Geometric theta-lifting for the dual pair $\mathrm{G}\mathbb{S}\mathrm{p}_{2n}$, $\mathrm{G}\mathbb{O}_{2m}$

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Abstract Let X be a smooth projective curve over an algebraically closed field of characteristic > 2. Consider the dual pair $H = \mathrm{G}\mathbb{O}_{2m}$, $G = \mathrm{G}\mathbb{S}\mathrm{p}_{2n}$ over X, where H splits over an étale two-sheeted covering $\pi : \tilde X \to X$. Write Bun_G and Bun_H for the stacks of G-torsors and H-torsors on X. We show that for $m \le n$ (respectively, for $m > n$) the theta-lifting functor $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ (respectively, $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$) commutes with Hecke functors with respect to a morphism of the corresponding L-groups involving the SL_2 of Arthur. In two particular cases $n = m$ and $m = n + 1$ this becomes the geometric Langlands functoriality for the corresponding dual pair.

As an application, we prove a particular case of the geometric Langlands conjectures. Namely, we construct the automorphic Hecke eigensheaves on $\mathrm{Bun}_{\mathrm{G}\mathbb{S}\mathrm{p}_4}$ corresponding to the endoscopic local systems on X.

1. Introduction

1.1 The classical theta correspondence for the dual reductive pair $(\mathrm{G}\mathbb{S}\mathrm{p}_{2n}, \mathrm{G}\mathbb{O}_{2m})$ is known to satisfy a version of strong Howe duality (cf. [12]). In this paper, which is a continuation of [7], we develop the geometric theory of theta-lifting for this dual pair in the everywhere unramified case.

The classical theta-lifting operators for this dual pair are as follows. Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$ (with q odd). Let $F = \mathbb{F}_q(X)$, A be the adèles ring of X, $\mathcal{O}$ the integer adèles. Write Ω for the canonical line bundle on X. Pick a rank $2n$-vector bundle M with symplectic form $\wedge^2 M \to \mathcal{A}$ with values in a line bundle $\mathcal{A}$ on X. Let G be the group scheme over X of automorphisms of the $\mathrm{G}\mathbb{S}\mathrm{p}_{2n}$-torsor $(M, \mathcal{A})$.

Let $\pi : \tilde X \to X$ be an étale two-sheeted covering with Galois group $\Sigma = \{1, \sigma\}$. Let $\mathcal{E}$ be the σ-anti-invariants in $\pi_*\mathcal{O}_{\tilde X}$. Fix a rank $2m$-vector bundle V on X with symmetric form $\mathrm{Sym}^2 V \to \mathcal{C}$ with values in a line bundle $\mathcal{C}$ on X together with a compatible trivialization $\gamma : \mathcal{C}^{-m} \otimes \det V \widetilde{\to} \mathcal{E}$. This means that $\gamma^2 : \mathcal{C}^{-2m} \otimes (\det V)^2 \widetilde{\to} \mathcal{O}$ is the trivialization induced by the symmetric form. Let $\tilde H$ be the group scheme over X of automorphisms of V preserving the symmetric form up to a multiple and fixing γ. This is a form of $\mathrm{G}\mathbb{O}^0_{2m}$, where $\mathrm{G}\mathbb{O}^0_{2m}$ is the connected component of unity of the split orthogonal similitude group. Assume given an isomorphism $\mathcal{A} \otimes \mathcal{C} \widetilde{\to} \Omega$.

Let G_{2nm} the group scheme of automorphisms of $M \otimes V$ preserving the symplectic form $\wedge^2(M \otimes V) \to \Omega$. Write $G\tilde H \subset G \times \tilde H$ for the group subscheme over X of pairs (g, h) such that $g \otimes h$ acts trivially on $\mathcal{A} \otimes \mathcal{C}$. The metaplectic cover $\widetilde{G}_{2nm}(\mathbb{A}) \to G_{2nm}(\mathbb{A})$ splits naturally after restriction under $G\tilde H(\mathbb{A}) \to G_{2nm}(\mathbb{A})$. Let S be the corresponding Weil representation of $G\tilde{H}(\mathbb{A})$. The space $S^{G\tilde{H}(\mathcal{O})}$ has a distinguished nonramified vector v_0. If $\theta : S \to \bar{\mathbb{Q}}_\ell$ is a theta-functional then $\phi_0 : G\tilde{H}(F)\backslash G\tilde{H}(\mathbb{A})/G\tilde{H}(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell$ given by $\phi_0(g,h) = \theta((g,h)v_0)$ is the classical theta-function. The theta-lifting operators

$$F_G : \mathrm{Funct}(\tilde{H}(F)\backslash\tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O})) \to \mathrm{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O}))$$

and

$$F_{\tilde{H}} : \mathrm{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O})) \to \mathrm{Funct}(\tilde{H}(F)\backslash\tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O}))$$

are the integral operators with kernel ϕ_0 for the diagram of projections

$$\begin{tikzcd}
 & G\tilde{H}(F)\backslash G\tilde{H}(\mathbb{A})/G\tilde{H}(\mathcal{O}) \arrow[dl, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}"] & \\
\tilde{H}(F)\backslash\tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O}) & & G(F)\backslash G(\mathbb{A})/G(\mathcal{O})
\end{tikzcd}$$

The following statement would be an analog of a theorem of Rallis [11] for similitude groups (the author have not found its proof in the litterature). If $m \le n$ (resp., $m > n$) then F_G (resp., $F_{\tilde{H}}$) commutes with the actions of global Hecke algebras $\mathcal{H}_G$, $\mathcal{H}_{\tilde{H}}$ with respect to certain homomorphism $\mathcal{H}_G \to \mathcal{H}_{\tilde{H}}$ (resp., $\mathcal{H}_{\tilde{H}} \to \mathcal{H}_G$). We prove a geometric version of this result (cf. Theorem 1). Its precise formulation in the geometric setting involves the SL_2 of Arthur (or rather its maximal torus). In the particular case $n = m$ (resp., $m = n + 1$) the SL_2 of Arthur dissapears, and the corresponding morphisms of Hecke algebras come from morphisms of L-groups $H^L \to G^L$ (resp, $G^L \to H^L$).

Our methods extend those of [7], the global results are derived from the corresponding local ones. Remind that $S \,\tilde{\to}\, \otimes'_{x\in X} S_x$ is the restricted tensor product of local Weil representations. Let F_x be the completion of F at $x \in X$, $\mathcal{O}_x \subset F_x$ the ring of integers. The geometric analog of the $G\tilde{H}(F_x)$-representation S_x is the Weil category $W(\widetilde{\mathcal{L}}_d(W_0(F_x)))$ (cf. Sections 3.1-3.2). Informally speaking, we work rather with the geometric analog of the compactly induced representation

$$\bar{S}_x = \text{c-ind}_{G\tilde{H}(F_x)}^{(G\times\tilde{H})(F_x)} S_x$$

Its manifestation is a family of categories $\mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F_x)))$ indexed by $a \in \mathbb{Z}$ (cf. Section 4.2).

Our main local result is Theorem 3. In classical terms, it compares the action of Hecke operators for G and $\tilde{H}$ on the natural nonramified vector in $\bar{S}_x$. As a byproduct, we also obtain some new results at the classical level of functions (Propositions A.1 and A.2). For a even they reduce to a result from [10], but for a odd they are new and amount to a calculation of $K \times \mathbb{SO}(\mathcal{O}_x)$-invariants in the Weil representation of $(\mathbb{S}\mathrm{p}_{2n} \times \mathbb{SO}_{2m})(F_x)$, where K is the nonstandard maximal compact subgroup of $\mathbb{S}\mathrm{p}_{2n}(F_x)$.

1.2 The most striking application of our Theorem 1 is a proof of the following particular case of the geometric Langlands conjecture for $G = \mathrm{G}\mathbb{S}\mathrm{p}_4$. Let E be an irreducible rank 2 smooth $\bar{\mathbb{Q}}_\ell$-sheaf on $\tilde{X}$ equipped with an isomorphism $\pi^*\chi \,\tilde{\to}\, \det E$, where χ is a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on X of rank one. Then $\pi_*(E^*)$ is equipped with a natural symplectic form $\wedge^2(\pi_*E^*) \to \chi^{-1}$, so can be viewed as a $\check{G}$-local system $E_{\check{G}}$ on X, where $\check{G}$ is the Langlands dual group over $\bar{\mathbb{Q}}_\ell$. We construct the automorphic sheaf K on Bun_G, which is a Hecke eigensheaf with respect to $E_{\check{G}}$ (cf. Corollary 1).

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2. MAIN RESULTS

2.1 NOTATION From now on k denotes an algebraically closed field of characteristic $p > 2$, all the schemes (or stacks) we consider are defined over k (except in Section 4.8.7.2).

Fix a prime $\ell \neq p$. For a scheme (or stack) S write $\mathrm{D}(S)$ for the bounded derived category of ℓ-adic étale sheaves on S, and $\mathrm{P}(S) \subset \mathrm{D}(S)$ for the category of perverse sheaves. Set $\mathrm{DP}(S) = \oplus_{i\in\mathbb{Z}} \mathrm{P}(S)[i] \subset \mathrm{D}(S)$. By definition, we let for $K, K' \in \mathrm{P}(S), i, j \in \mathbb{Z}$

$$\mathrm{Hom}_{\mathrm{DP}(S)}(K[i], K'[j]) = \left\{ \begin{array}{ll} \mathrm{Hom}_{\mathrm{P}(S)}(K, K'), & \text{for } \; i = j \\ 0, & \text{for } \; i \neq j \end{array} \right.$$

Since we are working over an algebraically closed field, we systematically ignore Tate twists (except in Section 4.8.7.2, where we work over a finite subfield $k_0 \subset k$. In this case we also fix a square root $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ of the sheaf $\bar{\mathbb{Q}}_\ell(1)$ over $\mathrm{Spec}\, k_0$). Fix a nontrivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathcal{L}_\psi$ the corresponding Artin-Shreier sheaf on $\mathbb{A}^1$.

If $V \to S$ and $V^* \to S$ are dual rank n vector bundles over a stack S, we normalize the Fourier transform $\mathrm{Four}_\psi : \mathrm{D}(V) \to \mathrm{D}(V^*)$ by $\mathrm{Four}_\psi(K) = (p_{V^*})_!(\xi^*\mathcal{L}_\psi \otimes p_V^* K)[n](\frac{n}{2})$, where p_V, p_{V^*} are the projections, and $\xi : V \times_S V^* \to \mathbb{A}^1$ is the pairing.

For a sheaf of groups G on a scheme S, $\mathcal{F}_G^0$ denotes the trivial G-torsor on S. For a representation V of G and a G-torsor $\mathcal{F}_G$ on S write $V_{\mathcal{F}_G} = V \times^G \mathcal{F}_G$ for the induced vector bundle on S. For a morphism of stacks $f : Y \to Z$ denote by $\dim.\mathrm{rel}(f)$ the function of connected component C of Y given by $\dim C - \dim C'$, where C' is the connected component of Z containing $f(C)$.

2.2 HECKE OPERATORS Let X be a smooth connected projective curve. For $r \geq 1$ write Bun_r for the stack of rank r vector bundles on X. The Picard stack Bun_1 is also denoted $\mathrm{Pic}\, X$. For a connected reductive group $\mathbb{G}$ over k, let $\mathrm{Bun}_{\mathbb{G}}$ denote the stack of $\mathbb{G}$-torsors on X.

Given a maximal torus and a Borel subgroup $\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$, we write $\Lambda_{\mathbb{G}}$ (resp., $\check{\Lambda}_{\mathbb{G}}$) for the coweights (resp., weights) lattice of $\mathbb{G}$. Let $\Lambda^+_{\mathbb{G}}$ (resp., $\check{\Lambda}^+_{\mathbb{G}}$) denote the set of dominant coweights (resp., dominant weights) of $\mathbb{G}$. Write $\check{\rho}_{\mathbb{G}}$ (resp., $\rho_{\mathbb{G}}$) for the half sum of the positive roots (resp., coroots) of $\mathbb{G}$, w_0 for the longuest element of the Weyl group of $\mathbb{G}$.

Set $K = k(X)$. For a closed point $x \in X$ let K_x be the completion of K at x, $\mathcal{O}_x \subset K_x$ be its ring of integers.

The following notations are borrowed from [7]. Write $\mathrm{Gr}_{\mathbb{G},x}$ for the affine grassmanian $\mathbb{G}(K_x)/\mathbb{G}(\mathcal{O}_x)$. This is an ind-scheme classifying a $\mathbb{G}$-torsor $\mathcal{F}_{\mathbb{G}}$ on X together with a trivialization $\beta : \mathcal{F}_{\mathbb{G}} \mid_{X-x} \,\widetilde{\to}\, \mathcal{F}^0_{\mathbb{G}} \mid_{X-x}$. For $\lambda \in \Lambda^+_{\mathbb{G}}$ write $\overline{\mathrm{Gr}}^\lambda_{\mathbb{G},x} \subset \mathrm{Gr}_{\mathbb{G},x}$ for the closed subscheme classifying $(\mathcal{F}_{\mathbb{G}}, \beta)$ for which $V_{\mathcal{F}^0_{\mathbb{G}}}(-\langle\lambda, \check{\lambda}\rangle x) \subset V_{\mathcal{F}_{\mathbb{G}}}$ for every $\mathbb{G}$-module V whose weights are $\leq \check{\lambda}$. The unique dense open $\mathbb{G}(\mathcal{O}_x)$-orbit in $\overline{\mathrm{Gr}}^\lambda_{\mathbb{G},x}$ is denoted $\mathrm{Gr}^\lambda_{\mathbb{G},x}$.

For $\theta \in \pi_1(\mathbb{G})$ the connected component $\mathrm{Gr}^{\theta}_{\mathbb{G}}$ of $\mathrm{Gr}_{\mathbb{G}}$ classifies pairs $(\mathcal{F}_{\mathbb{G}}, \beta)$ such that $V_{\mathcal{F}^0_{\mathbb{G}}}(-\langle \theta, \check{\lambda}\rangle) \,\widetilde{\to}\, V_{\mathcal{F}_{\mathbb{G}}}$ for every one-dimensional $\mathbb{G}$-module with highest weight $\check{\lambda}$.

Denote by $\mathcal{A}^{\lambda}_{\mathbb{G}}$ the intersection cohomology sheaf of $\overline{\mathrm{Gr}}^{\lambda}_{\mathbb{G}}$. Write $\check{\mathbb{G}}$ for the Langlands dual group to $\mathbb{G}$, this is a reductive group over $\bar{\mathbb{Q}}_\ell$ equipped with the dual maximal torus and Borel subgroup $\check{\mathbb{T}} \subset \check{\mathbb{B}} \subset \check{\mathbb{G}}$. Write $\mathrm{Sph}_{\mathbb{G}}$ for the category of $\mathbb{G}(\mathcal{O}_x)$-equivariant perverse sheaves on $\mathrm{Gr}_{\mathbb{G},x}$. This is a tensor category, and one has a canonical equivalence of tensor categories $\mathrm{Loc} : \mathrm{Rep}(\check{\mathbb{G}}) \,\widetilde{\to}\, \mathrm{Sph}_{\mathbb{G}}$, where $\mathrm{Rep}(\check{\mathbb{G}})$ is the category of finite-dimensional representations of $\check{\mathbb{G}}$ over $\bar{\mathbb{Q}}_\ell$ (cf. [9]).

For the definition of the Hecke functors

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}}, \mathrm{H}^{\to}_{\mathbb{G}} : \mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}(\mathrm{Bun}_{\mathbb{G}}) \to \mathrm{D}(X \times \mathrm{Bun}_{\mathbb{G}})$$

we refer the reader to ([7], Section 2.2.1). Write $* : \mathrm{Sph}_{\mathbb{G}} \,\widetilde{\to}\, \mathrm{Sph}_{\mathbb{G}}$ for the covariant equivalence induced by the map $\mathbb{G}(K_x) \to \mathbb{G}(K_x)$, $g \mapsto g^{-1}$. In view of Loc, the corresponding functor $* : \mathrm{Rep}(\check{\mathbb{G}}) \,\widetilde{\to}\, \mathrm{Rep}(\check{\mathbb{G}})$ sends an irreducible $\check{\mathbb{G}}$-module with h.w. λ to the irreducible $\check{\mathbb{G}}$-module with h.w. $-w_0(\lambda)$. For $\lambda \in \Lambda^+_{\mathbb{G}}$ we also write $\mathrm{H}^{\lambda}_{\mathbb{G}}(\cdot) = \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{A}^{\lambda}_{\mathbb{G}}, \cdot)$.

Set

$$\mathrm{D}\,\mathrm{Sph}_{\mathbb{G}} = \oplus_{r \in \mathbb{Z}} \mathrm{Sph}_{\mathbb{G}}[r] \subset \mathrm{D}(\mathrm{Gr}_{\mathbb{G}})$$

As in ([7], Section 2.2.2), we equip it with a structure of a tensor category in such a way that the Satake equivalence extends to an equivalence of tensor categories $\mathrm{Loc}^{\mathfrak{r}} : \mathrm{Rep}(\check{\mathbb{G}} \times \mathbb{G}_m) \,\widetilde{\to}\, \mathrm{D}\,\mathrm{Sph}_{\mathbb{G}}$. Our convention is that $\mathbb{G}_m$ acts on $\mathrm{Sph}_{\mathbb{G}}[r]$ by the character $x \mapsto x^{-r}$.

Now let $\pi : \tilde{X} \to X$ be a finite étale Galois covering with Galois group Σ. Given a homomorphism $\Sigma \to \mathrm{Aut}(\mathbb{G})$, let G be the group scheme on X obtained as the twisting of $\mathbb{G}$ by the Σ-torsor $\pi : \tilde{X} \to X$. Set $\tilde{K} = k(\tilde{X})$. For a closed point $\tilde{x} \in \tilde{X}$ write $K_{\tilde{x}}$ for the completion of $\tilde{K}$ at $\tilde{x}$, $\mathcal{O}_{\tilde{x}} \subset K_{\tilde{x}}$ for its ring of integers, and $\mathrm{Gr}_{\mathbb{G},\tilde{x}}$ for the affine grassmanian $\mathbb{G}(K_{\tilde{x}})/\mathbb{G}(\mathcal{O}_{\tilde{x}})$.

Write Bun_G for the stack of G-torsors on X. One defines Hecke functors

$$ {}_{\tilde{x}}\mathrm{H}^{\leftarrow}_G, {}_{\tilde{x}}\mathrm{H}^{\to}_G : \mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G) \tag{1}$$

as follows. Write ${}_{\tilde{x}}\mathcal{H}_G$ for the Hecke stack classifying G-torsors $\mathcal{F}_G, \mathcal{F}'_G$ on X and an isomorphism $\mathcal{F}_G \,\widetilde{\to}\, \mathcal{F}'_G \mid_{X - \pi(\tilde{x})}$. We have a diagram

$$\mathrm{Bun}_G \overset{h^{\leftarrow}}{\leftarrow} {}_{\tilde{x}}\mathcal{H}_G \overset{h^{\to}}{\to} \mathrm{Bun}_G,$$

where $h^{\leftarrow}$ (resp., $h^{\to}$) sends $(\mathcal{F}_G, \mathcal{F}'_G, \tilde{x})$ to $\mathcal{F}_G$ (resp., to $\mathcal{F}'_G$). Set $D_{\tilde{x}} = \mathrm{Spec}\,\mathcal{O}_{\tilde{x}}$. Let $\mathrm{Bun}_{G,\tilde{x}}$ be the stack classifying $\mathcal{F}_G \in \mathrm{Bun}_G$ together with a trivialization $\mathcal{F}_G \mid_{D_{\tilde{x}}} \,\widetilde{\to}\, \mathcal{F}^0_{\mathbb{G}}$. Write $\mathrm{id}^l, \mathrm{id}^r$ for the isomorphisms

$$ {}_{\tilde{x}}\mathcal{H}_G \,\widetilde{\to}\, \mathrm{Bun}_{G,\tilde{x}} \times^{\mathbb{G}(\mathcal{O}_{\tilde{x}})} \mathrm{Gr}_{G,\tilde{x}}$$

such that the projection to the first factor corresponds to $h^{\leftarrow}, h^{\to}$ respectively. To $\mathcal{S} \in \mathrm{Sph}_{\mathbb{G}}$, $K \in \mathrm{D}(\mathrm{Bun}_G)$ one attaches their twisted external product $(K \,\tilde{\boxtimes}\, \mathcal{S})^l$ and $(K \,\tilde{\boxtimes}\, \mathcal{S})^r$ on ${}_{\tilde{x}}\mathcal{H}_G$, they are normalized to be perverse for K, S perverse. The functors (1) are defined by

$$ {}_{\tilde{x}}\mathrm{H}^{\leftarrow}_G(\mathcal{S}, K) = h^{\leftarrow}_!(K \,\tilde{\boxtimes} * \mathcal{S})^r \quad \text{and} \quad {}_{\tilde{x}}\mathrm{H}^{\to}_G(\mathcal{S}, K) = h^{\to}_!(K \,\tilde{\boxtimes}\, \mathcal{S})^l$$

We have canonically ${}_{\tilde{x}}\mathrm{H}^{\leftarrow}_G(*\mathcal{S},K)\xrightarrow{\sim}{}_{\tilde{x}}\mathrm{H}^{\rightarrow}_G(\mathcal{S},K)$. Letting $\tilde{x}$ move along $\tilde{X}$, one similarly defines Hecke functors

$$\mathrm{H}^{\leftarrow}_G,\mathrm{H}^{\rightarrow}_G:\mathrm{Sph}_{\mathbb{G}}\times\mathrm{D}(\mathrm{Bun}_G)\to\mathrm{D}(\tilde{X}\times\mathrm{Bun}_G)$$

They are compatible with the tensor structure on $\mathrm{Sph}_{\mathbb{G}}$ and commute with the Verdier duality (cf. [3, 7]). The group Σ acts on $\mathrm{Gr}_{G,\tilde{x}}$, hence also on $\mathrm{Sph}_{\mathbb{G}}$ by transport of structure, and for $\sigma\in\Sigma$ we have isomorphisms of functors $(\sigma\times\mathrm{id})^*\circ\mathrm{H}^{\leftarrow}_G(\mathcal{S},\cdot)\xrightarrow{\sim}\mathrm{H}^{\leftarrow}_G(\sigma^*\mathcal{S},\cdot)$.

Assume that $\mathbb{T}$ is Σ-invariant then Σ acts on the root datum $\mathcal{R}=(\Lambda_{\mathbb{G}},R,\check{\Lambda}_{\mathbb{G}},\check{R})$ of $(\mathbb{G},\mathbb{T})$, here R and $\check{R}$ stand for coroots and roots of $\mathbb{G}$ respectively. Given an action of Σ on $(\check{\mathbb{G}},\check{\mathbb{T}})$ such that the composition $\Sigma\to\mathrm{Aut}(\check{\mathbb{G}},\check{\mathbb{T}})\to\mathbb{O}\mathrm{ut}(\mathbb{G})$ coincides with $\Sigma\to\mathrm{Aut}(\mathbb{G},\mathbb{T})\to\mathbb{O}\mathrm{ut}(\mathbb{G})$, we form the semi-direct product $G^L:=\check{\mathbb{G}}\ltimes\Sigma$ included into an exact sequence $1\to\check{\mathbb{G}}\to\check{\mathbb{G}}\ltimes\Sigma\to\Sigma\to 1$. This is a version of the L-group associated to G_F. Here G_F denotes the restriction of the group scheme G to the generic point $\mathrm{Spec}\,F\in X$ of X (cf. [5]).

2.3 THETA-LIFTING FUNCTORS The following notations are borrowed from [5]. Write Ω for the canonical line bundle on X. For $k\geq 1$ let G_k denote the sheaf of automorphisms of $\mathcal{O}_X^k\oplus\Omega^k$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^k\oplus\Omega^k)\to\Omega$. The stack Bun_{G_k} of G_k-torsors on X classifies $M\in\mathrm{Bun}_{2k}$ equipped with a symplectic form $\wedge^2 M\to\Omega$. Write $\mathcal{A}_{G_k}$ for the line bundle on Bun_{G_k} with fibre $\det\mathrm{R}\Gamma(X,M)$ at M, we view it as $\mathbb{Z}/2\mathbb{Z}$-graded of parity zero. Let $\widetilde{\mathrm{Bun}}_{G_k}\to\mathrm{Bun}_{G_k}$ denote the μ_2-gerb of square roots of $\mathcal{A}_{G_k}$. Write Aut for the perverse theta-sheaf on $\widetilde{\mathrm{Bun}}_{G_k}$ (cf. also [6]).

Let $n,m\in\mathbb{N}$ and $\mathbb{G}=G=\mathrm{G}\mathbb{S}\mathrm{p}_{2n}$. Pick a maximal torus and Borel subgroup $\mathbb{T}_{\mathbb{G}}\subset\mathbb{B}_{\mathbb{G}}\subset\mathbb{G}$. The stack Bun_G classifies $M\in\mathrm{Bun}_{2n},\mathcal{A}\in\mathrm{Bun}_1$ with symplectic form $\wedge^2 M\to\mathcal{A}$. Write $\mathcal{A}_G$ for the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on Bun_G with fibre $\det\mathrm{R}\Gamma(X,M)$ at $(M,\mathcal{A})$.

Write $\check{\omega}_0$ for the character of $\mathbb{G}$ such that $\mathcal{A}$ is obtained from $(M,\mathcal{A})$ by the extension of scalars $\check{\omega}_0:G\to\mathbb{G}_m$. Write ${}_a\mathrm{Sph}_{\mathbb{G}}\subset\mathrm{Sph}_{\mathbb{G}}$ for the full subcategory of objects that vanishh off the connected components $\mathrm{Gr}^{\theta}_{\mathbb{G}}$ satisfying $\langle\theta,\check{\omega}_0\rangle=-a$.

Let $\pi:\tilde{X}\to X$ be an étale degree 2 covering with Galois group $\Sigma=\{\mathrm{id},\sigma\}$. Let $\mathcal{E}$ be the σ-anti-invariants in $\pi_*\mathcal{O}$, it is equipped with a trivialization $\mathcal{E}^2\xrightarrow{\sim}\mathcal{O}_X$.

Let $\mathbb{H}=\mathrm{G}\mathbb{O}^0_{2m}$ be the connected component of unity of the split orthogonal similitude group $\mathrm{G}\mathbb{O}_{2m}$ over k. Pick a maximal torus and Borel subgroup $\mathbb{T}_{\mathbb{H}}\subset\mathbb{B}_{\mathbb{H}}\subset\mathbb{H}$. Pick $\tilde{\sigma}\in\mathbb{O}_{2m}(k)$ with $\tilde{\sigma}^2=1$ such that $\tilde{\sigma}\notin\mathrm{S}\mathbb{O}_{2m}(k)$. We assume in addition that $\tilde{\sigma}$ preserves $\mathbb{T}_{\mathbb{H}}$ and $\mathbb{B}_{\mathbb{H}}$, so for $m\geq 2$ it induces the unique[1] nontrivial automorphism of the Dynkin diagram of $\mathbb{H}$. For $m=1$ we identify $\mathbb{H}\xrightarrow{\sim}\mathbb{G}_m\times\mathbb{G}_m$ in such a way that $\tilde{\sigma}$ permutes the two copies of $\mathbb{G}_m$.

Realize $\mathbb{H}$ as the subgroup of GL_{2m} preserving up to a multiple the symmetric form given by the matrix

$$\left(\begin{array}{cc}0 & E_m\\ E_m & 0\end{array}\right),$$

where $E_m\in\mathrm{GL}_m$ is the unity. Take $\mathbb{T}_H$ to be the maximal torus of diagonal matrices, $\mathbb{B}_H$ the Borel subgroup preserving for $i=1,\dots,m$ the isotropic subspace generated by the first i base

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[1] except for $m=4$. The group $\mathrm{G}\mathbb{O}_8$ also has trilitarian outer forms, we do not consider them.

vectors $\{e_1, \dots, e_i\}$. Then one may take $\tilde{\sigma}$ interchanging e_m and e_{2m} and acting trivially on the orthogonal complement to $\{e_m, e_{2m}\}$.

Consider the corresponding Σ-action on $\mathbb{H}$ by conjugation. Let $\tilde{H}$ be the group scheme on X, the twisting of $\mathbb{H}$ by the Σ-torsor $\pi : \tilde{X} \to X$.

The stack $\operatorname{Bun}_{\tilde{H}}$ classifies: $V \in \operatorname{Bun}_{2m}$, $\mathcal{C} \in \operatorname{Bun}_1$, a nondegenerate symmetric form $\operatorname{Sym}^2 V \to \mathcal{C}$, and a compatible trivialization $\gamma : \mathcal{C}^{-m} \otimes \det V \widetilde{\to} \mathcal{E}$. This means that the composition

$$\mathcal{C}^{-2m} \otimes (\det V)^2 \overset{\gamma^2}{\to} \mathcal{E}^2 \widetilde{\to} \mathcal{O}$$

is the isomorphism induced by $V \widetilde{\to} V^* \otimes \mathcal{C}$.

Write $\check{\alpha}_0$ for the character of $\mathbb{H}$ such that $\mathcal{C}$ is the extension of scalars of $(V, \mathcal{C})$ under $\check{\alpha}_0 : \mathbb{H} \to \mathbb{G}_m$. Write ${}_a\operatorname{Sph}_{\mathbb{H}} \subset \operatorname{Sph}_{\mathbb{H}}$ for the full subcategory of objects that vanish off the connected components $\operatorname{Gr}_{\mathbb{H}}^{\theta}$ of $\operatorname{Gr}_{\mathbb{H}}$ satisfying $\langle \theta, \check{\alpha}_0 \rangle = -a$.

Let RCov^0 denote the stack classifying a line bundle $\mathcal{U}$ on X together with a trivialization $\mathcal{U}^{\otimes 2} \widetilde{\to} \mathcal{O}$. Its connected components are indexed by $\operatorname{H}^1_{et}(X, \mathbb{Z}/2\mathbb{Z})$, each connected component is isomorphic to the classifying stack $B(\mu_2)$.

Let Bun_H be the stack classifying $V \in \operatorname{Bun}_{2m}, \mathcal{C} \in \operatorname{Bun}_1$ and a symmetric form $\operatorname{Sym}^2 V \to \mathcal{C}$ such that the corresponding trivialization $(\mathcal{C}^{-m} \otimes \det V)^2 \widetilde{\to} \mathcal{O}$ lies in the component of RCov^0 given by $(\mathcal{E}, \kappa)$. Note that

$$\operatorname{Bun}_{\tilde{H}} \widetilde{\to} \operatorname{Spec} k \times_{\operatorname{RCov}^0} \operatorname{Bun}_H,$$

where the map $\operatorname{Spec} k \to \operatorname{RCov}^0$ is given by $(\mathcal{E}, \kappa)$. Write $\rho_H : \operatorname{Bun}_{\tilde{H}} \to \operatorname{Bun}_H$ for the projection.

Let $\mathcal{A}_H$ be the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on Bun_H with fibre $\det \mathrm{R}\Gamma(X, V)$ at $(V, \mathcal{C})$. Set

$$\operatorname{Bun}_{G,H} = \operatorname{Bun}_H \times_{\operatorname{Pic} X} \operatorname{Bun}_G,$$

where the map $\operatorname{Bun}_H \to \operatorname{Pic} X$ sends $(V, \mathcal{C}, \operatorname{Sym}^2 V \to \mathcal{C})$ to $\Omega \otimes \mathcal{C}^{-1}$, and $\operatorname{Bun}_G \to \operatorname{Pic} X$ sends $(M, \wedge^2 M \to \mathcal{A})$ to $\mathcal{A}$. So, we have an isomorphism $\mathcal{C} \otimes \mathcal{A} \widetilde{\to} \Omega$ for a point of $\operatorname{Bun}_{G,H}$. Write $\operatorname{Bun}_{G,\tilde{H}}$ for the stack obtained from $\operatorname{Bun}_{G,H}$ by the base change $\operatorname{Bun}_{\tilde{H}} \to \operatorname{Bun}_H$. Let

$$\tau : \operatorname{Bun}_{G,H} \to \operatorname{Bun}_{G_{2nm}}$$

be the map sending a point as above to $V \otimes M$ with the induced symplectic form $\wedge^2(V \otimes M) \to \Omega$.

By ([5], Proposition 2), for a point $(M, \mathcal{A}, V, \mathcal{C})$ of $\operatorname{Bun}_{G,H}$ there is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det \mathrm{R}\Gamma(X, V \otimes M) \widetilde{\to} \frac{\det \mathrm{R}\Gamma(X, V)^{2n} \otimes \det \mathrm{R}\Gamma(X, M)^{2m}}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{A})^{2nm}} \tag{2}$$

It yields a map $\tilde{\tau} : \operatorname{Bun}_{G,H} \to \widetilde{\operatorname{Bun}}_{G_{2nm}}$ sending $(\wedge^2 M \to \mathcal{A}, \operatorname{Sym}^2 V \to \mathcal{C}, \mathcal{A} \otimes \mathcal{C} \widetilde{\to} \Omega)$ to $(\wedge^2(M \otimes V) \to \Omega, \mathcal{B})$. Here

$$\mathcal{B} = \frac{\det \mathrm{R}\Gamma(X, V)^n \otimes \det \mathrm{R}\Gamma(X, M)^m}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{A})^{nm}},$$

and $\mathcal{B}^2$ is identified with $\det \mathrm{R}\Gamma(X, M \otimes V)$ via (2).

Definition 1. Set $\mathrm{Aut}_{G,H} = \tilde{\tau}^* \mathrm{Aut}[\dim.\mathrm{rel}(\tau)]$. For the diagram of projections

$$\mathrm{Bun}_H \xleftarrow{\mathfrak{q}} \mathrm{Bun}_{G,H} \xrightarrow{\mathfrak{p}} \mathrm{Bun}_G$$

define $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ by

$$F_G(K) = \mathfrak{p}_!(\mathrm{Aut}_{G,H} \otimes \mathfrak{q}^* K)[-\dim \mathrm{Bun}_H]$$

$$F_H(K) = \mathfrak{q}_!(\mathrm{Aut}_{G,H} \otimes \mathfrak{p}^* K)[-\dim \mathrm{Bun}_G]$$

Since $\mathfrak{p}$ and $\mathfrak{q}$ are not representable, F_G and F_H a priori may send a bounded complex to a complex, which is not bounded even over some open substack of finite type. Let also $F_{\tilde{H}}$ denote F_H followed by restriction under $\mathrm{Bun}_{\tilde{H}} \to \mathrm{Bun}_H$. Write $\mathrm{Aut}_{G,\tilde{H}}$ for the restriction of $\mathrm{Aut}_{G,H}$ under $\mathrm{Bun}_{G,\tilde{H}} \to \mathrm{Bun}_{G,H}$. By abuse of notation, the composition $F_G \circ (\rho_H)_!$ is also denoted F_G.

2.4 Morphism of L-groups For $m \geq 2$ let $i_{\mathbb{H}} \in \mathbb{S}\mathrm{pin}_{2m}$ be the central element of order 2 such that $\mathbb{S}\mathrm{pin}_{2m}/\{i_{\mathbb{H}}\} \,\widetilde{\to}\, \mathbb{SO}_{2m}$. Here $\mathbb{S}\mathrm{pin}_{2m}$ and $\mathbb{SO}_{2m}$ denote the corresponding split groups over $\mathrm{Spec}\, k$. For $m \geq 2$ denote by $\mathrm{G}\mathbb{S}\mathrm{pin}_{2m}$ the quotient of $\mathbb{G}_m \times \mathbb{S}\mathrm{pin}_{2m}$ by the subgroup generated by $(-1, i_{\mathbb{H}})$. Let us convent that $\mathrm{G}\mathbb{S}\mathrm{pin}_2 \,\widetilde{\to}\, \mathbb{G}_m \times \mathbb{G}_m$. The Langlands dual group is $\check{\mathbb{H}} \,\widetilde{\to}\, \mathrm{G}\mathbb{S}\mathrm{pin}_{2m}$. We also have $\check{\mathbb{G}} \,\widetilde{\to}\, \mathrm{G}\mathbb{S}\mathrm{pin}_{2n+1}$, where $\mathrm{G}\mathbb{S}\mathrm{pin}_{2n+1} := \mathbb{G}_m \times \mathbb{S}\mathrm{pin}_{2n+1}/\{(-1, i_{\mathbb{G}})\}$. Here $i_{\mathbb{G}} \in \mathbb{S}\mathrm{pin}_{2n+1}$ is the nontrivial central element.

Let $V_{\mathbb{H}}$ (resp., $V_{\mathbb{G}}$) denote the standard representation of $\mathbb{SO}_{2m}$ (resp., of $\mathbb{SO}_{2n+1}$).

CASE $m \leq n$. Pick an inclusion $V_{\mathbb{H}} \hookrightarrow V_{\mathbb{G}}$ compatible with symmetric forms. It yields an inclusion $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}}$, which we assume compatible with the corresponding maximal tori. Pick an element $\sigma_{\mathbb{G}} \in \mathbb{SO}(V_{\mathbb{G}}) \,\widetilde{\to}\, \check{\mathbb{G}}_{ad}$ normalizing $\check{\mathbb{T}}_{\mathbb{G}}$ and preserving $V_{\mathbb{H}}$ and $\check{\mathbb{T}}_{\mathbb{H}} \subset \check{\mathbb{B}}_{\mathbb{H}}$. Let $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$ be its restriction to $V_{\mathbb{H}}$. We assume that $\sigma_{\mathbb{H}}$ viewed as an automorphism of $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$ extends the action of Σ on the roots datum of $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$ defined in Section 2.3.

In concrete terms, one may take $V_{\mathbb{G}} = \bar{\mathbb{Q}}_\ell^{2n+1}$ with symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E_n \in \mathrm{GL}_n$ is the unity. Take $\check{\mathbb{T}}_{\mathbb{G}}$ to be the preimage of the torus of diagonal matrices under $\check{\mathbb{G}} \to \mathbb{SO}_{2n+1}$. Let $V_{\mathbb{H}} \subset V_{\mathbb{G}}$ be generated by $\{e_1, \ldots, e_m, e_{n+1}, \ldots, e_{n+m}\}$. Let $\check{\mathbb{T}}_{\mathbb{H}}$ be the preimage under $\check{\mathbb{H}} \to \mathbb{SO}(V_{\mathbb{H}})$ of the torus of diagonal matrices, and $\check{\mathbb{B}}_{\mathbb{H}}$ the Borel subgroup preserving for $i = 1, \ldots, m$ the isotropic subspace generated by $\{e_1, \ldots, e_i\}$. Then one may take $\sigma_{\mathbb{G}}$ permuting e_m and e_{n+m}, sending e_{2n+1} to $-e_{2n+1}$ and acting trivially on the other base vectors.

We let Σ act on $\check{\mathbb{H}}$ and $\check{\mathbb{G}}$ via the elements $\sigma_{\mathbb{H}}, \sigma_{\mathbb{G}}$. So, the inclusion $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}}$ is Σ-equivariant and yields a morphism of the L-groups $\tilde{H}^L \to G^L$.

CASE $m > n$. Pick an inclusion $V_{\mathbb{G}} \hookrightarrow V_{\mathbb{H}}$ compatible with symmetric forms. It yields an inclusion $\check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$, which we assume compatible with the corresponding maximal tori. Let $\sigma_{\mathbb{G}}$

be the identical automorphism of $V_{\mathbb{G}}$. Extend it to an element $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$ by requiring that $\sigma_{\mathbb{H}}$ preserves $\check{\mathbb{T}}_{\mathbb{H}} \subset \check{\mathbb{B}}_{\mathbb{H}}$ and $\sigma_{\mathbb{H}} \notin \mathbb{SO}(V_{\mathbb{H}})$, $\sigma_{\mathbb{H}}^2 = \mathrm{id}$.

In concrete terms, take the symmetric form on $V_{\mathbb{H}} = \bar{\mathbb{Q}}_\ell^{2m}$ given by the matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}$$

Let $V_{\mathbb{G}}$ be the subspace of $V_{\mathbb{H}}$ generated by $\{e_1, \dots, e_n; e_{m+1}, \dots, e_{m+n}; e_{n+1} + e_{m+n+1}\}$. Take $\check{\mathbb{T}}_{\mathbb{H}}$ to be the preimage under $\check{\mathbb{H}} \to \mathbb{SO}(V_{\mathbb{H}})$ of the torus of diagonal matrices, and $\check{\mathbb{B}}_{\mathbb{H}}$ the Borel subgroup preserving for $i = 1, \dots, m$ the isotropic subspace of $V_{\mathbb{H}}$ generated by $\{e_1, \dots, e_i\}$. Let $\check{\mathbb{T}}_{\mathbb{G}}$ be the preimage under $\check{\mathbb{G}} \to \check{\mathbb{H}}$ of $\check{\mathbb{T}}_{\mathbb{H}}$. Let $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$ permute e_m and e_{2m} and act trivially on the orthogonal complement to $\{e_m, e_{2m}\}$. Then $\sigma_{\mathbb{H}}$ lifts uniquely to an automorphism of the exact sequence $1 \to \mathbb{G}_m \to \check{\mathbb{H}} \to \mathbb{SO}(V_{\mathbb{H}}) \to 1$ that acts trivially on $\mathbb{G}_m$.

As above, the inclusion $\check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$ is Σ-equivariant and gives rise to a morphism of the L-groups $\check{\mathbb{G}} \times \Sigma = G^L \to \tilde{H}^L$.

Theorem 1. *1) For $m \le n$ there is a homomorphism $\kappa : \check{\mathbb{H}} \times \mathbb{G}_m \to \check{\mathbb{G}}$ with the following property. There exists an isomorphism*

$$(\pi \times \mathrm{id})^* \mathrm{H}^{\leftarrow}_G(\mathcal{S}, F_G(K)) \xrightarrow{\sim} (\mathrm{id} \boxtimes F_G)(\mathrm{H}^{\leftarrow}_{\tilde{H}}(\mathrm{gRes}^\kappa(\mathcal{S}), K)) \tag{3}$$

in $\mathrm{D}(\tilde{X} \times \mathrm{Bun}_G)$ functorial in $\mathcal{S} \in \mathrm{Sph}_{\mathbb{G}}$ and $K \in \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$. Here $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_G \to X \times \mathrm{Bun}_G$, and $\mathrm{id} \boxtimes F_G : \mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\tilde{H}}) \to \mathrm{D}(\tilde{X} \times \mathrm{Bun}_G)$ is the corresponding theta-lifting functor.
2) For $m > n$ there is a homomoprhism $\kappa : \check{\mathbb{G}} \times \mathbb{G}_m \to \check{\mathbb{H}}$ with the following property. There exists an isomorphism

$$\mathrm{H}^{\leftarrow}_{\tilde{H}}(\mathcal{S}, F_{\tilde{H}}(K)) \xrightarrow{\sim} (\pi \times \mathrm{id})^* (\mathrm{id} \boxtimes F_{\tilde{H}})(\mathrm{H}^{\rightarrow}_{\mathbb{G}}(\mathrm{gRes}^\kappa(*\mathcal{S}), K))$$

in $\mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\tilde{H}})$ functorial in $\mathcal{S} \in \mathrm{Sph}_{\mathbb{H}}$ and $K \in \mathrm{D}(\mathrm{Bun}_G)$. Here $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_{\tilde{H}} \to X \times \mathrm{Bun}_{\tilde{H}}$ and $\mathrm{id} \boxtimes F_{\tilde{H}} : \mathrm{D}(X \times \mathrm{Bun}_G) \to \mathrm{D}(X \times \mathrm{Bun}_{\tilde{H}})$ is the corresponding theta-lifting functor.

Remark 1. If $m = n$ or $m = n+1$ then the restriction of κ to $\mathbb{G}_m$ is trivial. The explicit formulas for κ are given in Section 4.8.9. If $m \le n$ then κ fits into the diagram

$$\begin{tikzcd}
\check{\mathbb{H}} \times \mathbb{G}_m \arrow[r, "\kappa"] \arrow[d] & \check{\mathbb{G}} \arrow[d] \\
\mathbb{SO}_{2m} \times \mathbb{G}_m \arrow[r, "\bar{\kappa}"] & \mathbb{SO}_{2n+1}
\end{tikzcd}$$

If $m > n$ then κ fits into the diagram

$$\begin{tikzcd}
\check{\mathbb{G}} \times \mathbb{G}_m \arrow[r, "\kappa"] \arrow[d] & \check{\mathbb{H}} \arrow[d] \\
\mathbb{SO}_{2n+1} \times \mathbb{G}_m \arrow[r, "\bar{\kappa}"] & \mathbb{SO}_{2m},
\end{tikzcd}$$

In both cases $\bar{\kappa}$ is the map from ([7], Theorem 3).

For $a \in \mathbb{Z}$ let ${}^a\operatorname{Bun}_{G,\tilde{H}}$ be the stack classifying $\tilde{x} \in \tilde{X}$, $(M, \mathcal{A}) \in \operatorname{Bun}_G$, $(V, \mathcal{C}, \gamma) \in \operatorname{Bun}_{\tilde{H}}$, and an isomorphism $\mathcal{A} \otimes \mathcal{C} \,\widetilde{\to}\, \Omega(a\pi(\tilde{x}))$. We have the Hecke functors defined as in Section 2.2

$$\mathrm{H}^{\leftarrow}_G : {}_{-a}\operatorname{Sph}_{\mathbb{G}} \times \mathrm{D}(\operatorname{Bun}_{G,\tilde{H}}) \to \mathrm{D}({}^a\operatorname{Bun}_{G,\tilde{H}})$$

and

$$\mathrm{H}^{\leftarrow}_{\tilde{H}} : {}_{-a}\operatorname{Sph}_{\mathbb{H}} \times \mathrm{D}(\operatorname{Bun}_{G,\tilde{H}}) \to \mathrm{D}({}^a\operatorname{Bun}_{G,\tilde{H}})$$

Set also $\mathrm{H}^{\rightarrow}_{\tilde{H}}(\mathcal{S}, \cdot) = \mathrm{H}^{\leftarrow}_{\tilde{H}}(*\mathcal{S}, \cdot)$. We will derive Theorem 1 from the following Hecke property of $\operatorname{Aut}_{G,\tilde{H}}$.

Theorem 2. *Let κ be as in Theorem 1.*
1) For $m \le n$ there exists an isomorphism

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, \operatorname{Aut}_{G,\tilde{H}}) \,\widetilde{\to}\, \mathrm{H}^{\leftarrow}_{\tilde{H}}(*\operatorname{gRes}^{\kappa}(\mathcal{S}), \operatorname{Aut}_{G,\tilde{H}}) \tag{4}$$

in $\mathrm{D}({}^a\operatorname{Bun}_{G,\tilde{H}})$ functorial in $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{G}}$.
2) For $m > n$ there exists an isomorphism

$$\mathrm{H}^{\leftarrow}_{\tilde{H}}(\mathcal{S}, \operatorname{Aut}_{G,\tilde{H}}) \,\widetilde{\to}\, \mathrm{H}^{\leftarrow}_G(\operatorname{gRes}^{\kappa}(*\mathcal{S}), \operatorname{Aut}_{G,\tilde{H}}) \tag{5}$$

in $\mathrm{D}({}^a\operatorname{Bun}_{G,\tilde{H}})$ functorial in $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{H}}$.

2.5 APPLICATION: AUTOMORPHIC SHEAVES ON $\operatorname{Bun}_{\mathbb{G}\mathrm{Sp}_4}$.

Keep the notation of Section 2.3 assuming $m = n = 2$, so $G = \mathbb{G}\mathrm{Sp}_4$. Let $\tilde{E}$ be an irreducible rank two smooth $\bar{\mathbb{Q}}_\ell$-sheaf on $\tilde{X}$, χ a rank one local system on X equipped with an isomorphism $\pi^*\chi \,\widetilde{\to}\, \det \tilde{E}$. To this data one associates the perverse sheaf $K_{\tilde{E},\chi,\tilde{H}}$ on $\operatorname{Bun}_{\tilde{H}}$ introduced in ([7], Section 5.1). The local system $\pi_*\tilde{E}^*$ is equipped with a natural symplectic form $\wedge^2(\pi_*\tilde{E}^*) \to \chi^{-1}$, so gives rise to a $\check{\mathbb{G}}$-local system $E_{\check{G}}$ on X. Since $K_{\tilde{E},\chi,\tilde{H}}$ is a Hecke eigensheaf, Theorem 1 implies the following.

Corollary 1. *The complex $F_G(\rho_{H\,!}K_{\tilde{E},\chi,\tilde{H}}) \in \mathrm{D}(\operatorname{Bun}_G)$ is a Hecke eigensheaf corresponding to the $\check{G}$-local system $E_{\check{G}}$.*

Remark 2. i) We expect that for each open substack of finite type $\mathcal{U} \subset \operatorname{Bun}_G$ the restriction of $F_G(\rho_{H\,!}K_{\tilde{E},\chi,\tilde{H}})$ to $\mathcal{U}$ is a bounded complex. We also expect it to be perverse.
ii) If $\tilde{X}$ splits fix a numbering of connected components of $\tilde{X}$. Then $\tilde{E}$ becomes a pair of irreducible rank 2 local systems E_1, E_2 on X with the isomorphisms $\det E_1 \,\widetilde{\to}\, \det E_2 \,\widetilde{\to}\, \chi$.

3. LOCAL THEORY

3.1 BACKGROUND ON NON-RAMIFIED WEIL CATEGORY

Remind the following constructions from [8]. Let W be a symplectic Tate space over k. By definition ([2], 4.2.13), W is a complete topological k-vector space having a base of neighbourhoods of 0 consisting of commesurable vector subspaces (i.e., $\dim U_1/(U_1 \cap U_2) < \infty$ for any U_1, U_2 from this base). It is equipped with a (continuous) symplectic form $\wedge^2 W \to k$ (it induces a topological isomorphism $W \widetilde{\to} W^*$).

For a k-subspace $L \subset W$ write $L^\perp = \{w \in W \mid \langle w, l\rangle = 0 \text{ for all } l \in L\}$. Write $\mathcal{L}_d(W)$ for the scheme of discrete lagrangian lattices in W. For a c-lattice $R \subset W$ let $\mathcal{L}_d(W)_R \subset \mathcal{L}_d(W)$ be the open subscheme of $L \in \mathcal{L}_d(W)$ satisfying $L \cap R = 0$.

For a k-point $L \in \mathcal{L}_d(W)$ one defines the category $\mathcal{H}_L$ as in ([8], Section 6.1). Let us remind the definition. For a c-lattice $R \subset R^\perp \subset W$ with $R \cap L = 0$ we have a lagrangian subspace $L_R := L \cap R^\perp \in \mathcal{L}(R^\perp/R)$ and the Heisenberg group $H_R = R^\perp/R \oplus k$. Let $\mathcal{H}_{L_R}$ be the category of perverse sheaves on H_R, which are $(\bar{L}_R, \chi_{L,R})$-equivariant under the left multiplication on H_R. Here $\bar{L}_R = L_R \times \mathbb{A}^1 \subset H_R$ and $\chi_{L,R}$ is the local system $\mathrm{pr}^* \mathcal{L}_\psi$ for the projection $\mathrm{pr} : \bar{L}_R \to \mathbb{A}^1$ sending (l, a) to a. Let $D\mathcal{H}_{L_R} \subset \mathrm{D}(H_R)$ be the full subcategory of objects whose all perverse cohomologies lie in $\mathcal{H}_{L_R}$.

For another c-lattice $S \subset R$ we have (an exact for the perverse t-structures) transition functor $T^L_{S,R} : \mathrm{D}\,\mathcal{H}_{L_R} \to \mathrm{D}\,\mathcal{H}_{L_S}$ (cf. *loc.cit.*, Section 6.1). Now $\mathcal{H}_L$ is the inductive 2-limit of $\mathcal{H}_{L_R}$ over the partially ordered set of c-lattices $R \subset R^\perp$ such that $R \cap L = 0$.

Given a c-lattice M in W, we have a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on $\mathcal{L}_d(W)$, whose fibre at L is $\det(M : L)$. Remind that

$$\det(M : L) = \det(M \oplus L \to W),$$

where the complex $M \oplus L \to W$ is placed in cohomological degrees 0 and 1. If $S \subset M \subset S^\perp$ is a c-lattice with $S \cap L = 0$ then $\det(M : L) \widetilde{\to} \det(M/S) \otimes \det L_S$, where $L_S := L \cap S^\perp$. Note that $\det(M : L) \widetilde{\to} \det(M^\perp : L)$ canonically. If $M' \subset W$ is another c-lattice then we have $\det(M : L) \widetilde{\to} \det(M : M') \otimes \det(M' : L)$ canonically. If $R' \subset W$ is a lagrangian c-lattice then, as $\mathbb{Z}/2\mathbb{Z}$-graded, $\det(M : L)$ is of parity $\dim(R' : M) \mod 2$.

Fix a one-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded space $\mathcal{J}_W$ placed in degree $\dim(R' : M) \mod 2$. Let $\mathcal{A}_d$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero line bundle on $\mathcal{L}_d(W)$ with fibre $\mathcal{J}_W \otimes \det(M : L)$ at L. Let $\widetilde{\mathcal{L}}_d(W)$ be the μ_2-gerb of square roots of $\mathcal{A}_d$.

For k-points $N^0, L^0 \in \widetilde{\mathcal{L}}_d(W)$ one associates to them in a canonical way a functor $\mathcal{F}_{N^0,L^0} : \mathrm{D}\,\mathcal{H}_L \to \mathrm{D}\,\mathcal{H}_N$ sending $\mathcal{H}_L$ to $\mathcal{H}_N$ (defined as in [8], Section 6.2). Let us precise some details. For a c-lattice $R \subset R^\perp$ in W we have the projection

$$\mathcal{L}_d(W)_R \to \mathcal{L}(R^\perp/R)$$

sending L to L_R. Let $\mathcal{A}_R$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero line bundle on $\mathcal{L}(R^\perp/R)$ whose fibre at L_1 is $\det L_1 \otimes \det(M : R) \otimes \mathcal{J}_W$. Its restriction to $\mathcal{L}_d(W)_R$ identifies canonically with $\mathcal{A}_d$, hence a morphism of stacks

$$\widetilde{\mathcal{L}}_d(W)_R \to \widetilde{\mathcal{L}}(R^\perp/R) \tag{6}$$

where $\widetilde{\mathcal{L}}(R^\perp/R)$ is the gerb of square roots of $\mathcal{A}_R$. Write N^0_R, L^0_R for the images of N^0, L^0 under (6). By definition, the enchanced structure on L_R and N_R is given by one-dimensional spaces

$\mathcal{B}_L, \mathcal{B}_N$ equipped with

$$\mathcal{B}_L^2 \widetilde{\to} \det L_R \otimes \det(M : R) \otimes \mathcal{J}_W, \;\; \mathcal{B}_N^2 \widetilde{\to} \det N_R \otimes \det(M : R) \otimes \mathcal{J}_W,$$

hence an isomorphism $\mathcal{B}^2 \widetilde{\to} \det L_R \otimes \det N_R$ for $\mathcal{B} := \mathcal{B}_L \otimes \mathcal{B}_N \otimes \det(M : R)^{-1} \otimes \mathcal{J}_W^{-1}$. Write

$$\mathcal{F}_{N_R^0, L_R^0} : \mathrm{D}\,\mathcal{H}_{L_R} \to \mathrm{D}\,\mathcal{H}_{N_R} \tag{7}$$

for the canonical interwining functor corresponding to $(N_R, L_R, \mathcal{B})$ (as in *loc.cit*, Section 6.2). Then $\mathcal{F}_{N^0,L^0}$ is defined as the limit of the functors (7) over the partially ordered set of c-lattices $R \subset R^\perp$ such that $N, R \in \mathcal{L}_d(W)_R$.

The proof of (Theorem 2, [8]) holds through, so for a k-point $L^0 \in \widetilde{\mathcal{L}}_d(W)$ we have the functor $\mathcal{F}_{L^0} : \mathrm{D}\,\mathcal{H}_L \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W))$ exact for the perverse t-structures. For two k-points $L^0, N^0 \in \widetilde{\mathcal{L}}_d(W)$ the diagram is canonically 2-commutative

$$\begin{tikzcd}
\mathrm{D}\,\mathcal{H}_L \arrow[r, "\mathcal{F}_{L^0}"] \arrow[d, "\mathcal{F}_{N^0,L^0}"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W)) \\
\mathrm{D}\,\mathcal{H}_N \arrow[ur, "\mathcal{F}_{N^0}"'] &
\end{tikzcd}$$

The non-ramified Weil category $W(\widetilde{\mathcal{L}}_d(W))$ is defined as the essential image of $\mathcal{F}_{L^0} : \mathcal{H}_L \to \mathrm{P}(\widetilde{\mathcal{L}}_d(W))$ for any k-point $L^0 \in \widetilde{\mathcal{L}}_d(W)$.

3.2 Let $\mathcal{O}$ be a complete discrete valuation k-algebra, F its fraction field. Write Ω for the completed module of relative differentials of $\mathcal{O}$ over k. For a free $\mathcal{O}$-module V of finite rank write $V(r) \subset V \otimes F$ for the $\mathcal{O}$-submodule $t^{-r}V$, where $t \in \mathcal{O}$ is any uniformizer.

For $r \in \mathbb{Z}$ let W_r be a free $\mathcal{O}$-module of rank $2n$ with symplectic form $\wedge^2 W_r \to \Omega(r)$. Then $W_r(F)$ is a symplectic Tate space with the form $\wedge^2 W_r(F) \to \Omega(F) \overset{\mathrm{Res}}{\to} k$. Set

$$\mathcal{L}_d^{ex} = \sqcup_{r \in \mathbb{Z}} \mathcal{L}_d(W_r(F))$$

Let $\mathcal{G}_{b,a}$ be the set of F-linear isomorphisms $g : W_a(F) \to W_b(F)$ of symplectic F-spaces. Let $G_a = \mathbb{S}\mathrm{p}(W_a)$ as a group scheme over $\mathcal{O}$.

Fix a $\mathbb{Z}/2\mathbb{Z}$-graded line $\mathcal{J}_r$ placed in degree $nr \mod 2$. Let $\mathcal{A}_{d,r}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero line bundle on $\mathcal{L}_d(W_r(F))$ whose fibre at L is $\mathcal{J}_r \otimes \det(W_r : L)$. Let $\widetilde{\mathcal{L}}_d(W_r(F))$ be the μ_2-gerb of square roots of $\mathcal{A}_{d,r}$.

Let $\widetilde{\mathcal{G}}_{b,a}$ be the μ_2-gerb over $\mathcal{G}_{b,a}$ classifying $g \in \mathcal{G}_{b,a}$, a one-dimensional space $\mathcal{B}$ and an isomorphism $\mathcal{B}^2 \widetilde{\to} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$. The composition $\mathcal{G}_{c,b} \times \mathcal{G}_{b,a} \to \mathcal{G}_{c,a}$ lifts to a morphism $\widetilde{\mathcal{G}}_{c,b} \times \widetilde{\mathcal{G}}_{b,a} \to \widetilde{\mathcal{G}}_{c,a}$ sending $(g_2, \mathcal{B}_2) \in \widetilde{\mathcal{G}}_{c,b}$, $(g_1, \mathcal{B}_1) \in \widetilde{\mathcal{G}}_{b,a}$ to $(g_2 g_1, \mathcal{B})$, where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$.

Consider the action map

$$\widetilde{\mathcal{G}}_{b,a} \times \widetilde{\mathcal{L}}_d(W_a(F)) \to \widetilde{\mathcal{L}}_d(W_b(F))$$

sending $(g, \mathcal{B}) \in \widetilde{\mathcal{G}}_{b,a}$ and $(L, \mathcal{B}_L) \in \widetilde{\mathcal{L}}_d(W_a(F))$ to $(gL, \mathcal{B}_1)$, where $\mathcal{B}_1 = \mathcal{B} \otimes \mathcal{B}_L$ is equipped with the induced isomorphism

$$\mathcal{B}_1^2 \widetilde{\to} \mathcal{J}_b \otimes \det(W_b : gL)$$

In this way $\widetilde{\mathcal{G}}^{ex} := \sqcup_{a,b\in\mathbb{Z}} \widetilde{\mathcal{G}}_{b,a}$ becomes a groupoid acting on

$$\widetilde{\mathcal{L}}^{ex}_d := \sqcup_{r\in\mathbb{Z}} \widetilde{\mathcal{L}}_d(W_r(F))$$

The gerb $\widetilde{\mathcal{G}}_{a,a} \to \mathcal{G}_{a,a}$ has a canonical section over $G_a(\mathcal{O}) \subset \mathcal{G}_{a,a}$ sending $g \in G_a(\mathcal{O})$ to $(g, \mathcal{B} = k)$ equipped with $\mathrm{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(W_a : W_a)$. One can define the equivariant derived category $\mathrm{D}_{G_a(\mathcal{O})}(\widetilde{\mathcal{L}}_d(W_a(F)))$ as in ([7], Section 8.2.2).

For $g \in \mathcal{G}_{b,a}$ and a c-lattice $R \subset R^\perp \subset W_a(F)$ we have $(gR)^\perp = g(R^\perp)$, and g induces an isomorphism of symplectic spaces

$$g : R^\perp/R \xrightarrow{\sim} (gR)^\perp/(gR) \tag{8}$$

If $L \in \mathcal{L}_d(W_a(F))_R$ then g yields an equivalence $\mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}$ sending K to g_*K for the map $g : H_R \xrightarrow{\sim} H_{gR}$. Passing to the limit by R, we further get an equivalence $g : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$.

Proposition 1. *Let $a, b \in \mathbb{Z}$, $\tilde g \in \widetilde{\mathcal{G}}_{b,a}$ over $g \in \mathcal{G}_{b,a}$ and $L^0 \in \widetilde{\mathcal{L}}_d(W_a(F))$ be k-points. Then the diagram is canonically 2-commutative*

$$\begin{tikzcd}
\mathrm{D}\,\mathcal{H}_L \arrow[r, "\mathcal{F}_{L^0}"] \arrow[d, "g"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F))) \arrow[d, "\tilde g"] \\
\mathrm{D}\,\mathcal{H}_{gL} \arrow[r, "\mathcal{F}_{\tilde g L^0}"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_b(F)))
\end{tikzcd}$$

Proof Let $R \subset R^\perp \subset W_a(F)$ be a c-lattice with $R \cap L = 0$. We get an equivalence $g : \mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}$. Let $\mathcal{A}_R$ be the line bundle on $\mathcal{L}(R^\perp/R)$ whose fibre at L_1 is

$$\mathcal{J}_a \otimes \det(W_a : R) \otimes \det L_1$$

Let $\widetilde{\mathcal{L}}(R^\perp/R)$ be the μ_2-gerb of square roots of $\mathcal{A}_R$. We have the projection

$$\widetilde{\mathcal{L}}_d(W_a(F))_R \to \widetilde{\mathcal{L}}(R^\perp/R)$$

sending L^0 to L^0_R. As in ([8], Section 6.4), we have the functors $\mathcal{F}_{L^0_R} : \mathcal{H}_{L_R} \to \mathrm{P}(\widetilde{\mathcal{L}}(R^\perp/R))$. It suffices to show that the diagram is canonically 2-commutative

$$\begin{tikzcd}
\mathcal{H}_{L_R} \arrow[r, "\mathcal{F}_{L^0_R}"] \arrow[d, "g"] & \mathrm{P}(\widetilde{\mathcal{L}}(R^\perp/R)) \arrow[d, "\tilde g"] \\
\mathcal{H}_{gL_{gR}} \arrow[r, "\mathcal{F}_{\tilde g L^0_{gR}}"] & \mathrm{P}(\widetilde{\mathcal{L}}((gR)^\perp/gR))
\end{tikzcd} \tag{9}$$

The above expression $\tilde g L^0_{gR}$ is the image of $\tilde g(L^0)$ under $\widetilde{\mathcal{L}}_d(W_b(F))_{gR} \to \widetilde{\mathcal{L}}((gR)^\perp/(gR))$. Note that $\tilde g L^0_{gR} = \tilde g(L^0_R)$, where

$$\tilde g : \widetilde{\mathcal{L}}(R^\perp/R) \xrightarrow{\sim} \widetilde{\mathcal{L}}((gR)^\perp/gR)$$

sends $(L_1, \mathcal{B})$ to $(gL_1, \mathcal{B} \otimes \mathcal{B}_0)$. Here $\tilde g = (g, \mathcal{B}_0)$.

Remind that H_R denotes the Heisenberg group $R^\perp/R \times \mathbb{A}^1$. For the isomorphism

$$\tilde{g} : \widetilde{\mathcal{L}}(R^\perp/R) \times \widetilde{\mathcal{L}}(R^\perp/R) \times H_R \widetilde{\to} \widetilde{\mathcal{L}}((gR)^\perp/gR) \times \widetilde{\mathcal{L}}((gR)^\perp/gR) \times H_{gR}$$

we have $\tilde{g}^*F \widetilde{\to} F$ canonically, where F is the CIO sheaf on each side (introduced in [8], Theorem 1). The 2-commutativity of (9) follows. □

By Proposition 1, each $\tilde{g} \in \mathcal{G}_{b,a}$ yields an equivalence $\tilde{g} : W(\widetilde{\mathcal{L}}_d(W_a(F))) \widetilde{\to} W(\widetilde{\mathcal{L}}_d(W_b(F)))$.

3.3 Now assume that we are given for each $a \in \mathbb{Z}$ a decomposition $W_a = U_a \oplus U_a^* \otimes \Omega(a)$, where U_a is a free $\mathcal{O}$-module of rank n, U_a and $U_a^* \otimes \Omega(a)$ are lagrangians, and the form $\omega : \wedge^2 W_a \to \Omega(a)$ is given by $\omega\langle u, u^*\rangle = \langle u, u^*\rangle$ for $u \in U_a, u^* \in U_a^* \otimes \Omega(a)$, where $\langle\cdot,\cdot\rangle$ is the canonical pairing between U_a and $U_a^* \otimes \Omega(a)$.

Remark 3. If U_1 is a free $\mathcal{O}$-module of finite rank and $U_2 \subset U_1(F)$ is a $\mathcal{O}$-lattice then there is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\det(U_2 : U_1)^* \widetilde{\to} \det(U_1^* \otimes \Omega : U_2^* \otimes \Omega)$.

For $a, b \in \mathbb{Z}$ let $\mathcal{U}_{b,a}$ be the set of F-linear isomorphisms $U_a(F) \to U_b(F)$. We have an inclusion $\mathcal{U}_{b,a} \hookrightarrow \mathcal{G}_{b,a}$ given by $g \mapsto (g, ({}^tg)^{-1})$. Here ${}^tg \in \mathrm{GL}(U^* \otimes \Omega)(F)$ is the adjoint operator. By Remark 3, for $g \in \mathcal{U}_{b,a}$ we have canonically

$$\det(W_b : gW_a) \widetilde{\to} \det(U_b : gU_a)^2 \otimes (\det U_{a,x})^a \otimes (\det U_{b,x})^{-b} \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^n[n(b-a)]$$

Assume in addition that n is even. Assume given a one-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero vector space $\mathcal{J}_{U,a}$ equipped with $\mathcal{J}_{U,a}^2 \widetilde{\to} \mathcal{J}_a \otimes \det(U_{a,x})^{-a}$. This yields a section $\rho_{b,a} : \mathcal{U}_{b,a} \to \widetilde{\mathcal{G}}_{b,a}$ defined as follows. We send $g \in \mathcal{U}_{b,a}$ to $(g, \mathcal{B})$, where

$$\mathcal{B} = \mathcal{J}_{U,b} \otimes \mathcal{J}_{U,a}^{-1} \otimes \det(U_b : gU_a) \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^{n/2}$$

is equipped with the induced isomorphism

$$\mathcal{B}^2 \widetilde{\to} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$$

The section ρ is compatible with the groupoïd structures on $\widetilde{\mathcal{G}}^{ex}$ and $\mathcal{U}^{ex} = \sqcup_{a,b} \mathcal{U}_{b,a}$. We let $\mathcal{U}^{ex}$ act on $\widetilde{\mathcal{L}}_d^{ex}$ via ρ.

Proposition 2. *For $a \in \mathbb{Z}$ there is a canonical functor $\mathcal{F}_{U_a(F)} : \mathrm{D}(U_a^* \otimes \Omega(F)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$ exact for the perverse t-structures. For $g \in \mathcal{U}_{b,a}$ and $\tilde{g} = \rho_{b,a}(g)$ the diagram is canonically 2-commutative*

$$\begin{tikzcd}
\mathrm{D}(U_a^* \otimes \Omega(F)) \arrow[r, "\mathcal{F}_{U_a(F)}"] \arrow[d, "g"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F))) \arrow[d, "\tilde{g}"] \\
\mathrm{D}(U_b^* \otimes \Omega(F)) \arrow[r, "\mathcal{F}_{U_b(F)}"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_b(F))),
\end{tikzcd} \qquad (10)$$

Proof

Step 1. Let $R_1 \subset R_2 \subset U_a(F)$ be c-lattices. Write $\langle\cdot,\cdot\rangle_a$ for the symplectic form on the Tate space $W_a(F)$. For a c-lattice $R \subset U_a(F)$ set $R' = \{w \in U_a^* \otimes \Omega(F) \mid \langle w, r\rangle_a = 0 \text{ for all } r \in R\}$, this is a c-lattice in $U_a^* \otimes \Omega(F)$.

Set $R = R_1 \oplus R_2'$ then $R^\perp = R_2 \oplus R_1'$. Let $U_R = R_2/R_1$ then $U_R \in \mathcal{L}(R^\perp/R)$. Set $U_R^0 = (U_R, \mathcal{B})$ equipped with the canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\mathcal{B}^2 \widetilde{\to} \mathcal{J}_a \otimes \det(U_R) \otimes \det(W_a : R),$$

where $\mathcal{B} = \mathcal{J}_{U,a} \otimes \det(U_a : R_1) \otimes \det(\mathcal{O}(-a) : \mathcal{O})^{n/2}$.

Remind the line bundle $\mathcal{A}_R$ on $\mathcal{L}(R^\perp/R)$ with fibre $\mathcal{J}_a \otimes \det L_1 \otimes \det(W_a : R)$ at L_1. Let $\widetilde{\mathcal{L}}(R^\perp/R)$ be the gerb of square roots of $\mathcal{A}_R$. So, $U_R^0 \in \widetilde{\mathcal{L}}(R^\perp/R)$.

Write H_R for the Heisenberg group $R^\perp/R \times \mathbb{A}^1$ and $\mathcal{H}_{U_R}$ for the corresponding category of $(\bar{U}_R, \chi_{U,R})$-equivariant perverse sheaves on H_R. Here $\bar{U}_R = U_R \times \mathbb{A}^1$ and $\chi_{U,R}$ is the local system $\operatorname{pr}^* \mathcal{L}_\psi$ on $\bar{U}_R$, where $\operatorname{pr} : \bar{U}_R \to \mathbb{A}^1$ is the projection.

Let $\mathcal{F}_{U_R^0} : \mathrm{D}\,\mathcal{H}_{U_R} \to \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R))$ be the corresponding functor (defined as in [8], Section 3.6). The lattice $gR \subset W_b(F)$ satisfies the same assumptions, so we have $U_{gR} = gR_2/gR_1 \in \mathcal{L}(gR^\perp/gR)$, and $g(R^\perp) = (gR)^\perp$. Further, $U_{gR}^0 = (U_{gR}, \mathcal{B}_1)$ with

$$\mathcal{B}_1 = \mathcal{J}_{U,b} \otimes \det(U_b : gR_1) \otimes \det(\mathcal{O}(-b) : \mathcal{O})^{n/2}$$

equipped with the canonical isomorphism $\mathcal{B}_1^2 \widetilde{\to} \mathcal{J}_b \otimes \det(U_{gR}) \otimes \det(W_b : gR)$.

We have $\tilde{g} = (g, \mathcal{B}_0)$, where

$$\mathcal{B}_0 = \mathcal{J}_{U,b} \otimes \mathcal{J}_{U,a}^{-1} \otimes \det(U_b : gU_a) \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^{n/2}$$

is equipped with $\mathcal{B}_0^2 \widetilde{\to} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$. It follows that $\tilde{g}(U_R^0) \widetilde{\to} U_{gR}^0$ canonically.

Further, g yields an equivalence $g : \mathrm{D}\,\mathcal{H}_{U_R} \widetilde{\to} \mathrm{D}\,\mathcal{H}_{U_{gR}}$, and the diagram is canonically 2-commutative

$$\begin{tikzcd}
\mathrm{D}\,\mathcal{H}_{U_R} \arrow[r, "\mathcal{F}_{U_R^0}"] \arrow[d, "g"] & \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)) \arrow[d, "\tilde{g}"] \\
\mathrm{D}\,\mathcal{H}_{U_{gR}} \arrow[r, "\mathcal{F}_{U_{gR}^0}"] & \mathrm{D}(\widetilde{\mathcal{L}}(gR^\perp/gR))
\end{tikzcd} \tag{11}$$

Indeed, this is a consequence of the following isomorphism. We have

$$\tilde{g} : \widetilde{\mathcal{L}}(R^\perp/R) \times \widetilde{\mathcal{L}}(R^\perp/R) \times H_R \widetilde{\to} \widetilde{\mathcal{L}}(gR^\perp/gR) \times \widetilde{\mathcal{L}}(gR^\perp/gR) \times H_{gR},$$

and for this isomorphism $\tilde{g}^* F \widetilde{\to} F$ canonically, where F on both sides is the corresponding CIO sheaf (introduced in [8], Theorem 1).

Step 2. Given c-lattices $S_1 \subset R_1 \subset R_2 \subset S_2$ in $U_a(F)$, similarly define $S = S_1 \oplus S_2'$ and $U_S^0 \in \widetilde{\mathcal{L}}(S^\perp/S)$ for $S \subset S^\perp \subset W_a(F)$. We have a canonical transition functor $T_{S,R}^U : \mathrm{D}\,\mathcal{H}_{U_R} \to \mathrm{D}\,\mathcal{H}_{U_S}$

defined as in ([8], Section 6.6). Let $j : \mathcal{L}(S^\perp/S)_R \subset \mathcal{L}(S^\perp/S)$ be the open subscheme of L satisfying $L \cap (R/S) = 0$. We have a projection

$$p_{R/S} : \widetilde{\mathcal{L}}(S^\perp/S)_R \to \widetilde{\mathcal{L}}(R^\perp/R)$$

sending $(L, \mathcal{B}_S)$ to $(L_R, \mathcal{B}_S)$, where $L_R := L \cap R^\perp$. It is understood that $\mathcal{B}_S$ is equipped with

$$\mathcal{B}_S^2 \widetilde{\to} \mathcal{J}_a \otimes \det L \otimes \det(W_a : S),$$

and we used the canonical isomorphism $\det L \otimes \det(W_a : S) \widetilde{\to} \det L_R \otimes \det(W_a : R)$.

Set $P_{R/S} = p_{R/S}^*[\dim.\mathrm{rel}(p_{R/S})]$. Then the following diagram is canonically 2-commutative

$$\begin{tikzcd}
\mathrm{D}\,\mathcal{H}_{U_R} \arrow[r, "\mathcal{F}_{U^0_R}"] \arrow[d, "T^U_{S,R}"] & \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)) \arrow[r, "P_{R/S}"] & \mathrm{D}(\widetilde{\mathcal{L}}(S^\perp/S)_R) \\
\mathrm{D}\,\mathcal{H}_{U_S} \arrow[r, "\mathcal{F}_{U^0_S}"] & \mathrm{D}(\widetilde{\mathcal{L}}(S^\perp/S)) \arrow[ur, "j^*"] &
\end{tikzcd}$$

Define ${}_R\mathcal{F}_{U_a(F)}$ as the composition

$$\mathrm{D}\,\mathcal{H}_{U_R} \overset{\mathcal{F}_{U^0_R}}{\to} \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F))_R),$$

where the second arrow is the restriction (exact for the perverse t-structures) with respect to the projection $\widetilde{\mathcal{L}}_d(W_a(F))_R \to \widetilde{\mathcal{L}}(R^\perp/R)$.

The above diagram shows that the following diagram is also 2-commutative

$$\begin{tikzcd}
\mathrm{D}\,\mathcal{H}_{U_R} \arrow[r, "{}_R\mathcal{F}_{U_a(F)}"] \arrow[d, "T^U_{S,R}"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F))_R) \\
\mathrm{D}\,\mathcal{H}_{U_S} \arrow[r, "{}_S\mathcal{F}_{U_a(F)}"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F))_S), \arrow[u, "j^*_{S,R}"]
\end{tikzcd}$$

where $j_{S,R} : \widetilde{\mathcal{L}}_d(W_a(F))_R \subset \widetilde{\mathcal{L}}_d(W_a(F))_S$ is the natural open immersion.

So, define

$$\mathcal{F}_{U_a(F),R} : \mathrm{D}\,\mathcal{H}_{U_R} \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$$

as the functor sending K_1 to the following object K_2. For c-lattices $S_1 \subset R_1 \subset R_2 \subset S_2$ as above and $S = S_1 \oplus S_2'$ declare the restriction of K_2 to $\widetilde{\mathcal{L}}_d(W_a(F))_S$ to be

$$({}_S\mathcal{F}_{U_a(F)} \circ T^U_{S,R})(K_1)$$

The corresponding projective system (indexed by such S) defines an object $K_2 \in \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$.

Further, passing to the limit by R (of the above form) the functors $\mathcal{F}_{U_a(F),R}$ yield the desired functor $\mathcal{F}_{U_a(F)} : \mathrm{D}(U_a^* \otimes \Omega(F)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$.

Finally, the commutativity of (10) follows from the commutativity of (11). $\square$

Remark 4. We could also argue differently in Proposition 2. For each $a \in \mathbb{Z}$ and $L^0 \in \widetilde{\mathcal{L}}_d(W_a(F))$ we could first construct an equivalence $\mathcal{F}_{U_a(F),L^0} : \mathrm{D}(U_a^* \otimes \Omega(F)) \xrightarrow{\sim} \mathrm{D}\,\mathcal{H}_L$ as in ([8], Proposition 5) such that for any $g \in \mathcal{U}_{b,a}$ the diagram is 2-commutative

$$\begin{tikzcd}
\mathrm{D}(U_a^* \otimes \Omega(F)) \arrow[r, "\mathcal{F}_{U_a(F),L^0}"] \arrow[d, "g"] & \mathrm{D}\,\mathcal{H}_L \arrow[d, "g"] \\
\mathrm{D}(U_b^* \otimes \Omega(F)) \arrow[r, "\mathcal{F}_{U_b(F),\tilde{g}(L^0)}"] & \mathrm{D}\,\mathcal{H}_{gL}
\end{tikzcd}$$

with $\tilde{g} = \rho_{b,a}(g)$. Here $\tilde{g}(L^0) \in \widetilde{\mathcal{L}}_d(W_b(F))$. Then we could define $\mathcal{F}_{U_a(F)}$ as the composition

$$\mathrm{D}(U_a^* \otimes \Omega(F)) \xrightarrow{\mathcal{F}_{U_a(F),L^0}} \mathrm{D}\,\mathcal{H}_L \xrightarrow{\mathcal{F}_{L^0}} \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$$

The resulting functor would be (up to a canonical isomorphism) independent of $L^0 \in \widetilde{\mathcal{L}}_d(W_a(F))$.

4. DUAL PAIR $\mathrm{G}\mathbb{S}\mathrm{p}_{2n}, \mathrm{G}\mathbb{O}_{2m}$

4.1 As in Section 3.2, let $\mathcal{O}$ be a complete discrete valuation k-algebra, F its fraction field, Ω the completed module of relative differentials of $\mathcal{O}$ over k. For a free $\mathcal{O}$-module M we write $M_x = M \otimes_{\mathcal{O}} k$ for its geometric fibre.

Fix free $\mathcal{O}$-modules M_a of rank $2n$, V_a of rank $2m$, and A_a, C_a of rank one with symplectic form $\wedge^2 M_a \to A_a$, a nondegenerate symmetric form $\mathrm{Sym}^2 V_a \to C_a$, and a compatible trivialization $\det V_a \xrightarrow{\sim} C_a^m$. Assume also given an isomorphism $A_a \otimes C_a \xrightarrow{\sim} \Omega(a)$ for each $a \in \mathbb{Z}$.

Set $W_a = M_a \otimes V_a$, it is equipped with the symplectic form $\wedge^2 W_a \to \Omega(a)$. For $a \in \mathbb{Z}$ set $\mathcal{J}_a = C_{a,x}^{-anm}$, which is of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded. Define $\widetilde{\mathcal{L}}_d(W_a(F))$, $\mathcal{G}_{b,a}$, G_a and $\widetilde{\mathcal{G}}_{b,a}$ as in Section 3.2.

Let $\mathbb{G} = \mathrm{G}\mathbb{S}\mathrm{p}_{2n}$ be the symplectic similitude group over k of semisimple rank n. Let $\mathbb{H}$ be the connected component of unity of the split orthogonal similitude group $\mathrm{G}\mathbb{O}_{2m}$ of semisimple rank m. We may view (M_a, A_a) (resp., (V_a, C_a)) as a $\mathbb{G}$-torsor (resp., $\mathbb{H}$-torsor) on $\mathrm{Spec}\,\mathcal{O}$.

Let $\mathbb{G}_{b,a}$ be the set of isomorphisms $M_a(F) \to M_b(F)$ of $\mathbb{G}$-torsors over $\mathrm{Spec}\,F$. Let $\mathbb{H}_{b,a}$ be the set of isomorphisms $V_a(F) \to V_b(F)$ of $\mathbb{H}$-torsors over $\mathrm{Spec}\,F$. Let $\mathcal{T}_{b,a}$ be the set of pairs $g = (g_1, g_2)$, where $g_1 \in \mathbb{G}_{b,a}$, $g_2 \in \mathbb{H}_{b,a}$ such that $g \in \mathcal{G}_{b,a}$. That is, the composition

$$\Omega(F) \xrightarrow{\sim} A_a \otimes C_a(F) \xrightarrow{g_1 \otimes g_2} A_b \otimes C_b(F) \xrightarrow{\sim} \Omega(F)$$

must equal to the identity. The natural composition map $\mathcal{T}_{c,b} \times \mathcal{T}_{b,a} \to \mathcal{T}_{c,a}$ makes $\mathcal{T} = \sqcup_{a,b} \mathcal{T}_{b,a}$ into a groupoïd.

Lemma 1. *Let M_i, V be a free $\mathcal{O}_x$-modules of finite rank, where $M_2 \subset M_1(F_x)$ is a $\mathcal{O}_x$-lattice. Set $\dim(M_1 : M_2) = \dim(M_1/R) - \dim(M_2/R)$ for any $\mathcal{O}_x$-lattice $R \subset M_1 \cap M_2$. Then we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism*[2]

$$\det(M_1 \otimes V : M_2 \otimes V) \xrightarrow{\sim} \det(M_1 : M_2)^{\mathrm{rk}\,V} \otimes (\det V_x)^{\dim(M_1 : M_2)}[\dim(M_1 : M_2)\,\mathrm{rk}\,V]$$

---

[2] there may be sign problems, the corresponding isomorphism is well defined at least up to a sign

Proof Pick a $\mathcal{O}_x$-lattice $R \subset M_1 \cap M_2$. It suffices to establish a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det(M_1 \otimes V : R \otimes V) \stackrel{\sim}{\to} \det(M_1/R)^{\operatorname{rk} V} \otimes (\det V_x)^{\dim(M_1/R)}[\dim(M_1/R) \operatorname{rk} V]$$

To do so, it suffices to pick a flag $R = R_0 \subset R_1 \subset \ldots \subset R_s = M_1$ of $\mathcal{O}_x$-lattices with $\dim(R_i/R_{i-1}) = 1$. $\square$

For $e \in \mathbb{Z}$ set $\mathbb{G}^e_{b,a} = \{g \in \mathbb{G}_{b,a} \mid gA_a = A_b(e)\}$ and $\mathbb{H}^e_{b,a} = \{g \in \mathbb{H}_{b,a} \mid gC_a = C_b(e)\}$.

Let us construct a canonical section $\nu_{b,a} : \mathcal{T}_{b,a} \to \widetilde{\mathcal{G}}_{b,a}$ compatible with the groupoïds structures. Let $g = (g_1, g_2) \in \mathcal{T}_{b,a}$ with $g_1 \in \mathbb{G}^e_{b,a}$, $g_2 \in \mathbb{H}^c_{b,a}$, so $e + c = a - b$. Using Lemma 1 we get a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\begin{aligned}
&\det(M_b \otimes V_b : (g_1 M_a) \otimes (g_2 V_a)) \stackrel{\sim}{\to} \\
&\quad \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes (\det V_b)_x^{\dim(M_b : g_1 M_a)} \otimes (\det M_a)_x^{\dim(V_b : g_2 V_a)} \stackrel{\sim}{\to} \\
&\qquad \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes C_{b,x}^{-mne} \otimes A_{a,x}^{-mnc} \stackrel{\sim}{\to} \\
&\qquad\quad \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes C_{b,x}^{-mne} \otimes \mathcal{C}_{a,x}^{mnc} \otimes \mathcal{O}((1-a)c)_x^{mn}
\end{aligned}$$

We used that $\dim(M_b : g_1 M_a) = -ne$, $\dim(V_b : g_2 V_a) = -mc$. Identifying further $C_a \overset{g_2}{\to} C_b(c)$, we get

$$\mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a) \stackrel{\sim}{\to} \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes C_{b,x}^{2cnm} \otimes \mathcal{O}(c(1+c))_x^{nm}$$

Let $\nu_{b,a}(g) = (g, \mathcal{B})$, where

$$\mathcal{B} = \det(M_b : g_1 M_a)^m \otimes \det(V_b : g_2 V_a)^n \otimes C_{b,x}^{cnm} \otimes \mathcal{O}(c(1+c)/2)_x^{nm}$$

is equipped with the induced isomorphism $\mathcal{B}^2 \stackrel{\sim}{\to} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$.

We let $\mathcal{T}$ act on $\widetilde{\mathcal{L}}_d^{ex}$ via ν.

4.2 Let $\mathbb{G}_a = \mathrm{G}\mathbb{S}\mathrm{p}(M_a)$ and $\mathbb{H}_a = \mathrm{G}\mathbb{O}^0(V_a)$, the connected component of unity of the group scheme $\mathrm{G}\mathbb{O}(V_a)$ over $\operatorname{Spec} \mathcal{O}$. Set

$$\mathcal{T}_a = \{(g_1, g_2) \in (\mathbb{G}_a \times \mathbb{H}_a)(\mathcal{O}) \mid g_1 \otimes g_2 \text{ acts trivially on } A_a \otimes C_a\}$$

The line bundle on $\mathcal{L}_d(W_a(F))$ with fibre $\mathcal{J}_a \otimes \det(W_a : L)$ at L is naturally $\mathcal{T}_a$-equivariant (we underline that $\mathcal{T}_a$ acts nontrivially on $\mathcal{J}_a$). So, it can be seen as a line bundle on the quotient stack ${}^a\mathcal{XL} := \mathcal{L}_d(W_a(F))/\mathcal{T}_a$. We also have the corresponding μ_2-gerb

$${}^a\widetilde{\mathcal{XL}} := \widetilde{\mathcal{L}}_d(W_a(F))/\mathcal{T}_a$$

of square roots of this line bundle. The derived category $\mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$ is defined as in ([7], Section 8.2.2).

The stack ${}^a\mathcal{XL}$ classifies: a $\mathbb{G}$-torsor (M, A) over $\operatorname{Spec}\mathcal{O}$, a $\mathbb{H}$-torsor (V, C) over $\operatorname{Spec}\mathcal{O}$ (so, we have a compatible isomorphism $\det V \xrightarrow{\sim} C^m$), an isomorphism $A \otimes C \xrightarrow{\sim} \Omega(a)$, and a discrete lagrangian subspace $L \subset M \otimes V(F)$.

Let ${}^a\mathcal{A}_{\mathcal{XL}}$ be the line bundle over ${}^a\mathcal{XL}$ whose fibre at (M, A, V, C, L) is $C_x^{-anm} \otimes \det(M \otimes V : L)$. It is of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded. Then ${}^a\widetilde{\mathcal{XL}}$ is the μ_2-gerb of square roots of ${}^a\mathcal{A}_{\mathcal{XL}}$.

4.3.1 HECKE OPERATORS Denote by ${}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{XL}}$ the stack classifying: a point $(L, M, A, V, C) \in {}^a\mathcal{XL}$, a lattice $M' \subset M(F)$ such that for $A' = A(a' - a)$ the induced form is regular and nondegenerate $\wedge^2 M' \to A'$. We get a diagram

$$
{}^a\mathcal{XL} \overset{h^{\leftarrow}}{\leftarrow} {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{XL}} \overset{h^{\rightarrow}}{\rightarrow} {}^{a'}\mathcal{XL}, \tag{12}
$$

where $h^{\leftarrow}$ (resp., $h^{\rightarrow}$) sends a point of ${}^{a,a'}\mathcal{H}_{\mathcal{XL}}$ to (L, M, A, V, C) (resp., to (L, M', A', V, C)).

Lemma 2. *For a point (L, M, A, M', A', V, C) of ${}^{a,a'}\mathcal{H}_{\mathcal{XL}}$ there is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism*

$$
C_x^{-a'nm} \otimes \det(M' \otimes V : L) \xrightarrow{\sim} C_x^{-anm} \otimes \det(M \otimes V : L) \otimes \det(M' : M)^{2m} \qquad \square
$$

Let ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}} \overset{\tilde{h}^{\rightarrow}}{\rightarrow} {}^{a'}\widetilde{\mathcal{XL}}$ be map obtained from $h^{\rightarrow}$ by the base change ${}^{a'}\widetilde{\mathcal{XL}} \to {}^{a'}\mathcal{XL}$. By Lemma 2, we get a diagram

$$
{}^a\widetilde{\mathcal{XL}} \overset{\tilde{h}^{\leftarrow}}{\leftarrow} {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}} \overset{\tilde{h}^{\rightarrow}}{\rightarrow} {}^{a'}\widetilde{\mathcal{XL}} \tag{13}
$$

Here a point of ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}}$ is given by a collection $(L, M, A, M', A', V, C) \in {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{XL}}$ together with a one-dimensional space $\mathcal{B}$ equipped with $\mathcal{B}^2 \xrightarrow{\sim} C_x^{-a'nm} \otimes \det(M' \otimes V : L)$. The map $\tilde{h}^{\leftarrow}$ sends this point to $(L, M, A, V, C) \in {}^a\mathcal{XL}$ together with the one-dimensional space $\mathcal{B}_1 = \mathcal{B} \otimes \det(M : M')^m$ with the induced isomorphism $\mathcal{B}_1^2 \xrightarrow{\sim} C_x^{-anm} \otimes \det(M \otimes V : L)$.

The affine grassmanian $\operatorname{Gr}_{\mathbb{G}_a} = \mathbb{G}_a(F)/\mathbb{G}_a(\mathcal{O})$ is the ind-scheme classifying $\mathcal{O}$-lattices $R \subset M_a(F)$ such that for some $r \in \mathbb{Z}$ the induced form $\wedge^2 R \to A_a(r)$ is regular and nondegenerate. Write $\operatorname{Gr}^r_{\mathbb{G}_a}$ for the connected component of $\operatorname{Gr}_{\mathbb{G}_a}$ given by fixing such r.

Trivializing a point of ${}^{a'}\mathcal{XL}$ (resp., of ${}^a\mathcal{XL}$) one gets isomorphisms

$$
\mathrm{id}^r : {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{XL}} \xrightarrow{\sim} (\mathcal{L}_d(W_{a'}(F)) \times \operatorname{Gr}^{a-a'}_{\mathbb{G}_{a'}})/\mathcal{T}_{a'}
$$

and

$$
\mathrm{id}^l : {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{XL}} \xrightarrow{\sim} (\mathcal{L}_d(W_a(F)) \times \operatorname{Gr}^{a'-a}_{\mathbb{G}_a})/\mathcal{T}_a,
$$

where the corresponding action of $\mathcal{T}_{a'}$ (resp., of $\mathcal{T}_a$) is diagonal. They lift naturally to a $\mathcal{T}_{a'}$-torsor

$$
\widetilde{\mathcal{L}}_d(W_{a'}(F)) \times \operatorname{Gr}^{a-a'}_{\mathbb{G}_{a'}} \to {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}}
$$

and a $\mathcal{T}_a$-torsor

$$
\widetilde{\mathcal{L}}_d(W_a(F)) \times \operatorname{Gr}^{a'-a}_{\mathbb{G}_a} \to {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}}
$$

So, for $K \in \mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$, $K' \in \mathrm{D}_{\mathcal{T}_{a'}}(\widetilde{\mathcal{L}}_d(W_{a'}(F)))$, $\mathcal{S} \in \mathrm{Sph}_{\mathbb{G}_a}$, $\mathcal{S}' \in \mathrm{Sph}_{\mathbb{G}_{a'}}$ we can form their external products

$$(K \tilde{\boxtimes} \mathcal{S})^l, (K' \tilde{\boxtimes} \mathcal{S}')^r$$

on ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}}$. The Hecke functor

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}} : \mathrm{Sph}_{\mathbb{G}_{a'}} \times \mathrm{D}_{\mathcal{T}_{a'}}(\widetilde{\mathcal{L}}_d(W_{a'}(F))) \to \mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$$

is defined by

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}', K') = (\tilde{h}^{\leftarrow})_!(K' \tilde{\boxtimes} * \mathcal{S}')^r$$

It is understood that this informal definition should be made rigorous in a way similar to ([7], Section 4.3).

Write ${}_b\mathrm{Sph}_{\mathbb{G}_{a'}} \subset \mathrm{Sph}_{\mathbb{G}_{a'}}$ for the full subcategory of objects that vanish off $\mathrm{Gr}^b_{\mathbb{G}_{a'}}$. The first agrument of $\mathrm{H}^{\leftarrow}_{\mathbb{G}}$ actually lies in ${}_{a'-a}\mathrm{Sph}_{\mathbb{G}_{a'}}$.

4.3.2 Let ${}^{a,a'}\mathcal{H}_{\mathbb{H},\mathcal{XL}}$ be the stack classifying: a point $(L, M, A, V, C) \in {}^a\mathcal{XL}$, a lattice $V' \subset V(F)$ such that for $C' = C(a'-a)$ the induced form $\mathrm{Sym}^2 V' \to C'$ is regular and nondegenerate (we also get the isomorphism $C'^{-m} \otimes \det V' \widetilde{\to} C^{-m} \otimes \det V \widetilde{\to} \mathcal{O}$). As for $\mathbb{G}$, we get a diagram

$$\begin{tikzcd}
{}^a\widetilde{\mathcal{XL}} \arrow[d] & {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{H},\mathcal{XL}} \arrow[l, "\tilde{h}^{\leftarrow}"'] \arrow[r, "\tilde{h}^{\rightarrow}"] \arrow[d] & {}^{a'}\widetilde{\mathcal{XL}} \arrow[d] \\
{}^a\mathcal{XL} & {}^{a,a'}\mathcal{H}_{\mathbb{H},\mathcal{XL}} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] & {}^{a'}\mathcal{XL},
\end{tikzcd} \tag{14}$$

where $h^{\leftarrow}$ (resp. $h^{\rightarrow}$) sends (L, M, A, V, C, V', C') to (L, M, A, V, C) (resp., to (L, M, A, V', C')), the vertical arrows are μ_2-gerbs, and the right square is cartesian (thus defining the stack ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{H},\mathcal{XL}}$).

A point of ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{H},\mathcal{XL}}$ is given by $(L, M, A, V, C, V', C') \in {}^{a,a'}\mathcal{H}_{\mathbb{H},\mathcal{XL}}$ and a one-dimensional space $\mathcal{B}$ equipped with

$$\mathcal{B}^2 \widetilde{\to} (C'_x)^{-a'nm} \otimes \det(M \otimes V' : L)$$

The map $\tilde{h}^{\leftarrow}$ sends this point to $(L, M, A, V, C) \in {}^a\mathcal{XL}$, the one-dimensional space $\mathcal{B}_1$ with $\mathcal{B}_1^2 \widetilde{\to} C_x^{-anm} \otimes \det(M \otimes V : L)$, where

$$\mathcal{B}_1 = \mathcal{B} \otimes C_x^{nm(a'-a)} \otimes \det(V : V')^n \otimes \mathcal{O}(\frac{1}{2}nm(a-a')(a-a'-1))_x$$

The affine grassmanian $\mathrm{Gr}_{\mathbb{H}_a}$ classifies lattices $V' \subset V_a(F)$ such that the induced symmetric form $\mathrm{Sym}^2 V' \to C_a(b)$ is regular and nondegenerate for some $b \in \mathbb{Z}$. Write $\mathrm{Gr}^b_{\mathbb{H}_a}$ for the locus of $\mathrm{Gr}_{\mathbb{H}_a}$ given by fixing this b. For $m \geq 2$ there is an exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathbb{H}_a) \to \mathbb{Z} \to 0$, so if $m \geq 2$ then $\mathrm{Gr}^b_{\mathbb{H}_a}$ is a union of two connected components of $\mathrm{Gr}_{\mathbb{H}_a}$. Write ${}_b\mathrm{Sph}_{\mathbb{H}_a} \subset \mathrm{Sph}_{\mathbb{H}_a}$ for the full subcategory of objects that vanish off $\mathrm{Gr}^b_{\mathbb{H}_a}$.

The Hecke functor

$$\mathrm{H}^{\leftarrow}_{\mathbb{H}} : \mathrm{Sph}_{\mathbb{H}_{a'}} \times \mathrm{D}_{\mathcal{T}_{a'}}(\widetilde{\mathcal{L}}_d(W_{a'}(F))) \to \mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$$

is defined as in Section 4.3.1 using the diagram (14).

For each $a \in \mathbb{Z}$ a trivialization α of the $\mathbb{G}$-torsor (M_a, A_a) on $\operatorname{Spec} \mathcal{O}$ yields an isomorphism $\bar{\alpha} : \operatorname{Gr}_{\mathbb{G}_a} \widetilde{\to} \operatorname{Gr}_{\mathbb{G}}$. The induced equivalences $\bar{\alpha}^* : \operatorname{Sph}_{\mathbb{G}} \widetilde{\to} \operatorname{Sph}_{\mathbb{G}_a}$ are canonically 2-isomorphic for different α's. In what follows we sometimes identify these two categories in this way. Similarly, we identify $\operatorname{Sph}_{\mathbb{H}_a} \widetilde{\to} \operatorname{Sph}_{\mathbb{H}}$.

4.4 Let $S_{W_0(F)} \in \mathrm{P}_{\mathcal{T}_0}(\widetilde{\mathcal{L}}_d(W_0(F)))$ be the theta-sheaf introduced in ([8], Section 6.5). This is a $\mathcal{T}_0$-equivariant object of the Weil category $W(\widetilde{\mathcal{L}}_d(W_0(F)))$. Here is the main result of Section 4.

Theorem 3. *1) Assume $m \le n$. There is a homomorphism $\kappa : \check{\mathbb{H}} \times \mathbb{G}_m \to \check{\mathbb{G}}$ such that for the corresponding geometric restriction functor* $\mathrm{gRes}^{\kappa} : \operatorname{Sph}_{\mathbb{G}} \to \mathrm{D}\operatorname{Sph}_{\mathbb{H}}$ *we have an isomorphism in* $\mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, S_{W_0(F)}) \widetilde{\to} \mathrm{H}^{\leftarrow}_{\mathbb{H}}(* \,\mathrm{gRes}^{\kappa}(\mathcal{S}), S_{W_0(F)})$$

functorial in $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{G}}$.
2) Assume $m > n$. There is a homomorphism $\kappa : \check{\mathbb{G}} \times \mathbb{G}_m \to \check{\mathbb{H}}$ such that for the corresponding geometric restriction functor $\mathrm{gRes}^{\kappa} : \operatorname{Sph}_{\mathbb{H}} \to \mathrm{D}\operatorname{Sph}_{\mathbb{G}}$ *we have an isomorphism in* $\mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_a(W_a(F)))$

$$\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, S_{W_0(F)}) \widetilde{\to} \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathrm{gRes}^{\kappa}(*\mathcal{S}), S_{W_0(F)})$$

functorial in $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{H}}$.

The proof occupies the rest of Section 4. The explisit formulas for κ are found in Section 4.8.9.

4.5 Assume given a decomposition $M_a = L_a \oplus (L_a^* \otimes A_a)$, where L_a is a free $\mathcal{O}$-module of rank n, L_a and $L_a^* \otimes A_a$ are lagrangians, and the form is given by the canonical pairing between L_a and L_a^*. Assume given a similar decomposition $V_a = U_a \oplus (U_a^* \otimes C_a)$ for V_a, here U_a is a free $\mathcal{O}$-module of rank m.

Write $Q(\mathbb{G}_a) \subset \mathbb{G}_a$ and $Q(\mathbb{H}_a)$ for the Levi subgroups preserving the above decompositions. Set

$$Q\mathbb{G}\mathbb{H}_a = \{g = (g_1, g_2) \in Q(\mathbb{G}_a) \times Q(\mathbb{H}_a) \mid g \in \mathcal{T}_a\}$$

$$\mathbb{G}Q\mathbb{H}_a = \{g = (g_1, g_2) \in \mathbb{G}_a \times Q(\mathbb{H}_a) \mid g \in \mathcal{T}_a\}$$

$$\mathbb{H}Q\mathbb{G}_a = \{g = (g_1, g_2) \in \mathbb{H}_a \times Q(\mathbb{G}_a) \mid g \in \mathcal{T}_a\}$$

We view all of them as group schemes over $\operatorname{Spec} \mathcal{O}$. We also pick Levi subgroups $Q(\mathbb{G}) \subset \mathbb{G}$ and $Q(\mathbb{H}) \subset \mathbb{H}$ which identify with the above over $\operatorname{Spec} \mathcal{O}$.

The affine grassmanian $\operatorname{Gr}_{Q(\mathbb{G}_a)}$ classifies pairs of lattices $L' \subset L_a(F)$, $A' \subset A_a(F)$. For $b \in \mathbb{Z}$ write $\operatorname{Gr}^b_{Q(\mathbb{G}_a)}$ for the locus of $\operatorname{Gr}_{Q(\mathbb{G}_a)}$ given by $A' = A_a(b)$. Write ${}_b\operatorname{Sph}_{Q(\mathbb{G}_a)} \subset \operatorname{Sph}_{Q(\mathbb{G}_a)}$ for the full subcategory of objects that vanish off $\operatorname{Gr}^b_{Q(\mathbb{G}_a)}$. As in Section 4.4, we identify canonically $\operatorname{Sph}_{Q(\mathbb{G})} \widetilde{\to} \operatorname{Sph}_{Q(\mathbb{G}_a)}$. The geometric restriction $\mathrm{gRes} : \operatorname{Sph}_{\mathbb{G}} \to \operatorname{Sph}_{Q(\mathbb{G})}$ corresponding to the inlcusion of the Langlands dual groups $\check{Q}(\mathbb{G}) \hookrightarrow \check{\mathbb{G}}$ yields a faithful functor ${}_b\operatorname{Sph}_{\mathbb{G}} \to {}_b\operatorname{Sph}_{Q(\mathbb{G})}$ for each b. And similarly for $\mathbb{H}$.

For $b, a \in \mathbb{Z}$ write $Q(\mathbb{G}_{b,a})$ for the set of isomorphisms $(L_a(F) \to L_b(F),\ A_a(F) \to A_b(F))$ of $\mathrm{GL}_n \times \mathbb{G}_m$-torsors over $\operatorname{Spec} F$. Let $Q(\mathbb{H}_{b,a})$ be the set of isomorphisms $(U_a(F) \to U_b(F),\ C_a(F) \to C_b(F))$ of $\mathrm{GL}_m \times \mathbb{G}_m$-torsors over $\operatorname{Spec} F$. Set

$$Q\mathbb{G}\mathbb{H}_{b,a} = \{g = (g_1, g_2) \in Q(\mathbb{G}_{b,a}) \times Q(\mathbb{H}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

$$\mathbb{G}Q\mathbb{H}_{b,a} = \{g = (g_1, g_2) \in \mathbb{G}_{b,a} \times Q(\mathbb{H}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

$$\mathbb{H}Q\mathbb{G}_{b,a} = \{g = (g_1, g_2) \in \mathbb{H}_{b,a} \times Q(\mathbb{G}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

Set $\Upsilon_a = L_a^* \otimes A_a \otimes V_a$ and $\Pi_a = U_a^* \otimes C_a \otimes M_a$. For $a \in \mathbb{Z}$ and any $L^0 \in \widetilde{\mathcal{L}}_d(W_a(F))$ we have the equivalences

$$\mathcal{F}_{L_a \otimes V_a(F), L^0} : \mathrm{D}(\Upsilon_a(F)) \widetilde{\to} \mathrm{D}\,\mathcal{H}_L$$

and $\mathcal{F}_{U_a \otimes M_a(F), L^0} : \mathrm{D}(\Pi_a(F)) \widetilde{\to} \mathrm{D}\,\mathcal{H}_L$ defined as in Remark 4.

Remind that for free $\mathcal{O}$-modules of finite type $\mathcal{V}, \mathcal{U}$ one has the partial Fourier transform

$$\mathrm{Four}_\psi : \mathrm{D}(\mathcal{V}(F) \oplus \mathcal{U}(F)) \widetilde{\to} \mathrm{D}(\mathcal{V}^* \otimes \Omega(F) \oplus \mathcal{U}(F))$$

normalized to preserve perversity and purity (cf. [7], Section 4.8 for the definition). Thus, the decompositions

$$\Pi_a \widetilde{\to} U_a^* \otimes C_a \otimes L_a \ \oplus\ U_a^* \otimes L_a^* \otimes \Omega(a)$$

and

$$\Upsilon_a \widetilde{\to} L_a^* \otimes A_a \otimes U_a \ \oplus\ U_a^* \otimes L_a^* \otimes \Omega(a)$$

yield the partial Fourier transform, which we denote

$$\zeta_a : \mathrm{D}(\Upsilon_a(F)) \widetilde{\to} \mathrm{D}(\Pi_a(F))$$

One checks that ζ_a is canonically isomorphic to the functor $\mathcal{F}^{-1}_{U_a \otimes M_a(F), L^0} \circ \mathcal{F}_{L_a \otimes V_a(F), L^0}$ for any $L^0 \in \widetilde{\mathcal{L}}_d(W_a(F))$.

4.6.1 It is convenient to denote $\bar{\Upsilon}_a = L_a \otimes V_a$ and $\bar{\Pi}_a = U_a \otimes M_a$. For the decomposition $W_a = \bar{\Pi}_a \oplus \bar{\Pi}_a^* \otimes \Omega(a)$ we define a $\mathbb{Z}/2\mathbb{Z}$-graded line (purely of parity zero)

$$\mathcal{J}_{\bar{\Pi},a} = \mathcal{O}((1-a)a/2)_x^{nm} \otimes (\det U_{a,x})^{-na}$$

equipped with a natural $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\mathcal{J}^2_{\bar{\Pi},a} \widetilde{\to} \mathcal{J}_a \otimes (\det \bar{\Pi}_a)_x^{-a}$$

It yields a section ${}_{\bar{\Pi}}\rho_{b,a} : \mathbb{G}Q\mathbb{H}_{b,a} \to \widetilde{\mathcal{G}}_{b,a}$ defined as in Section 3.3.

For the decomposition

$$W_a = \bar{\Upsilon}_a \oplus \bar{\Upsilon}_a^* \otimes \Omega(a)$$

define a $\mathbb{Z}/2\mathbb{Z}$-graded line (purely of parity zero)

$$\mathcal{J}_{\bar{\Upsilon},a} = C_{a,x}^{-mna} \otimes (\det L_{a,x})^{-ma}$$

equipped with a natural $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\mathcal{J}^2_{\bar{\Upsilon},a} \,\widetilde{\to}\, \mathcal{J}_a \otimes (\det \bar{\Upsilon}_{a,x})^{-a}$$

It yields a section ${}_{\bar{\Upsilon}}\rho_{b,a} : \mathbb{H}Q\mathbb{G}_{b,a} \to \widetilde{\mathcal{G}}_{b,a}$ defined as in Section 3.3.

From definitions one derives the following.

Lemma 3. *For $a, b \in \mathbb{Z}$ the following diagrams are canonically 2-commutative*

$$\begin{tikzcd}
\mathcal{T}_{b,a} \arrow[r, "\nu_{b,a}"] & \widetilde{\mathcal{G}}_{b,a} \\
\mathbb{G}Q\mathbb{H}_{b,a} \arrow[u] \arrow[ur, "{}_{\bar{\Pi}}\rho_{b,a}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}_{b,a} \arrow[r, "\nu_{b,a}"] & \widetilde{\mathcal{G}}_{b,a} \\
\mathbb{H}Q\mathbb{G}_{b,a} \arrow[u] \arrow[ur, "{}_{\bar{\Upsilon}}\rho_{b,a}"'] &
\end{tikzcd}$$

□

For $a \in \mathbb{Z}$ we have the functors $\mathcal{F}_{\bar{\Upsilon}_a(F)} : \mathrm{D}(\Upsilon_a(F)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$ and $\mathcal{F}_{\bar{\Pi}_a(F)} : \mathrm{D}(\Pi_a(F)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F)))$ defined in Proposition 2. Note that the diagram is canonically 2-commutative

$$\begin{tikzcd}
\mathrm{D}(\Upsilon_a(F)) \arrow[r, "\mathcal{F}_{\bar{\Upsilon}_a(F)}"] \arrow[d, "\zeta_a"] & \mathrm{D}(\widetilde{\mathcal{L}}_d(W_a(F))) \\
\mathrm{D}(\Pi_a(F)) \arrow[ur, "\mathcal{F}_{\bar{\Pi}_a(F)}"'] &
\end{tikzcd}$$

Remark 5. The following structure emerge. For each $g \in \mathcal{T}_{b,a}$ we get functors that fit into a 2-commutative diagram

$$\begin{tikzcd}
\mathrm{D}(\Pi_a(F)) \arrow[r, "g"] & \mathrm{D}(\Pi_b(F)) \\
\mathrm{D}(\Upsilon_a(F)) \arrow[u, "\zeta_a"] \arrow[r, "g"] & \mathrm{D}(\Upsilon_b(F)) \arrow[u, "\zeta_b"]
\end{tikzcd}$$

They are compatible with the groupoid structure on $\mathcal{T}$. Indeed, one first defines these functors separately for $\mathbb{G}Q\mathbb{H}_{b,a} \subset \mathcal{T}_{b,a}$ and for $\mathbb{H}Q\mathbb{G}_{b,a} \subset \mathcal{T}_{b,a}$ using the models Π and Υ respectively. This is sufficient because any $g \in \mathcal{T}_{b,a}$ writes as a composition $g = g'' \circ g'$ with $g'' \in \mathbb{H}Q\mathbb{G}_{b,b}$ and $g' \in \mathbb{G}Q\mathbb{H}_{b,a}$. The arrows in the above diagram are equivalences.

4.7 We have the full subcategories (stable under subquotients)

$$\mathrm{P}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \subset \mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F)) \subset \mathrm{P}(\Upsilon_a(F))$$

$$\mathrm{P}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \subset \mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \subset \mathrm{P}(\Pi_a(F)),$$

and ζ_a yields an equivalence $\zeta_a : \mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F)) \,\widetilde{\to}\, \mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$.

Definition 2. For $a \in \mathbb{Z}$ let $\mathcal{W}\mathrm{eil}_a$ be the category of triples $(\mathcal{F}_1, \mathcal{F}_2, \beta)$, where

$$\mathcal{F}_1 \in \mathrm{P}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)), \quad \mathcal{F}_2 \in \mathrm{P}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)),$$

and $\beta : \zeta_a(\mathcal{F}_1) \,\widetilde{\to}\, \mathcal{F}_2$ is an isomorphism in $\mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$. Write $\mathrm{D}\mathcal{W}\mathrm{eil}_a$ for the category obtained by replacing everywhere in the above definition P by DP.

Clearly, $\mathcal{W}\mathrm{eil}_a$ is an abelian category, and the forgetful functors $f_{\mathbb{H}} : \mathcal{W}\mathrm{eil}_a \to \mathrm{P}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$ and $f_{\mathbb{G}} : \mathcal{W}\mathrm{eil}_a \to \mathrm{P}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$ are full embeddings. By Proposition 2, we get a functor

$$\mathcal{F}_{\mathcal{W}\mathrm{eil}_a} : \mathcal{W}\mathrm{eil}_a \to \mathrm{P}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$$

sending $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ to $\mathcal{F}_{\tilde{\Upsilon}_a(F)}(\mathcal{F}_1)$.

Let $I_0 \in \mathrm{P}_{\mathbb{H}Q\mathbb{G}_0(\mathcal{O})}(\Upsilon_0(F))$ denote the constant perverse sheaf on Υ_0 extended by zero to $\Upsilon_0(F)$. Remind that $\zeta_0(I_0)$ is the constant perverse sheaf on Π_0 extended by zero to $\Pi_0(F)$. The object $\zeta_0(I_0)$ will also be denoted I_0 by abuse of notation. So, $I_0 \in \mathcal{W}\mathrm{eil}_0$ naturally.

By definition of the theta-sheaf, we have canonically $\mathcal{F}_{\mathcal{W}\mathrm{eil}_0}(I_0) \,\tilde{\to}\, S_{W_0(F)}$ in $\mathrm{P}_{\mathcal{T}_0}(\widetilde{\mathcal{L}}_d(W_0(F)))$.

4.8 MORE HECKE OPERATORS

4.8.1 For $a \in \mathbb{Z}$ let ${}^a\mathcal{X}\Pi$ be the stack classifying: a $\mathrm{GL}_m \times \mathbb{G}_m$-torsor (U, C) over $\operatorname{Spec} \mathcal{O}$, $\mathbb{G}$-torsor $(M, A, \wedge^2 M \to A)$ over $\operatorname{Spec} \mathcal{O}$, an isomorphism $A \otimes C \,\tilde{\to}\, \Omega(a)$, and a section $s \in U^* \otimes M^* \otimes \Omega(F)$.

Informally, we may view $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$ as the derived category on ${}^a\mathcal{X}\Pi$. For $a, a' \in \mathbb{Z}$ we are going to define a Hecke functor

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}} : {}_{a'-a}\mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F)) \to \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \tag{15}$$

To do so, consider the stack ${}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi}$ classifying: a point of ${}^a\mathcal{X}\Pi$ as above, a lattice $M' \subset M(F)$ such that for $A' = A(a' - a)$ the induced form $\wedge^2 M' \to A'$ is regular and nondegenerate.

We get a diagram

$${}^a\mathcal{X}\Pi \overset{h^{\leftarrow}}{\leftarrow} {}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi} \overset{h^{\rightarrow}}{\to} {}^{a'}\mathcal{X}\Pi,$$

where $h^{\leftarrow}$ sends the above collection to (U, C, M, A, s), the map $h^{\rightarrow}$ sends the above collection to (U, C, M', A', s'), where s' is the image of s under $U^* \otimes M^* \otimes \Omega(F) \,\tilde{\to}\, U^* \otimes M'^* \otimes \Omega(F)$.

Trivializing a point of ${}^{a'}\mathcal{X}\Pi$ (resp., of ${}^a\mathcal{X}\Pi$), one gets isomorphisms

$$\mathrm{id}^r : {}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi} \,\tilde{\to}\, (\Pi_{a'}(F) \times \mathrm{Gr}^{a-a'}_{\mathbb{G}_{a'}})/\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})$$

and

$$\mathrm{id}^l : {}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi} \,\tilde{\to}\, (\Pi_a(F) \times \mathrm{Gr}^{a'-a}_{\mathbb{G}_a})/\mathbb{G}Q\mathbb{H}_a(\mathcal{O})$$

So for

$$K \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)),\ K' \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F)),\ \mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{G}},\ \mathcal{S}' \in {}_{a-a'}\mathrm{Sph}_{\mathbb{G}}$$

one can form the twisted exteriour products $(K \tilde{\boxtimes} \mathcal{S})^l$ and $(K' \tilde{\boxtimes} \mathcal{S}')^r$ on ${}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi}$. The functor (15) is defined by

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}', K') \,\tilde{\to}\, h^{\leftarrow}_!(K' \tilde{\boxtimes} * \mathcal{S}')^r$$

It is understood that this informal definition should be made rigorous in the same way as in ([7], Section 4.3).

4.8.2 For $a \in \mathbb{Z}$ let ${}^a\mathcal{X}\Upsilon$ be the stack classifying: a $\mathrm{GL}_n \times \mathbb{G}_m$-torsor (L, A) over $\operatorname{Spec}\mathcal{O}$, a $\mathbb{H}$-torsor (V, C) over $\mathcal{O}$ (so, we are also given a compatible trivialization $\det V \widetilde{\to} C^m$), an isomorphism $A \otimes C \widetilde{\to} \Omega(a)$, and a section $s \in L^* \otimes V^* \otimes \Omega(F)$.

We may view $\mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$ as the derived category on ${}^a\mathcal{X}\Upsilon$. For $a, a' \in \mathbb{Z}$ we define a Hecke functor

$$\mathrm{H}^{\leftarrow}_{\mathbb{H}} : {}_{a'-a}\operatorname{Sph}_{\mathbb{H}} \times \mathrm{D}_{\mathbb{H}Q\mathbb{G}_{a'}(\mathcal{O})}(\Upsilon_{a'}(F)) \to \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \qquad (16)$$

as follows. Let ${}^{a,a'}\mathcal{H}_{\mathcal{X}\Upsilon}$ be the stack classifying: a point of ${}^a\mathcal{X}\Upsilon$ as above, a lattice $V' \subset V(F)$ such that for $C' = C(a' - a)$ the induced form $\operatorname{Sym}^2 V' \to C'$ is regular and nondegenerate (we also get a compatible trivialization

$$C'^{-m} \otimes \det V' \widetilde{\to} C^{-m} \otimes \det V \widetilde{\to} \mathcal{O},$$

so (V', C') is a $\mathbb{H}$-torsor over $\operatorname{Spec}\mathcal{O}$).

As in Section 4.8.1, we get a diagram

$$\begin{tikzcd} {}^a\mathcal{X}\Upsilon & {}^{a,a'}\mathcal{H}_{\mathcal{X}\Upsilon} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] & {}^{a'}\mathcal{X}\Upsilon \end{tikzcd}$$

and the desired functor (16).

4.8.3 We need the following lemma. Write ${}^a\mathcal{X}\check{\Pi}$ for the stack classifying: a $\mathrm{GL}_m \times \mathbb{G}_m$-torsor (U, C) over $\operatorname{Spec}\mathcal{O}$, a $\mathbb{G}$-torsor $(M, A, \wedge^2 M \to A)$ over $\operatorname{Spec}\mathcal{O}$, an isomorphism $A \otimes C \widetilde{\to} \Omega(a)$, and a section $s_1 \in U \otimes M(F)$. View $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F))$ as the derived category on ${}^a\mathcal{X}\check{\Pi}$.

For $a, a' \in \mathbb{Z}$ define the Hecke functor

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}} : {}_{a'-a}\operatorname{Sph}_{\mathbb{G}} \times \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}}(\Pi_{a'}^* \otimes \Omega(F)) \to \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a^* \otimes \Omega(F)) \qquad (17)$$

as follows. Let ${}^{a,a'}\mathcal{H}_{\mathcal{X}\check{\Pi}}$ be the stack classifying: a point of ${}^a\mathcal{X}\check{\Pi}$ as above, a lattice $M' \subset M(F)$ such that for $A' = A(a' - a)$ the induced form $\wedge^2 M' \to A'$ is regular and nondegenerate.

As above, we get a diagram

$$\begin{tikzcd} {}^a\mathcal{X}\check{\Pi} & {}^{a,a'}\mathcal{H}_{\mathcal{X}\check{\Pi}} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] & {}^{a'}\mathcal{X}\check{\Pi}, \end{tikzcd}$$

where $h^{\leftarrow}$ sends the above point to (U, C, M, A, s_1), the map $h^{\rightarrow}$ sends the above point to (U, C, M', A', s_1'), where s_1' is the image of s_1 under $U \otimes M(F) \widetilde{\to} U \otimes M'(F)$. Now (17) is defined in a way similar to (15).

Write $\operatorname{Four}_\psi : \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}}(\Pi_{a'}(F)) \widetilde{\to} \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}}(\Pi_{a'}^* \otimes \Omega(F))$ for the Fourier transform (normalized as in Section 2.1). The following is standard (cf. also [7], Lemma 11).

Lemma 4. *We have a canonical isomorphism in* $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a^* \otimes \Omega(F))$

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, \operatorname{Four}_\psi(K)) \widetilde{\to} \operatorname{Four}_\psi \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, K)$$

functorial in $\mathcal{S} \in {}_{a'-a}\operatorname{Sph}_{\mathbb{G}}$, $K \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}}(\Pi_{a'}(F))$. $\square$

4.8.4 Write $P_{\mathbb{H}_a}\subset\mathbb{H}_a$ (resp., $P^-_{\mathbb{H}_a}\subset\mathbb{H}_a$) for the parabolic subgroup preserving U_a (resp., $U_a^*\otimes C_a$). Let $U_{\mathbb{H}_a}\subset P_{\mathbb{H}_a}$ and $U^-_{\mathbb{H}_a}\subset P^-_{\mathbb{H}_a}$ denote their unipotent radicals. We view all of them as group schemes over $\operatorname{Spec}\mathcal{O}$. Then $U_{\mathbb{H}_a}\,\widetilde{\to}\, C_a^*\otimes\wedge^2 U_a$ and $U^-_{\mathbb{H}_a}\,\widetilde{\to}\, C_a\otimes\wedge^2 U_a^*$ canonically.

Similarly, let $P_{\mathbb{G}_a}\subset\mathbb{G}_a$ (resp., $P^-_{\mathbb{G}_a}\subset\mathbb{G}_a$) be the parabolic subgroup preserving L_a (resp., $L_a^*\otimes A_a$). Write $U_{\mathbb{G}_a}\subset P_{\mathbb{G}_a}$ and $U^-_{\mathbb{G}_a}\subset P^-_{\mathbb{G}_a}$ for their unipotent radicals. All of them are group schemes over $\operatorname{Spec}\mathcal{O}$. We have canonically

$$U_{\mathbb{G}_a}\,\widetilde{\to}\, A_a^*\otimes\operatorname{Sym}^2 L_a,\qquad U^-_{\mathbb{G}_a}\,\widetilde{\to}\, A_a\otimes\operatorname{Sym}^2 L_a^*$$

View $v\in\Pi_a(F)$ as a map $v: C_a^*\otimes U_a(F)\to M_a(F)$. For $v\in\Pi_a(F)$ let $s_\Pi(v)$ denote the composition

$$\wedge^2(U_a\otimes C_a^{-1})(F)\overset{\wedge^2 v}{\to}\wedge^2 M_a(F)\to A_a(F)$$

Let $\operatorname{Char}(\Pi_a)\subset\Pi_a(F)$ denote the ind-subscheme of $v\in\Pi_a(F)$ such that $s_\Pi(v): C_a^*\otimes\wedge^2 U_a\to\Omega$ is regular. An object $K\in\mathrm{P}(\Upsilon_a(F))$ is $U_{\mathbb{H}_a}(\mathcal{O})$-equivariant iff $\zeta_a(K)$ is the extension by zero from $\operatorname{Char}(\Pi_a)$.

View $v\in\Upsilon_a(F)$ as a map $v: L_a\otimes A_a^*(F)\to V_a(F)$. For $v\in\Upsilon_a(F)$ let $s_\Upsilon(v)$ denote the composition

$$\operatorname{Sym}^2(A_a^*\otimes L_a)\overset{\operatorname{Sym}^2 v}{\to}\operatorname{Sym}^2 V_a(F)\to C_a(F)$$

Write $\operatorname{Char}(\Upsilon_a)\subset\Upsilon_a(F)$ for the ind-subscheme of $v\in\Upsilon_a(F)$ such that $s_\Upsilon(v): A_a^*\otimes\operatorname{Sym}^2 L_a\to\Omega$ is regular. An object $K\in\mathrm{P}(\Pi_a(F))$ is $U_{\mathbb{G}_a}(\mathcal{O})$-equivariant iff $\zeta_a^{-1}(K)$ is the extension by zero from $\operatorname{Char}(\Upsilon_a)$.

The next result follows from ([7], Lemma 13).

Lemma 5. *The full subcategory* $\mathrm{P}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))\subset\mathrm{P}(\Upsilon_a(F))$ *is the intersection of the full subcategories*

$$\mathrm{P}_{U_{\mathbb{H}_a}(\mathcal{O})}(\Upsilon_a(F))\,\cap\,\mathrm{P}_{U^-_{\mathbb{H}_a}(\mathcal{O})}(\Upsilon_a(F))\,\cap\,\mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F))$$

inside $\mathrm{P}(\Upsilon_a(F))$. $\square$

Proposition 3. *For* $a\in\mathbb{Z}$ *the functor* ${}_{-a}\operatorname{Sph}_{\mathbb{G}}\to\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$ *sending* $\mathcal{S}$ *to* $\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S},I_0)$ *factors naturally into*

$${}_{-a}\operatorname{Sph}_{\mathbb{G}}\to\mathrm{D}\,\mathcal{W}\mathrm{eil}_a\to\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$$

For $a\in\mathbb{Z}$ *the functor* ${}_{-a}\operatorname{Sph}_{\mathbb{H}}\to\mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$ *sending* $\mathcal{S}$ *to* $\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S},I_0)$ *factors naturally into*

$${}_{-a}\operatorname{Sph}_{\mathbb{H}}\to\mathrm{D}\,\mathcal{W}\mathrm{eil}_a\to\mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$$

Proof The argument is similar or both claims, we prove only the first one. For a finite subfield $k'\subset k$ we may pick a k'-structure on $\mathcal{O}$. Then I_0 admits a k'-structure and, as such, is pure of weight zero. So, by the decomposition theorem ([1]), one has $\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S},I_0)\in\mathrm{D}\,\mathrm{P}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$.

It remains to show that each perverse cohomology sheaf K of $\zeta_a^{-1}\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S},I_0)$ lies in the full subcategory $\mathrm{P}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$ of $\mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F))$.

By definition of the Hecke functors, $\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)$ is the extension by zero from $\mathrm{Char}(\Pi_a)$, so $\zeta_a(K)$ also satisfies this property. This yields a $U_{\mathbb{H}_a}(\mathcal{O})$-action on K.

To get a $U^-_{\mathbb{H}_a}(\mathcal{O})$-action on K, consider the commutative diagram of equivalences

$$\begin{tikzcd}
\mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F)) \arrow[r, "\zeta_a"] \arrow[d, "\zeta_{1,a}"] & \mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \arrow[dl, "\mathrm{Four}_\psi"] \\
\mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F)), &
\end{tikzcd}$$

where Four_ψ is the complete Fourier transform, and $\zeta_{1,a}$ is the corresponding partial one.

For $v \in \Pi_a^* \otimes \Omega(F)$ write $s_{\check{\Pi}}(v)$ for the composition

$$\wedge^2 U_a^*(F) \overset{\wedge^2 v}{\to} \wedge^2 M_a(F) \to A_a(F)$$

Write $\mathrm{Char}(\Pi_a^* \otimes \Omega) \subset \Pi_a^* \otimes \Omega(F)$ for the ind-subscheme of v such that $s_{\check{\Pi}}(v) : C_a \otimes \wedge^2 U_a^* \to \Omega$ is regular. The $U^-_{\mathbb{H}_a}(\mathcal{O})$-equivariance of K is equivalent to the fact that $\zeta_{1,a}(K)$ is the extension by zero from $\mathrm{Char}(\Pi_a^* \otimes \Omega)$.

By Lemma 4, we have $\mathrm{Four}_\psi \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0) \,\widetilde{\to}\, \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, \check{I}_0)$, where $\check{I}_0 := \mathrm{Four}_\psi(I_0)$ is the constant perverse sheaf on $\Pi_0^* \otimes \Omega$ extended by zero to $\Pi_0^* \otimes \Omega(F)$. Clearly, $\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, \check{I}_0)$ is the extension by zero from $\mathrm{Char}(\Pi_a^* \otimes \Omega)$, and our assertion follows. $\square$

According to Proposition 3, in what follows we will write $\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\cdot, I_0) : {}_{-a}\mathrm{Sph}_{\mathbb{G}} \to \mathrm{D}\,\mathcal{W}\mathrm{eil}_a$ and $\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\cdot, I_0) : {}_{-a}\mathrm{Sph}_{\mathbb{H}} \to \mathrm{D}\,\mathcal{W}\mathrm{eil}_a$ for the corresponding functors. From Proposition 2 one derives the following.

Corollary 2. *For $a \in \mathbb{Z}$, $\mathcal{S} \in {}_{-a}\mathrm{Sph}_{\mathbb{G}}$, $\mathcal{T} \in {}_{-a}\mathrm{Sph}_{\mathbb{H}}$ there are canonical isomorphisms*

$$\mathcal{F}_{\mathcal{W}\mathrm{eil}_a} \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0) \,\widetilde{\to}\, \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, S_{W_0(F)})$$

and

$$\mathcal{F}_{\mathcal{W}\mathrm{eil}_a} \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, I_0) \,\widetilde{\to}\, \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, S_{W_0(F)})$$

in $\mathrm{D}\,\mathrm{P}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F)))$.

Thus, Theorem 3 is reduced to the following.

Theorem 4. *Let the maps κ be as in Theorem 3.*
1) Assume $m \le n$. The two functors ${}_{-a}\mathrm{Sph}_{\mathbb{G}} \to \mathrm{D}\,\mathcal{W}\mathrm{eil}_a$ given by

$$\mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0) \quad \textit{and} \quad \mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_{\mathbb{H}}(*\,\mathrm{gRes}^\kappa(\mathcal{S}), I_0)$$

are isomorphic.
2) Assume $m > n$. The two functors ${}_{-a}\mathrm{Sph}_{\mathbb{H}} \to \mathrm{D}\,\mathcal{W}\mathrm{eil}_a$ given by

$$\mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, I_0) \quad \textit{and} \quad \mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathrm{gRes}^\kappa(*\mathcal{T}), I_0)$$

are isomorphic.

Remark 6. For $a = 0$ Theorem 4 is nothing but ([7], Theorem 7).

4.8.5 Hecke operators for Levi subgroups

For $a \in \mathbb{Z}$ set $Q\Pi_a = U_a^* \otimes C_a \otimes L_a \subset \Pi_a$ and $Q\Upsilon_a = L_a^* \otimes A_a \otimes U_a \subset \Upsilon_a$.

We are going to define for $a, a' \in \mathbb{Z}$ Hecke functors

$$\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})} : {}_{a'-a}\operatorname{Sph}_{Q(\mathbb{G})} \times \mathrm{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Pi_{a'}(F)) \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)) \tag{18}$$

in a way compatible with the functors defined in Section 4.8.

Let ${}^a\mathcal{X}Q\Pi$ be the stack classifying a $Q(\mathbb{H})$-torsor (U, C) over $\operatorname{Spec}\mathcal{O}$, a $Q(\mathbb{G})$-torsor (L, A) over $\operatorname{Spec}\mathcal{O}$, an isomorphism $A \otimes C \,\widetilde{\to}\, \Omega(a)$, and a section $s \in U^* \otimes C \otimes L(F)$.

Informally, we think of $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$ as the derived category on ${}^a\mathcal{X}Q\Pi$. Consider the stack ${}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})}$ classifying: a point of ${}^a\mathcal{X}Q\Pi$ as above, a lattice $L' \subset L(F)$, for which we set $A' = A(a' - a)$. We get a diagram

$${}^a\mathcal{X}Q\Pi \overset{h^{\leftarrow}}{\leftarrow} {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})} \overset{h^{\rightarrow}}{\rightarrow} {}^{a'}\mathcal{X}Q\Pi,$$

where $h^{\leftarrow}$ sends the above collection to (U, C, L, A, s), the map $h^{\rightarrow}$ sends the above collection to (U, C, L', A', s'), where s' is the image of s under $U^* \otimes C \otimes L(F) \,\widetilde{\to}\, U^* \otimes C \otimes L'(F)$.

Trivializing a point of ${}^{a'}\mathcal{X}Q\Pi$ (resp., of ${}^a\mathcal{X}Q\Pi$), one gets isomorphisms

$$\mathrm{id}^r : {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})} \,\widetilde{\to}\, (Q\Pi_{a'}(F) \times \mathrm{Gr}^{a-a'}_{Q(\mathbb{G}_{a'})})/Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})$$

and

$$\mathrm{id}^l : {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})} \,\widetilde{\to}\, (Q\Pi_a(F) \times \mathrm{Gr}^{a'-a}_{Q(\mathbb{G}_a)})/Q\mathbb{G}\mathbb{H}_a(\mathcal{O})$$

So, for

$$K \in \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)),\ K' \in \mathrm{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Pi_{a'}(F))$$

and $\mathcal{S} \in {}_{a'-a}\operatorname{Sph}_{Q(\mathbb{G})}$, $\mathcal{S}' \in {}_{a-a'}\operatorname{Sph}_{Q(\mathbb{G})}$ one can form their twisted exteriour products $(K \,\tilde{\boxtimes}\, \mathcal{S})^l$ and $(K' \,\tilde{\boxtimes}\, \mathcal{S}')^r$ on ${}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})}$. The functor (18) is defined by

$$\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathcal{S}', K') = h^{\leftarrow}_!(K' \,\tilde{\boxtimes}\, * \mathcal{S}')^r$$

Let ${}^a\mathcal{X}Q\Upsilon$ be the stack classifying a $Q(\mathbb{H})$-torsor (U, C) over $\operatorname{Spec}\mathcal{O}$, a $Q(\mathbb{G})$-torsor (L, A) over $\operatorname{Spec}\mathcal{O}$, an isomorphism $A \otimes C \,\widetilde{\to}\, \Omega(a)$, and a section $s \in U \otimes A \otimes L^*(F)$. Informally, we think of $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ as the derived category on ${}^a\mathcal{X}Q\Upsilon$. One defines the Hecke functor

$$\mathrm{H}^{\leftarrow}_{Q(\mathbb{H})} : {}_{a'-a}\operatorname{Sph}_{Q(\mathbb{H}))} \times \mathrm{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Upsilon_{a'}(F)) \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \tag{19}$$

using a similar diagram

$${}^a\mathcal{X}Q\Upsilon \overset{h^{\leftarrow}}{\leftarrow} {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Upsilon, Q(\mathbb{H})} \overset{h^{\rightarrow}}{\rightarrow} {}^{a'}\mathcal{X}Q\Upsilon$$

By abuse of notation, we also write I_0 for the constant perverse sheaf on $Q\Upsilon_0$ and on $Q\Pi_0$, the exact meaning is easily understood from the context. The next result is a straightforward consequence of ([7], Corollary 4).

Proposition 4. *1) Assume $m > n$. The functor*

$${}_{-a}\mathrm{Sph}_{Q(\mathbb{G})} \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$$

given by $\mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathcal{S}, I_0)$ takes values in $\mathrm{P}^{ss}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$ and induces an equivalence

$${}_{-a}\mathrm{Sph}_{Q(\mathbb{G})} \widetilde{\to} \mathrm{P}^{ss}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$$

2) Assume $m \leq n$. The functor

$${}_{-a}\mathrm{Sph}_{Q(\mathbb{H})} \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$$

given by $\mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathcal{S}, I_0)$ takes values in $\mathrm{P}^{ss}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ and induces an equivalence

$${}_{-a}\mathrm{Sph}_{Q(\mathbb{H})} \widetilde{\to} \mathrm{P}^{ss}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$$

□

4.8.5.2 For $a, a' \in \mathbb{Z}$ we will use in Section 4.8.9 the following Hecke functor

$$\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})} : {}_{a'-a}\mathrm{Sph}_{Q(\mathbb{G})} \times \mathrm{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Upsilon_{a'}(F)) \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \tag{20}$$

Consider the stack ${}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Upsilon, Q(\mathbb{G})}$ classifying: a point (U, C, L, A, s) of ${}^{a}\mathcal{X}Q\Upsilon$ as above, a lattice $L' \subset L(F)$ for which we set $A' = A(a' - a)$. We get a diagram

$${}^{a}\mathcal{X}Q\Upsilon \overset{h^{\leftarrow}}{\leftarrow} {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Upsilon, Q(\mathbb{G})} \overset{h^{\rightarrow}}{\to} {}^{a'}\mathcal{X}Q\Upsilon,$$

where $h^{\leftarrow}$ sends the above collection to (U, C, L, A, s), and $h^{\rightarrow}$ sends the same collection to (U, C, L', A', s'), where s' is the image of s under $U \otimes A \otimes L^*(F) \widetilde{\to} U \otimes A' \otimes L'^*(F)$. The functor (20) is defined as in Section 4.8.5 for the above diagram.

The following is a consequence of ([7], Lemma 11).

Lemma 6. *For $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{Q(\mathbb{G})}$ the diagram of functors is canonically 2-commutative*

$$\begin{tikzcd}
\mathrm{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Upsilon_{a'}(F)) \arrow[r, "\mathrm{Four}_{\psi}"] \arrow[d, "\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathcal{S}, \cdot)"] & \mathrm{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Pi_{a'}(F)) \arrow[d, "\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathcal{S}, \cdot)"] \\
\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \arrow[r, "\mathrm{Four}_{\psi}"] & \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))
\end{tikzcd}$$

4.8.6 Weak Jacquet functors

As in ([7], Section 4.7) for each $a \in \mathbb{Z}$ we define the weak Jacquet functors

$$J^*_{P_{\mathbb{H}_a}}, J^!_{P_{\mathbb{H}_a}} : \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \tag{21}$$

and

$$J^*_{P_{\mathbb{G}_a}}, J^!_{P_{\mathbb{G}_a}} : \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)) \tag{22}$$

Both definitions being similar, we recall the definition of (21) only.

For a free $\mathcal{O}$-module of finite type M and $N, r \in \mathbb{Z}$ with $N + r \geq 0$ write ${}_{N,r}M = M(N)/M(-r)$.

For $N + r \geq 0$ consider the natural embedding $i_{N,r} : {}_{N,r}Q\Upsilon_a \hookrightarrow {}_{N,r}\Upsilon_a$. Set

$$PQ\mathbb{G}_a = \{g = (g_1, g_2) \in P_{\mathbb{H}_a} \times Q(\mathbb{G}_a) \mid g \in \mathcal{T}_a\},$$

this is a group scheme over $\operatorname{Spec}\mathcal{O}$. We have a diagram of stack quotients

$$\begin{tikzcd}
PQ\mathbb{G}_a(\mathcal{O}/t^{N+r})\backslash({}_{N,r}Q\Upsilon_a) \arrow[r, "i_{N,r}"] \arrow[d, "q"] & PQ\mathbb{G}_a(\mathcal{O}/t^{N+r})\backslash({}_{N,r}\Upsilon_a) \arrow[r, "p"] & \mathbb{H}Q\mathbb{G}_a(\mathcal{O}/t^{N+r})\backslash({}_{N,r}\Upsilon_a) \\
Q\mathbb{G}\mathbb{H}_a(\mathcal{O}/t^{N+r})\backslash({}_{N,r}Q\Upsilon_a), & &
\end{tikzcd}$$

where $t \in \mathcal{O}$ is a uniformizer, p comes from the inclusion $P_{\mathbb{H}_a} \subset \mathbb{H}_a$, and q is the natural quotient map. First, define functors

$$J^*_{P_{\mathbb{H}_a}}, J^!_{P_{\mathbb{H}_a}} : \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O}/t^{N+r})}({}_{N,r}\Upsilon_a) \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O}/t^{N+r})}({}_{N,r}Q\Upsilon_a) \tag{23}$$

by

$$q^* \circ J^*_{P_{\mathbb{H}_a}}[\dim.\mathrm{rel}(q)] = i^*_{N,r}p^*[\dim.\mathrm{rel}(p) - rnm]$$

$$q^* \circ J^!_{P_{\mathbb{H}_a}}[\dim.\mathrm{rel}(q)] = i^!_{N,r}p^*[\dim.\mathrm{rel}(p) + rnm]$$

Since

$$q^*[\dim.\mathrm{rel}(q)] : \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O}/t^{N+r})}({}_{N,r}Q\Upsilon_a) \to \mathrm{D}_{PQ\mathbb{G}_a(\mathcal{O}/t^{N+r})}({}_{N,r}Q\Upsilon_a)$$

is an equivalence (exact for the perverse t-structures), the functors (23) are well-defined. Further, (23) are compatible with the transition functors in the definition of the corresponding derived categories, so give rise to the functors (21) in the limit as N, r go to infinity. Note that for (21) we get $\mathbb{D} \circ J^*_{P_{\mathbb{H}_a}} \xrightarrow{\sim} J^!_{P_{\mathbb{H}_a}} \circ \mathbb{D}$ naturally.

We identify $\mathbb{H} \xrightarrow{\sim} \mathbb{H}_0$ and $Q(\mathbb{H}) \xrightarrow{\sim} Q(\mathbb{H}_0)$. Let $\check{\mu}_{\mathbb{H}} = \det V_0$ and $\check{\nu}_{\mathbb{H}} = \det U_0$ viewed as characters of $\mathbb{H}$ or, equivalently, as cocharacters of the center $Z(\check{Q}(\mathbb{H}))$ of the Langlands dual group $\check{Q}(\mathbb{H})$ of $Q(\mathbb{H})$. Let $\kappa_{\mathbb{H}} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \to \check{\mathbb{H}}$ be the map, whose first component is the natural inclusion of the Levi subgroup, and the second one is $2(\check{\rho}_{\mathbb{H}} - \check{\rho}_{Q(\mathbb{H})}) + n(\check{\mu}_{\mathbb{H}} - \check{\nu}_{\mathbb{H}})$. The corresponding geometric restriction functor is denoted $\mathrm{gRes}^{\kappa_{\mathbb{H}}}$.

Lemma 7. *For $a, a' \in \mathbb{Z}$ and $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{H}}$, $K \in \mathrm{D}_{\mathbb{H}Q\mathbb{G}_{a'}(\mathcal{O})}(\Upsilon_{a'}(F))$ there is a filtration in the derived category $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ on*

$$J^*_{P_{\mathbb{H}_a}} \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, K)$$

such that the corresponding graded object identifies with $\mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{S}), J^*_{P_{\mathbb{H}_{a'}}}(K))$.

Proof The proof is quite similar to ([7], Lemma 10), we only have to determine the corresponding map κ. To do so, it suffices to perform the calculation for a particular K. Let $I_{a'}$ be the constant perverse sheaf on $\Upsilon_{a'}$ extended by zero to $\Upsilon_{a'}(F)$. Take $K = I_{a'}$.

For $s_1, s_2 \geq 0$ let ${}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a} \subset \operatorname{Gr}_{\mathbb{H}_a}$ be the closed subscheme of $h\mathbb{H}_a(\mathcal{O}) \in \operatorname{Gr}_{\mathbb{H}_a}$ such that

$$V_a(-s_1) \subset hV_a \subset V_a(s_2)$$

Assume that s_1, s_2 are large enough so that $\mathcal{S}$ is the extension by zero from ${}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a}$. Then $\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, I_{a'}) \in \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}({}_{s_2,s_1}\Upsilon_a)$ is as follows. Write ${}_{0,s_1}\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a}$ for the scheme classifying pairs

$$h\mathbb{H}_a(\mathcal{O}) \in {}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a},\ v \in L_a^* \otimes A_a \otimes (hV_a)/V_a(-s_1)$$

Let $\pi : {}_{0,s_1}\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a} \to {}_{s_2,s_1}\Upsilon_a$ be the map sending $(h\mathbb{H}_a(\mathcal{O}), v)$ to v. By definition,

$$\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, I_{a'}) \xrightarrow{\sim} \pi_!(\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{S}), \tag{24}$$

where $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{S}$ is normalized to be perverse. If $\theta \in \pi_1(\mathbb{H})$ then ${}_{0,s_1}\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a}$ is a vector bundle over ${}_{s_1,s_2}\operatorname{Gr}^{\theta}_{\mathbb{H}_a}$ of rank $2s_1nm - \langle\theta, n\check{\mu}_{\mathbb{H}}\rangle$.

Let ${}_{s_1,s_2}P_{\mathbb{H}_a} = \{p \in P_{\mathbb{H}_a}(F) \mid V_a(-s_1) \subset pV_a \subset V_a(s_2)\}$. Then

$${}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}} = ({}_{s_1,s_2}P_{\mathbb{H}_a}(F))/P_{\mathbb{H}_a}(\mathcal{O})$$

is closed in $\operatorname{Gr}_{\mathbb{P}_{\mathbb{H}_a}}$. The natural map ${}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}} \to {}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a}$ at the level of reduced schemes yields a stratification of ${}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a}$ by the connected components of ${}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}}$. Calculate (24) with respect to this stratification. Denote by ${}_{s_1,s_2}\operatorname{Gr}_{Q(\mathbb{H}_a)} \subset \operatorname{Gr}_{Q(\mathbb{H}_a)}$ the closed subscheme of $hQ(\mathbb{H}_a) \in \operatorname{Gr}_{Q(\mathbb{H}_a)}$ satisfying

$$U_a(-s_1) \subset hU_a \subset U_a(s_2),$$

write $\mathfrak{t}_P : \operatorname{Gr}_{P_{\mathbb{H}_a}} \to \operatorname{Gr}_{Q(\mathbb{H}_a)}$ for the natural map. We have the diagram

$$\begin{tikzcd}
{}_{0,s_1}Q\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{Q(\mathbb{H}_a)} \arrow[dr, "\pi_Q"] & {}_{0,s_1}Q\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}} \arrow[l, "\mathrm{id}\times\mathfrak{t}_P"'] \arrow[r, hook] \arrow[d] & {}_{0,s_1}\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}} \arrow[r] \arrow[d] & {}_{0,s_1}\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{\mathbb{H}_a} \arrow[dl, "\pi"] \\
 & {}_{s_2,s_1}Q\Upsilon_a \arrow[r, hook] & {}_{s_2,s_1}\Upsilon_a, &
\end{tikzcd}$$

where the square is cartesian. Here ${}_{0,s_1}\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}}$ is the scheme classifying pairs

$$hP_{\mathbb{H}_a(\mathcal{O})} \in {}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}}, v \in L_a^* \otimes A_a \otimes (hV_a)/V_a(-s_1),$$

and ${}_{0,s_1}Q\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{P_{\mathbb{H}_a}}$ is its closed subscheme given by the condition $v \in L_a^* \otimes A_a \otimes (hU_a)/U_a(-s_1)$.

By definition, for $\mathcal{T} \in {}_{a'-a}\operatorname{Sph}_{Q(\mathbb{H})}$ we have

$$\mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathcal{T}, I_{a'}) \xrightarrow{\sim} \pi_{Q!}(\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{T}),$$

where $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{T}$ is normalized to be perverse. If $\theta \in \pi_1(Q(\mathbb{H}))$ then ${}_{0,s_1}Q\Upsilon\tilde{\times}_{s_1,s_2}\operatorname{Gr}_{Q(\mathbb{H}_a)}$ is a vector bundle over ${}_{s_1,s_2}\operatorname{Gr}^{\theta}_{Q(\mathbb{H}_a)}$ of rank $s_1nm - \langle\theta, n\check{\nu}_{\mathbb{H}}\rangle$. Our assertion follows. $\square$

We identify $\mathbb{G}\,\widetilde{\to}\,\mathbb{G}_0$, $Q(\mathbb{G})\,\widetilde{\to}\,Q(\mathbb{G}_0)$. Write $\check{\mu}_{\mathbb{G}} = \det M_0$ and $\check{\nu}_{\mathbb{G}} = \det L_0$ as cocharacters of the center $Z(\check{Q}(\mathbb{G}))$ of the Langlands dual group $\check{Q}(\mathbb{G})$ of $Q(\mathbb{G})$. Let $\kappa_{\mathbb{G}} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \to \check{\mathbb{G}}$ be the map whose first component is the natural inclusion of the Levi subgroup, and the second one is $2(\check{\rho}_{\mathbb{G}} - \check{\rho}_{Q(\mathbb{G})}) + m(\check{\mu}_{\mathbb{G}} - \check{\nu}_{\mathbb{G}})$. The corresponding geometric restriction functor is denoted $\mathrm{gRes}^{\kappa_{\mathbb{G}}}$.

Lemma 8. *For $a, a' \in \mathbb{Z}$ and $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{G}}$, $K \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F))$ there is a filtration in the derived category $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$ on*

$$J^*_{P_{\mathbb{G}_a}} \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, K)$$

such that the corresponding graded object identifies with $\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathrm{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), J^*_{P_{\mathbb{G}_{a'}}}(K))$. $\square$

We will use Lemmas 7 and 8 in the following form (the proof is as in [7], Corollary 3).

Corollary 3. *Let For $a, a' \in \mathbb{Z}$ and $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{H}}$. Assume that $K \in \mathrm{P}_{\mathbb{H}Q\mathbb{G}_{a'}(\mathcal{O})}(\Upsilon_{a'}(F))$ admits a k_0-structure for some finite subfield $k_0 \subset k$ and, as such, is pure of weight zero. Then $J^*_{P_{\mathbb{H}_{a'}}}(K)$ is also pure of weight zero over k_0, and there is an isomorphism in $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$*

$$J^*_{P_{\mathbb{H}_a}} \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, K)\,\widetilde{\to}\,\mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{S}), J^*_{P_{\mathbb{H}_{a'}}}(K))$$

(Similar strengthened version of Lemma 8 also holds.)

4.8.7 Action of $\mathrm{Sph}_{\mathbb{G}}$

Pick a maximal torus and a Borel subgroup $T_{\mathbb{G}} \subset B_{\mathbb{G}} \subset \mathbb{G}$, and similarly for $\mathbb{H}$. We assume $T_{\mathbb{G}} \subset Q(\mathbb{G})$ and $T_{\mathbb{H}} \subset Q(\mathbb{H})$. A trivialization of the $\mathbb{G}_a$-torsor (M_a, A_a) over $\mathrm{Spec}\,\mathcal{O}$ yields a maximal torus and a Borel subgroup in $\mathbb{G}_a$, hence also an equivalence $\mathrm{Sph}_{\mathbb{G}_a}\,\widetilde{\to}\,\mathrm{Sph}_{\mathbb{G}}$ and a bijection $\Lambda^+_{\mathbb{G}_a}\,\widetilde{\to}\,\Lambda^+_{\mathbb{G}}$ as in Section 4.3.2 (and similarly for $\mathbb{H}$ and $\mathbb{H}_a$).

Write $\check{\omega}_i$ for the h.w. of the fundamental representation of $\mathbb{G}_a$ that appear in $\wedge^i M_a$ for $i = 1, \ldots, n$. All the weights of $\wedge^i M_a$ are $\le \check{\omega}_i$. Write $\check{\omega}_0$ for the h.w. of the $\mathbb{G}_a$-module A_a.

For $\lambda \in \Lambda^+_{\mathbb{G}}$ set $a = \langle \lambda, \check{\omega}_0 \rangle$ then $\mathcal{A}^{\lambda}_{\mathbb{G}} \in {}_{-a}\mathrm{Sph}_{\mathbb{G}}$. By definition, the complex

$$\mathrm{H}^{\lambda}_{\mathbb{G}}(I_0) = \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{A}^{\lambda}, I_0) \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$$

is as follows. Set $r = \langle \lambda, \check{\omega}_1 \rangle$ and $N = \langle -w_0^{\mathbb{G}}(\lambda), \check{\omega}_1 \rangle$. Let ${}_{0,r}\Pi \tilde{\times} \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{G}_a}$ be the scheme classifying $g \in \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{G}_a}$, $x \in U_a^* \otimes C_a \otimes ((gM_a)/M_a(-r))$. Let

$$\pi : {}_{0,r}\Pi \tilde{\times} \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{G}_a} \to {}_{N,r}\Pi_a \tag{25}$$

be the map sending $(x, g\mathbb{G}_a(\mathcal{O}))$ to x. Then $\mathrm{H}^{\lambda}_{\mathbb{G}}(I_0)\,\widetilde{\to}\,\pi_!(\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^{\lambda}_{\mathbb{G}})$ canonically (recall that $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^{\lambda}_{\mathbb{G}}$ is normalized to be perverse).

Define the closed subscheme ${}_{\lambda}\Pi_a \subset \Pi_a(N)$ as follows. A point $v \in \Pi_a(N)$ lies in ${}_{\lambda}\Pi_a$ if the following conditions hold:

C1) $v \in \mathrm{Char}(\Pi_a)$;

C2) for $i = 1, \dots, n$ the map $\wedge^i v : \wedge^i(U_a \otimes C_a^{-1}) \to (\wedge^i M_a)(\langle -w_0^{\mathbb{G}}(\lambda), \check{\omega}_i\rangle)$ is regular.

The subscheme ${}_{\lambda}\Pi_a$ is stable under translations by $\Pi_a(-r)$, so there is a closed subscheme ${}_{\lambda,N}\Pi_a \subset {}_{N,r}\Pi_a$ such that ${}_{\lambda}\Pi_a$ is the preimage of ${}_{\lambda,N}\Pi_a$ under the projection $\Pi_a(N) \to {}_{N,r}\Pi_a$. Since all the weights of $\wedge^i M_a$ are $\le \check{\omega}_i$, the map (25) factors through the closed subscheme ${}_{\lambda,N}\Pi_a \subset {}_{N,r}\Pi_a$.

For each $v \in \mathrm{Char}(\Pi_a)$ let us define a $\mathcal{O}$-lattice $M_v \subset M_a(F)$ as follows. View v as a map $U_a \otimes C_a^{-1} \to M_a(F)$. For a $\mathcal{O}$-lattice $R \subset M_a(F)$ set

$$R^{\perp} = \{m \in M_a(F) \mid \langle m, x\rangle \in A_a(-a) \text{ for all } x \in R\}$$

Consider two cases.

CASE: a is even. For $v \in \mathrm{Char}(\Pi_a)$ set $R_v = v(U_a \otimes C_a^{-1}) + M_a(-\frac{a}{2})$ and $M_v = v(U_a \otimes C_a^{-1}) + R_v^{\perp}$. Then $R_v^{\perp} \subset M_v \subset R_v$, and the induced form $\wedge^2 M_v \to A_a(-a)$ is regular and nondegenerate. So, $M_v \in \mathrm{Gr}_{\mathbb{G}_a}^{-a}$.

CASE: a is odd. Let $b = (-a-1)/2$. Note that $(M_a(b))^{\perp} = M_a(b+1)$. Set $R_v = v(U_a \otimes C_a^{-1}) + M_a(b+1)$ and $M_v = v(U_a \otimes C_a^{-1}) + R_v^{\perp}$. Clearly, the induced form $\wedge^2 M_v \to A_a(-a)$ is regular, but still can be degenerate. We call v *generic* if the form $\wedge^2 M_v \to A_a(-a)$ is nondegenerate. In this case $M_v \in \mathrm{Gr}_{\mathbb{G}_a}^{-a}$.

For a even we get a stratification of $\mathrm{Char}(\Pi_a)$ indexed by $\{\lambda \in \Lambda_{\mathbb{G}}^{+} \mid \langle \lambda, \check{\omega}_0\rangle = a\}$, the stratum ${}_{\lambda}\mathrm{Char}(\Pi_a)$ is given by the condition that $M_v \in \mathrm{Gr}_{\mathbb{G}_a}^{\lambda}$. This condition is also equivalent to requiring that there is an isomorphism of $\mathcal{O}$-modules

$$R_v/(M_a(-a/2)) \widetilde{\to} \mathcal{O}/t^{a_1 - \frac{a}{2}} \oplus \ldots \oplus \mathcal{O}/t^{a_n - \frac{a}{2}},$$

where $t \in \mathcal{O}$ is a uniformizer.

Clearly, ${}_{\lambda}\mathrm{Char}(\Pi_a) \subset {}_{\lambda}\Pi_a$. There is a unique open subscheme ${}_{\lambda,N}\Pi_a^0 \subset {}_{\lambda,N}\Pi_a$ whose preimage under the projection ${}_{\lambda}\Pi_a \to {}_{\lambda,N}\Pi_a$ equals ${}_{\lambda}\mathrm{Char}(\Pi_a)$.

We say that a morphism of free $\mathcal{O}$-modules $M_1 \to M_2$ is *maximal* if it does not factor through $M_2(-1) \subset M_2$.

For a odd define ${}_{\lambda}\mathrm{Char}(\Pi_a) \subset {}_{\lambda}\Pi_a$ as the open subscheme given by the condition that each map $\wedge^i v$ in C2) is maximal. Then there is an open subscheme ${}_{\lambda,N}\Pi_a^0 \subset {}_{\lambda,N}\Pi_a$ whose preimage under the projection ${}_{\lambda}\Pi_a \to {}_{\lambda,N}\Pi_a$ equals ${}_{\lambda}\mathrm{Char}(\Pi_a)$. One checks that any $v \in {}_{\lambda}\mathrm{Char}(\Pi_a)$ is generic and the corresponding lattice M_v satisfies $M_v \in \mathrm{Gr}_{\mathbb{G}_a}^{\lambda}$. Note that for $v \in {}_{\lambda}\mathrm{Char}(\Pi_a)$ we have an isomorphism of $\mathcal{O}$-modules

$$R_v/(M_a(b+1)) \widetilde{\to} \mathcal{O}/t^{a_1 - (a+1)/2} \oplus \ldots \oplus \mathcal{O}/t^{a_n - (a+1)/2}$$

for any uniformizer $t \in \mathcal{O}$.

Write $\mathrm{IC}({}_{\lambda,N}\Pi_a^0)$ for the intersection cohomology sheaf of ${}_{\lambda,N}\Pi_a^0$.

Proposition 5. *Let* $\lambda \in \Lambda^+_{\mathbb{G}}$ *with* $\langle \lambda, \check{\omega}_0 \rangle = a$.
1) The map

$$\pi : {}_{0,r}\Pi \tilde{\times} \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{G}_a} \to {}_{\lambda,N}\Pi_a$$

is an isomorphism over the open subscheme ${}_{\lambda,N}\Pi^0_a$.
2) Assume $m > n$ *then one has a canonical isomorphism* $\mathrm{H}^{\lambda}_{\mathbb{G}}(I_0) \widetilde{\to} \mathrm{IC}({}_{\lambda,N}\Pi^0_a)$.

Proof 1) The fibre of π over $v \in {}_{\lambda,N}\Pi^0_a$ is the scheme classifying lattices $M' \in \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{G}_a}$ such that $v(U_a \otimes C_a^{-1}) \subset M'$. Given such a lattice M' let us show that $M_v = M'$.

Consider first the case of a odd. The inclusion $R_v \subset M' + M_a(b+1)$ must be an equality, because for $M' \in \mathrm{Gr}^{\mu}_{\mathbb{G}_a}$ with $\mu \le \lambda$ we have

$$\dim(M' + M_a(b+1))/(M_a(b+1)) = \epsilon(\mu) \le \epsilon(\lambda) = \dim R_v/(M_a(b+1))$$

We have denoted here $\epsilon(\mu) = \langle \mu, \check{\omega}_n \rangle - \frac{n}{2}(a+1)$. So, $M_v = v(U_a \otimes C_a^{-1}) + (M' \cap M_a(b)) \subset M'$ is also an equality, because both M_v and M' have symplectic forms with values in $A_a(-a)$.

The case of a even is quite similar to ([7], Lemma 15). Namely, the inclusion $R_v \subset M' + M_a(-\frac{a}{2})$ must be an equality, because for $M' \in \mathrm{Gr}^{\mu}_{\mathbb{G}_a}$ with $\mu \le \lambda$ we get

$$\dim(M' + M_a(-a/2))/(M_a(-a/2)) = \epsilon(\mu) \le \epsilon(\lambda) = \dim R_v/(M_a(-a/2))$$

Here for a even we have set $\epsilon(\mu) = \langle \mu, \check{\omega}_n \rangle - \frac{n}{2}a$. So,

$$M_v = v(U_a \otimes C_a^{-1}) + (M' \cap (M_a(-a/2))) \subset M'$$

is also an equality. The first assertion follows.

2) For $m \ge n$ the scheme ${}_{\lambda,N}\Pi^0_a$ is nonempty, so $\mathrm{IC}({}_{\lambda,N}\Pi^0_a)$ appears in $\mathrm{H}^{\lambda}_{\mathbb{G}}(I_0)$ with multiplicity one. So, it suffices to show that

$$\mathrm{Hom}(\mathrm{H}^{\lambda}_{\mathbb{G}}(I_0), \mathrm{H}^{\lambda}_{\mathbb{G}}(I_0)) = \bar{\mathbb{Q}}_{\ell},$$

where Hom is taken in the derived category $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$. By adjointness,

$$\mathrm{Hom}(\mathrm{H}^{\lambda}_{\mathbb{G}}(I_0), \mathrm{H}^{\lambda}_{\mathbb{G}}(I_0)) \widetilde{\to} \mathrm{Hom}(\mathrm{H}^{-w_0^{\mathbb{G}}(\lambda)}_{\mathbb{G}} \mathrm{H}^{\lambda}_{\mathbb{G}}(I_0), I_0),$$

where Hom in the RHS is taken in $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_0(\mathcal{O})}(\Pi_0(F))$. We are reduced to show that for any $0 \ne \mu \in \Lambda^+_{\mathbb{G}}$ with $\langle \mu, \check{\omega}_0 \rangle = 0$ one has

$$\mathrm{Hom}(\mathrm{H}^{\mu}_{\mathbb{G}}(I_0), I_0) = 0$$

in $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_0(\mathcal{O})}(\Pi_0(F))$. The latter assertion is true for $m > n$, it is proved in ([7], part 2) of Lemma 15). $\square$

Remark 7. For any $a, b \in \mathbb{Z}$ let us construct an equivalence $\mathcal{W}\mathrm{eil}_a \widetilde{\to} \mathcal{W}\mathrm{eil}_{a+2b}$. Pick isomorphisms of $\mathcal{O}$-modules

$$L_a(b) \widetilde{\to} L_{a+2b}, \;\; A_a(2b) \widetilde{\to} A_{a+2b}, \;\; U_a \widetilde{\to} U_{a+2b}, \tag{26}$$

They yield isomorphisms $C_a \widetilde{\to} C_{a+2b}$, $V_a \widetilde{\to} V_{a+2b}$, $M_a(b) \widetilde{\to} M_{a+2b}$. Hence, also isomorphisms $Q(\mathbb{G}_a) \widetilde{\to} Q(\mathbb{G}_{a+2b})$, $\mathbb{G}_a \widetilde{\to} \mathbb{G}_{a+2b}$ of group schemes over $\operatorname{Spec} \mathcal{O}$ (and similarly for $\mathbb{H}$). We also get isomorphisms of group schemes over $\operatorname{Spec} \mathcal{O}$

$$Q\mathbb{G}\mathbb{H}_a \widetilde{\to} Q\mathbb{G}\mathbb{H}_{a+2b}, \;\; \mathbb{G}Q\mathbb{H}_a \widetilde{\to} \mathbb{G}Q\mathbb{H}_{a+2b}, \;\; \mathbb{H}Q\mathbb{G}_a \widetilde{\to} \mathbb{H}Q\mathbb{G}_{a+2b}$$

The isomorphisms (26) also yield $\Pi_a(b) \widetilde{\to} \Pi_{a+2b}$ and $\Upsilon_a(b) \widetilde{\to} \Upsilon_{a+2b}$. In turn, we get equivalences

$$\mathrm{P}_{\mathbb{H}Q\mathbb{G}_a}(\Upsilon_a(F)) \widetilde{\to} \mathrm{P}_{\mathbb{H}Q\mathbb{G}_{a+2b}}(\Upsilon_{a+2b}(F)), \;\; \mathrm{P}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F)) \widetilde{\to} \mathrm{P}_{\mathbb{G}Q\mathbb{H}_{a+2b}}(\Pi_{a+2b}(F))$$

which yield the desired equivalence $\mathcal{W}\mathrm{eil}_a \widetilde{\to} \mathcal{W}\mathrm{eil}_{a+2b}$. The diagram commutes

$$\begin{tikzcd}
{}_{-a}\,\mathrm{Sph}_{\mathbb{G}} \arrow[r] \arrow[d, "\epsilon"] & \mathcal{W}\mathrm{eil}_a \arrow[d, "\wr"] \\
{}_{-a-2b}\,\mathrm{Sph}_{\mathbb{G}} \arrow[r] & \mathcal{W}\mathrm{eil}_{a+2b},
\end{tikzcd}$$

where the horizontal arrows are given by $\mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)$, and ϵ, at the level of representations of $\check{G}$, is given by $V \mapsto V \otimes V^{b\omega}$. Here V^{ω} is the one-dimensional representation of $\check{\mathbb{G}}$ with h.w. ω such that $\langle \omega, \check{\omega}_0 \rangle = 2$. So, the case of a even in Proposition 5 also follows from ([7], Lemma 15).

4.8.7.2 Let $k_0 \subset k$ be a finite subfield. In this subsection we assume that all the objects of Sections 4 are defined over k_0.

Write $\mathcal{W}\mathrm{eil}_{a,k_0}$ for the category of triples $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ as in Definition 2 of $\mathcal{W}\mathrm{eil}_a$ but with a k_0-structure and, as such, pure of weight zero. It is understood that the Fourier transform functors are normalized to preserve purity. Note that for any $(\mathcal{F}_1, \mathcal{F}_2, \beta) \in \mathcal{W}\mathrm{eil}_a$ the perverse sheaf $\mathcal{F}_1$ is $\mathbb{G}_m$-equivariant with respect to the homotheties on $\Upsilon_a(F)$.

Denote by $\mathrm{D}\,\mathcal{W}\mathrm{eil}_{a,k_0}$ the category of complexes as in the definition of $\mathrm{D}\,\mathcal{W}\mathrm{eil}_a$ but, in addition, with a k_0-structure and, as such, pure of weight zero. So, for an object of $\mathrm{D}\,\mathcal{W}\mathrm{eil}_{a,k_0}$ its semi-simplification is a bounded complex of the form $\oplus_{i \in \mathbb{Z}} F_i[i](\frac{i}{2})$ with $F_i \in \mathcal{W}\mathrm{eil}_{a,k_0}$.

Write F_0 for k_0-valued points of F. For a totally disconnected locally compact space Y write $\mathcal{S}(Y)$ for the Schwarz space of locally constant $\bar{\mathbb{Q}}_\ell$-valued functions on Y with compact support. Write $\mathcal{W}\mathrm{eil}_a(k_0)$ for the $\bar{\mathbb{Q}}_\ell$-vector space of pairs $(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{F}_1 \in \mathcal{S}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F_0))$, $\mathcal{F}_2 \in \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$ with $\zeta_a(\mathcal{F}_1) = \mathcal{F}_2$.

Write $\mathcal{P}$ for the composition of functors

$$\mathrm{D}\,\mathcal{W}\mathrm{eil}_a \overset{f_{\mathbb{H}}}{\to} \mathrm{D}\,\mathrm{P}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \overset{J^*_{P_{\mathbb{H}_a}}}{\to} \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)),$$

where $f_{\mathbb{H}}$ sends $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ to $\mathcal{F}_1$. By abuse of notation, we also write $\mathcal{P} : \mathrm{D}\,\mathcal{W}\mathrm{eil}_{a,k_0} \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_0))$ for the similary defined functor over k_0.

Proposition 6. *For $i = 1, 2$ let $K_i \in \mathrm{D}\,\mathcal{W}\mathrm{eil}_{a,k_0}$. If $\mathcal{P}(K_1) \widetilde{\to} \mathcal{P}(K_2)$ then $K_1 \widetilde{\to} K_2$ in $\mathrm{D}\,\mathcal{W}\mathrm{eil}_a$.*

Proof Write K_{k_0} (resp., DK_{k_0}) for the Grothendieck group of the category $\mathcal{W}\mathrm{eil}_{a,k_0}$ (resp., of $\mathrm{D}\,\mathcal{W}\mathrm{eil}_{a,k_0}$). Note that $DK_{k_0} \widetilde{\to} K_{k_0} \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$. Write ΥK_{k_0} for the Grothendieck group of the category of pure complexes of weight zero on $Q\Upsilon_a(F_0)$, whose all perverse cohomologies lie in $\mathrm{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_0))$. The functor $J^*_{P_{\mathbb{H}_a}}$ yields a homomoprhism $J^*_{P_{\mathbb{H}_a}} : DK_{k_0} \to \Upsilon K_{k_0}$. Let us show that it is injective. Let F be an objects in its kernel. For any finite subfield $k_0 \subset k_1 \subset k$ the map tr_{k_1} trace of Frobenius over k_1 fits into the diagram

$$\begin{tikzcd}
DK_{k_0} \arrow[r, "J^*_{P_{\mathbb{H}_a}}"] \arrow[d, "\mathrm{tr}_{k_1}"] & \Upsilon K_{k_0} \arrow[d, "\mathrm{tr}_{k_1}"] \\
\mathcal{W}\mathrm{eil}_a(k_1) \arrow[r, "J_{k_1}"] & \mathcal{S}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_1)),
\end{tikzcd} \tag{27}$$

where F_1 denotes the k_1-valued points of F. By (Lemma 11, Appendix A), J_{k_1} is injective, so $\mathrm{tr}_{k_1}(F) = 0$ for any finite extension $k_0 \subset k_1$. By the result of Laumon ([4], Theorem 1.1.2) this implies $F = 0$ in DK_{k_0}. Finally, if $K_1 = K_2$ in DK_{k_0} then $K_1 \widetilde{\to} K_2$ in $\mathrm{D}\,\mathcal{W}\mathrm{eil}_a$. $\square$

The following result will not be used in this paper, its proof is found in Appendix A.

Proposition A.1. *Assume $m > n$. The map $K_0({}_{-a}\,\mathrm{Sph}_{\mathbb{G}}) \otimes \bar{\mathbb{Q}}_\ell \to \mathcal{W}\mathrm{eil}_a(k_0)$ given by $\mathcal{S} \mapsto \mathrm{tr}_{k_0} \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)$ is an isomorphism of $\bar{\mathbb{Q}}_\ell$-vector spaces.*

Write $\mathcal{W}\mathrm{eil}^{ss}_a \subset \mathcal{W}\mathrm{eil}_a$ for the full subcategory of semi-simple objects.

Conjecture 1. *Assume $m > n$. The functor ${}_{-a}\,\mathrm{Sph}_{\mathbb{G}} \to \mathcal{W}\mathrm{eil}^{ss}_a$ given by $\mathcal{S} \to \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)$ is an equivalence of categories.*

4.8.8 Action of $\mathrm{Sph}_{\mathbb{H}}$

We write $V^{\check{\lambda}}$ for the irreducible $\mathbb{H}$-module with h.w. $\check{\lambda}$. Assume that V_0 is a $2m$-dimensional k-vector space with nondegenerate symmetric form $\mathrm{Sym}^2 V_0 \to C_0$, and $\mathbb{H}$ is the connected component of unity of $\mathrm{G}\mathbb{O}(V_0)$. Write $\check{\alpha}_0$ for the h.w. of the $\mathbb{H}$-module C_0. For $0 < i < m$ let $\check{\alpha}_i$ denote the h.w. of the irreducible $\mathbb{H}$-module $\wedge^i V_0$. Remind that

$$\wedge^m V_0 \widetilde{\to} V^{\check{\alpha}_m} \oplus V^{\check{\alpha}'_m}$$

is a direct sum of two irreducible representations, this is our definition of $\check{\alpha}_m, \check{\alpha}'_m$. Say that a maximal isotropic subspace $\mathcal{L} \subset V_0$ is *$\check{\alpha}_m$-oriented* (resp., *$\check{\alpha}'_m$-oriented*) if $\wedge^m \mathcal{L} \subset V^{\check{\alpha}_m}$ (resp., $\wedge^m \mathcal{L} \subset V^{\check{\alpha}'_m}$). The group $\mathbb{H}$ has two orbits on the set of maximal isotropic subspaces in V_0 given by the orientation.

Remind that $\mathrm{Gr}^b_{\mathbb{H}}$ classifies lattices $V' \subset V_0(F)$ such that the induced form $\mathrm{Sym}^2 V' \to C(b)$ is regular and nondegenerate, here $C = C_0(\mathcal{O})$.

Let $\lambda \in \Lambda^+_{\mathbb{H}}$, set $a = \langle \lambda, \check{\alpha}_0 \rangle$. Remind that $\mathcal{A}^\lambda_{\mathbb{H}} \in \mathrm{Sph}_{\mathbb{H}}$ denotes the IC-sheaf of $\overline{\mathrm{Gr}}^\lambda_{\mathbb{H}}$, so $\mathcal{A}^\lambda_{\mathbb{H}} \in {}_{-a}\,\mathrm{Sph}_{\mathbb{H}}$. By definition, the complex

$$\mathrm{H}^\lambda_{\mathbb{H}}(I_0) = \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{A}^\lambda_{\mathbb{H}}, I_0) \in \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a}(\Upsilon_a(F))$$

is as follows. Set $r = \langle \lambda, \check{\alpha}_1 \rangle$ and $N = \langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}_1 \rangle$. Let ${}_{0,r}\Upsilon \tilde{\times} \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{H}_a}$ be the scheme classifying $h \in \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{H}_a}$, $x \in L_a^* \otimes A_a \otimes ((hV_a)/V_a(-r))$. Let

$$\pi : {}_{0,r}\Upsilon \tilde{\times} \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{H}_a} \to {}_{N,r}\Upsilon_a \tag{28}$$

be the map sending $(x, h\mathbb{H}_a(\mathcal{O}))$ to x. Then $\mathrm{H}^{\lambda}_{\mathbb{H}}(I_0) \,\tilde{\to}\, \pi_!(\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^{\lambda}_{\mathbb{H}})[b]$ canonically, where b is the unique integer such that $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^{\lambda}_{\mathbb{H}}[b]$ is perverse.

View a point of $\Upsilon_a(F)$ as a map $L_a \otimes A_a^* \to V_a(F)$. Define a closed subscheme ${}_{\lambda}\Upsilon_a \subset \Upsilon_a(N)$ as follows. A point $v \in \Upsilon_a(N)$ lies in ${}_{\lambda}\Upsilon_a$ if the following conditions hold:

C1) $v \in \mathrm{Char}(\Upsilon_a)$;

C2) for $1 \le i < m$ the map $\wedge^i v : \wedge^i(L_a \otimes A_a^*) \to (\wedge^i V_a)(\langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}_i \rangle)$ is regular;

C3) the map $(v_m, v'_m) : \wedge^m(L_a \otimes A_a^*) \to V_a^{\check{\alpha}_m}(\langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}_m \rangle) \oplus V_a^{\check{\alpha}'_m}(\langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}'_m \rangle)$ induced by $\wedge^m v$ is regular.

The scheme ${}_{\lambda}\Upsilon_a$ is stable under translations by $\Upsilon_a(-r)$, so there is a closed subscheme ${}_{\lambda,N}\Upsilon_a \subset {}_{N,r}\Upsilon_a$ such that ${}_{\lambda}\Upsilon_a$ is the preimage of ${}_{\lambda,N}\Upsilon_a$ under the projection $\Upsilon_a(N) \to {}_{N,r}\Upsilon_a$. Clearly, the map (28) factors through the closed subscheme ${}_{\lambda,N}\Upsilon_a \subset {}_{N,r}\Upsilon_a$.

For each $v \in \mathrm{Char}(\Upsilon_a)$ define a $\mathcal{O}$-lattice $V_v \subset V_a(F)$ as follows. For a $\mathcal{O}$-lattice $R \subset V_a(F)$ set

$$R^{\perp} = \{x \in V_a(F) \mid \langle x, y \rangle \in C_a(-a) \quad \text{for all } \; y \in R\}$$

Consider two cases.

CASE: a is even. For $v \in \mathrm{Char}(\Upsilon_a)$ set $R_v = v(L_a \otimes A_a^*) + V_a(-\frac{a}{2})$ and $V_v = v(L_a \otimes A_a^*) + R_v^{\perp}$. Then $V_v \in \mathrm{Gr}^{-a}_{\mathbb{H}}$. In this case we get a stratification of $\mathrm{Char}(\Upsilon_a)$ by locally closed subschemes ${}_{\lambda}\mathrm{Char}(\Upsilon_a)$ indexed by $\{\lambda \in \Lambda^+_{\mathbb{H}} \mid \langle \lambda, \check{\alpha}_0 \rangle = a\}$. Namely, $v \in \mathrm{Char}(\Upsilon_a)$ lies in ${}_{\lambda}\mathrm{Char}(\Upsilon_a)$ iff $V_v \in \mathrm{Gr}^{\lambda}_{\mathbb{H}}$.

Clearly, ${}_{\lambda}\mathrm{Char}(\Upsilon_a) \subset {}_{\lambda}\Upsilon_a$. There is a unique open subscheme ${}_{\lambda,N}\Upsilon^0_a \subset {}_{\lambda,N}\Upsilon_a$ whose preimage under the projection ${}_{\lambda}\Upsilon_a \to {}_{\lambda,N}\Upsilon_a$ equals ${}_{\lambda,N}\Upsilon^0_a$.

CASE: a is odd. Let $b = (-a-1)/2$. We have $(V_a(b+1))^{\perp} = V_a(b)$. Set $R_v = v(L_a \otimes A_a^*) + V_a(b+1)$ and $V_v = v(L_a \otimes A_a^*) + R_v^{\perp}$. Then the induced form $\mathrm{Sym}^2 V_v \to C_a(-a)$ is regular, but still can be degenerate. We call v *generic* if the form $\mathrm{Sym}^2 V_v \to C_a(-a)$ is nondegenerate. In this case $V_v \in \mathrm{Gr}^{-a}_{\mathbb{H}}$.

For a odd define an open subscheme ${}_{\lambda}\mathrm{Char}(\Upsilon_a) \subset {}_{\lambda}\Upsilon_a$ as follows. Note that $\langle w_0^{\mathbb{H}}(\lambda), \check{\alpha}_m - \check{\alpha}'_m \rangle \neq 0$. A point $v \in {}_{\lambda}\Upsilon_a$ lies in ${}_{\lambda}\mathrm{Char}(\Upsilon_a)$ if the following conditions hold:

- the maps in C2) are maximal;
- if $\langle w_0^{\mathbb{H}}(\lambda), \check{\alpha}_m - \check{\alpha}'_m \rangle < 0$ then v_m in C3) is maximal, otherwise v'_m in C3) is maximal.

There is a unique open subscheme ${}_{\lambda,N}\Upsilon^0_a \subset {}_{\lambda,N}\Upsilon_a$ whose preimage under the projection ${}_{\lambda}\Upsilon_a \to {}_{\lambda,N}\Upsilon_a$ equals ${}_{\lambda}\operatorname{Char}(\Upsilon_a)$.

Write $\mathrm{IC}({}_{\lambda,N}\Upsilon^0_a)$ for the intersection cohomology sheaf of ${}_{\lambda,N}\Upsilon^0_a$.

Proposition 7. *Let* $\lambda \in \Lambda^+_{\mathbb{H}}$ *with* $\langle \lambda, \check{\alpha}_0\rangle = a$.
1) The map

$$\pi : {}_{0,r}\Upsilon \tilde{\times} \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{H}_a} \to {}_{\lambda,N}\Upsilon_a$$

is an isomorphism over the open subscheme ${}_{\lambda,N}\Upsilon^0_a$.
2) Assume $m \le n$ *then one has a canonical isomorphism* $\mathrm{H}^{\lambda}_{\mathbb{H}}(I_0) \xrightarrow{\sim} \mathrm{IC}({}_{\lambda,N}\Upsilon^0_a)$.

Proof 1) Let $v \in {}_{\lambda,N}\Upsilon^0_a$. The fibre of π over v is the scheme classifying lattices $V' \in \overline{\mathrm{Gr}}^{\lambda}_{\mathbb{H}_a}$ such that $v(L_a \otimes A^*_a) \subset V'$. Given such a lattice V' let us show that $V_v = V'$.

In view of Remark 7 the case of a even is reduced to the case $a = 0$, and the latter is done in ([7], Lemma 14).

Consider the case of a odd. The inclusion $R_v \subset V' + V_a(b+1)$ must be an equality, because for $V' \in \mathrm{Gr}^{\mu}_{\mathbb{H}_a}$ with $\mu \le \lambda$ we have

$$\dim(V' + V_a(b+1))/(V_a(b+1)) = \epsilon(\mu) \le \epsilon(\lambda) = \dim R_v/(V_a(b+1))$$

We have denoted here $\epsilon(\mu) = -m(b+1) + \max\{\langle -w^{\mathbb{H}}_0(\mu), \check{\alpha}_m\rangle, \langle -w^{\mathbb{H}}_0(\mu), \check{\alpha}'_m\rangle\}$.

It follows that $V_v = v(L_a \otimes A^{-1}_a) + (V' \cap V_a(b)) \subset V'$. To prove that $V' = V_v$, it suffices to show that v is generic. This follows from the fact that $(v(L_a \otimes A^{-1}_a) + R^{\perp}_v)/R^{\perp}_v$ is a maximal isotropic subspace in $R_v/R^{\perp}_v$.

2) For $m \le n$ the scheme ${}_{\lambda,N}\Upsilon^0_a$ is nonempty, so $\mathrm{IC}({}_{\lambda,N}\Upsilon^0_a)$ appears in $\mathrm{H}^{\lambda}_{\mathbb{H}}(I_0)$ with multiplicity one. Now it remains to show that

$$\mathrm{Hom}(\mathrm{H}^{\lambda}_{\mathbb{H}}(I_0), \mathrm{H}^{\lambda}_{\mathbb{H}}(I_0)) = \bar{\mathbb{Q}}_{\ell},$$

where Hom is taken in the derived category $\mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$. By adjointness,

$$\mathrm{Hom}(\mathrm{H}^{\lambda}_{\mathbb{H}}(I_0), \mathrm{H}^{\lambda}_{\mathbb{H}}(I_0)) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{H}^{-w^{\mathbb{H}}_0(\lambda)}_{\mathbb{H}} \mathrm{H}^{\lambda}_{\mathbb{H}}(I_0), I_0),$$

where Hom in the RHS is taken in $\mathrm{D}_{\mathbb{H}Q\mathbb{G}_0(\mathcal{O})}(\Upsilon_0(F))$. We are reduced to show that for any $0 \ne \mu \in \Lambda^+_{\mathbb{H}}$ with $\langle \mu, \check{\alpha}_0\rangle = 0$ one has

$$\mathrm{Hom}(\mathrm{H}^{\mu}_{\mathbb{H}}(I_0), I_0) = 0$$

in $\mathrm{D}_{\mathbb{H}Q\mathbb{G}_0(\mathcal{O})}(\Upsilon_0(F))$. For $m \le n$ this is proved in ([7], part 2) of Lemma 14). $\square$

As in the case $m > n$, assume for a moment that $k_0 \subset k$ is a finite subfield, and all the objects introduced in Section 4 have a k_0-structure. The following result is analogous to Proposition A.1, its proof is omited.

Proposition A.2. *Assume* $m \le n$. *Then the map* $K_0({}_{-a}\mathrm{Sph}_{\mathbb{H}}) \otimes \bar{\mathbb{Q}}_{\ell} \to \mathcal{W}\mathrm{eil}_a(k_0)$ *given by* $\mathcal{S} \mapsto \mathrm{tr}_{k_0} \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, I_0)$ *is an isomorphism of* $\bar{\mathbb{Q}}_{\ell}$*-vector spaces.*

Conjecture 2. *Assume $m \le n$. The functor ${}_a\mathrm{Sph}_{\mathbb{H}} \to \mathcal{W}\mathrm{eil}^{ss}_a$ given by $\mathcal{S} \mapsto \mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{S}, I_0)$ is an equivalence of categories.*

4.8.9 *Proof of Theorem 4*

Use the notations of Section 4.8.7 and 4.8.8. Assume that U_0 is $\check{\alpha}_m$-oriented. Below we identify $\check{\omega}_0 : \mathbb{G}_m \widetilde{\to} \check{\mathrm{G}}\mathrm{L}(A_0)$ and $\check{\alpha}_0 : \mathbb{G}_m \widetilde{\to} \check{\mathrm{G}}\mathrm{L}(C_0)$.

For $m > n$ fix a decomposition $U_0 \widetilde{\to} {}_1U \oplus {}_2U$ into direct sum of free $\mathcal{O}$-modules, where ${}_1U$ is of rank n, and ${}_2U$ is of rank $m-n$, fix also an isomorphism ${}_1U \widetilde{\to} L_0$ of $\mathcal{O}$-modules. We assume that these choices are compatible with the maximal tori chosen before. For $m > n$ let $\kappa_0 : \check{\mathrm{G}}\mathrm{L}(L_0) \times \mathbb{G}_m \to \check{\mathrm{G}}\mathrm{L}(U_0)$ be the composition

$$\check{\mathrm{G}}\mathrm{L}(L_0) \times \mathbb{G}_m \overset{\tau\times\mathrm{id}}{\to} \check{\mathrm{G}}\mathrm{L}(L_0) \times \mathbb{G}_m = \check{\mathrm{G}}\mathrm{L}({}_1U) \times \mathbb{G}_m \overset{\mathrm{id}\times 2\check{\rho}_{\mathrm{GL}({}_2U)}}{\to} \check{\mathrm{G}}\mathrm{L}({}_1U) \times \check{\mathrm{G}}\mathrm{L}({}_2U) \overset{\mathrm{Levi}}{\to} \check{\mathrm{G}}\mathrm{L}(U_0),$$

where τ is an automorphism of $\check{\mathrm{G}}\mathrm{L}(L_0)$ inducing the functor $* : \mathrm{Sph}_{\mathrm{GL}(L_0)} \widetilde{\to} \mathrm{Sph}_{\mathrm{GL}(L_0)}$.

Let $\kappa_Q : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \to \check{Q}(\mathbb{H})$ be the map

$$\check{\mathrm{G}}\mathrm{L}(L_0) \times \check{\mathrm{G}}\mathrm{L}(A_0) \times \mathbb{G}_m \to \check{\mathrm{G}}\mathrm{L}(U_0) \times \check{\mathrm{G}}\mathrm{L}(C_0)$$

given by $(x, y, z) \mapsto (\kappa_0(x, z), y\omega_m(x))$. Here ω_m is the unique coweight of the center of $\mathrm{GL}(L_0)$ such that $\langle \omega_m, \check{\omega}_1 \rangle = 1$.

Write $\kappa_{Q,ex} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \to \check{Q}(\mathbb{H}) \times \mathbb{G}_m$ for the map (κ_Q, pr), where $\mathrm{pr} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \to \mathbb{G}_m$ is the projection.

For $m \le n$ fix a decomposition $L_0 \widetilde{\to} {}_1L \oplus {}_2L$ into direct sum of free $\mathcal{O}$-modules, where ${}_1L$ is of rank m, and ${}_2L$ is of rank $n-m$, fix also an isomorphism $U_0 \widetilde{\to} {}_1L$ of $\mathcal{O}$-modules. We assume that these choices are compatible with the maximal tori chosen before. For $m \le n$ we let $\kappa_0 : \check{\mathrm{G}}\mathrm{L}(U_0) \times \mathbb{G}_m \to \check{\mathrm{G}}\mathrm{L}(L_0)$ be the composition

$$\check{\mathrm{G}}\mathrm{L}(U_0) \times \mathbb{G}_m = \check{\mathrm{G}}\mathrm{L}({}_1L) \times \mathbb{G}_m \overset{\mathrm{id}\times 2\check{\rho}_{\mathrm{GL}({}_2L)}}{\to} \check{\mathrm{G}}\mathrm{L}({}_1L) \times \check{\mathrm{G}}\mathrm{L}({}_2L) \overset{\mathrm{Levi}}{\to} \check{\mathrm{G}}\mathrm{L}(L_0) \overset{\tau}{\to} \check{\mathrm{G}}\mathrm{L}(L_0),$$

here τ is an automorphism inducing the functor $* : \mathrm{Sph}_{\mathrm{GL}(L_0)} \widetilde{\to} \mathrm{Sph}_{\mathrm{GL}(L_0)}$.

Let $\kappa_Q : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \to \check{Q}(\mathbb{G})$ be the map

$$\check{\mathrm{G}}\mathrm{L}(U_0) \times \check{\mathrm{G}}\mathrm{L}(C_0) \times \mathbb{G}_m \to \check{\mathrm{G}}\mathrm{L}(L_0) \times \check{\mathrm{G}}\mathrm{L}(A_0)$$

given by $(x, y, z) \mapsto (\kappa_0(x, z), y\alpha_m(x))$. Here α_m is the unique coweight of the center of $\mathrm{GL}(U_0)$ such that $\langle \alpha_m, \check{\alpha}_1 \rangle = 1$.

Define $\kappa_{Q,ex} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \to \check{Q}(\mathbb{G}) \times \mathbb{G}_m$ as (κ_Q, pr). The following is a consequence of ([7], Corollary 5).

Proposition 8. *1) For $m > n$ the two functors ${}_{-a}\mathrm{Sph}_{Q(\mathbb{H})} \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ given by*

$$\mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathcal{T}, I_0) \quad \textit{and} \quad \mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathrm{gRes}^{\kappa_Q}(\mathcal{T}), I_0)$$

are isomorphic.
2) For $m \le n$ the two functors ${}_{-a}\mathrm{Sph}_{Q(\mathbb{G})} \to \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ given by

$$\mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathcal{T}, I_0) \quad \textit{and} \quad \mathcal{T} \mapsto \mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathrm{gRes}^{\kappa_Q}(\mathcal{T}), I_0)$$

are isomorphic. $\square$

As in ([7], Theorem 7), for each $a \in \mathbb{Z}$ the diagram of functors is canonically 2-commutative

$$\begin{tikzcd} & \mathrm{D}\mathcal{W}\mathrm{eil}_a \arrow[dl, "f_{\mathbb{H}}"] \arrow[dr, "f_{\mathbb{G}}"] & \\ \mathrm{DP}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \arrow[d, "J^*_{P_{\mathbb{H}_a}}"] & & \mathrm{DP}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \arrow[d, "J^*_{P_{\mathbb{G}_a}}"] \\ \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \arrow[rr, "\mathrm{Four}_\psi"] & & \mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)), \end{tikzcd}$$

where $f_{\mathbb{H}}$ (resp., $f_{\mathbb{G}}$) sends $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ to $\mathcal{F}_1$ (resp., to $\mathcal{F}_2$).

Remind the maps $\kappa_{\mathbb{H}} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \to \check{H}$ and $\kappa_{\mathbb{G}} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \to \check{\mathbb{G}}$ from Section 4.8.6. The restriction of $\kappa_{\mathbb{H}}$ and of $\kappa_{\mathbb{G}}$ to $\mathbb{G}_m$ equals

$$2(\check{\rho}_{\mathbb{H}} - \check{\rho}_{Q(\mathbb{H})}) + nm\check{\alpha}_0 - n\check{\alpha}_m$$

and $2(\check{\rho}_{\mathbb{G}} - \check{\rho}_{Q(\mathbb{G})}) + mn\check{\omega}_0 - m\check{\omega}_n$ respectively. From definitions one gets

$$\left\{ \begin{array}{l} 2(\check{\rho}_{\mathbb{H}} - \check{\rho}_{Q(\mathbb{H})}) = (m-1)\check{\alpha}_m - \frac{m(m-1)}{2}\check{\alpha}_0 \\ \\ 2(\check{\rho}_{\mathbb{G}} - \check{\rho}_{Q(\mathbb{G})}) = (n+1)\check{\omega}_n - \frac{n(n+1)}{2}\check{\omega}_0 \end{array} \right.$$

Write $\kappa_{\mathbb{H},ex} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \to \check{H} \times \mathbb{G}_m$ for the map, whose first component is $\kappa_{\mathbb{H}}$ and the second $\check{Q}(\mathbb{H}) \times \mathbb{G}_m \to \mathbb{G}_m$ is the projection, and similarly for $\kappa_{\mathbb{G},ex}$.

By Corollary 3, for $\mathcal{T} \in {}_{-a}\mathrm{Sph}_{\mathbb{H}}$ and $\mathcal{S} \in {}_{-a}\mathrm{Sph}_{\mathbb{G}}$ we get isomorphisms

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, I_0)) \widetilde{\to} \mathrm{H}^{\leftarrow}_{Q(\mathbb{H})}(\mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{T}), I_0) \tag{29}$$

and

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)) \widetilde{\to} \mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathrm{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), I_0) \tag{30}$$

in $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$. The Hecke functors in the RHS of (29) and (30) are from $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_0(\mathcal{O})}(Q\Upsilon_0(F))$ to the category $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$.

CASE $m > n$. Proposition 8 together with (29) yields an isomorphism

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, I_0)) \widetilde{\to} \mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathrm{gRes}^{\kappa_{Q,ex}} \mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{T}), I_0)$$

We will define an automorphism $\tau_{\mathbb{H}}$ of $\check{\mathbb{H}}$ inducing $* : \mathrm{Rep}(\check{\mathbb{H}}) \widetilde{\to} \mathrm{Rep}(\check{\mathbb{H}})$ and κ making the following diagram commutative

$$\begin{tikzcd} \check{\mathbb{G}} \times \mathbb{G}_m \arrow[r, "\tau_{\mathbb{H}} \circ \kappa"] & \check{\mathbb{H}} \\ \check{Q}(\mathbb{G}) \times \mathbb{G}_m \arrow[u, "\kappa_{\mathbb{G},ex}"] \arrow[r, "\kappa_{Q,ex}"] & \check{Q}(\mathbb{H}) \times \mathbb{G}_m \arrow[u, "\kappa_{\mathbb{H}}"] \end{tikzcd} \tag{31}$$

The above diagram together with (30) yield isomorphisms

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)\,\widetilde{\to}\,\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathrm{gRes}^{\kappa_{\mathbb{G}}}\,\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)\,\widetilde{\to}\,\mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathrm{gRes}^{\kappa_{Q,ex}}\,\mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{T}), I_0)$$

Thus, we get an isomorphism

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0))\,\widetilde{\to}\,\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, I_0))$$

By Proposition 6, it lifts to the desired isomorphism in $\mathrm{D}\,\mathcal{W}\mathrm{eil}_a$

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)\,\widetilde{\to}\,\mathrm{H}^{\leftarrow}_{\mathbb{H}}(\mathcal{T}, I_0)$$

Note that $m \geq 2$. Let $W_0 = \bar{\mathbb{Q}}_\ell^m$, let W_1 (resp., W_2) be the subspace of W_1 spanned by the first n (resp., last $m-n$) base vectors. Equip $W_0 \oplus W_0^*$ with the symmetric form given by the matrix

$$\left(\begin{array}{cc} 0 & E_m \\ E_m & 0 \end{array}\right),$$

where $E_m \in \mathrm{GL}(\bar{\mathbb{Q}}_\ell)$ is the unity. Let $i_{\mathbb{H}} \in \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)$ be the unique central element such that

$$\mathbb{SO}(W_0 \oplus W_0^*)\,\widetilde{\to}\,\mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)/\{i_{\mathbb{H}}\}$$

Realize $\check{\mathbb{H}}$ as $\mathrm{G}\mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*) := \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)/\{(-1, i_{\mathbb{H}})\}$. There is a unique automorphism τ' of $\mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)$ that preserves $i_{\mathbb{H}}$ and induces the automorphism $g \mapsto {}^t g^{-1}$ on $\mathbb{SO}(W_0 \oplus W_0^*)$. The automorphism $(a, g) \mapsto (a^{-1}, \tau'(g))$ of $\mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)$ descends to an automorphism of $\check{\mathbb{H}}$ that we denote $\tau_{\mathbb{H}}$.

Let $\bar{W} \subset W_2 \oplus W_2^*$ be the subspace spanned by $e_{n+1} + e_{n+1}^*$. Equip $W_1 \oplus W_1^* \oplus \bar{W}$ with the induced form. Write $i_{\mathbb{G}}$ for the central element of $\mathbb{S}\mathrm{pin}(W_1 \oplus W_1^* \oplus \bar{W})$. Realize $\check{\mathbb{G}}$ as

$$\mathrm{G}\mathbb{S}\mathrm{pin}(W_1 \oplus W_1^* \oplus \bar{W}) := \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^* \oplus \bar{W})/\{(-1, i_{\mathbb{G}})\}$$

There is a unique inclusion $\epsilon_0 : \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^* \oplus \bar{W}) \hookrightarrow \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)$ extending the natural inclusion $\mathbb{SO}(W_1 \oplus W_1^* \oplus \bar{W}) \hookrightarrow \mathbb{SO}(W_0 \oplus W_0^*)$ and sending $i_{\mathbb{G}}$ to $i_{\mathbb{H}}$. The map

$$\mathrm{id} \times \epsilon_0 : \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^* \oplus \bar{W}) \to \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^*)$$

gives rise to an inclusion $i_\kappa : \check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$. Finally, there is a unique $\alpha_\kappa : \mathbb{G}_m \to \check{\mathbb{H}}$ such that for $\kappa := (i_\kappa, \alpha_\kappa)$ the diagram (31) commutes. The map $\tau_{\mathbb{H}} \circ \kappa : \check{T}_{\mathbb{G}} \to \check{T}_{\mathbb{H}}$ is uniquely defined by the formulas

$$\left\{\begin{array}{l} \check{\omega}_i \mapsto -\check{\alpha}_i + i\check{\alpha}_0, \quad 1 \leq i \leq n \\ \check{\omega}_0 \mapsto \check{\alpha}_0 \end{array}\right.$$

Using these formulas, one checks that

$$\tau_{\mathbb{H}}(\alpha_\kappa) = 2\check{\rho}_{\mathrm{GL}({}_2U)} + (m-1-n)(\check{\alpha}_m - \check{\alpha}_n) + (\frac{n(n+1)}{2} - \frac{m(m-1)}{2} - (n+1-m)n)\check{\alpha}_0$$

If $m = n+1$ then α_κ is trivial.

CASE $m \le n$. Proposition 8 together with (30) yields an isomorphism

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)) \widetilde{\to} \mathrm{H}^{\leftarrow}_{\check{Q}(\mathbb{H})}(\mathrm{gRes}^{\kappa_{Q,ex}} \mathrm{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), I_0)$$

We will define an automorphism $\tau_{\mathbb{H}}$ of $\check{\mathbb{H}}$ inducing $* : \mathrm{Rep}(\check{\mathbb{H}}) \widetilde{\to} \mathrm{Rep}(\check{\mathbb{H}})$ and κ making the following diagram commutative

$$\begin{tikzcd}
\check{\mathbb{H}} \times \mathbb{G}_m \arrow[r, "\tau_{\mathbb{H}} \times \mathrm{id}"] & \check{\mathbb{H}} \times \mathbb{G}_m \arrow[r, "\kappa"] & \check{\mathbb{G}} \\
\check{Q}(\mathbb{H}) \times \mathbb{G}_m \arrow[u, "\kappa_{\mathbb{H},ex}"'] \arrow[rr, "\kappa_{Q,ex}"] & & \check{Q}(\mathbb{G}) \times \mathbb{G}_m \arrow[u, "\kappa_{\mathbb{G}}"']
\end{tikzcd} \tag{32}$$

The above diagram together with (29) yield isomorphisms

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{H}}(* \,\mathrm{gRes}^{\kappa}(\mathcal{S}), I_0)) \widetilde{\to} \mathrm{H}^{\leftarrow}_{\check{Q}(\mathbb{H})}(\mathrm{gRes}^{\kappa_{\mathbb{H}}}(* \,\mathrm{gRes}^{\kappa}(\mathcal{S})), I_0) \widetilde{\to} \mathrm{H}^{\leftarrow}_{\check{Q}(\mathbb{H})}(\mathrm{gRes}^{\kappa_{Q,ex}} \mathrm{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), I_0)$$

Thus, we get

$$\mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{H}}(* \,\mathrm{gRes}^{\kappa}(\mathcal{S}), I_0)) \widetilde{\to} \mathcal{P}(\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0))$$

By Proposition 6, it lifts to the desired isomorphism in $\mathrm{D}\,\mathcal{W}\mathrm{eil}_a$

$$\mathrm{H}^{\leftarrow}_{\mathbb{H}}(* \,\mathrm{gRes}^{\kappa}(\mathcal{S}), I_0) \widetilde{\to} \mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, I_0)$$

For $m = 1$ the map $\kappa_{\mathbb{H},ex}$ is an isomorphism, so there is a unique κ making (32) commutative. Now assume $m > 1$. Let $W_0 = \bar{\mathbb{Q}}_\ell^n$, let W_1 (resp., W_2) be the subspace of W_0 generated by the first m (resp., last $n - m$) vectors. Equip $W_0 \oplus W_0 \oplus \bar{\mathbb{Q}}_\ell$ with the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E_n \in \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ is the unity. Write $i_{\mathbb{G}}$ for nontrivial the central element of $\mathbb{S}\mathrm{pin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$. Realize $\check{\mathbb{G}}$ as

$$\mathrm{G}\mathbb{S}\mathrm{pin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell) := \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)/\{(-1, i_{\mathbb{G}})\}$$

Equip the subspace $W_1 \oplus W_1^* \subset W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell$ with the induced symmetric form. Write $i_{\mathbb{H}}$ for the unique central element of $\mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*)$ such that $\mathbb{SO}(W_1 \oplus W_1^*) \widetilde{\to} \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*)/\{i_{\mathbb{H}}\}$. Realize $\check{\mathbb{H}}$ as

$$\mathrm{G}\mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*) := \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*)/\{(-1, i_{\mathbb{H}})\}$$

There is a unique automorphism τ' of $\mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*)$ preserving $i_{\mathbb{H}}$ and inducing the map $g \mapsto {}^t g^{-1}$ on $\mathbb{SO}(W_1 \oplus W_1^*)$. The automorphism $(a, g) \mapsto (a^{-1}, \tau'(g))$ of $\mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*)$ descends to an automorphism of $\check{\mathbb{H}}$ that we denote $\tau_{\mathbb{H}}$.

There is a unique inclusion

$$\epsilon_0 : \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*) \hookrightarrow \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$$

sending $i_{\mathbb{H}}$ to $i_{\mathbb{G}}$ and extending the natural inclusion $\mathbb{S}\mathbb{O}(W_1 \oplus W_1^*) \hookrightarrow \mathbb{S}\mathbb{O}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$. The map

$$\mathrm{id} \times \epsilon_0 : \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_1 \oplus W_1^*) \to \mathbb{G}_m \times \mathbb{S}\mathrm{pin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$$

gives rise to an inclusion $i_\kappa : \check{\mathbb{H}} \to \check{\mathbb{G}}$. Now there is a unique $\alpha_\kappa : \mathbb{G}_m \to \check{\mathbb{G}}$ such that for $\kappa = (i_\kappa, \alpha_\kappa)$ the diagram (32) commutes. The map $\kappa \circ \tau_{\mathbb{H}} : \check{T}_{\mathbb{H}} \to \check{T}_{\mathbb{G}}$ is uniquely defined by the formulas

$$\begin{cases} \check{\alpha}_i \mapsto -\check{\omega}_i + i\check{\omega}_0, \quad 1 \le i \le m \\ \check{\alpha}_0 \mapsto \check{\omega}_0 \end{cases}$$

From this formulas one gets that

$$\alpha_\kappa = (n+1-m)(\check{\omega}_n - \check{\omega}_m) + (mn - \frac{m(m-1)}{2} - \frac{n(n+1)}{2})\check{\omega}_0 - 2\check{\rho}_{GL({}_2L)}$$

In particular, α_κ is trivial for $m = n$. Theorem 4 is proved.

5. GLOBAL THEORY

5.1 In Sections 5.1-5.2 we derive Theorem 2 from Theorem 3. To simplify notations, fix a closed point $\tilde{x} \in \tilde{X}$. Let ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$ be obtained from ${}^{a}\mathrm{Bun}_{G,\tilde{H}}$ by the base change $\tilde{x} \to \tilde{X}$. We will establish isomorphisms (4) and (5) over ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$. The fact that these isomorphisms depend on $\tilde{x}$ as expected is left to the reader. Set $x = \pi(\tilde{x})$.

Recall the line bundle $\mathcal{E}$ from Section 2.3, we have $\pi^*\mathcal{E} \widetilde{\rightarrow} \mathcal{O}_{\tilde{X}}$ canonically. So, the above choice of $\tilde{x}$ yields a trivialization $\mathcal{E} \widetilde{\rightarrow} \mathcal{O} \mid_{D_x}$ over $D_x = \mathrm{Spec}\, \mathcal{O}_x$. The corresponding trivialization for $\sigma\tilde{x}$ is the previous one multiplied by -1. We will apply Theorem 3 for $\mathcal{O} = \mathcal{O}_x$.

Recall the stack ${}^{a}\mathcal{X}\mathcal{L}$ and a line bundle ${}^{a}\mathcal{A}_{\mathcal{X}\mathcal{L}}$ on it introduced in Section 4.2. A point of ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$ is given by a collection: $(M, \mathcal{A}) \in \mathrm{Bun}_G$, $(V, \mathcal{C}) \in \mathrm{Bun}_{\tilde{H}}$, and an isomorphism $\mathcal{A} \otimes \mathcal{C} \widetilde{\rightarrow} \Omega(ax)$. Let ${}^{a}\xi : {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \to {}^{a}\mathcal{X}\mathcal{L}$ be the map sending $(M, \mathcal{A}, V, \mathcal{C})$ to $(M, \mathcal{A}, V, \mathcal{C}) \mid_{D_x}$ together with the discrete lagrangian subspace $L = \mathrm{H}^0(X - x, M \otimes V) \subset M \otimes V(F_x)$.

Lemma 9. *For a point $(\mathcal{M}, \mathcal{A}, \mathcal{V}, \mathcal{C})$ of ${}^{a\tilde{x}}\mathrm{Bun}_{\mathbb{G},\mathbb{H}}$ there is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism*

$$\det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V}) \otimes \mathcal{C}_x^{-anm} \widetilde{\rightarrow} \frac{\det \mathrm{R}\Gamma(X, \mathcal{M})^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{V})^{2n}}{\det \mathrm{R}\Gamma(X, \mathcal{C})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm}}$$

Here $\mathcal{C}_x$ is of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded.

Proof By ([5], Lemma 1), we get a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V}) \widetilde{\rightarrow} \frac{\det \mathrm{R}\Gamma(X, \mathcal{M})^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{V})^{2n}}{\det \mathrm{R}\Gamma(X, \mathcal{A}^n) \otimes \det \mathrm{R}\Gamma(X, \det \mathcal{V})} \otimes \frac{\det \mathrm{R}\Gamma(X, \mathcal{A}^n \otimes \det \mathcal{V})}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{4nm-1}}$$

Applying this to $\mathcal{M} = \mathcal{O}^n \oplus \mathcal{A}^n$ with natural symplectic form $\wedge^2 \mathcal{M} \to \mathcal{A}$, we get

$$\frac{\det \mathrm{R}\Gamma(X, \mathcal{V} \otimes \mathcal{A})^n}{\det \mathrm{R}\Gamma(X, \mathcal{V})^n} \widetilde{\rightarrow} \frac{\det \mathrm{R}\Gamma(X, \mathcal{A})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{A}^n \otimes \det \mathcal{V})}{\det \mathrm{R}\Gamma(X, \mathcal{A}^n) \otimes \det \mathrm{R}\Gamma(X, \det \mathcal{V}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm-1}}$$

Since $\mathcal{A}\otimes\mathcal{C}\,\widetilde{\to}\,\Omega(ax)$ and $\mathcal{V}\,\widetilde{\to}\,\mathcal{V}^*\otimes\mathcal{C}$, the LHS of the above formula idetifies with

$$\det\mathrm{R}\Gamma(X,V/V(-ax))^{-n}\,\widetilde{\to}\,(\det V_x)^{-an}\otimes\det(\mathcal{O}/\mathcal{O}(-ax))^{-2mn}$$

We have used a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det(V/V(-ax))\,\widetilde{\to}\,(\det V_x)^a\otimes(\det(\mathcal{O}/\mathcal{O}(-ax))^{2m}$$

Since $\det V_x\,\widetilde{\to}\,\mathcal{C}_x^m$, we get

$$\det\mathrm{R}\Gamma(X,\mathcal{M}\otimes\mathcal{V})\,\widetilde{\to}\,\frac{\det\mathrm{R}\Gamma(X,\mathcal{M})^{2m}\otimes\det\mathrm{R}\Gamma(X,\mathcal{V})^{2n}}{\det\mathrm{R}\Gamma(X,\mathcal{A})^{2nm}\otimes\det\mathrm{R}\Gamma(X,\mathcal{O})^{2nm}\otimes\det\mathrm{R}\Gamma(X,\mathcal{O}/\mathcal{O}(-ax))^{2nm}}\otimes\mathcal{C}_x^{-anm}$$

To simplify the above expression, note that $\det\mathrm{R}\Gamma(X,\mathcal{A})\,\widetilde{\to}\,\det\mathrm{R}\Gamma(X,\mathcal{C}(-ax))$ and

$$\det\mathrm{R}\Gamma(X,\mathcal{C})\,\widetilde{\to}\,\det\mathrm{R}\Gamma(X,\mathcal{C}(-ax))\otimes\mathcal{C}_x^a\otimes\det\mathrm{R}\Gamma(X,\mathcal{O}/\mathcal{O}(-ax))$$

Our assertion follows. □

Let ${}^a\mathcal{A}$ be the line bundle on ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$ with fibre $\det\mathrm{R}\Gamma(X,M\otimes V)\otimes C_x^{-anm}$ at $(M,\mathcal{A},V,\mathcal{C})$. We have canonically $({}^a\xi)^*({}^a\mathcal{A}_{\mathcal{XL}})\,\widetilde{\to}\,{}^a\mathcal{A}$. Extend ${}^a\xi$ to a morphism ${}^a\tilde{\xi}:{}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}\to{}^a\widetilde{\mathcal{XL}}$ sending $(M,\mathcal{A},V,\mathcal{C})$ to its image under ${}^a\xi$ together with the one-dimensional space

$$\mathcal{B}=\frac{\det\mathrm{R}\Gamma(X,\mathcal{M})^m\otimes\det\mathrm{R}\Gamma(X,\mathcal{V})^n}{\det\mathrm{R}\Gamma(X,\mathcal{C})^{nm}\otimes\det\mathrm{R}\Gamma(X,\mathcal{O})^{nm}}$$

equipped with the isomorphism $\mathcal{B}^2\,\widetilde{\to}\,\det\mathrm{R}\Gamma(X,\mathcal{M}\otimes\mathcal{V})\otimes\mathcal{C}_x^{-anm}$ of Lemma 9.

5.2 Let ${}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}}$ be the stack classifying collections: a point of the Hecke stack $(M,\mathcal{A},M',\mathcal{A}',\beta)\in{}_x\mathcal{H}_G$ such that the isomorphism β of the G-torsors $(M,\mathcal{A})$ and $(M',\mathcal{A}')$ over $X-x$ induces an isomorphism $\mathcal{A}(-ax)\,\widetilde{\to}\,\mathcal{A}'$; a $\tilde{H}$-torsor $(V,\mathcal{C})\in\mathrm{Bun}_{\tilde{H}}$, and an isomorphism $\mathcal{A}'\otimes\mathcal{C}\,\widetilde{\to}\,\Omega$. We have the diagram

$${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}\xleftarrow{h^{\leftarrow}}{}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}}\xrightarrow{h^{\rightarrow}}\mathrm{Bun}_{G,\tilde{H}},$$

where $h^{\rightarrow}$ (resp., $h^{\leftarrow}$) sends the above point of ${}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}}$ to $(M',\mathcal{A}',V,\mathcal{C})\in\mathrm{Bun}_{G,\tilde{H}}$ (resp., to $(M,\mathcal{A},V,\mathcal{C})\in{}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$).

Restriction to D_x gives rise to the diagram

$$\begin{tikzcd}
{}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \arrow[d, "{}^a\xi"] & {}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] \arrow[d, "{}^a\xi_G"] & \mathrm{Bun}_{G,\tilde{H}} \arrow[d, "{}^0\xi"] \\
{}^a\mathcal{XL} & {}^{a,0}\mathcal{H}_{\mathbb{G},\mathcal{XL}} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] & {}^0\mathcal{XL},
\end{tikzcd} \tag{33}$$

where the low row is the diagram (12) for $a'=0$. Now Lemma 9 allows to extend (33) to the following diagram, where both squares are cartesian

$$\begin{tikzcd}
{}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \arrow[d, "{}^a\tilde{\xi}"] & {}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] \arrow[d, "{}^a\tilde{\xi}_G"] & \mathrm{Bun}_{G,\tilde{H}} \arrow[d, "{}^0\tilde{\xi}"] \\
{}^a\widetilde{\mathcal{XL}} & {}^{a,0}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{XL}} \arrow[l, "\tilde{h}^{\leftarrow}"'] \arrow[r, "\tilde{h}^{\rightarrow}"] & {}^0\widetilde{\mathcal{XL}},
\end{tikzcd}$$

and the low row is the diagram (13) for $a' = 0$ from Section 4.3.1. This provides an isomorphism

$$\mathrm{H}^{\leftarrow}_G(\mathcal{S}, ({}^0\tilde{\xi})^*K) \,\widetilde{\to}\, ({}^a\tilde{\xi})^*\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, K)$$

functorial in $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{G}}$ and $K \in \mathrm{D}_{\mathcal{T}_0}(\widetilde{\mathcal{L}}_d(W_0(F_x)))$. Here the functors

$$({}^a\tilde{\xi})^* : \mathrm{D}_{\mathcal{T}_a}(\widetilde{\mathcal{L}}_d(W_a(F_x))) \to \mathrm{D}({}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}})$$

are defined as in ([8], Section 7.2).

Let ${}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G}$ be the stack clasifying collections: a point of the Hecke stack $(V, \mathcal{C}, V', \mathcal{C}', \beta) \in {}_x\mathcal{H}_{\tilde{H}}$ such that the isomorphism β of $\tilde{H}$-torsors $(V, \mathcal{C})$ and $(V', \mathcal{C}')$ over $X - x$ induces an isomorphism $\mathcal{C}(-ax) \,\widetilde{\to}\, cC'$; a G-torsor $(M, \mathcal{A})$ on X and an ismorphism $\mathcal{A} \otimes \mathcal{C}' \,\widetilde{\to}\, \Omega$. As above, we get a diagram

$${}^{a\tilde{x}}\,\mathrm{Bun}_{G,\tilde{H}} \overset{h^{\leftarrow}}{\leftarrow} {}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G} \overset{h^{\rightarrow}}{\to} \mathrm{Bun}_{G,\tilde{H}},$$

where $h^{\rightarrow}$ (resp., $h^{\leftarrow}$) sends the above point of ${}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G}$ to $(M, \mathcal{A}, V', \mathcal{C}')$ (resp., to $(M, \mathcal{A}, V, \mathcal{C})$).

As in the case of the Hecke functor for G, we get the diagram, where both squares are cartesian

$$\begin{tikzcd}
{}^{a\tilde{x}}\,\mathrm{Bun}_{G,\tilde{H}} \arrow[d, "{}^a\tilde{\xi}"] & {}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G} \arrow[l, "h^{\leftarrow}"'] \arrow[r, "h^{\rightarrow}"] \arrow[d, "{}^a\tilde{\xi}_{\tilde{H}}"] & \mathrm{Bun}_{G,\tilde{H}} \arrow[d, "{}^0\tilde{\xi}"] \\
{}^a\widetilde{\mathcal{XL}} & {}^{a,0}\widetilde{\mathcal{H}}_{\mathbb{H},\mathcal{XL}} \arrow[l, "\tilde{h}^{\leftarrow}"'] \arrow[r, "\tilde{h}^{\rightarrow}"] & {}^0\widetilde{\mathcal{XL}},
\end{tikzcd}$$

and the low row is the diagram (14) for $a' = 0$ from Section 4.3.2. This provides an isomorphism

$$\mathrm{H}^{\leftarrow}_{\tilde{H}}(\mathcal{S}, ({}^0\tilde{\xi})^*K) \,\widetilde{\to}\, ({}^a\tilde{\xi})^*\mathrm{H}^{\leftarrow}_{\tilde{H}}(\mathcal{S}, K)$$

functorial in $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{H}}$ and $K \in \mathrm{D}_{\mathcal{T}_0}(\widetilde{\mathcal{L}}_d(W_0(F_x)))$. By ([8], Proposition 6), we have $({}^0\tilde{\xi})^*S_{W_0(F)} \,\widetilde{\to}\, \mathrm{Aut}_{G,\tilde{H}}$ canonically. Now Theorem 2 from Theorem 3 by applying the functor $({}^a\tilde{\xi})^*$. Theorem 2 is proved.

5.3 In this subsection we derive Theorem 1 from Theorem 2. We give the argument only for $m \le n$ (the case $m > n$ is completely similar).

Let $a \in \mathbb{Z}$. It suffices to establish the isomorphism (3) for any $\mathcal{S} \in {}_{-a}\operatorname{Sph}_{\mathbb{G}}$. By base change theorem, for $K \in \mathrm{D}(\mathrm{Bun}_{\tilde{H}})$ we get

$$(\pi \times \mathrm{id})^*\mathrm{H}^{\leftarrow}_G(\mathcal{S}, F_G(K)) \,\widetilde{\to}\, ({}^a\mathfrak{p})_!({}^a\mathfrak{q}^*K \otimes \mathrm{H}^{\leftarrow}_G(\mathcal{S}, \mathrm{Aut}_{G,\tilde{H}}))[-\dim \mathrm{Bun}_{\tilde{H}}],$$

where ${}^a\mathfrak{q} : {}^a\,\mathrm{Bun}_{G,\tilde{H}} \to \mathrm{Bun}_{\tilde{H}}$ and ${}^a\mathfrak{p} : {}^a\,\mathrm{Bun}_{G,\tilde{H}} \to \tilde{X} \times \mathrm{Bun}_G$ send a collection $(\tilde{x} \in \tilde{X}, M, \mathcal{A}, V, \mathcal{C}) \in {}^a\,\mathrm{Bun}_{G,\tilde{H}}$ to $(V, \mathcal{C})$ and $(\tilde{x}, M, \mathcal{A})$ respectively.

By Theorem 2, the latter complex identifies with

$$({}^a\mathfrak{p})_!({}^a\mathfrak{q}^*K \otimes \mathrm{H}^{\rightarrow}_{\tilde{H}}(\mathrm{gRes}^{\kappa}(\mathcal{S}), \mathrm{Aut}_{G,\tilde{H}}))[-\dim \mathrm{Bun}_{\tilde{H}}] \tag{34}$$

Consider the diagram

$$\begin{tikzcd}
\tilde{X}\times \mathrm{Bun}_{\tilde H} & & {}^a\mathcal{H}_{\tilde H} \arrow[ll, "\mathrm{supp}\times h^{\leftarrow}"'] \arrow[rr, "h^{\rightarrow}"] & & \mathrm{Bun}_{\tilde H} \\
\tilde{X}\times \mathrm{Bun}_{G,\tilde H} \arrow[u, "\mathrm{id}\times\mathfrak{q}"'] \arrow[drr, "\mathrm{id}\times\mathfrak{p}"] & & {}^{a}_{\tilde X}\mathcal{H}_{\tilde H,G} \arrow[ll, "\mathrm{supp}\times h^{\leftarrow}"'] \arrow[rr, "\mathrm{supp}\times h^{\rightarrow}"] \arrow[u] & & {}^a\,\mathrm{Bun}_{G,\tilde H} \arrow[u, "{}^a\mathfrak{q}"'] \arrow[dll, "{}^a\mathfrak{p}"'] \\
& & \tilde{X}\times \mathrm{Bun}_G, & &
\end{tikzcd}$$

where ${}^a\mathcal{H}_{\tilde H}$ is the stack classifying $\tilde{x}\in\tilde{X}$, $\tilde{H}$-torsors $(V,\mathcal{C})$ and $(V',\mathcal{C}')$ on X identified via an isomorphism β over $X-\pi(\tilde{x})$ so that β yields $\mathcal{C}'\tilde{\to}\mathcal{C}(a\pi(\tilde{x}))$. The map $\mathrm{supp}\times h^{\leftarrow}$ (resp., $h^{\rightarrow}$) in the top row sends this point to $(\tilde{x},V,\mathcal{C})$ (resp., to $(V',\mathcal{C}')$).

The stack ${}^{a}_{\tilde X}\mathcal{H}_{\tilde H,G}$ is the above diagram classifies collections: $(\tilde{x},V,\mathcal{C},V',\mathcal{C}',\beta)\in{}^a\mathcal{H}_{\tilde H}$, a G-torsor $(M,\mathcal{A})$ on X, and an isomorphism $\mathcal{A}\otimes\mathcal{C}\tilde{\to}\Omega$. The map $\mathrm{supp}\times h^{\leftarrow}$ (resp., $\mathrm{supp}\times h^{\rightarrow}$) is the middle row sends this collection to $(\tilde{x},M,\mathcal{A},V,\mathcal{C})$ (resp., to $(\tilde{x},M,\mathcal{A},V',\mathcal{C}')$).

By the projection formulas, now (34) identifies with

$$(\mathrm{id}\times\mathfrak{p})_!(\mathrm{Aut}_{G,\tilde H}\otimes(\mathrm{id}\times\mathfrak{q})^*\mathrm{H}^{\leftarrow}_{\tilde H}(\mathrm{gRes}^{\kappa}(\mathcal{S}),K))[-\dim\mathrm{Bun}_{\tilde H}]$$

Theorem 1 is proved.

Appendix A. Invariants in the classical setting

A.1 In this appendix we assume that $k_0\subset k$ is a finite subfield, and all the objects introduced in Section 4 are defined over k_0. Write F_0 for k_0-valued points of F. Our purpose is to prove Proposition A.1 formulated in Section 4.8.7.2.

Lemma 10. *Let G be a group scheme over $\mathrm{Spec}\,\mathcal{O}$, $P\subset G$ be a parabolic and $U\subset P$ its unipotent radical. Let V be a smooth $\bar{\mathbb{Q}}_\ell$-representation of $G(F)$. Then the natural map $V^{G(\mathcal{O})}\to V_{U(F)}$ is injective, here $V_{U(F)}$ denotes the corresponding Jacquet module.*

Proof The author thanks J.-F. Dat for the following proof communicated to me. Pick a Borel subgroup $B\subset P$, write $I\subset G(\mathcal{O})$ for the corresponding Iwahori subgroup. It suffices to show that $V^I\to V_{U(F)}$ is injective.

Let $v\in V^I$ vanish in $V_{U(F)}$. Then one may find a semisimple $t\in B(F)$ such that the characteristic function ϕ of ItI annihilates v (it suffices that the action of t on $U(F)$ be sufficiently contracting). However, ϕ is invertible in the Iwahori-Hecke algebra of $(G(F),I)$, so $v=0$. $\square$

Lemma 11. *The maps $J^*_{P_{\mathbb{H}_a}}:\mathcal{W}\mathrm{eil}_a(k_0)\to\mathcal{S}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_0))$ and*

$$J^*_{P_{\mathbb{G}_a}}:\mathcal{W}\mathrm{eil}_a(k_0)\to\mathcal{S}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F_0)) \qquad (35)$$

are injective.

Proof Both claims being similar, we prove only the second one. Apply Lemma 10 for the parabolic $P_{\mathbb{H}_a} \subset \mathbb{H}_a$ and the representation $\mathcal{S}(\Pi_a(F))$ of $\mathcal{T}_a(F)$. Remind that $\mathcal{T}_a = \{(g_1, g_2) \in \mathbb{G}_a \times \mathbb{H}_a \mid (g_1, g_2)$ acts trivially on $A_a \otimes C_a\}$, and $U_{\mathbb{H}_a} \subset P_{\mathbb{H}_a}$ is the unipotent radical.

For $v \in \Pi_a(F)$ let $s_\Pi(v) : C_a^* \otimes \wedge^2 U_a(F) \to \Omega(F)$ be the map introduced in Section 4.8.4. Write $\mathrm{Cr}(\Pi_a)$ for the space of $v \in \Pi_a(F)$ such that $s_\Pi(v) = 0$. By ([10], page 72), the Jacquet module $\mathcal{S}(\Pi_a(F))_{U_{\mathbb{H}_a}(F)}$ identifies with the Schwarz space $\mathcal{S}(\mathrm{Cr}(\Pi_a))$, and the projection

$$\mathcal{S}(\Pi_a(F)) \to \mathcal{S}(\Pi_a(F))_{U_{\mathbb{H}_a}(F)}$$

identifies with the restriction map $\mathcal{S}(\Pi_a(F)) \to \mathcal{S}(\mathrm{Cr}(\Pi_a))$. We learn that the restriction map $\mathcal{W}\mathrm{eil}_a(k_0) \to \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\mathrm{Cr}(\Pi_a))$ is injective. So, (35) is also injective. $\square$

Proof of Proposition A.1

For $b \in \mathbb{Z}$ set ${}_b\mathcal{H}_{\mathbb{G}} = K_0({}_b\mathrm{Sph}_{\mathbb{G}}) \otimes \bar{\mathbb{Q}}_\ell$ and ${}_b\mathcal{H}_{Q(\mathbb{G})} = K_0({}_b\mathrm{Sph}_{Q(\mathbb{G})}) \otimes \bar{\mathbb{Q}}_\ell$. So,

$$\mathcal{H}_{\mathbb{G}} = \underset{b \in \mathbb{Z}}{\oplus} {}_b\mathcal{H}_{\mathbb{G}}, \quad \mathcal{H}_{Q(\mathbb{G})} = \underset{b \in \mathbb{Z}}{\oplus} {}_b\mathcal{H}_{Q(\mathbb{G})}$$

are the Hecke algebras for $\mathbb{G}$ and $Q(\mathbb{G})$ respectively. From Proposition 4, we learn that the map

$${}_{-a}\mathcal{H}_{Q(\mathbb{G})} \to \mathcal{S}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F_0))$$

given by $\mathcal{S} \mapsto \mathrm{tr}_{k_0} \mathrm{H}^{\leftarrow}_{Q(\mathbb{G})}(\mathcal{S}, I_0)$ is an isomorphism of $\bar{\mathbb{Q}}_\ell$-vector spaces. Write ${}_{-a}W \subset {}_{-a}\mathcal{H}_{Q(\mathbb{G})}$ for the image of the map (35). We get a $\mathbb{Z}$-graded subspace $W := \underset{a \in \mathbb{Z}}{\oplus} {}_aW \subset \mathcal{H}_{Q(\mathbb{G})}$.

For $a, a' \in \mathbb{Z}$ we have the Hecke operators

$$\mathrm{H}^{\leftarrow}_{\mathbb{G}} : {}_{a'-a}\mathcal{H}_{\mathbb{G}} \times \mathcal{S}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F_0)) \to \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$$

defined as in Section 4.8.1. We claim that for $\mathcal{S} \in {}_{a'-a}\mathcal{H}_{\mathbb{G}}$ the operator $\mathrm{H}^{\leftarrow}_{\mathbb{G}}(\mathcal{S}, \cdot)$ sends $\mathcal{W}\mathrm{eil}_{a'}(k_0)$ to the subspace $\mathcal{W}\mathrm{eil}_a(k_0) \subset \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$. This follows from the fact the actions of the groupoids $\mathbb{G}Q\mathbb{H}$ and $\mathbb{H}Q\mathbb{G}$ on the spaces $\mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$ commute with each other.

More precisely, for $a, b \in \mathbb{Z}$ given $g = (g_1, g_2) \in \mathcal{T}_{b,a}$ such that $g_2 : V_a \tilde{\to} V_b$ is an isomorphism of $Q(\mathbb{H})$-torsors over $\mathrm{Spec}\,\mathcal{O}$, let $h = (h_1, h_2) \in \mathcal{T}_b$ be any element such that $h_1 : M_b \tilde{\to} M_b$ is a scalar automorphism of the $\mathbb{G}$-torsor M_b over $\mathrm{Spec}\,\mathcal{O}$. Here h_2 is an automorphism of the $\mathbb{H}$-torsor V_b over $\mathrm{Spec}\,\mathcal{O}$. Set $h'_2 = g_2^{-1} h_2 g_2$, so h'_2 is an automorphism of the $\mathbb{H}$-torsor V_a over $\mathrm{Spec}\,\mathcal{O}$. Set $h'_1 = h_1$ then $h' = (h_1, h_2) \in \mathcal{T}_a$. The equality $gh' = hg$ in $\mathcal{T}$ shows that $g : \mathcal{S}(\Pi_a(F)) \to \mathcal{S}(\Pi_b(F))$ sends $\mathbb{H}_a(\mathcal{O})$-equivariant objects to $\mathbb{H}_b(\mathcal{O})$-equivariant objects. We have used the action of the groupoid $\mathcal{T}$ on the spaces $\mathcal{S}(\Pi_a(F))$ obtained as in Remark 5.

Thus, W is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\mathcal{H}_{\mathbb{G}}$. We also know from ([7], Proposition 2) that ${}_0W = {}_0\mathcal{H}_{\mathbb{G}}$. Our statement is reduced to Lemma 12 below. $\square$

Remind that we have picked a maximal torus $T_{\mathbb{G}} \subset Q(\mathbb{G})$. Write W (resp., W_Q) for the Weyl group of $(\mathbb{G}, T_{\mathbb{G}})$ (resp., of $(Q(\mathbb{G}), T_{\mathbb{G}})$). Then

$$\mathcal{H}_{Q(\mathbb{G})} \tilde{\to} \bar{\mathbb{Q}}_\ell[\check{T}_{\mathbb{G}}]^{W_Q}, \qquad \mathcal{H}_{\mathbb{G}} \tilde{\to} \bar{\mathbb{Q}}_\ell[\check{T}_{\mathbb{G}}]^{W}$$

The homomorphism $\mathrm{Res}^{\kappa_{\mathbb{G}}} : \mathcal{H}_{\mathbb{G}} \to \mathcal{H}_{Q(\mathbb{G})}$ (cf. Section 4.8.6) comes from the map $f^{\kappa_{\mathbb{G}}} : \check{T}_{\mathbb{G}}^{W_Q} \to \check{T}_{\mathbb{G}}^{W}$ obtained by taking the Weil group invariants of the map $\check{T}_{\mathbb{G}} \to \check{T}_{\mathbb{G}}$, $t \mapsto t\nu(q^{1/2})$, where ν is some coweight of the center $Z(\check{Q}(\mathbb{G}))$, and q is the number of elements of k_0.

Lemma 12. *View $\mathcal{H}_{Q(\mathbb{G})}$ as a $\mathbb{Z}$-graded $\mathcal{H}(\mathbb{G})$-module via $\mathrm{Res}^{\kappa_{\mathbb{G}}} : \mathcal{H}_{\mathbb{G}} \to \mathcal{H}_{Q(\mathbb{G})}$. Let $W = \underset{a\in\mathbb{Z}}{\oplus} {}_aW \subset \mathcal{H}_{Q(\mathbb{G})} = \underset{a\in\mathbb{Z}}{\oplus} {}_a\mathcal{H}_{Q(\mathbb{G})}$ be a $\mathbb{Z}$-graded submodule over the $\mathbb{Z}$-graded ring $\mathcal{H}_{\mathbb{G}}$. Assume that ${}_0W = {}_0\mathcal{H}_{\mathbb{G}}$. Then $W = \mathcal{H}_{\mathbb{G}}$.*

Proof Given $x \in {}_aW$, pick a nonzero $h \in {}_{-a}\mathcal{H}_{\mathbb{G}}$ then $hx \in {}_0\mathcal{H}_{\mathbb{G}}$. So, x is a rational function on $\check{T}_{\mathbb{G}}^{W}$ which becomes everywhere regular after restriction under $f^{\kappa_{\mathbb{G}}} : \check{T}_{\mathbb{G}}^{W_Q} \to \check{T}_{\mathbb{G}}^{W}$. Since $\check{T}_{\mathbb{G}}^{W}$ is normal by Remark 8 below, and x is entire over $\bar{\mathbb{Q}}_\ell[\check{T}_{\mathbb{G}}]^W$, it follows that $x \in \bar{\mathbb{Q}}_\ell[\check{T}_{\mathbb{G}}]^W$. □

Remark 8. Let A be an entire normal ring, W be a finite group acting on A. Assuming that A is finite over A^W, one checks that A^W is normal.

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