n$ is the projective limit

$$\widetilde{V} = \lim_{\overleftarrow{N}} V/\mathscr{I}_N$$

with the projective limit topology induced from the discrete topology on each $V/\mathscr{I}_N$. The topology on $\widetilde{V}$ coincides with the linear topology defined by $\widetilde{\mathscr{I}}_N$, where $\widetilde{\mathscr{I}}_N$ is the kernel of the natural map $\widetilde{V} \to V/\mathscr{I}_N$. The following assertion is well-known (see e.g. [53]).

Theorem A.1.1. *Let V a vector space equipped with the linear topology defined by a decreasing sequence $\{\mathscr{I}_n\}$ of subspaces. Let $U \subset V$ and give U and V/U the induced topology. Then we have the exact sequence $0 \to \widetilde{U} \to \widetilde{V} \to \widetilde{U/V} \to 0$ and $\widetilde{U}$ coincides the closure of the image of U in $\widetilde{V}$.*

A.2. Compatible degreewise complete algebras. Following [54], by a *compatible degreewise topological algebra* we mean a $\mathbb{Z}$-graded algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ satisfying the following:

(i) each A_d equipped with a topology and the multiplication map $A_d \times A_{d'} \to A_{d+d'}$ is continuous with respect to this topology;
(ii) the topology of A_d coincides with the linear topology define by $\mathscr{I}_N(A_d)$ with $N \geq \max(0, d)$, where $\mathscr{I}_N(A_d)$ is the closure of $\sum_{r>N} A_{d-r}A_r$ in A_d.

Further, if A is *degreewise complete*, that is, if each A_d is complete, then A is called a *compatible degreewise complete algebra*. The *degreewise completion* of a compatible degreewise topological algebra is $\widetilde{A} = \bigoplus_{d\in\mathbb{Z}} \widetilde{A}_d$, where each $\widetilde{A}_d$ is the completion of A_d:

$$\widetilde{A}_d = \lim_{\overleftarrow{N}} A_d/\mathscr{I}_N(A_d).$$

Then $\widetilde{A}$ is a compatible degreewise complete algebra. By definition A is degreewise complete if and only if $\widetilde{A} = A$ as compatible degreewise topological algebras.

Let $A = \bigoplus_{d\in\mathbb{Z}} A_d$ be any $\mathbb{Z}$-graded algebra. The *standard degreewise topology* of A is the the linear topology defined by the sequence $\bigoplus_{d\in\mathbb{Z}}(\sum_{r\geq N} A_{d-r}A_r)$. This makes A a compatible degreewise topological algebra. The corresponding degreewise completion $\widetilde{A}$ of A is called the *standard degreewise completion*.

Let $J = \bigoplus_{d\in\mathbb{Z}} J_d$ be a graded subspace of a compatible degreewise topological algebra A. The sum $\widetilde{J}$ of the closures $\widetilde{J}_d$ of J_d in A_d is called the *degreewise closure* of J. If J is an ideal then $\widetilde{J}$ is also an ideal of A. Further, if A is complete then $\widetilde{J}$ is the degreewise completion of J with respect to the relative topology induced from A, and the quotient algebra $A/\widetilde{J}$ with the quotient topology is also degreewise complete (Theorem A.1.1).

References

1. Abe, T., Buhl, G., Dong, C.: Rationality, regularity, and C_2-cofiniteness. Trans. Amer. Math. Soc. **356**(8), 3391–3402 (2004)
2. Arakawa, T.: Vanishing of cohomology associated to quantized Drinfeld–Sokolov reduction. Int. Math. Res. Not. **15**, 729–767 (2004)
3. Arakawa, T.: Representation theory of superconformal algebras and the Kac–Roan–Wakimoto conjecture. Duke Math. J. **130**(3), 435–478 (2005)
4. Backelin, E.: Representation of the category $\mathcal{O}$ in Whittaker categories. Internat. Math. Res. Not. **4**, 153–172 (1997)
5. Belavin, A., Polyakov, A., Zamolodchikov, A.: Infinite conformal symmetry in two dimensional quantum field theory. Nucl. Phys. B **241**(2), 333–380 (1984)
6. Bernstein, I.N., Gelfand, I.M., Gelfand, S.I.: A certain category of $\mathfrak{g}$-modules. Funkts. Anal. Pril. **10**(2), 1–8 (1976)
7. Bershadsky, M., Ooguri, H.: Hidden $SL(n)$ symmetry in conformal field theories. Commun. Math. Phys. **126**(1), 49–83 (1989)
8. Borcherds, R.E.: Vertex algebras, Kac–Moody algebras, and the Monster. Proc. Nat. Acad. Sci. USA **83**(10), 3068–3071 (1986)
9. Bouwknegt, P., McCarthy, J., Pilch, K.: The $\mathcal{W}_3$ algebra. Lecture Notes in Physics. New Series m: Monographs, vol. 42. Springer, Berlin (1996)
10. Bouwknegt, P., Schoutens, K.: $\mathcal{W}$-symmetry in conformal field theory. Phys. Rep. **223**(4), 183–276 (1993)
11. Bouwknegt, P., Schoutens, K. (eds.): $\mathcal{W}$-symmetry. Adv. Ser. Math. Phys., vol. 22. World Scientific Publishing Co., Inc., River Edge, NJ (1995)
12. Brundan, J., Kleshchev, K.: Representations of shifted Yangians and finite W-algebras. math.RT/0508003
13. Casian, L.: Proof of the Kazhdan–Lusztig conjecture for Kac–Moody algebras (the characters ch $L_{\omega\rho-\rho}$). Adv. Math. **119**(2), 207–281 (1996)
14. de Boer, J., Tjin, T.: The relation between quantum $\mathscr{W}$-algebras and Lie algebras. Commun. Math. Phys. **160**(2), 317–332 (1994)
15. Deodhar, V.V., Gabber, O., Kac, V.G.: Structure of some categories of representations of infinite-dimensional Lie algebras. Adv. Math. **45**(1), 92–116 (1982)
16. Fateev, V.A., Lukyanov, S.L.: The models of two-dimensional conformal quantum field theory with Z_n symmetry. Int. J. Mod. Phys. A **3**(2), 507–520 (1988)
17. Fateev, V.A., Zamolodchikov, A.B.: Conformal quantum field theory models in two dimensions having Z_3 symmetry. Nucl. Phys. B **280**(4), 644–660 (1987)
18. Feigin, B.L.: Semi-infinite homology of Lie, Kac–Moody and Virasoro algebras. Uspekhi Mat. Nauk **39**(2), 195–196 (1984)
19. Feigin, B.L., Frenkel, E.: Affine Kac–Moody algebras, bosonization and resolutions. Phys. Lett. **19**, 307–317 (1990)
20. Feigin, B.L., Frenkel, E.: Duality in $\mathscr{W}$-algebras. Int. Math. Res. Not. **1991**(6), 75–82 (1991)
21. Feigin, B.L., Frenkel, E.: Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras. Infinite analysis. Part A (Kyoto, 1991) In: Adv. Ser. Math. Phys., vol. 16, pp. 197–215. World Scientific (1992)
22. Feigin, B.L., Fuchs, D.B.: Skew-symmetric invariant differential operators on the line and Verma modules over the Virasoro algebra. Funkts. Anal. Prilozh. **16**(2), 47–63, 96 (1982)
23. Feigin, B.L., Fuchs, D.B.: Representation of the Virasoro Algebra. In: Adv. Stud. Contemp. Math., vol. 7, pp. 465–554. Gordon and Breach, New York (1990)

24. Frenkel, E.: $\mathcal{W}$-algebras and Langlands–Drinfeld correspondence. In: New Symmetry Principles in Quantum Field Theory (Cargse, 1991). NATO Adv. Sci. Inst. Ser. B Phys., vol. 295, pp. 433–447. Plenum, New York (1992)
25. Frenkel, E., Ben-Zvi, D.: Vertex algebras and algebraic curves. Math. Surv. Monogr., vol. 88. American Mathematical Society, Providence, RI (2004)
26. Frenkel, I.B., Huang, Y.-Z., Lepowsky, J.: On Axiomatic Approaches to Vertex Operator Algebras and Modules. Mem. Amer. Math. Soc., vol. 104(494). Amer. Math. Soc., Providence, RI (1993)
27. Frenkel, E., Kac, V.G., Wakimoto, M.: Characters and fusion rules for W-algebras via quantized Drinfeld–Sokolov reduction. Commun. Math. Phys. **147**(2), 295–328 (1992)
28. Frenkel, I.B., Lepowsky, J., Meurman, A.: Vertex operator algebras and the Monster. Pure Appl. Math., vol. 134. Academic Press, Boston, MA (1988)
29. Frenkel, I.B., Zhu, Y.: Vertex operator algebras associated to representations of affine and Virasoro algebras. Duke Math. J. **66**(1), 123–168 (1992)
30. Gan, W.-L., Ginzburg, V.: Quantization of Slodowy slices. Int. Math. Res. Not. **5**, 243–255 (2002)
31. Kac, V.G.: Contravariant form for infinite dimensional Lie algebras and superalgebras. Lect. Notes Phys. **94**, 441–445 (1974)
32. Kac, V.G.: Infinite-Dimensional Lie Algebras. Cambridge University Press, Cambridge (1990)
33. Kac, V.G.: Vertex Algebras for Beginners. University Lect. Ser., vol. 10. American Mathematical Society, Providence, RI (1998)
34. Kac, V.G., Kazhdan, D.A.: Structure of representations with highest weight of infinite-dimensional Lie algebras. Adv. Math. **34**(1), 97–108 (1979)
35. Kac, V.G., Raina, A.K.: Bombay lectures on highest weight representations of infinite-dimensional Lie algebras. In: Adv. Ser. Math. Phys., vol. 2. World Scientific Publishing Co., Teaneck, NJ (1987)
36. Kac, V.G., Roan, S.-S., Wakimoto, M.: Quantum reduction for affine superalgebras. Commun. Math. Phys. **241**(2–3), 307–342 (2003)
37. Kac, V.G., Wakimoto, M.: Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. Proc. Nat. Acad. Sci. USA **85**(14), 4956–4960 (1988)
38. Kac, V.G., Wakimoto, M.: Classification of modular invariant representations of affine algebras. In: Infinite-Dimensional Lie Algebras and Groups (Luminy–Marseille, 1988). Adv. Ser. Math. Phys., vol. 7, pp. 138–177. World Scientific Publishing, Teaneck, NJ (1989)
39. Kac, V.G., Wakimoto, M.: Branching functions for winding subalgebras and tensor products. Acta Appl Math. **21**(1–2), 3–39 (1990)
40. Kac, V.G., Wakimoto, M.: Quantum reduction and representation theory of superconformal algebras. Adv. Math. **185**(2), 400–45 (2004)
41. Kashiwara, M., Tanisaki, T.: Kazhdan–Lusztig conjecture for affine Lie algebras with negative level. Duke Math. J. **77**, 21–62 (1995)
42. Kashiwara, M., Tanisaki, T.: Kazhdan–Lusztig conjecture for affine Lie algebras with negative level. II. Nonintegral case. Duke Math. J. **84**(3), 771–813 (1996)
43. Kashiwara, M., Tanisaki, T.: Kazhdan–Lusztig conjecture for symmetrizable Kac–Moody Lie algebras. III. Positive rational case. Mikio Sato; a great Japanese mathematician of the twentieth century. Asian J. Math. **2**(4), 779–832 (1998)
44. Kashiwara, M., Tanisaki, T.: Characters of irreducible modules with non-critical highest weights over affine Lie algebras. In: Representations and Quantizations (Shanghai, 1998). pp. 275–296, China High. Educ. Press, Beijing (2000)
45. Kostant, B.: On Whittaker vectors and representation theory. Invent. Math. **48**(2), 101–184 (1978)
46. Kostant, B., Sternberg, S.: Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Ann. Phys. **176**(1), 49–113 (1987)
47. Kumar, S.: Extension of the category $\mathcal{O}^{g}$ and a vanishing theorem for the Ext functor for Kac–Moody algebras. J. Algebra **108**(2), 472–491 (1987)

48. Lepowsky, J., Li, H.: Introduction to vertex operator algebras and their representations. In: Prog. Math., vol. 227. Birkhäuser Boston Inc., Boston, MA (2004)
49. Li, H.: Vertex algebras and vertex Poisson algebras. Commun. Contemp. Math. **6**(1), 61–110 (2004)
50. Lukyanov, S.L.: Quantization of the Gelfand–Dikii bracket. Funkts. Anal. Prilozh. **22**, 1–10, 96 (1988); translation in Funct. Anal. Appl. **22**(4), 255–262 (1988)
51. Lukyanov, S.L., Fateev, V.A.: Conformally invariant models of two-dimensional quantum field theory with Z_n-symmetry. Soviet Phys. JETP **67**(3), 447–454 (1988)
52. Nagatomo, K., Tsuchiya, A.: Conformal field theories associated to regular chiral vertex operator algebras I; theories over the projective line. Duke Math. J. **128**(3), 393–471 (2005)
53. Matsumura, H.: Commutative Algebra. W.A. Benjamin Inc., New York (1970)
54. Matsuo, A., Nagatomo, K., Tsuchiya, A.: Quasi-finite algebras graded by Hamiltonian and vertex operator algebras. math.QA/050507
55. Matumoto, H.: Whittaker vectors and the Goodman–Wallach operators. Acta Math. **161**(3–4), 183–241 (1988)
56. Moody, R.V., Pianzola, A.: Lie algebras with triangular decompositions. Can. Math. Soc. Ser. Monogr. Adv. Texts. John Wiley & Sons Inc., New York (1995)
57. Premet, A.: Special transverse slices and their enveloping algebras. With an appendix by Serge Skryabin. Adv. Math. **170**(1), 1–55 (2002)
58. Soergel, W.: Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3**(2), 421–445 (1990)
59. Zamolodchikov, A.: Infinite additional symmetries in two-dimensional conformal field theory. Theor. Math. Phys. **65**, 1205–1213 (1985)
60. Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc. **9**(1), 237–302 (1996)