

Geometric Weil representation: local field case

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ABSTRACT Let k be an algebraically closed field of characteristic > 2 , $F = k((t))$ and $G = \mathrm{Sp}_{2d}$. In this paper we propose a geometric analog of the Weil representation of the metaplectic group $\tilde{G}(F)$. This is a category of certain perverse sheaves on some stack, on which $\tilde{G}(F)$ acts by functors. This construction will be used in [11] (and subsequent publications) for the proof of the geometric Langlands functoriality for some dual reductive pairs.

1. INTRODUCTION

1.1 This paper followed by [11] form a series, where we prove the geometric Langlands functoriality for the dual reductive pair $\mathrm{Sp}_{2n}, \mathrm{SO}_{2m}$ (in the everywhere nonramified case).

Let $k = \mathbb{F}_q$ with q odd, set $\mathcal{O} = k[[t]] \subset F = k((t))$. Write Ω for the completed module of relative differentials of \mathcal{O} over k . Let M be a free \mathcal{O} -module of rank $2d$ with symplectic form $\wedge^2 M \rightarrow \Omega$, set $G = \mathrm{Sp}(M)$. The group $G(F)$ admits a nontrivial metaplectic extension

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{G}(F) \rightarrow G(F) \rightarrow 1$$

(defined up to a unique isomorphism). Let $\psi : k \rightarrow \bar{\mathbb{Q}}_\ell^*$ be a nontrivial additive character, let $\chi : \Omega(F) \rightarrow \bar{\mathbb{Q}}_\ell^*$ be given by $\chi(\omega) = \psi(\mathrm{Res} \omega)$. Write $H = M \oplus \Omega$ for the Heisenberg group of M with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\omega\langle m_1, m_2 \rangle) \quad m_i \in M, a_i \in \Omega$$

Denote by \mathcal{S}_ψ the Weil representation of $H(M)(F)$ with central character χ . As a representation of $\tilde{G}(F)$, it decomposes $\mathcal{S}_\psi \xrightarrow{\sim} \mathcal{S}_{\psi, \text{odd}} \oplus \mathcal{S}_{\psi, \text{even}}$ into a direct sum of two irreducible smooth representations, where the even (resp., the odd) part is unramified (resp., ramified).

The discovery of this representation by A. Weil in [14] had a major influence on the theory of automorphic forms (among numerous developpements and applications are Howe duality for reductive dual pairs, particular cases of classical Langlands functoriality, Siegel-Weil formulas, relation with L-functions, representation-theoretic approach to the theory of theta-series. We refer the reader to [3], [9], [7], [12], [13] for history and details).

In this paper we introduce a geometric analog of the Weil representation \mathcal{S}_ψ . The pioneering work in this direction is due to P. Deligne [2], where a geometric approach to the Weil

representation of a symplectic group over a finite field was set up. It was further extended by Gurevich-Hadani in [4, 5]. The point of this paper is to develop the geometric theory in the case when a finite field is replaced by a local non-archimedean field.

First, we introduce a k -scheme $\mathcal{L}_d(M(F))$ of discrete lagrangian lattices in $M(F)$ and a certain μ_2 -gerb $\tilde{\mathcal{L}}_d(M(F))$ over it. We view the metaplectic group $\tilde{G}(F)$ as a group stack over k . We construct a category

$$W(\tilde{\mathcal{L}}_d(M(F)))$$

of certain perverse sheaves on $\tilde{\mathcal{L}}_d(M(F))$, which provides a geometric analog of $\mathcal{S}_{\psi, \text{even}}$. The metaplectic group $\tilde{G}(F)$ acts on the category $W(\tilde{\mathcal{L}}_d(M(F)))$ by functors. This action is *geometric* in the sense that it comes from a natural action of $\tilde{G}(F)$ on $\tilde{\mathcal{L}}_d(M(F))$ (cf. Theorem 2).

The category $W(\tilde{\mathcal{L}}_d(M(F)))$ has a distinguished object $S_{M(F)}$ corresponding to the unique non-ramified vector of $\mathcal{S}_{\psi, \text{even}}$.

Our category $W(\tilde{\mathcal{L}}_d(M(F)))$ is obtained from Weil representations of symplectic groups $\text{Sp}_{2r}(k)$ by some limit procedure. This uses a construction of geometric canonical intertwining operators for such representations. A similar result has been announced by Gurevich and Hadani in [4] and proved for $d = 1$ in [5]. We give a proof for any d (cf. Theorem 1). When this paper has already been written we learned about a new preprint [6], where a result similar to our Theorem 1 is claimed to be proved for all d . However, the sheaves of canonical intertwining operators constructed in *loc.cit.* and in this paper live on different bases.

Finally, in Section 7 we give a global application. Let X be a smooth projective curve. Write Ω_X for the canonical line bundle on X . Let G denote the sheaf of automorphisms of $\mathcal{O}_X^d \oplus \Omega_X^d$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^d \oplus \Omega_X^d) \rightarrow \Omega_X$.

Our Theorem 3 relates $S_{M(F)}$ with the theta-sheaf Aut on the moduli stack $\widetilde{\text{Bun}}_G$ of metaplectic bundles on X introduced in [10]. This result will play an important role in [11].

1.2 NOTATION In Section 2 we let $k = \mathbb{F}_q$ of characteristic $p > 2$. Starting from Section 3 we assume k either finite as above or algebraically closed with a fixed inclusion $\mathbb{F}_q \hookrightarrow k$. All the schemes (or stacks) we consider are defined over k .

Fix a prime $\ell \neq p$. For a scheme (or stack) S write $\text{D}(S)$ for the bounded derived category of ℓ -adic étale sheaves on S , and $\text{P}(S) \subset \text{D}(S)$ for the category of perverse sheaves.

Fix a nontrivial character $\psi : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell^*$, write \mathcal{L}_ψ for the corresponding Artin-Schreier sheaf on \mathbb{A}^1 . Fix a square root $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ of the sheaf $\bar{\mathbb{Q}}_\ell(1)$ on $\text{Spec } \mathbb{F}_q$. Isomorphism classes of such correspond to square roots of q in $\bar{\mathbb{Q}}_\ell$.

If $V \rightarrow S$ and $V^* \rightarrow S$ are dual rank n vector bundles over a stack S , we normalize the Fourier transform $\text{Four}_\psi : \text{D}(V) \rightarrow \text{D}(V^*)$ by $\text{Four}_\psi(K) = (p_{V^*})_!(\xi^* \mathcal{L}_\psi \otimes p_V^* K)[n](\frac{n}{2})$, where p_V, p_{V^*} are the projections, and $\xi : V \times_S V^* \rightarrow \mathbb{A}^1$ is the pairing.

Our conventions about $\mathbb{Z}/2\mathbb{Z}$ -gradings are those of [10].

2. CANONICAL INTERWINING OPERATORS: FINITE FIELD CASE

2.1 Let M be a symplectic k -vector space of dimension $2d$. The symplectic form on M is denoted $\omega\langle \cdot, \cdot \rangle$. The Heisenberg group $H = M \times \mathbb{A}^1$ with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\omega\langle m_1, m_2 \rangle) \quad m_i \in M, a_i \in \mathbb{A}^1$$

is algebraic over k . Set $G = \mathrm{Sp}(M)$. Write $\mathcal{L}(M)$ for the variety of lagrangian subspaces in M . Fix a one-dimensional k -vector space \mathcal{J} (purely of degree $d \bmod 2$ as $\mathbb{Z}/2\mathbb{Z}$ -graded). Let \mathcal{A} be the (purely of degree zero as $\mathbb{Z}/2\mathbb{Z}$ -graded) line bundle over $\mathcal{L}(M)$ with fibre $\mathcal{J} \otimes \det L$ at $L \in \mathcal{L}(M)$. Write $\tilde{\mathcal{L}}(M)$ for the gerb of square roots of \mathcal{A} . The line bundle \mathcal{A} is G -equivariant, so G acts naturally on $\tilde{\mathcal{L}}(M)$.

For a k -point $L \in \mathcal{L}(M)$ write L^0 for a k -point of $\tilde{\mathcal{L}}(M)$ over L . Write

$$\bar{L} = L \oplus k,$$

this is a subgroup of $H(k)$ equipped with the character $\chi_L : \bar{L} \rightarrow \bar{\mathbb{Q}}_\ell^*$ given by $\chi_L(l, a) = \psi(a)$, $l \in L, a \in k$. Write

$$\mathcal{H}_L = \{f : H(k) \rightarrow \bar{\mathbb{Q}}_\ell \mid f(\bar{l}h) = \chi_L(\bar{l})f(h), \text{ for } \bar{l} \in \bar{L}, h \in H\}$$

This is a representation of $H(k)$ by right translations. Write $\mathcal{S}(H)$ for the space of all $\bar{\mathbb{Q}}_\ell$ -valued functions on $H(k)$. The group G acts naturally in $\mathcal{S}(H)$. For $L \in \mathcal{L}(M), g \in G$ we have an isomorphism $\mathcal{H}_L \rightarrow \mathcal{H}_{gL}$ sending f to gf .

The purpose of Sections 2 and 3 is to study the canonical interwining operators (and their geometric analogs) between various models \mathcal{H}_L of the Weil representation. The corresponding results for a finite field were formulated by Gurevich and Hadani [4] without a proof (we give all proofs for the sake of completeness). Besides, our setting is a bit different from *loc.cit*, we work with gerbs instead of the total space of the corresponding line bundles.

2.2 For k -points $L^0, N^0 \in \tilde{\mathcal{L}}(M)$ we will define a canonical interwining operator

$$F_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$$

They will satisfy the properties

- $F_{L^0, L^0} = \mathrm{id}$
- $F_{R^0, N^0} \circ F_{N^0, L^0} = F_{R^0, L^0}$ for any $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)$
- for any $g \in G$ we have $g \circ F_{N^0, L^0} \circ g^{-1} = F_{gN^0, gL^0}$.
- under the natural action of μ_2 on the set $\tilde{\mathcal{L}}(M)(k)$ of (isomorphism classes of) k -points, F_{N^0, L^0} is odd as a function of N^0 and of L^0 .

In (Remark 2, Section 3.1) we will define a function F^{cl} on the set of k -points of $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$, which we denote $F_{N^0, L^0}(h)$ for $h \in H$. It will realize the operator F_{N^0, L^0} by

$$(F_{N^0, L^0} f)(h_1) = \int_{h_2 \in H} F_{N^0, L^0}(h_1 h_2^{-1}) f(h_2) dh_2$$

All our measures on finite sets are normalized by requiring the volume of a point to be one. Given two functions $f_1, f_2 : H \rightarrow \bar{\mathbb{Q}}_\ell$ their convolution $f_1 * f_2 : H \rightarrow \bar{\mathbb{Q}}_\ell$ is defined by

$$(f_1 * f_2)(h) = \int_{v \in H} f_1(hv^{-1}) f_2(v) dv \quad h \in H$$

The function F_{N^0, L^0} will satisfy the following:

- $F_{N^0, L^0}(\bar{n}h\bar{l}) = \chi_N(\bar{n})\chi_L(\bar{l})F_{N^0, L^0}(h)$ for $\bar{l} \in \bar{L}, \bar{n} \in \bar{N}, h \in H$.
- $F_{gN^0, gL^0}(gh) = F_{N^0, L^0}(h)$ for $g \in G, h \in H$.
- Convolution property: $F_{R^0, L^0} = F_{R^0, N^0} * F_{N^0, L^0}$ for any $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)$.

2.3 First, we define the non-normalized function $\tilde{F}_{N, L} : H \rightarrow \bar{\mathbb{Q}}_\ell$, it will depend only on $N, L \in \mathcal{L}(M)$, not of their inanced structure.

Given $N, L \in \mathcal{L}(M)$ let $\chi_{NL} : \bar{N}\bar{L} \rightarrow \bar{\mathbb{Q}}_\ell$ be the function given by

$$\chi_{NL}(\bar{n}\bar{l}) = \chi_N(\bar{n})\chi_L(\bar{l}),$$

it is correctly defined. Note that $\bar{N}\bar{L} = \bar{L}\bar{N}$ but $\chi_{NL} \neq \chi_{LN}$ in general. Set

$$\tilde{F}_{N, L}(h) = \begin{cases} \chi_{NL}(h), & \text{if } h \in \bar{N}\bar{L} \\ 0, & \text{otherwise} \end{cases}$$

Note that $\chi_{LL} = \chi_L$.

Given $L, R, N \in \mathcal{L}(M)$ with $N \cap L = N \cap R = 0$, define $\theta(R, N, L) \in \bar{\mathbb{Q}}_\ell$ as follows. There is a unique map $b : L \rightarrow N$ such that $R = \{l + b(l) \in L \oplus N \mid l \in L\}$. Set

$$\theta(R, N, L) = \int_{l \in L} \psi\left(\frac{1}{2}\omega\langle l, b(l) \rangle\right) dl$$

This expression has been considered in ([10], Appendix B).

Lemma 1. 1) Let $L, N \in \mathcal{L}(M)$. If $L \cap N = 0$ then $\tilde{F}_{L, N} * \tilde{F}_{N, L} = q^{2d+1}\tilde{F}_{L, L}$.

2) Let $L, R, N \in \mathcal{L}(M)$ with $N \cap L = N \cap R = 0$. Then $\tilde{F}_{R, N} * \tilde{F}_{N, L} = q^{d+1}\theta(R, N, L)\tilde{F}_{R, L}$

Proof 2) Using $L \oplus N = N \oplus R = M$, for $h \in H$ we get

$$(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) = q^{d+1} \int_{v \in \tilde{N} \setminus H} \chi_{RN}(hv^{-1}) \chi_{NL}(v) dv = q^{d+1} \int_{r \in R} \chi_{RN}(h(-r, 0)) \chi_{NL}(r, 0) dr$$

Because of the equivariance property of $\tilde{F}_{R,N} * \tilde{F}_{N,L}$, we may assume $h = (n, 0), n \in N$. We get

$$\begin{aligned} (\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) &= q^{d+1} \int_{r \in R} \chi_{RN}((n, 0)(-r, 0)) \chi_{NL}(r, 0) dr \\ &= q^{d+1} \int_{r \in R} \psi(\omega\langle r, n \rangle) \chi_{NL}(r, 0) dr \quad (1) \end{aligned}$$

The latter formula essentially says that the resulting function on N is the Fourier transform of some local system on R (the symplectic form on M induces an isomorphism $R \xrightarrow{\sim} N^*$). This will be used for geometrization in Lemma 2.

There is a unique map $b : L \rightarrow N$ such that $R = \{l + b(l) \in L \oplus N \mid l \in L\}$. So, the above integral rewrites

$$\begin{aligned} (\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) &= q^{d+1} \int_{l \in L} \psi(\omega\langle l, n \rangle) \chi_{NL}((l + b(l), 0)) dl = \\ &= q^{d+1} \int_{l \in L} \psi(\omega\langle l, n \rangle) \chi_{NL}((b(l), \frac{1}{2}\omega\langle l, b(l) \rangle)(l, 0)) dl = q^{d+1} \int_{l \in L} \psi(\omega\langle l, n \rangle + \frac{1}{2}\omega\langle l, b(l) \rangle) dl \quad (2) \end{aligned}$$

Note that if $R = L$ then $b = 0$ and the latter formula yields 1).

Let us identify $N \xrightarrow{\sim} L^*$ via the map sending $n \in N$ to the linear functional $l \mapsto \omega\langle l, n \rangle$. Denote by $\langle \cdot, \cdot \rangle$ the symmetric pairing between L and L^* . By Sublemma 1 below, the value (2) vanishes unless $n \in (R + L) \cap N = \text{Im } b$. In the latter case pick $l_1 \in L$ with $b(l_1) = n$. Then

$$\chi_{RL}(n, 0) = \psi(-\frac{1}{2}\omega\langle l_1, b(l_1) \rangle)$$

So, we get for $L' = \text{Ker } b$

$$(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) = q^{d+1+\dim L'} \chi_{RL}(h) \int_{l \in L/L'} \psi(\frac{1}{2}\omega\langle l, b(l) \rangle) dl$$

We are done. \square

Sublemma 1. *Let L be a d -dimensional k -vector space, $b \in \text{Sym}^2 L^*$ and $u \in L^*$. View b as a map $b : L \rightarrow L^*$, let L' be the kernel of b . Then*

$$\int_{l \in L} \psi(\langle l, u \rangle + \frac{1}{2}\langle l, b(l) \rangle) dl \quad (3)$$

is supported at $u \in (L/L')^$ and there equals*

$$q^{\dim L'} \psi(-\frac{1}{2}\langle b^{-1}u, u \rangle) \int_{L/L'} \psi(\frac{1}{2}\langle l, b(l) \rangle) dl,$$

where $b : L/L' \xrightarrow{\sim} (L/L')^*$, so that $b^{-1}u \in L/L'$. (Here the scalar product is between L and L^* , so is symmetric).

Proof Let $L' \subset L$ denote the kernel of $b : L \rightarrow L^*$. Integrating first along the fibres of the projection $L \rightarrow L/L'$ we will get zero unless $u \in (L/L')^*$. For any $l_0 \in L$ the integral (3) equals

$$\int_{l \in L} \psi(\langle l+l_0, u \rangle + \frac{1}{2} \langle l+l_0, b(l)+b(l_0) \rangle) dl = \psi(\langle l_0, u \rangle + \frac{1}{2} \langle l_0, b(l_0) \rangle) \int_{l \in L} \psi(\langle l, u+b(l_0) \rangle + \frac{1}{2} \langle l, b(l) \rangle) dl$$

Assuming $u \in (L/L')^*$ take l_0 such that $u = -b(l_0)$. Then (3) becomes

$$\psi(\frac{1}{2} \langle l_0, u \rangle) \int_{l \in L} \psi(\frac{1}{2} \langle l, b(l) \rangle) dl$$

We are done. \square

Remark 1. The expression (3) is the Fourier transform from L to L^* . In the geometric setting we will use 2) of Lemma 1 only under the additional assumption $R \cap L = 0$.

3. GEOMETRIZATION

3.1 Let $M, H, \mathcal{L}(M)$ and $\tilde{\mathcal{L}}(M)$ be as in Section 2.1. Remind that $G = \mathrm{Sp}(M)$. For each $L \in \mathcal{L}(M)$ we have a rank one local system χ_L on $\bar{L} = L \times \mathbb{A}^1$ defined by $\chi_L = \mathrm{pr}^* \mathcal{L}_\psi$, where $\mathrm{pr} : L \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the projection. Let \mathcal{H}_L denote the category of perverse sheaves on H which are (\bar{L}, χ_L) -equivariant under the left multiplication, this is a full subcategory in $\mathrm{P}(H)$. Write $\mathrm{D}\mathcal{H}_L \subset \mathrm{D}(H)$ for the full subcategory of objects whose all perverse cohomologies lie in \mathcal{H}_L .

Denote by $C \rightarrow \mathcal{L}(M)$ (resp., $\bar{C} \rightarrow \mathcal{L}(M)$) the vector bundle whose fibre over $L \in \mathcal{L}(M)$ is L (resp., $\bar{L} = L \times \mathbb{A}^1$). Its inverse image to $\mathcal{L}(M)$ is denoted by the same symbol.

Write $\chi_{\bar{C}}$ for the local system $p^* \mathcal{L}_\psi$ on \bar{C} , where $p : \bar{C} \rightarrow \mathbb{A}^1$ is the projection on the center sending $(L \in \mathcal{L}(M), (l, a) \in \bar{L})$ to a . Consider the maps

$$\mathrm{pr}, \mathrm{act}_{lr} : \bar{C} \times \bar{C} \times H \rightarrow \mathcal{L}(M) \times \mathcal{L}(M) \times H \times H$$

where act_{lr} sends $(\bar{n} \in \bar{N}, \bar{l} \in \bar{L}, h)$ to $(N, L, \bar{n}h\bar{l})$, and pr sends the above point to (N, L, h) . We say that a perverse sheaf K on $\mathcal{L}(M) \times \mathcal{L}(M) \times H$ is act_{lr} -equivariant if it admits an isomorphism

$$\mathrm{act}_{lr}^* K \xrightarrow{\sim} \mathrm{pr}^* K \otimes \mathrm{pr}_1^* \chi_{\bar{C}} \otimes \mathrm{pr}_2^* \chi_{\bar{C}}$$

satisfying the usual associativity condition and whose restriction to the unit section is the identity (such isomorphism is unique if it exists). One has a similar definition for $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$.

Let

$$\mathrm{act}_G : G \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$$

be the action map sending (g, N^0, L^0, h) to

$$(gN^0, gL^0, gh)$$

For this map we have a usual notion of a G -equivariant perverse sheaf on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$. As G is connected, a perverse sheaf on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ admits at most one G -equivariant structure.

If S is a stack then for $K, F \in \mathbf{D}(S \times H)$ define their convolution $K * F \in \mathbf{D}(S \times H)$ by

$$K * F = \text{mult}_!(\text{pr}_1^* K \otimes \text{pr}_2^* F) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{d+1-2\dim \mathcal{L}(M)},$$

here $\text{pr}_i : S \times H \times H \rightarrow S \times H$ is the projection to the i -th component in the pair $H \times H$ (and the identity on S). The multiplication map $\text{mult} : H \times H \rightarrow H$ sends (h_1, h_2) to $h_1 h_2$.

Let

$$(\mathcal{L}(M) \times H)_\Delta \hookrightarrow \mathcal{L}(M) \times H \tag{4}$$

be the closed subscheme of those $(L \in \mathcal{L}(M), h \in H)$ for which $h \in \bar{L}$. Let

$$\alpha_\Delta : (\mathcal{L}(M) \times H)_\Delta \rightarrow \mathbb{A}^1$$

be the map sending (L, h) to a , where $h = (l, a)$, $l \in L, a \in \mathbb{A}^1$. Define a perverse sheaf

$$\tilde{F}_\Delta = \alpha_\Delta^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{d+1+\dim \mathcal{L}(M)},$$

which we extend by zero under (4).

Since $\tilde{\mathcal{L}}(M) \rightarrow \mathcal{L}(M)$ is a μ_2 -gerb, μ_2 acts on each $K \in \mathbf{D}(\tilde{\mathcal{L}}(M))$, and we say that K is *genuine* if $-1 \in \mu_2$ acts on K as -1 .

Theorem 1. *There exists an irreducible perverse sheaf F on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ (pure of weight zero) with the following properties:*

- for the diagonal map $i : \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ the complex $i^* F$ identifies canonically with the inverse image of

$$\tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

under the projection $\tilde{\mathcal{L}}(M) \times H \rightarrow \mathcal{L}(M) \times H$.

- F is act_{l_r} -equivariant;
- F is G -equivariant;
- F is genuine in the first and the second variable;
- convolution property for F holds, namely for the ij -th projections

$$q_{ij} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$$

inside the triple $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M)$ we have $(q_{12}^* F) * (q_{23}^* F) \xrightarrow{\sim} q_{13}^* F$ canonically.

The proof of Theorem 1 is given in Sections 3.2-3.4.

Remark 2. In the case $k = \mathbb{F}_q$ define F^{cl} as the trace of the geometric Frobenius on F .

3.2 Let $U \subset \mathcal{L}(M) \times \mathcal{L}(M)$ be the open subset of pairs $(N, L) \in \mathcal{L}(M) \times \mathcal{L}(M)$ such that $N \cap L = 0$. Define a perverse sheaf \tilde{F}_U on $U \times H$ as follows. Let

$$\alpha_U : U \times H \rightarrow \mathbb{A}^1$$

be the map sending (N, L, h) to $a + \frac{1}{2}\omega\langle l, n \rangle$, where $l \in L, n \in N, a \in \mathbb{A}^1$ are uniquely defined by $h = (n + l, a)$. Set

$$\tilde{F}_U = \alpha_U^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim H + 2 \dim \mathcal{L}(M)} \quad (5)$$

Write $U \times_{\mathcal{L}(M)} U \subset \mathcal{L}(M) \times \mathcal{L}(M) \times \mathcal{L}(M)$ for the open subscheme classifying (R, N, L) with $N \cap L = N \cap R = 0$. Let

$$q_i : U \times_{\mathcal{L}(M)} U \rightarrow U$$

be the projection on the i -th factor, so q_1 (resp., q_2) sends (R, N, L) to (R, N) (resp., to (N, L)). Let $q : U \times_{\mathcal{L}(M)} U \rightarrow \mathcal{L}(M) \times \mathcal{L}(M)$ be the map sending (R, N, L) to (R, L) . Write

$$(U \times_{\mathcal{L}(M)} U)_0 = q^{-1}(U)$$

The geometric analog of $\theta(R, N, L)$ is the following (shifted) perverse sheaf Θ on $U \times_{\mathcal{L}(M)} U$. Let $\pi_C : C_3 \rightarrow U \times_{\mathcal{L}(M)} U$ be the vector bundle whose fibre over (R, N, L) is L . We have a map $\beta : C_3 \rightarrow \mathbb{A}^1$ defined as follows. Given a point $(R, N, L) \in U \times_{\mathcal{L}(M)} U$, there is a unique map $b : L \rightarrow N$ such that $R = \{l + b(l) \in L \oplus N = M \mid l \in L\}$. Set $\beta(R, N, L, l) = \frac{1}{2}\omega\langle l, b(l) \rangle$. Set

$$\Theta = (\pi_C)_! \beta^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^d$$

Write $Y = \mathcal{L}(M) \times \mathcal{L}(M)$, let \mathcal{A}_Y be the ($\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) line bundle on Y whose fibre at (R, L) is $\det R \otimes \det L$. Write \tilde{Y} for the gerb of square roots of \mathcal{A}_Y . Note that \mathcal{A}_Y is G -equivariant, so G acts on \tilde{Y} naturally.

The following perverse sheaf S_M on \tilde{Y} was introduced in ([10], Definition 2). Let $Y_i \subset Y$ be the locally closed subscheme given by $\dim(R \cap L) = i$ for $(R, L) \in Y_i$. The restriction of \mathcal{A}_Y to each Y_i admits the following G -equivariant square root. For a point $(R, L) \in Y_i$ we have an isomorphism $L/(R \cap L) \xrightarrow{\sim} (R/(R \cap L))^*$ sending l to the functional $r \mapsto \omega\langle r, l \rangle$. It induces a $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism $\det R \otimes \det L \xrightarrow{\sim} \det(R \cap L)^2$.

So, for the restriction \tilde{Y}_i of the gerb $\tilde{Y} \rightarrow Y$ to Y_i we get a trivialization

$$\tilde{Y}_i \xrightarrow{\sim} Y_i \times B(\mu_2) \quad (6)$$

Write W for the nontrivial local system of rank one on $B(\mu_2)$ corresponding to the covering $\text{Spec } k \rightarrow B(\mu_2)$.

Definition 1. Let $S_{M,g}$ (resp., $S_{M,s}$) denote the intermediate extension of

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim Y}$$

from \tilde{Y}_0 to \tilde{Y} (resp., of $(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim Y-1}$ from \tilde{Y}_1 to \tilde{Y}). Set $S_M = S_{M,g} \oplus S_{M,s}$.

Let

$$\pi_Y : U \times_{\mathcal{L}(M)} U \rightarrow \tilde{Y}$$

be the map sending (R, N, L) to

$$(R, L, \mathcal{B}, \epsilon : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L),$$

where $\mathcal{B} = \det L$ and ϵ is the isomorphism induced by ϵ_0 . Here $\epsilon_0 : L \xrightarrow{\sim} R$ is the isomorphism sending $l \in L$ to $l + b(l) \in R$. In other words, ϵ_0 sends l to the unique $r \in R$ such that $r = l \bmod N \in M/N$. Write also $\tilde{U} = \tilde{Y}_0$.

Define $\mathcal{E} \in \mathbf{D}(\mathrm{Spec} k)$ by

$$\mathcal{E} = \mathrm{R}\Gamma_c(\mathbb{A}^1, \beta_0^* \mathcal{L}_\psi) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2}),$$

where $\beta_0 : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ sends x to x^2 . Then \mathcal{E} is a 1-dimensional vector space placed in cohomological degree zero. The geometric Frobenius $\mathrm{Fr}_{\mathbb{F}_q}$ acts on \mathcal{E}^2 by 1 if $-1 \in (\mathbb{F}_q^*)^2$ and by -1 otherwise. A choice of $\sqrt{-1} \in k$ yields an isomorphism $\mathcal{E}^2 \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$, so $\mathcal{E}^4 \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$ canonically.

As in ([10], Proposition 5), one gets a canonical isomorphism

$$\pi_Y^*(S_{M,g} \otimes \mathcal{E}^d \oplus S_{M,s} \otimes \mathcal{E}^{d-1}) \xrightarrow{\sim} \Theta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M)} \quad (7)$$

Since $d \geq 1$, the restriction $\pi_Y : (U \times_{\mathcal{L}(M)} U)_0 \rightarrow \tilde{U}$ is smooth of relative dimension $\dim \mathcal{L}(M)$, with geometrically connected fibres. It is convenient to introduce a rank one local system Θ_U on \tilde{U} equipped with a canonical isomorphism

$$\Theta \xrightarrow{\sim} \pi_Y^* \Theta_U \quad (8)$$

over $(U \times_{\mathcal{L}(M)} U)_0$. The local system Θ_U is defined up to a unique isomorphism.

Let $i_U : U \rightarrow U \times_{\mathcal{L}(M)} U$ be the map sending (L, N) to (L, N, L) . Let $p_1 : U \rightarrow \mathcal{L}(M)$ be the projection sending (L, N) to L .

Lemma 2. 1) *The complex*

$$(q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

is an irreducible perverse sheaf on $U \times_{\mathcal{L}(M)} U \times H$ pure of weight zero. We have canonically

$$i_U^*((q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U)) \xrightarrow{\sim} p_1^* \tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

over $U \times H$.

2) There is a canonical isomorphism

$$(q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \xrightarrow{\sim} q^* \tilde{F}_U \otimes \Theta$$

over $(U \times_{\mathcal{L}(M)} U)_0 \times H$.

Proof 1) Follows from the properties of the Fourier transform as in Lemma 1, formula (1).

2) The proof of Lemma 1 goes through in the geometric setting. Our additional assumption that $(R, N, L) \in (U \times_{\mathcal{L}(M)} U)_0$ means that $b : L \rightarrow N$ is an isomorphism (it simplifies the argument a little). \square

Remark 3. Let $i_\Delta : \mathcal{L}(M) \rightarrow \tilde{Y}$ be the map sending L to $(L, L, \mathcal{B} = \det L)$ equipped with the isomorphism $\text{id} : \mathcal{B}^2 \xrightarrow{\sim} \det L \otimes \det L$. The commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{i_U} & U \times_{\mathcal{L}(M)} U \\ \downarrow p_1 & & \downarrow \pi_Y \\ \mathcal{L}(M) & \xrightarrow{i_\Delta} & \tilde{Y} \end{array} \quad (9)$$

together with (7) yield a canonical isomorphism

$$i_\Delta^* S_M \xrightarrow{\sim} \begin{cases} \mathcal{E}^{-d} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M) - d}, & d \text{ is even} \\ \mathcal{E}^{1-d} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M) - d}, & d \text{ is odd} \end{cases}$$

3.3 Consider the following diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{q}_1} & (U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_2} & \tilde{U} \\ & & \downarrow \tilde{q} & & \\ & & \tilde{U} & & \end{array}$$

Here \tilde{q} is the restriction of π_Y , and the map \tilde{q}_i is the lifting of q_i defined as follows. We set $\tilde{q}_1(R, N, L) = \tilde{q}(R, L, N)$ and $\tilde{q}_2(R, N, L) = \tilde{q}(N, R, L)$.

The following property is a geometric counterpart of the way the Maslov index of (R, N, L) changes under permutations of three lagrangian subspaces.

Lemma 3. 1) For $i = 1, 2$ we have canonically over $(U \times_{\mathcal{L}(M)} U)_0$

$$\tilde{q}_i^* \Theta_U \otimes \tilde{q}^* \Theta_U \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$$

2) We have $\Theta_U^2 \xrightarrow{\sim} \mathcal{E}^{2d}$ canonically, so $\Theta_U^4 \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$ canonically.

Proof 1) The two isomorphisms are obtained similarly, we consider only the case $i = 2$. For a point $(R, N, L) \in (U \times_{\mathcal{L}(M)} U)_0$ we have isomorphisms $b : L \xrightarrow{\sim} N$ and $b_0 : L \xrightarrow{\sim} R$ such that

$R = \{l + b(l) \mid l \in L\}$ and $N = \{l + b_0(l) \mid l \in L\}$. Clearly, $b_0(-l) = l + b(l)$ for $l \in L$. Let $\beta_2 : L \times L \rightarrow \mathbb{A}^1$ be the map sending (l, l_0) to $\frac{1}{2}\omega\langle l, b(l)\rangle + \frac{1}{2}\omega\langle l, b_0(l)\rangle$. We must show that

$$\mathrm{R}\Gamma_c(L \times L, \beta_2^* \mathcal{L}_\psi) \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell[2d](d)$$

The quadratic form $(l, l_0) \mapsto \omega\langle l, b(l)\rangle - \omega\langle l_0, b(l_0)\rangle$ is hyperbolic on $L \oplus L$. Consider the isotopic subspace $Q = \{(l, l) \in L \times L \mid l \in L\}$. Integrating first along the fibres of the projection $L \times L \rightarrow (L \times L)/Q$ and then over $(L \times L)/Q$, one gets the desired isomorphism.

2) This follows from (7). \square

Define a perverse sheaf F_U on $\tilde{U} \times H$ by

$$F_U = \mathrm{pr}_1^* \Theta_U \otimes \tilde{F}_U,$$

it is understood that we take the inverse image of \tilde{F}_U under the projection $\tilde{U} \times H \rightarrow U \times H$ is the above formula. Let F be the intermediate extension of F_U under the open immersion $\tilde{U} \times H \subset \tilde{Y} \times H$.

Remark 4. In the case $d = 0$ we have $H = \mathbb{A}^1$ and $\tilde{Y} = B(\mu_2)$. In this case by definition $F = W \boxtimes \mathcal{L}_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})$ over $\tilde{Y} \times H = B(\mu_2) \times \mathbb{A}^1$.

Combining Lemma 3 and 2) of Lemma 2, we get the following.

Lemma 4. *We have canonically $(\tilde{q}_1^* F_U) * (\tilde{q}_2^* F_U) \xrightarrow{\sim} \tilde{q}^* F_U \otimes \mathcal{E}^{2d}$ over $(U \times_{\mathcal{L}(M)} U)_0 \times H$.*

We have a map $\xi : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \rightarrow \tilde{Y}$ sending $(\mathcal{B}_1, N, \mathcal{B}_1^2 \xrightarrow{\sim} \mathcal{J} \otimes \det N; \mathcal{B}_2, L, \mathcal{B}_2^2 \xrightarrow{\sim} \mathcal{J} \otimes \det L)$ to (N, L, \mathcal{B}) , where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{J}^{-1}$ is equipped with the natural isomorphism $\mathcal{B}^2 \xrightarrow{\sim} \det N \otimes \det L$. The restriction of F under

$$\xi \times \mathrm{id} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{Y} \times H$$

is also denoted by F . Clearly, F is an irreducible perverse sheaf of weight zero.

Consider the cartesian square

$$\begin{array}{ccc} (U \times_{\mathcal{L}(M)} U)_0 \times H & \hookrightarrow & (U \times_{\mathcal{L}(M)} U) \times H \\ \downarrow \pi_Y \times \mathrm{id} & & \downarrow \pi_Y \times \mathrm{id} \\ \tilde{U} \times H & \hookrightarrow & \tilde{Y} \times H \end{array}$$

This diagram together with Lemma 2 yield a canonical isomorphism over $(U \times_{\mathcal{L}(M)} U) \times H$

$$(\pi_Y \times \mathrm{id})^* F \xrightarrow{\sim} (q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \tag{10}$$

by intermediate extension from $(U \times_{\mathcal{L}(M)} U)_0 \times H$. This gives an explicit formula for F .

Consider the diagram

$$\begin{array}{ccc} U \times H & \xrightarrow{i_U \times \mathrm{id}} & U \times_{\mathcal{L}(M)} U \times H \\ \downarrow p_1 \times \mathrm{id} & & \downarrow \pi_Y \times \mathrm{id} \\ \mathcal{L}(M) \times H & \xrightarrow{i_\Delta \times \mathrm{id}} & \tilde{Y} \times H \end{array}$$

obtained from (9) by multiplication with H . By Lemma 2 and (10), we get canonically

$$(p_1 \times \text{id})^*(i_\Delta \times \text{id})^*F \xrightarrow{\sim} (p_1 \times \text{id})^*\tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

Since \tilde{F}_Δ is perverse and p_1 has connected fibres, this isomorphism descends to a uniquely defined isomorphism

$$(i_\Delta \times \text{id})^*F \xrightarrow{\sim} \tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

By construction, F is act_{l_r} -equivariant and G -equivariant (this holds for F_U and this property is preserved by the intermediate extension).

3.4 To finish the proof of Theorem 1, it remains to establish the convolution property of F . We actually prove it in the following form.

Write $\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$ for the stack classifying $R, N, L \in \mathcal{L}(M)$, one dimensional k -vector spaces $\mathcal{B}_1, \mathcal{B}_2$ equipped with isomorphisms $\mathcal{B}_1^2 \xrightarrow{\sim} \det R \otimes \det N$ and $\mathcal{B}_2^2 \xrightarrow{\sim} \det N \otimes \det L$. We have a diagram

$$\begin{array}{ccccc} \tilde{Y} & \xleftarrow{\tau_1} & \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y} & \xrightarrow{\tau_2} & \tilde{Y} \\ & & \downarrow \tau & & \\ & & \tilde{Y} & & \end{array}$$

where τ_1 (resp., τ_2) sends the above collection to $(R, N, \mathcal{B}_1) \in \tilde{Y}$ (resp., $(N, L, \mathcal{B}_2) \in \tilde{Y}$). The map τ sends the above collection to (R, L, \mathcal{B}) , where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes (\det N)^{-1}$ is equipped with $\mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L$.

Proposition 1. *There is a canonical isomorphism over $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$*

$$(\tau_1^*F) * (\tau_2^*F) \xrightarrow{\sim} \tau^*F \tag{11}$$

Proof

Step 1. Consider the diagram

$$\begin{array}{ccc} (U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_1 \times \tilde{q}_2} & (\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \\ & \searrow \tilde{q} & \downarrow \tau \\ & & \tilde{U} \end{array}$$

It becomes 2-commutative over $\text{Spec } \mathbb{F}_q(\sqrt{-1})$. More precisely, for $K \in \text{D}(\tilde{U})$ we have a canonical isomorphism functorial in K

$$\tilde{q}^*K \otimes \mathcal{E}^{2d} \xrightarrow{\sim} (\tilde{q}_1 \times \tilde{q}_2)^* \tau^*K$$

Indeed, let (R, N, L) be a k -point of $(U \times_{\mathcal{L}(M)} U)_0$, let $(R, N, L, \mathcal{B}_1, \mathcal{B}_2)$ be its image under $\tilde{q}_1 \times \tilde{q}_2$. So, $\mathcal{B}_1 = \det N$ and $\pi_Y(R, L, N) = (R, N, \mathcal{B}_1)$, $\mathcal{B}_2 = \det L$ and $\pi_Y(N, R, L) = (N, L, \mathcal{B}_2)$. Write

$$\tau(R, N, L, \mathcal{B}_1, \mathcal{B}_2) = (R, L, \mathcal{B}, \delta : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L)$$

Write $\tilde{q}(R, N, L) = (R, L, \mathcal{B}, \delta_0 : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L)$. It suffices to show that $\delta_0 = (-1)^d \delta$.

Let $\epsilon_1 : N \xrightarrow{\sim} R$ be the isomorphism sending $n \in N$ to $r \in R$ such that $r = n \pmod L$. Write $\epsilon_2 : L \xrightarrow{\sim} N$ for the isomorphism sending $l \in L$ to $n \in N$ such that $l = n \pmod R$. Let $\epsilon_0 : L \xrightarrow{\sim} R$ be the isomorphism sending $l \in L$ to $r \in R$ such that $r = l \pmod N$. We get two isomorphisms

$$\text{id} \otimes \det \epsilon_0, \det \epsilon_1 \otimes \det \epsilon_2 : \det N \otimes \det L \xrightarrow{\sim} \det R \otimes \det N$$

We must show that $\text{id} \otimes \det \epsilon_0 = (-1)^d \det \epsilon_1 \otimes \det \epsilon_2$. Pick a base $\{n_1, \dots, n_d\}$ in N . Define $r_i \in R, l_i \in L$ by $n_i = r_i + l_i$. Then

$$\epsilon_1(n_i) = r_i, \quad \epsilon_2(l_i) = n_i, \quad \epsilon_0(l_i) = -r_i$$

So, $\epsilon_0(l_1 \wedge \dots \wedge l_d) = (-1)^d r_1 \wedge \dots \wedge r_d$. On the other hand, $\det \epsilon_1 \otimes \det \epsilon_2$ sends

$$(n_1 \wedge \dots \wedge n_d) \otimes (l_1 \wedge \dots \wedge l_d)$$

to $(r_1 \wedge \dots \wedge r_d) \otimes (n_1 \wedge \dots \wedge n_d)$.

Step 2. The isomorphism (6) for $i = 0$ yields $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \xrightarrow{\sim} (U \times_{\mathcal{L}(M)} U)_0 \times B(\mu_2) \times B(\mu_2)$. The corresponding 2-automorphisms $\mu_2 \times \mu_2$ of $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y})$ act in the same way on both sides of (11). Now from Step 1 it follows that the isomorphism of Lemma 4 descends under $\tilde{q}_1 \times \tilde{q}_2$ to the desired isomorphism (11) over $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H$.

Step 3. To finish the proof it suffices to show that $(\tau_1^* F) * (\tau_2^* F)$ is perverse, the intermediate extension under the open immersion

$$(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H \subset (\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$$

Let us first explain the idea informally, at the level of functions. In this step for $(N, R, \mathcal{B}) \in \tilde{Y}$ we denote by $F_{N,R,\mathcal{B}} : H \rightarrow \bar{\mathbb{Q}}_\ell$ the function trace of Frobenius of the sheaf F .

Given $(R, N, \mathcal{B}_1) \in \tilde{Y}$ and $(N, L, \mathcal{B}_2) \in \tilde{Y}$ pick any $S, T \in \mathcal{L}(M)$ such that $(R, S, N) \in U \times_{\mathcal{L}(M)} U$, $(N, T, L) \in U \times_{\mathcal{L}(M)} U$ and $S \cap T = S \cap L = 0$. Assuming

$$(R, N, \mathcal{B}_1) = \pi_Y(R, S, N) \quad \text{and} \quad (N, L, \mathcal{B}_2) = \pi_Y(N, T, L),$$

by (10) we get

$$\begin{aligned} F_{R,N,\mathcal{B}_1} * F_{N,L,\mathcal{B}_2} &= (\tilde{F}_{R,S} * \tilde{F}_{S,N}) * (\tilde{F}_{N,T} * \tilde{F}_{T,L}) = q^{d+1} \theta(S, N, T) \tilde{F}_{R,S} * \tilde{F}_{S,T} * \tilde{F}_{T,L} \\ &= q^{2d+2} \theta(S, N, T) \theta(S, T, L) \tilde{F}_{R,S} * \tilde{F}_{S,L} = q^{2d+2} \theta(S, N, T) \theta(S, T, L) F_{R,L,\mathcal{B}}, \end{aligned}$$

where $(R, L, \mathcal{B}) = \pi_Y(R, S, L)$. Now we turn back to the geometric setting.

Step 4. Consider the scheme \mathcal{W} classifying $(R, S, N) \in U \times_{\mathcal{L}(M)} U$ and $(N, T, L) \in U \times_{\mathcal{L}(M)} U$ such that $S \cap T = S \cap L = 0$. Let

$$\kappa : \mathcal{W} \rightarrow \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$$

be the map sending the above point to $(R, N, L, \mathcal{B}_1, \mathcal{B}_2)$, where $(R, N, \mathcal{B}_1) = \pi_Y(R, S, N)$ and $(N, L, \mathcal{B}_2) = \pi_Y(N, T, L)$. The map κ is smooth and surjective. It suffices to show that

$$\kappa^*((\tau_1^*F) * (\tau_2^*F))$$

is a shifted perverse sheaf, the intermediate extension from $\kappa^{-1}(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0$.

Let $\mu : \mathcal{W} \rightarrow U \times_{\mathcal{L}(M)} U$ be the map sending a point of \mathcal{W} to (R, S, L) . Applying (10) several times as in Step 3, we learn that there is a local system of rank one and order two, say \mathcal{I} on \mathcal{W} such that

$$\kappa^*((\tau_1^*F) * (\tau_2^*F)) \xrightarrow{\sim} \mathcal{I} \otimes \mu^* \pi_Y^* F$$

Since F is an irreducible perverse sheaf, our assertion follows. \square

Thus, Theorem 1 is proved.

3.5 Now given k -points $N^0, L^0 \in \tilde{\mathcal{L}}(M)$, let $F_{N^0, L^0} \in \mathbf{D}(H)$ be the $*$ -restriction of F under $(N^0, L^0) \times \text{id} : H \hookrightarrow \tilde{Y} \times H$. Define the functor $\mathcal{F}_{N^0, L^0} : \mathbf{D}\mathcal{H}_L \rightarrow \mathbf{D}\mathcal{H}_N$ by

$$\mathcal{F}_{N^0, L^0}(K) = F_{N^0, L^0} * K$$

To see that it preserves perversity we can pick $S^0 \in \tilde{\mathcal{L}}(M)$ with $N \cap S = L \cap S = 0$ and use $\mathcal{F}_{N^0, L^0} = \mathcal{F}_{N^0, S^0} \circ \mathcal{F}_{S^0, L^0}$. This reduces the question to the case $N \cap L = 0$, in the latter case \mathcal{F}_{N^0, L^0} is nothing but the Fourier transform.

By Theorem 1, for $N^0, L^0, R^0 \in \tilde{\mathcal{L}}(M)$ the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathbf{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{R^0, L^0}} & \mathbf{D}\mathcal{H}_R \\ & \searrow \mathcal{F}_{N^0, L^0} & \downarrow \mathcal{F}_{N^0, R^0} \\ & & \mathbf{D}\mathcal{H}_N \end{array}$$

3.6 NONRAMIFIED WEIL CATEGORY

For a k -point $L^0 \in \tilde{\mathcal{L}}(M)$ let $i_{L^0} : \tilde{\mathcal{L}}(M) \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ be the map sending N^0 to $(N^0, L^0, 0)$. We get a functor $\mathcal{F}_{L^0} : \mathbf{D}\mathcal{H}_L \rightarrow \mathbf{D}(\tilde{\mathcal{L}}(M))$ sending K to the complex

$$i_{L^0}^*(F * \text{pr}_3^* K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M) - 2d - 1}$$

For any k -points $L^0, N^0 \in \tilde{\mathcal{L}}(M)$ the diagram commutes

$$\begin{array}{ccc} \mathbf{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathbf{D}(\tilde{\mathcal{L}}(M)) \\ & \searrow \mathcal{F}_{L^0, N^0} & \uparrow \mathcal{F}_{N^0} \\ & & \mathbf{D}\mathcal{H}_N \end{array} \tag{12}$$

One checks that \mathcal{F}_{L^0} is exact for the perverse t-structure.

Definition 2. *The non-ramified Weil category $W(\tilde{\mathcal{L}}(M))$ is the essential image of $\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow \mathbf{P}(\tilde{\mathcal{L}}(M))$. This is a full subcategory in $\mathbf{P}(\tilde{\mathcal{L}}(M))$ independent of L^0 , because (12) is commutative.*

The group G acts naturally on $\tilde{\mathcal{L}}(M)$, hence also on $\mathbf{P}(\tilde{\mathcal{L}}(M))$. This action preserves the full subcategory $W(\tilde{\mathcal{L}}(M))$.

At the classical level, for $L \in \mathcal{L}(M)$ the G -representation $\mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{L,odd} \oplus \mathcal{H}_{L,even}$ is a direct sum of two irreducible ones consisting of odd and even functions respectively. The category $W(\tilde{\mathcal{L}}(M))$ is a geometric analog of the space $\mathcal{H}_{L,even}$. The geometric analog of the whole Weil representation \mathcal{H}_L is as follows.

Definition 3. Let $\text{act}_l : \bar{C} \times H \rightarrow \tilde{\mathcal{L}}(M) \times H$ be the map sending $(L^0, h, \bar{l} \in \bar{L})$ to $(L^0, \bar{l}h)$. A perverse sheaf $K \in \mathbf{P}(\tilde{\mathcal{L}}(M) \times H)$ is $(\bar{C}, \chi_{\bar{C}})$ -equivariant if it is equipped with an isomorphism

$$\text{act}_l^* K \xrightarrow{\sim} \text{pr}_1^* K \otimes \text{pr}_1^* \chi_{\bar{C}}$$

satisfying the usual associativity property, and whose restriction to the unit section is the identity.

The complete Weil category $W(M)$ is the category of pairs (K, σ) , where $K \in \mathbf{P}(\tilde{\mathcal{L}}(M) \times H)$ is a $(\bar{C}, \chi_{\bar{C}})$ -equivariant perverse sheaf, and

$$\sigma : F * \text{pr}_{23}^* K \xrightarrow{\sim} \text{pr}_{13}^* K$$

is an isomorphism for the projections $\text{pr}_{13}, \text{pr}_{23} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times H$. The map σ must be compatible with the associativity constraint and the unit section constraint of F .

The group G acts on $\tilde{\mathcal{L}}(M) \times H$ sending $(g \in G, L^0, h)$ to (gL^0, gh) . This action extends to an action of G on the category $W(M)$.

4. COMPATIBILITY PROPERTY

4.1 In this section we establish the following additional property of the canonical intertwining operators. Let $V \subset M$ be an isotropic subspace, $V^\perp \subset M$ its orthogonal complement. Let $\mathcal{L}(M)_V \subset \mathcal{L}(M)$ be the open subscheme of $L \in \mathcal{L}(M)$ such that $L \cap V = 0$. Set $M_0 = V^\perp/V$. We have a map $p_V : \mathcal{L}(M)_V \rightarrow \mathcal{L}(M_0)$ sending L to $L_V := L \cap V^\perp$.

Write $Y = \mathcal{L}(M) \times \mathcal{L}(M)$ and $Y_V = \mathcal{L}(M)_V \times \mathcal{L}(M)_V$. The gerb \tilde{Y} is defined as in Section 3.2, write \tilde{Y}_V for its restriction to Y_V . Set $Y_0 = \mathcal{L}(M_0) \times \mathcal{L}(M_0)$, we have the corresponding gerb \tilde{Y}_0 defined as in Section 3.2. We extend the map $p_V \times p_V$ to a map

$$\pi_V : \tilde{Y}_V \rightarrow \tilde{Y}_0$$

sending $(L_1, L_2, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det L_1 \otimes \det L_2)$ to

$$(L_{1,V}, L_{2,V}, \mathcal{B}_0, \mathcal{B}_0^2 \xrightarrow{\sim} \det L_{1,V} \otimes \det L_{2,V})$$

Here $L_{i,V} = L_i \cap V^\perp$ and $\mathcal{B}_0 = \mathcal{B} \otimes \det V$. We used the exact sequences

$$0 \rightarrow L_{i,V} \rightarrow L_i \rightarrow M/V^\perp \rightarrow 0$$

yielding canonical ($\mathbb{Z}/2\mathbb{Z}$ -graded) isomorphisms $\det L_{i,V} \otimes \det V^* \xrightarrow{\sim} \det L_i$.

Write $H_0 = M_0 \oplus k$ for the Heisenberg group of M_0 . For $L \in \mathcal{L}(M)_V$ we have the categories \mathcal{H}_L and \mathcal{H}_{L_V} of certain perverse sheaves on H and H_0 respectively. To such L we associate a transition functor $T^L : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_L$ which will be fully faithful and exact for the perverse t-structures.

Write for brevity $H^V = V^\perp \times \mathbb{A}^1$. First, at the level of functions, given $f \in \mathcal{H}_{L_V}$ consider it as a function on H^V via the composition $H^V \xrightarrow{\alpha_V} H_0 \xrightarrow{f} \bar{\mathbb{Q}}_\ell$, where α_V sends (v, a) to $(v \bmod V, a)$. Then there is a unique $f_1 \in \mathcal{H}_L$ such that $f_1(m) = q^{\dim V} f(m)$ for all $m \in H^V$. We use the property $V^\perp + L = M$. We set

$$(T^L)(f) = f_1 \tag{13}$$

The image of T^L is

$$\{f_1 \in \mathcal{H}_L \mid f(h(v, 0)) = f(h), \quad h \in H, v \in V\}$$

Note that $H^V \subset H$ is a subgroup, and $V = \{(v, 0) \in H^V \mid v \in V\} \subset H^V$ is a normal subgroup lying in the center of H^V . The operator $T^L : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_L$ commutes with the action of H^V . It is understood that on \mathcal{H}_{L_V} this group acts via its quotient $H^V \xrightarrow{\alpha_V} H_0$.

On the geometric level, consider the map $s : L \times H^V \rightarrow H$ sending $(l, (v, a))$ to the product in the Heisenberg group $(l, 0)(v, a) \in H$. Note that s is smooth and surjective, an affine fibration of rank $\dim L_V$. Given $K \in \mathcal{H}_{L_V}$ there is a (defined up to a unique isomorphism) perverse sheaf $T^L K \in \mathcal{H}_L$ equipped with

$$s^*(T^L K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim L_V} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell \boxtimes \alpha_V^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim V + \dim L}$$

The *compatibility property* of the canonical intertwining operators is as follows.

Proposition 2. *Let $(L, N, \mathcal{B}) \in \tilde{Y}_V$, write $(L_V, N_V, \mathcal{B}_0)$ for the image of (L, N, \mathcal{B}) under π_V . Write $\mathcal{F}_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$ and $\mathcal{F}_{N_V^0, L_V^0} : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_{N_V}$ for the corresponding functors defined as in Section 3.5. Then the diagram of categories is canonically 2-commutative*

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^L} & \mathcal{H}_L \\ \downarrow \mathcal{F}_{N_V^0, L_V^0} & & \downarrow \mathcal{F}_{N^0, L^0} \\ \mathcal{H}_{N_V} & \xrightarrow{T^N} & \mathcal{H}_N \end{array}$$

One may also replace \mathcal{H} by $D\mathcal{H}$ in the above diagram.

4.2 First, we realize the functors T^L by a universal kernel, namely, we define a perverse sheaf T on $\mathcal{L}(M)_V \times H \times H_0$ as follows.

Remind the vector bundle $\bar{C} \rightarrow \mathcal{L}(M)$, its fibre over L is $\bar{L} = L \times \mathbb{A}^1$. Write \bar{C}_V for the restriction of \bar{C} to the open subscheme $\mathcal{L}(M)_V$. We have a closed immersion

$$i_0 : \bar{C}_V \times H^V \rightarrow \mathcal{L}(M)_V \times H \times H_0$$

sending $(\bar{l} \in \bar{L}, u \in H^V)$ to $(L, \bar{l}u, \alpha_V(u))$, where the product $\bar{l}u$ is taken in H . The perverse sheaf T is defined by

$$T = (i_0)_! \operatorname{pr}_1^* \chi_{\bar{C}} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \bar{C} + \dim V + \dim H_0},$$

here $\operatorname{pr}_1 : \bar{C}_V \times H^V \rightarrow \bar{C}_V$ is the projection, and $\chi_{\bar{C}}$ was defined in 3.1.

For $L \in \mathcal{L}(M)_V$ let T_L be the $*$ -restriction of T under $(L, \operatorname{id}) : H \times H_0 \rightarrow \mathcal{L}(M)_V \times H \times H_0$. Define $T^L : D\mathcal{H}_{L_V} \rightarrow D\mathcal{H}_L$ by

$$T^L(K) \xrightarrow{\sim} \operatorname{pr}_{1!}(T_L \otimes \operatorname{pr}_2^* K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim V - d - \dim \mathcal{L}(M)} \quad (14)$$

for the diagram of projections $H \xleftarrow{\operatorname{pr}_1} H \times H_0 \xrightarrow{\operatorname{pr}_2} H_0$. It is exact for the perverse t-structures.

The sheaf T has the following properties. At the level of functions, the corresponding function $T_L : H \times H_0 \rightarrow \bar{\mathbb{Q}}_\ell$ satisfies

$$T_L(\bar{l}h, \bar{l}_0 h_0) = \chi_L(\bar{l}) \chi_{L_V}(\bar{l}_0)^{-1} T_L(h, h_0), \quad \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V$$

The geometric analog is as follows. Let ${}^0\bar{C} \rightarrow \mathcal{L}(M)_V$ be the vector bundle, whose fibre over $L \in \mathcal{L}(M)_V$ is $\bar{L} \times \bar{L}_V$. Consider the diagram

$$\mathcal{L}(M)_V \times H \times H_0 \xleftarrow{\operatorname{pr}^V} {}^0\bar{C} \times H \times H_0 \xrightarrow{\operatorname{act}_{r'}^V} \mathcal{L}(M)_V \times H \times H_0,$$

where pr^V is the projection, and $\operatorname{act}_{r'}^V$ sends

$$(L \in \mathcal{L}(M)_V, \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V, h \in H, h_0 \in H_0)$$

to $(L, \bar{l}h, \bar{l}_0 h_0)$. Let ${}^0p : {}^0\bar{C} \rightarrow \mathbb{A}^1$ be the map sending

$$(L \in \mathcal{L}(M)_V, \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V)$$

to $p(\bar{l}) - p(\bar{l}_0)$. Here $p : \bar{L} \rightarrow \mathbb{A}^1$ and $p : \bar{L}_V \rightarrow \mathbb{A}^1$ are the projections on the center. Set ${}^0\chi = ({}^0p)^* \mathcal{L}_\psi$. Then T is $\operatorname{act}_{r'}^V$ -equivariant, that is, it admits an isomorphism

$$(\operatorname{act}_{r'}^V)^* T \xrightarrow{\sim} (\operatorname{pr}^V)^* T \otimes \operatorname{pr}_1^* ({}^0\chi),$$

satisfying the usual associativity property, and its restriction to the unit section is the identity.

4.3 We will prove a geometric version of the equality (up to an explicit power of q)

$$\int_{u \in H} F_{N^0, L^0}(hu^{-1}) T_L(u, h_0) du = \int_{v \in H_0} T_N(h, v) F_{N_V^0, L_V^0}(vh_0^{-1}) dv$$

for $h \in H, h_0 \in H_0$. Here $(N^0, L^0) \in \tilde{Y}_V$ and

$$(N_V^0, L_V^0) = \pi_V(N^0, L^0)$$

Write $\text{inv} : H \xrightarrow{\sim} H$ for the map sending h to h^{-1} , set $\text{inv}^* F = (\text{id} \times \text{inv})^* F$ for $\text{id} \times \text{inv} : \tilde{Y} \times H \rightarrow \tilde{Y} \times H$. For $i = 1, 2$ write $p_i : \tilde{Y}_V \rightarrow \mathcal{L}(M)_V$ for the projection on the i -th factor. Let q_0 denote the composition

$$\tilde{Y}_V \times H \times H_0 \xrightarrow{\text{pr}_{13}} \tilde{Y}_V \times H_0 \xrightarrow{\pi_V \times \text{id}} \tilde{Y}_0 \times H_0$$

Proposition 2 is an immediate consequence of the following.

Lemma 5. *There is a canonical isomorphism over $\tilde{Y}_V \times H \times H_0$*

$$(\text{pr}_{12}^* F) *_H (p_2 \times \text{id})^* T \xrightarrow{\sim} (q_0^*(\text{inv}^* F)) *_H (p_1 \times \text{id})^* T$$

where $\text{pr}_{12} : \tilde{Y}_V \times H \times H_0 \rightarrow \tilde{Y}_V \times H$ and $p_1 \times \text{id}, p_2 \times \text{id} : \tilde{Y}_V \times H \times H_0 \rightarrow \mathcal{L}(M)_V \times H \times H_0$.

Let $i_V : H^V \hookrightarrow H$ be the natural closed immersion. It is elementary to check that Lemma 5 is equivalent to the following.

Lemma 6. *There is a canonical isomorphism of (shifted) perverse sheaves*

$$(\text{id} \times \alpha_V)_! i_V^* F \xrightarrow{\sim} (\pi_V \times \text{id})^* F \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\pi_V) + \dim V} \quad (15)$$

for the diagram

$$\begin{array}{ccc} \tilde{Y}_V \times H^V & \xrightarrow{i_V} & \tilde{Y}_V \times H \\ \downarrow \text{id} \times \alpha_V & & \\ \tilde{Y}_0 \times H_0 & \xleftarrow{\pi_V \times \text{id}} & \tilde{Y}_V \times H_0 \end{array}$$

Proof Write $U(M_0)$ for the scheme U constructed out of the symplectic space M_0 , it classifies pairs of lagrangian subspaces in M_0 that do not intersect. We have a 2-commutative diagram

$$\begin{array}{ccccc} U(M_0) \times_{\mathcal{L}(M_0)} U(M_0) & \xleftarrow{\pi_W} & W_V & \xrightarrow{i_W} & U \times_{\mathcal{L}(M)} U \\ \downarrow \pi_{Y_0} & & \downarrow \pi_{Y,V} & \swarrow \pi_Y & \\ \tilde{Y}_0 & \xleftarrow{\pi_V} & \tilde{Y}_V & & \end{array}$$

where the square is cartesian thus defining W_V, π_W , and $\pi_{Y,V}$. The map i_W is a locally closed immersion. Write a point of W_V as a triple $(N, R, L) \in \mathcal{L}(M)$ such that $N, L \in \mathcal{L}(M)_V$, $V \subset R \subset V^\perp$, and $N \cap R = R \cap L = 0$. The map π_W sends (N, R, L) to (N_V, R_V, L_V) with $R_V = R/V$.

Let us establish the isomorphism (15) after restriction under $\pi_{Y,V} \times \alpha_V : W_V \times H^V \rightarrow \tilde{Y}_V \times H_0$. We first give the argument at the level of functions and then check that it holds through in the geometric setting.

Consider a point of W_V given by a triple $(N, R, L) \in \mathcal{L}(M)$, so $N, L \in \mathcal{L}(M)_V$, $V \subset R \subset V^\perp$, and $N \cap R = R \cap L = 0$. We have $V^\perp = R \oplus L_V$. Let $h \in H^V$, write $h = (r, a)(l_1, 0)$ for uniquely

defined $r \in R, l_1 \in L_V, a \in k$. Write $(N^0, L^0) \in \tilde{Y}_V$ for the image of (N, R, L) under $\pi_{Y,V}$. Using (10), we get

$$\begin{aligned} \int_{v \in V} F_{N^0, L^0}(h(v, 0)) dv &= q^{\dim \mathcal{L}(M) - \frac{d+1}{2}} \int_{v \in V, u \in H} \tilde{F}_{N,R}(u) \tilde{F}_{R,L}(u^{-1}h(v, 0)) dv du = \\ &= q^{\dim \mathcal{L}(M) + \frac{d+1}{2}} \int_{v \in V, u \in H/\tilde{R}} \tilde{F}_{N,R}(u) \tilde{F}_{R,L}(u^{-1}(r, a)(v, 0)) dv du = \\ &= q^{\dim \mathcal{L}(M) + \frac{d+1}{2}} \int_{v \in V, l \in L} \tilde{F}_{N,R}(l, 0) \tilde{F}_{R,L}((-l, 0)(r, a)(v, 0)) dv dl \end{aligned}$$

Since $(-l, 0)(r + v, a) = (r + v, a + \omega\langle r + v, l \rangle)(-l, 0)$, the latter expression equals

$$q^{-\frac{d}{2}} \int_{v \in V, l \in L} \tilde{F}_{N,R}(l, 0) \psi(a + \omega\langle r + v, l \rangle) dv dl = q^{\dim V - \frac{d}{2}} \int_{l \in L_V} \tilde{F}_{N,R}(l, 0) \psi(a + \omega\langle r, l \rangle) dl$$

For $l \in L_V$ we get $\tilde{F}_{N,R}(l, 0) = q^{\dim \mathcal{L}(M_0) - \dim \mathcal{L}(M) - \dim V} \tilde{F}_{N_V, R_V}(l, 0)$. Indeed, since $V^\perp = R \oplus N_V$, there are unique $r_1 \in R, n_1 \in N_V$ such that $l = n_1 + r_1$. For $\bar{r}_1 = r_1 \bmod V \in M_0$ we get

$$\begin{aligned} \tilde{F}_{N,R}(l, 0) &= q^{-\dim \mathcal{L}(M) - \frac{2d+1}{2}} \chi_{NR}(l, 0) = q^{-\dim \mathcal{L}(M) - \frac{2d+1}{2}} \psi\left(\frac{1}{2}\omega\langle r_1, n_1 \rangle\right) = \\ &= q^{-\dim \mathcal{L}(M) - \frac{2d+1}{2}} \chi_{N_V R_V}(\bar{r}_1 + n_1, 0) = q^{\dim \mathcal{L}(M_0) - \dim \mathcal{L}(M) - \dim V} \tilde{F}_{N_V, R_V}(l, 0) \end{aligned}$$

Further, we claim that

$$\tilde{F}_{R_V, L_V}((-l, 0)\alpha_V(h)) = q^{-\dim \mathcal{L}(M_0) - \frac{\dim H_0}{2}} \psi(a + \omega\langle r, l \rangle)$$

This follows from definition (5) of \tilde{F}_U and the formula $(-l, 0)(r, a) = (r, a + \omega\langle r, l \rangle)(-l, 0)$.

Combing the above we get

$$\begin{aligned} \int_{v \in V} F_{N^0, L^0}(h(v, 0)) dv &= q^c \int_{l \in L_V} \tilde{F}_{N_V, R_V}(l, 0) \tilde{F}_{R_V, L_V}((-l, 0)\alpha_V(h)) dl = \\ &= q^{c + \dim V - d - 1} \int_{u \in H_0} \tilde{F}_{N_V, R_V}(u) \tilde{F}_{R_V, L_V}(u^{-1}\alpha_V(h)) du \end{aligned}$$

with $c = \frac{\dim H_0 - d}{2} + 2 \dim \mathcal{L}(M_0) - \dim \mathcal{L}(M)$. By (10), the latter expression identifies with $F_{N_V^0, L_V^0}(h)$ up to an explicit power of q .

The argument holds through in the geometric setting yielding the desired isomorphism γ over $W_V \times H^V$. For any point $(N_V, L_V \mathcal{B}_0) \in \tilde{Y}_0$ such that $N_V \neq L_V$ the fibre of π_{Y_0} over this point is geometrically connected. So, for $\dim V < d$ the isomorphism γ descends to a uniquely defined isomorphism (15). The case $\dim V = d$ is easier and is left to the reader. \square

Remark 5. Let $i_H : \text{Spec } k \hookrightarrow H$ denote the zero section. Arguing as in Lemma 6, for the map $\text{id} \times i_H : \tilde{Y} \rightarrow \tilde{Y} \times H$ one gets a canonical isomorphism

$$(\text{id} \times i_H)^* F \xrightarrow{\sim} (S_{M,g} \otimes \mathcal{E}^d \oplus S_{M,s} \otimes \mathcal{E}^{d-1}) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim H},$$

it will not be used in this paper.

4.4 The functors T^L satisfy the following transitivity property. Assume that $V_1 \subset V$ is another isotropic subspace in M . Let $M_1 = V_1^\perp/V_1$ and $H_1 = M_1 \times \mathbb{A}^1$ be the corresponding Heisenberg group. Then for $L \in \mathcal{L}(M)_V$ we also have $L_{V_1} := L \cap V_1^\perp$ and the category $\mathcal{H}_{L_{V_1}}$ of certain perverse sheaves on H_1 . Then the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^{L_{V_1}}} & \mathcal{H}_{L_{V_1}} \\ & \searrow T^L & \downarrow T^L \\ & & \mathcal{H}_L \end{array}$$

4.5 We will need also one more compatibility property of the canonical interwining operators. Let $V \subset V^\perp \subset M$ be as in 4.1. Write $i_{0,V} : \mathcal{L}(M_0) \rightarrow \mathcal{L}(M)$ for the closed immersion sending L_0 to the preimage of L_0 under $V^\perp \rightarrow V^\perp/V$.

For $L \in \mathcal{L}(M)$ with $V \subset L$ set $L_V = L/V \in \mathcal{L}(M_0)$. Let $(\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\sim$ denote the restriction of the gerb \tilde{Y} under

$$\mathcal{L}(M_0) \times \mathcal{L}(M)_V \xrightarrow{i_{0,V} \times \text{id}} \mathcal{L}(M) \times \mathcal{L}(M)_V \subset Y$$

Define $\pi_{0,V} : (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\sim \rightarrow \tilde{Y}_0$ as the map sending $(L, N, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det L \otimes \det N)$ to

$$(L_V, N_V, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det L_V \otimes \det N_V)$$

Here $L \in \mathcal{L}(M)$ with $V \subset L$. We have used the canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism $\det L \otimes \det N \xrightarrow{\sim} \det L_V \otimes \det N_V$.

Remind the closed immersion $i_V : H^V \hookrightarrow H$. For $L \in \mathcal{L}(M)$ with $V \subset L$ define the transition functor $T^L : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_L$ by

$$T^L(K) = i_{V!} \alpha_V^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim V}$$

The proof of the following is similar to that of Proposition 2 and is left to the reader.

Proposition 3. *Let $(L, N, \mathcal{B}) \in (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\sim$, let (L_V, N_V, \mathcal{B}) denote its image under $\pi_{0,V}$. Write $\mathcal{F}_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$ and $\mathcal{F}_{N_V^0, L_V^0}$ for the corresponding functors defined as in Section 3.5. Then the diagram of categories is canonically 2-commutative*

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^L} & \mathcal{H}_L \\ \downarrow \mathcal{F}_{N_V^0, L_V^0} & & \downarrow \mathcal{F}_{N^0, L^0} \\ \mathcal{H}_{N_V} & \xrightarrow{T^N} & \mathcal{H}_N \end{array}$$

One may also replace \mathcal{H} by $D\mathcal{H}$ in the above diagram. \square

5. DISCRETE LAGRANGIAN LATTICES AND THE METAPLECTIC GROUP

5.1 Set $\mathcal{O} = k[[t]] \subset F = k((t))$. Denote by Ω the completed module of relative differentials of \mathcal{O} over k . Let M be a free \mathcal{O} -module of rank $2d$ with symplectic form $\wedge^2 M \rightarrow \Omega$. Write G for the group scheme over $\text{Spec } \mathcal{O}$ of automorphisms of M preserving the symplectic form. Consider the Tate space $M(F)$ (cf. [1], 4.2.13 for the definition), it is equipped with the symplectic form $(m_1, m_2) \mapsto \text{Res } \omega \langle m_1, m_2 \rangle$.

For a k -subspace $L \subset M(F)$ write

$$L^\perp = \{m \in M(F) \mid \text{Res } \omega \langle m, l \rangle = 0 \text{ for all } l \in L\}$$

For two k -subspaces $L_1, L_2 \subset M$ we get $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$. For a finite-dimensional symplectic k -vector space U write $\mathcal{L}(U)$ for the variety of lagrangian subspaces in U .

As in *loc.cit*, we say that an \mathcal{O} -submodule $R \subset M(F)$ is a *c-lattice* if $M(-N) \subset R \subset M(N)$ for some integer N . A *lagrangian d-lattice* in $M(F)$ is a k -vector subspace $L \subset M(F)$ such that $L^\perp = L$ and there exists a c-lattice R with $R \cap L = 0$. Note that the condition $R \cap L = 0$ implies $R^\perp + L = M(F)$. Let $\mathcal{L}_d(M(F))$ denote the set of lagrangian d-lattices in $M(F)$.

For a given c-lattice $R \subset M(F)$ write

$$\mathcal{L}_d(M(F))_R = \{L \in \mathcal{L}_d(M(F)) \mid L \cap R = 0\}$$

If R is a c-lattice in $M(F)$ with $R \subset R^\perp$ then $\mathcal{L}_d(M(F))_R$ is a naturally a k -scheme (not of finite type over k). Indeed, for each c-lattice $R_1 \subset R$ we have the variety

$$\mathcal{L}(R_1^\perp/R_1)_R := \{L_1 \in \mathcal{L}(R_1^\perp/R_1) \mid L_1 \cap R/R_1 = 0\}$$

For $R_2 \subset R_1 \subset R$ we get a map $p_{R_2, R_1} : \mathcal{L}(R_2^\perp/R_2)_R \rightarrow \mathcal{L}(R_1^\perp/R_1)_R$ sending L_2 to

$$L_1 = (L_2 \cap (R_1^\perp/R_2)) + R_1$$

The map p_{R_2, R_1} is a composition of two affine fibrations of constant rank. Then $\mathcal{L}_d(M(F))_R$ is the inverse limit of $\mathcal{L}(R_1^\perp/R_1)_R$ over the partially ordered set of c-lattices $R_1 \subset R$.

If $R' \subset R$ is another c-lattice then $\mathcal{L}_d(M(F))_R \subset \mathcal{L}_d(M(F))_{R'}$ is an open immersion (as it is an open immersion on each term of the projective system). So, $\mathcal{L}_d(M(F))$ is a k -scheme that can be seen as the inductive limit of $\mathcal{L}_d(M(F))_R$.

Let us define the categories $\mathbf{P}(\mathcal{L}_d(M(F)))$ and $\mathbf{P}_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$ of perverse sheaves and $G(\mathcal{O})$ -equivariant perverse sheaves on $\mathcal{L}_d(M(F))$.

For $r \geq 0$ set

$${}_r\mathcal{L}_d(M(F)) = \mathcal{L}_d(M(F))_{M(-r)},$$

the group $G(\mathcal{O})$ acts on ${}_r\mathcal{L}_d(M(F))$ naturally. First, define the category $\mathbf{D}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$ as follows.

For $N + r \geq 0$ set ${}_{N,r}M = t^{-N}M/t^rM$. For $N \geq r \geq 0$ the action of $G(\mathcal{O})$ on ${}_r\mathcal{L}({}_{N,N}M) := \mathcal{L}({}_{N,N}M)_{M(-r)}$ factors through $G(\mathcal{O}/t^{2N})$. For $r_1 \geq 2N$ the kernel

$$\text{Ker}(G(\mathcal{O}/t^{r_1})) \rightarrow G(\mathcal{O}/t^{2N})$$

is unipotent, so that we have an equivalence (exact for the perverse t-structures)

$$D_{G(\mathcal{O}/t^{2N})}({}_r\mathcal{L}_{(N,N)}M) \xrightarrow{\sim} D_{G(\mathcal{O}/t^{r_1})}({}_r\mathcal{L}_{(N,N)}M)$$

Define $D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N,N)}M)$ as $D_{G(\mathcal{O}/t^{r_1})}({}_r\mathcal{L}_{(N,N)}M)$ for any $r_1 \geq 2N$. It is equipped with the perverse t-structure.

For $N_1 \geq N \geq r \geq 0$ the fibres of the above projection

$$p : {}_r\mathcal{L}_{(N_1, N_1)}M \rightarrow {}_r\mathcal{L}_{(N, N)}M$$

are isomorphic to affine spaces of fixed dimension, and p is smooth and surjective. Hence, this map yields transition functors (exact for the perverse t-structures and fully faithful embeddings)

$$D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N, N)}M) \rightarrow D_{G(\mathcal{O})}({}_{r_1}\mathcal{L}_{(N_1, N_1)}M)$$

and

$$D({}_r\mathcal{L}_{(N, N)}M) \rightarrow D({}_{r_1}\mathcal{L}_{(N_1, N_1)}M)$$

We define $D_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$ as the inductive 2-limit of $D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N, N)}M)$ as N goes to plus infinity. The category $D({}_r\mathcal{L}_d(M(F)))$ is defined similarly. Both they are equipped with perverse t-structures.

If $N_1 \geq N \geq r_1 \geq r \geq 0$ we have a diagram

$$\begin{array}{ccc} {}_r\mathcal{L}_{(N_1, N_1)}M & \xrightarrow{p} & {}_r\mathcal{L}_{(N, N)}M \\ \downarrow j & & \downarrow j \\ {}_{r_1}\mathcal{L}_{(N_1, N_1)}M & \xrightarrow{p} & {}_{r_1}\mathcal{L}_{(N, N)}M, \end{array}$$

where j are natural open immersions. The restriction functors $j^* : D_{G(\mathcal{O})}({}_{r_1}\mathcal{L}_{(N, N)}M) \rightarrow D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N, N)}M)$ yield (in the limit as N goes to plus infinity) the functors

$$j_{r_1, r}^* : D_{G(\mathcal{O})}({}_{r_1}\mathcal{L}_d(M(F))) \rightarrow D_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$$

of restriction with respect to the open immersion $j_{r_1, r} : {}_r\mathcal{L}_d(M(F)) \hookrightarrow {}_{r_1}\mathcal{L}_d(M(F))$. Define $D_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$ as the projective 2-limit of

$$D_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$$

as r goes to plus infinity. Similarly, $P_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$ is defined as the projective 2-limit of $P_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$. Along the same lines, one defines the categories $P(\mathcal{L}_d(M(F)))$ and $D(\mathcal{L}_d(M(F)))$.

5.2 RELATIVE DETERMINANT For a pair of c-lattices M_1, M_2 in $M(F)$ define the relative determinant $\det(M_1 : M_2)$ as the following $\mathbb{Z}/2\mathbb{Z}$ -graded 1-dimensional k -vector space. If R is a c-lattice in $M(F)$ such that $R \subset M_1 \cap M_2$ then

$$\det(M_1 : M_2) \xrightarrow{\sim} \det(M_1/R) \otimes \det(M_2/R)^{-1},$$

it is defined up to a unique isomorphism.

Write Gr_G for the affine grassmanian $G(F)/G(\mathcal{O})$ of G (cf. [1], Section 4.5). For $R \in \text{Gr}_G, L \in \mathcal{L}_d(M(F))$ define the relative determinant $\det(R : L)$ as the following ($\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) 1-dimensional vector space. Pick a c-lattice $R_1 \subset R$ such that $R_1 \cap L = 0$. Then in R_1^\perp/R_1 one gets two lagrangian subspaces R/R_1 and $L_{R_1} := L \cap R_1^\perp$. Set

$$\det(R : L) = \det(R/R_1) \otimes \det(L_{R_1})$$

If $R_2 \subset R_1$ is another c-lattice then the exact sequence

$$0 \rightarrow L_{R_1} \rightarrow L \cap R_2^\perp \rightarrow R_2^\perp/R_1^\perp \rightarrow 0$$

yields a canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det(R/R_2) \otimes \det(L_{R_2}) \xrightarrow{\sim} \det(R_1/R_2) \otimes \det(R/R_1) \otimes \det(L_{R_1}) \otimes \det(R_2^\perp/R_1^\perp) \xrightarrow{\sim} \det(R/R_1) \otimes \det(L_{R_1})$$

So, $\det(R : L)$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded line defined up to a unique isomorphism. Another way to say is as follows. Consider the complex $R \oplus L \xrightarrow{s} M(F)$ placed in cohomological degrees 0 and 1, where $s(r, l) = r + l$. It has finite-dimensional cohomologies and

$$\det(R : L) = \det(R \oplus L \xrightarrow{s} M(F))$$

For $g \in G(F)$ we have canonically

$$\det(gR : gL) \xrightarrow{\sim} \det(R : L)$$

For $R_1, R_2 \in \text{Gr}_G, L \in \mathcal{L}_d(M(F))$ we have canonically

$$\det(R_1 : L) \xrightarrow{\sim} \det(R_1 : R_2) \otimes \det(R_2 : L)$$

5.3 Write \mathcal{A}_d for the line bundle on $\mathcal{L}_d(M(F))$ with fibre $\det(M : L)$ at $L \in \mathcal{L}_d(M(F))$. Clearly, \mathcal{A}_d is $G(\mathcal{O})$ -equivariant, so we may see \mathcal{A}_d as the line bundle on the stack quotient $\mathcal{L}_d(M(F))/G(\mathcal{O})$. Let $\tilde{\mathcal{L}}_d(M(F))$ denote the μ_2 -gerb of square roots of \mathcal{A}_d .

The categories of the corresponding perverse sheaves $\text{P}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$ and $\text{P}(\tilde{\mathcal{L}}_d(M(F)))$ are defined as above. Namely, first for $r \geq 0$ define

$$\text{D}_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$$

as follows. For $N \geq r$ take $r_1 \geq 2N$ and consider the stack quotient ${}_r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1})$. We have the line bundle, say \mathcal{A}_N on this stack whose fibre at L is $\det(M/M(-N)) \otimes \det L$. Here $L \subset {}_N, {}_N M$ is a Lagrangian subspace such that $L \cap (M(-r)/M(-N)) = 0$. Write

$$({}_r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\vee$$

for the gerb of square roots of this line bundle. Let $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M})$ denote the category

$$D((r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim)$$

for any $r_1 \geq 2N$ (we have canonical equivalences exact for the perverse t-structures between such categories for various r_1).

Assume $N_1 \geq N \geq r$ and $r_1 \geq 2N_1$. For the projection

$$p : r\mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}) \rightarrow r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1})$$

we have a canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism $p^* \mathcal{A}_N \xrightarrow{\sim} \mathcal{A}_{N_1}$. This yields a transition map

$$(r\mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}))^\sim \rightarrow (r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim$$

The corresponding inverse image yields a transition functor

$$D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M}) \rightarrow D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N_1,N_1)M}) \quad (16)$$

exact for the perverse t-structures (and a fully faithful embedding). We define $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$ as the inductive 2-limit of $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M})$ as N goes to plus infinity.

For $N \geq r' \geq r$ and $r_1 \geq 2N$ we have an open immersion

$$\tilde{j} : (r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim \subset (r'\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim$$

hence the *-restriction functors

$$\tilde{j}^* : D_{G(\mathcal{O})}(r'\tilde{\mathcal{L}}_{(N,N)M}) \rightarrow D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M})$$

compatible with the transition functors (16). Passing to the limit as N goes to plus infinity, we get the functors

$$\tilde{j}_{r',r}^* : D_{G(\mathcal{O})}(r'\tilde{\mathcal{L}}_d(M(F))) \rightarrow D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$$

Define $D_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$ as the projective 2-limit of $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$ as r goes to plus infinity, and similarly for $P_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$.

Along the same lines one defines the categories $P(\tilde{\mathcal{L}}_d(M(F)))$ and $D(\tilde{\mathcal{L}}_d(M(F)))$.

5.4 METAPLECTIC GROUP Let \mathcal{A}_G be the line bundle on the ind-scheme $G(F)$ whose fibre at g is $\det(M : gM)$. Write $\tilde{G}(F) \rightarrow G(F)$ for the gerb of square roots of \mathcal{A}_G . The stack $\tilde{G}(F)$ has a structure of a group stack. The product map $m : \tilde{G}(F) \times \tilde{G}(F) \rightarrow \tilde{G}(F)$ sends

$$(g_1, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \xrightarrow{\sim} \det(M : g_1M)), (g_2, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \xrightarrow{\sim} \det(M : g_2M))$$

to the collection $(g_1g_2, \mathcal{B}, \sigma : \mathcal{B}^2 \xrightarrow{\sim} \det(M : g_1g_2M))$, where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ and σ is the composition

$$\begin{aligned} (\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \xrightarrow{\sigma_1 \otimes \sigma_2} \det(M : g_1M) \otimes \det(M : g_2M) &\xrightarrow{\text{id} \otimes g_1} \det(M : g_1M) \otimes \det(g_1M : g_1g_2M) \\ &\xrightarrow{\sim} \det(M : g_1g_2M) \end{aligned}$$

Informally speaking, one may think of the exact sequence of group stacks

$$1 \rightarrow B(\mu_2) \rightarrow \tilde{G}(F) \rightarrow G(F) \rightarrow 1$$

We also have a canonical section $G(\mathcal{O}) \rightarrow \tilde{G}(F)$ sending g to

$$(g, \mathcal{B} = k, \text{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(M : M))$$

The group stack $\tilde{G}(F)$ acts naturally on $\tilde{\mathcal{L}}_d(M(F))$, the action map $\tilde{G}(F) \times \tilde{\mathcal{L}}_d(M(F)) \rightarrow \tilde{\mathcal{L}}_d(M(F))$ sends

$$(g, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \xrightarrow{\sim} \det(M : gM)), (L, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \xrightarrow{\sim} \det(M : L))$$

to the collection (gL, \mathcal{B}) , where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ is equipped with the isomorphism

$$(\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \xrightarrow{\sigma_1 \otimes \sigma_2} \det(M : gM) \otimes \det(M : L) \xrightarrow{\text{id} \otimes g} \det(M : gM) \otimes \det(gM : gL) \xrightarrow{\sim} \det(M : gL)$$

5.5 For $g \in G(F)$ and a c-lattice $R \subset R^\perp$ in $M(F)$ we have an isomorphism of symplectic spaces $g : R^\perp/R \xrightarrow{\sim} (gR)^\perp/gR$. For each c-lattice $R_1 \subset R$ we have a diagram

$$\begin{array}{ccc} \mathcal{L}(R_1^\perp/R_1)_R & \xrightarrow{g} & \mathcal{L}(gR_1^\perp/gR_1)_{gR} \\ \downarrow p & & \downarrow p \\ \mathcal{L}(R^\perp/R) & \xrightarrow{g} & \mathcal{L}(gR^\perp/gR) \end{array}$$

Let \mathcal{A}_{R_1} be the ($\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) line bundle on $\mathcal{L}(R_1^\perp/R_1)_R$ whose fibre at L is $\det L \otimes \det(M : R_1)$. Assume that $\tilde{g} = (g, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det(M : gM))$ is a k -point of $\tilde{G}(F)$ over g . It yields a diagram

$$\begin{array}{ccc} \tilde{\mathcal{L}}(R_1^\perp/R_1)_R & \xrightarrow{\tilde{g}} & \tilde{\mathcal{L}}(gR_1^\perp/gR_1)_{gR} \\ \downarrow p & & \downarrow p \\ \tilde{\mathcal{L}}(R^\perp/R) & \xrightarrow{\tilde{g}} & \tilde{\mathcal{L}}(gR^\perp/gR) \end{array}$$

Here the top horizontal arrow sends $(L, \mathcal{B}_1, \mathcal{B}_1^2 \xrightarrow{\sim} \det L \otimes \det(M : R_1))$ to

$$(gL, \mathcal{B}_2, \sigma : \mathcal{B}_2^2 \xrightarrow{\sim} \det(gL) \otimes \det(M : gR_1)),$$

where $\mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B}$ and σ is the composition

$$\begin{aligned} (\mathcal{B}_1 \otimes \mathcal{B})^2 &\xrightarrow{\sim} \det L \otimes \det(M : R_1) \otimes \det(M : gM) \xrightarrow{g \otimes \text{id}} \\ &\det(gL) \otimes \det(gM : gR_1) \otimes \det(M : gM) \xrightarrow{\sim} \det(gL) \otimes \det(M : gR_1) \end{aligned}$$

In the limit by R_1 the corresponding functors $\tilde{g}^* : \mathbb{P}(\tilde{\mathcal{L}}(gR_1^\perp/gR_1)_{gR}) \xrightarrow{\sim} \mathbb{P}(\tilde{\mathcal{L}}(R_1^\perp/R_1)_R)$ yield an equivalence

$$\tilde{g}^* : \mathbb{P}(\tilde{\mathcal{L}}_d(M(F))_{gR}) \xrightarrow{\sim} \mathbb{P}(\tilde{\mathcal{L}}_d(M(F))_R)$$

Taking one more limit by the partially ordered set of c-lattices R , one gets an equivalence

$$\tilde{g}^* : \mathbb{P}(\tilde{\mathcal{L}}_d(M(F))) \xrightarrow{\sim} \mathbb{P}(\tilde{\mathcal{L}}_d(M(F)))$$

In this sense $\tilde{G}(F)$ acts on $\mathbb{P}(\tilde{\mathcal{L}}_d(M(F)))$.

6. CANONICAL INTERWINING OPERATORS: LOCAL FIELD CASE

6.1 Keep notations of Section 5. Write $H = M \oplus \Omega$ for the Heisenberg group defined as in Section 2.1, this is a group scheme over $\text{Spec } \mathcal{O}$.

For $L \in \mathcal{L}_d(M(F))$ we have the subgroup $\bar{L} = L \oplus \Omega(F) \subset H(F)$ and the character $\chi_L : \bar{L} \rightarrow \bar{\mathbb{Q}}_\ell^*$ given by $\chi_L(l, a) = \chi(a)$. Here $\chi : \Omega(F) \rightarrow \bar{\mathbb{Q}}_\ell^*$ sends a to $\psi(\text{Res } a)$. In the classical setting we let \mathcal{H}_L denote the space of functions $f : H(F) \rightarrow \bar{\mathbb{Q}}_\ell$ satisfying

C1) $f(\bar{l}h) = \chi_L(\bar{l})f(h)$, for $h \in H, \bar{l} \in \bar{L}$;

C2) there exists a c-lattice $R \subset M(F)$ such that $f(h(r, 0)) = f(h)$ for $r \in R, h \in H$.

Note that such f has automatically compact support modulo \bar{L} . The group $H(F)$ acts on \mathcal{H}_L by right translations, this is a model of the Weil representation. Let us introduce a geometric analog of \mathcal{H}_L .

Given a c-lattice $R \subset M(F)$ such that $R \subset R^\perp$ write $H_R = (R^\perp/R) \oplus k$ for the Heisenberg group corresponding to the symplectic space R^\perp/R . If $L \in \mathcal{L}_d(M(F))_R$ then $L_R := L \cap R^\perp \subset R^\perp/R$ is lagrangian. Set $\bar{L}_R = L_R \oplus k \subset H_R$. Let $\chi_{L,R} : \bar{L}_R \rightarrow \bar{\mathbb{Q}}_\ell^*$ be the character sending (l, a) to $\psi(a)$. Set

$$\mathcal{H}_{L_R} = \{f : H_R \rightarrow \bar{\mathbb{Q}}_\ell \mid f(\bar{l}h) = \chi_{L,R}(\bar{l})f(h), h \in H_R, \bar{l} \in \bar{L}_R\}$$

Lemma 7. *There is a canonical embedding $T_R^L : \mathcal{H}_{L_R} \hookrightarrow \mathcal{H}_L$ whose image is the subspace of those $f \in \mathcal{H}_L$ which satisfy*

$$f(h(r, 0)) = f(h) \text{ for } r \in R, h \in H \tag{17}$$

Proof Set

$$'\mathcal{H}_{L_R} = \{\phi : R^\perp/R \rightarrow \bar{\mathbb{Q}}_\ell \mid \phi(r+l) = \chi(\frac{1}{2}\omega(r, l))\phi(r), r \in R^\perp/R, l \in L_R\}$$

We have an isomorphism $\mathcal{H}_{L_R} \xrightarrow{\sim} '\mathcal{H}_{L_R}$ sending f to ϕ given by $\phi(r) = f(r, 0)$. Given $f \in \mathcal{H}_L$ satisfying (17), we associate to f a function $\phi \in '\mathcal{H}_{L_R}$ given by

$$\phi(r) = q^{\frac{1}{2} \dim R^\perp/R} f(r, 0)$$

for $r \in R^\perp$. This defines the map T_R^L . \square

Assume that $S \subset R \subset M(F)$ are c-lattices and $R \cap L = 0$. Remind the operator $\mathcal{H}_{L_R} \xrightarrow{T^{L_S}}$ \mathcal{H}_{L_S} given by (13), it corresponds to the isotropic subspace $R/S \subset S^\perp/S$. The composition $\mathcal{H}_{L_R} \xrightarrow{T^{L_S}} \mathcal{H}_{L_S} \xrightarrow{T_S^L} \mathcal{H}_L$ equals T_R^L .

The geometric analog of \mathcal{H}_L is as follows. For a c-lattice R such that $R \cap L = 0$ and $R \subset R^\perp$ we have the category \mathcal{H}_{L_R} of perverse sheaves on H_R which are $(\bar{L}_R, \chi_{L,R})$ -equivariant, and the corresponding category $D\mathcal{H}_{L_R}$. For $S \subset R$ as above we have an (exact for the perverse structure and fully faithful) transition functor (14), which we now denote by

$$T_{S,R}^L : D\mathcal{H}_{L_R} \rightarrow D\mathcal{H}_{L_S}$$

Define \mathcal{H}_L (resp., $D\mathcal{H}_L$) as the inductive 2-limit of \mathcal{H}_{L_R} (resp., of $D\mathcal{H}_{L_R}$) over the partially ordered set of c-lattices R such that $R \cap L = 0$ and $R \subset R^\perp$. So, \mathcal{H}_L is abelian and $D\mathcal{H}_L$ is a triangulated category.

6.2 Let $R \subset R^\perp$ be a c-lattice in $M(F)$. We have a projection

$$\mathcal{L}_d(M(F))_R \rightarrow \mathcal{L}(R^\perp/R)$$

sending L to L_R . Let \mathcal{A}_R be the $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on $\mathcal{L}(R^\perp/R)$ whose fibre at L_1 is $\det L_1 \otimes \det(M : R)$. Write $\tilde{\mathcal{L}}(R^\perp/R)$ for the gerb of square roots of \mathcal{A}_R . The restriction of \mathcal{A}_R to $\mathcal{L}_d(M(F))_R$ identifies canonically with \mathcal{A}_d . The above projection lifts naturally to a morphism of gerbs

$$\tilde{\mathcal{L}}_d(M(F))_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R) \quad (18)$$

Given k -points $N^0, L^0 \in \tilde{\mathcal{L}}_d(M(F))$ we are going to associate to them in a canonical way a functor

$$\mathcal{F}_{N^0, L^0} : D\mathcal{H}_L \rightarrow D\mathcal{H}_N \quad (19)$$

sending \mathcal{H}_L to \mathcal{H}_N . To do so, consider a c-lattice $R \subset R^\perp$ in $M(F)$ such that $L, N \in \mathcal{L}_d(M(F))_R$. Write $N_R^0, L_R^0 \in \tilde{\mathcal{L}}(R^\perp/R)$ for the images of N^0 and L^0 under (18). By definition, the enhanced structure on L_R and N_R is given by one-dimensional vector spaces $\mathcal{B}_L, \mathcal{B}_N$ equipped with

$$\mathcal{B}_L^2 \xrightarrow{\sim} \det L_R \otimes \det(M : R), \quad \mathcal{B}_N^2 \xrightarrow{\sim} \det N_R \otimes \det(M : R),$$

hence an isomorphism $\mathcal{B}^2 \xrightarrow{\sim} \det L_R \otimes \det N_R$ for $\mathcal{B} := \mathcal{B}_L \otimes \mathcal{B}_N \otimes \det(M : R)^{-1}$. We denote by

$$\mathcal{F}_{N_R^0, L_R^0} : D\mathcal{H}_{L_R} \rightarrow D\mathcal{H}_{N_R}$$

the canonical intertwining functor defined in Section 3.5 corresponding to $(N_R, L_R, \mathcal{B}) \in \tilde{Y}$, here $Y = \mathcal{L}(R^\perp/R) \times \mathcal{L}(R^\perp/R)$. The following is an immediate consequence of Proposition 2.

Proposition 4. *Let $S \subset R \subset R^\perp \subset S^\perp$ be c-lattices such that $L^0, N^0 \in \tilde{\mathcal{L}}_d(M(F))_R$. Then the following diagram of categories is canonically 2-commutative*

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{T_{S,R}^L} & D\mathcal{H}_{L_S} \\ \downarrow \mathcal{F}_{N_R^0, L_R^0} & & \downarrow \mathcal{F}_{N_S^0, L_S^0} \\ D\mathcal{H}_{N_R} & \xrightarrow{T_{S,R}^N} & D\mathcal{H}_{N_S} \end{array}$$

Define (19) as the limit of functors $\mathcal{F}_{N_R^0, L_R^0}$ over the partially ordered set of c-lattices $R \subset R^\perp$ such that $L, N \in \mathcal{L}_d(M(F))_R$. As in Section 3.5, one shows that for $L^0, N^0, R^0 \in \tilde{\mathcal{L}}_d(M(F))$ the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{R^0, L^0}} & \mathrm{D}\mathcal{H}_R \\ & \searrow \mathcal{F}_{N^0, L^0} & \downarrow \mathcal{F}_{N^0, R^0} \\ & & \mathrm{D}\mathcal{H}_N \end{array}$$

Our main result in the local field case is as follows.

Theorem 2. *For each k -point $L^0 \in \tilde{\mathcal{L}}_d(M(F))$ there is a canonical functor*

$$\mathcal{F}_{L^0} : \mathrm{D}\mathcal{H}_L \rightarrow \mathrm{D}(\tilde{\mathcal{L}}_d(M(F))) \quad (20)$$

sending \mathcal{H}_L to $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$. For a pair of k -points (L^0, N^0) in $\tilde{\mathcal{L}}_d(M(F))$ the diagram

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathrm{D}(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow \mathcal{F}_{N^0, L^0} & \nearrow \mathcal{F}_{N^0} & \\ \mathrm{D}\mathcal{H}_N & & \end{array} \quad (21)$$

is canonically 2-commutative. Let $W(\tilde{\mathcal{L}}_d(M(F)))$ be the essential image of

$$\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))),$$

this is a full subcategory independent of L^0 . Besides, $W(\tilde{\mathcal{L}}_d(M(F)))$ is preserved under the natural action of $\tilde{G}(F)$ on $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$.

We will refer to $W(\tilde{\mathcal{L}}_d(M(F)))$ as *the non-ramified Weil category on $\tilde{\mathcal{L}}_d(M(F))$* . Remind that in the classical setting

$$\mathcal{H}_L = \mathcal{H}_{L, \text{odd}} \oplus \mathcal{H}_{L, \text{even}}$$

is a direct sum of two irreducible representations of the metaplectic group (consisting of odd and even functions respectively). The representation $\mathcal{H}_{L, \text{odd}}$ is ramified, whence $\mathcal{H}_{L, \text{even}}$ is not. The category $W(\tilde{\mathcal{L}}_d(M(F)))$ together with the action of $\tilde{G}(F)$ is a geometric counterpart of the representation $\mathcal{H}_{L, \text{even}}$. The proof of Theorem 2 is given in Sections 6.3-6.4.

6.3 Let L^0 be a k -point of $\tilde{\mathcal{L}}_d(M(F))$. Let $R \subset R^\perp$ be a c-lattice with $L \cap R = 0$. Write L_R^0 for the image of L^0 under (18). Applying the construction of Section 3.6 to the symplectic space R^\perp/R with $L_R^0 \in \tilde{\mathcal{L}}(R^\perp/R)$, one gets the functor

$$\mathcal{F}_{L_R^0} : \mathrm{D}\mathcal{H}_{L_R} \rightarrow \mathrm{D}(\tilde{\mathcal{L}}(R^\perp/R))$$

If N^0 is another k -point of $\tilde{\mathcal{L}}_d(M(F))_R$ then writing N_R^0 for the image of N^0 in $\tilde{\mathcal{L}}(R^\perp/R)$ we also get that the diagram

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L_R^0}} & \mathrm{D}(\tilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow \mathcal{F}_{N_R^0, L_R^0} & \nearrow \mathcal{F}_{N_R^0} & \\ \mathrm{D}\mathcal{H}_{N_R} & & \end{array} \quad (22)$$

is canonically 2-commutative.

Let now

$${}^R\mathcal{F}_{L^0} : D\mathcal{H}_{L_R} \rightarrow D(\tilde{\mathcal{L}}_d(M(F))_R)$$

denote the composition of \mathcal{F}_{L^0} with the (exact for the perverse t-structures) restriction functor $D(\tilde{\mathcal{L}}(R^\perp/R)) \rightarrow D(\tilde{\mathcal{L}}_d(M(F))_R)$ for the projection (18).

Let $S \subset R$ be another c-lattice. As in Section 5.3, for the open immersion $j_{S,R} : \tilde{\mathcal{L}}_d(M(F))_R \hookrightarrow \tilde{\mathcal{L}}_d(M(F))_S$ we have the restriction functors $j_{S,R}^* : D(\tilde{\mathcal{L}}_d(M(F))_S) \rightarrow D(\tilde{\mathcal{L}}_d(M(F))_R)$.

Lemma 8. *The diagram of functors is canonically 2-commutative*

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{{}^R\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}_d(M(F))_R) \\ \downarrow T_{S,R}^L & & \uparrow j_{S,R}^* \\ D\mathcal{H}_{L_S} & \xrightarrow{{}^S\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}_d(M(F))_S) \end{array}$$

Proof We have an open immersion $j : \tilde{\mathcal{L}}(S^\perp/S)_R \hookrightarrow \tilde{\mathcal{L}}(S^\perp/S)$ and a projection $p_{R/S} : \tilde{\mathcal{L}}(S^\perp/S)_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R)$. Set $P_{R/S} = p_{R/S}^* \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(p_{R/S})}$. It suffices to show that the diagram is canonically 2-commutative

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}(R^\perp/R)) & \xrightarrow{P_{R/S}} & D(\tilde{\mathcal{L}}(S^\perp/S)_R) \\ \downarrow T_{S,R}^L & & & \nearrow j^* & \\ D\mathcal{H}_{L_S} & \xrightarrow{\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}(S^\perp/S)) & & \end{array}$$

This follows from Lemma 5. \square

Define $\mathcal{F}_{L^0,R} : D\mathcal{H}_{L_R} \rightarrow D(\tilde{\mathcal{L}}_d(M(F)))$ as the functor sending K_1 to the following object K_2 . For a c-lattice $S \subset R$ we declare the restriction of K_2 to $\tilde{\mathcal{L}}_d(M(F))_S$ to be

$$({}^S\mathcal{F}_{L^0} \circ T_{S,R}^L)(K_1)$$

By Lemma 8, the corresponding projective system defines an object K_2 of $D(\tilde{\mathcal{L}}_d(M(F)))$.

Finally, for $S \subset R$ with $R \cap L = 0$ the diagram

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0,R}} & D(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow T_{S,R}^L & \nearrow \mathcal{F}_{L^0,S} & \\ D\mathcal{H}_{L_S} & & \end{array}$$

is canonically 2-commutative. We define (20) as the limit of the functors $\mathcal{F}_{L^0,R}$ over the partially ordered set of c-lattices $R \subset R^\perp$ such that $L \cap R = 0$. The commutativity of (21) follows from the commutativity of (22).

Definition 4. The non-ramified Weil category $W(\tilde{\mathcal{L}}_d(M(F)))$ is the essential image of the functor $\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow P(\tilde{\mathcal{L}}_d(M(F)))$. It does not depend on a choice of a k -point L^0 of $\tilde{\mathcal{L}}_d(M(F))$.

6.4 Let $R \subset R^\perp$ be a c-lattice in $M(F)$, let $\tilde{g} \in \tilde{G}(F)$ be a k -point, write g for its image in $G(F)$. As in Section 5.5, we have an isomorphism $g : H_R \xrightarrow{\sim} H_{gR}$ of algebraic groups over k sending $(x, a) \in (R^\perp/R) \times \mathbb{A}^1$ to $(gx, a) \in (gR^\perp/gR) \times \mathbb{A}^1$. For $L \in \mathcal{L}_d(M(F))_R$ it induces an equivalence

$$g : \mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}$$

If $L^0 \in \tilde{\mathcal{L}}_d(M(F))_R$ is a k -point then the G -equivariance of F implies that the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0_R}} & \mathrm{P}(\tilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0_{gR}}} & \mathrm{P}(\tilde{\mathcal{L}}(gR^\perp/gR)) \end{array}$$

This, in turn, implies that the diagram is 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0, R}} & \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0, gR}} & \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))) \end{array}$$

Thus, Theorem 2 is proved.

6.5 THETA-SHEAF Let $L \in \mathcal{L}_d(M(F))_M$, this is equivalent to saying that $L \subset M(F)$ is a lagrangian d-lattice such that $L \oplus M = M(F)$. Then the category \mathcal{H}_{L_M} has a distinguished object \mathcal{L}_ψ on $\mathbb{A}^1 = \mathrm{H}_M$. Write S_L for its image under $\mathcal{H}_{L_M} \rightarrow \mathcal{H}_L$. The line bundle \mathcal{A}_d over $\mathcal{L}_d(M(F))_M$ is canonically trivialized, so L has a distinguished enhanced structure

$$(L, \mathcal{B}) = L^0 \in \tilde{\mathcal{L}}_d(M(F))_M,$$

where $\mathcal{B} = k$ is equipped with $\mathrm{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(M : L)$. The *theta-sheaf* $S_{M(F)}$ over $\tilde{\mathcal{L}}_d(M(F))$ is defined as $\mathcal{F}_{L^0}(S_L)$. It does not depend on $L \in \mathcal{L}_d(M(F))_M$ in the sense that for another $N \in \mathcal{L}_d(M(F))_M$ the diagram (21) yields a canonical isomorphism $\mathcal{F}_{L^0}(S_L) \xrightarrow{\sim} \mathcal{F}_{N^0}(S_N)$. The perverse sheaf $S_{M(F)}$ has a natural $G(\mathcal{O})$ -equivariant structure.

6.6 RELATION WITH THE SCHRÖDINGER MODEL

Assume in addition that M is decomposed as $M \xrightarrow{\sim} U \oplus U^* \otimes \Omega$, where U is a free \mathcal{O} -module of rank d , both U and $U^* \otimes \Omega$ are isotropic, and the form $\omega : \wedge^2 M \rightarrow \Omega$ is given by $\omega\langle u, u^* \rangle = \langle u, u^* \rangle$ for $u \in U, u^* \in U^* \otimes \Omega$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between U and U^* . Let $\bar{U} = U(F) \oplus \Omega(F)$ viewed as a subgroup of $H(F)$, it is equipped with the character $\chi_U : \bar{U} \rightarrow \bar{\mathbb{Q}}_\ell^*$ given by $\chi_U(u, a) = \psi(\mathrm{Res} a), a \in \Omega(F), u \in U(F)$. Write

$$\mathrm{Shr}_U = \{f : H(F) \rightarrow \bar{\mathbb{Q}}_\ell \mid f(\bar{u}h) = \chi_U(\bar{u})f(h), \bar{u} \in \bar{U}, h \in H(F), f \text{ is smooth, of compact support modulo } \bar{U}\},$$

$H(F)$ acts on it by right translations. This is the Schrödinger model of the Weil representation, it identifies naturally with the Schwarz space $\mathcal{S}(U^* \otimes \Omega(F))$.

Remind the definition of the derived category $D(U^* \otimes \Omega)$ and its subcategory of perverse sheaves $P(U^* \otimes \Omega)$ given in ([11], Section 4). For $N, r \in \mathbb{Z}$ with $N + r \geq 0$ we write ${}_{N,r}U = t^{-N}U/t^rU$.

For $N_1 \geq N_2, r_1 \geq r_2$ we have a diagram

$${}_{N_2, r_2}(U^* \otimes \Omega) \xleftarrow{p} {}_{N_2, r_1}(U^* \otimes \Omega) \xrightarrow{i} {}_{N_1, r_1}(U^* \otimes \Omega),$$

where p is the smooth projection and i is a closed immersion. We have a transition functor

$$D({}_{N_2, r_2}(U^* \otimes \Omega)) \rightarrow D({}_{N_1, r_1}(U^* \otimes \Omega)) \quad (23)$$

sending K to $i_! p^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(p)}$, it is fully faithful and exact for the perverse t-structures. Then $D(U^* \otimes \Omega(F))$ (resp., $P(U^* \otimes \Omega(F))$) is defined as the inductive 2-limit of $D({}_{N,r}(U^* \otimes \Omega))$ (resp., of $P({}_{N,r}(U^* \otimes \Omega))$) as r, N go to infinity. The category $P(U^* \otimes \Omega(F))$ is the geometric analog of the space Shr_U .

In this section we prove the following.

Proposition 5. *For each k -point $L^0 \in \tilde{\mathcal{L}}_d(M(F))$ there is a canonical equivalence*

$$\mathcal{F}_{U(F), L^0} : D(U^* \otimes \Omega(F)) \rightarrow D\mathcal{H}_L \quad (24)$$

which identifies $P(U^* \otimes \Omega(F))$ with the category \mathcal{H}_L . For $L^0, N^0 \in \tilde{\mathcal{L}}_d(M(F))$ the diagram is canonically 2-commutative

$$\begin{array}{ccc} D(U^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U(F), L^0}} & D\mathcal{H}_L \\ \downarrow \mathcal{F}_{U(F), N^0} & \nearrow \mathcal{F}_{L^0, N^0} & \\ D\mathcal{H}_N & & \end{array}$$

For $N \geq 0$ consider the c-lattice $R = t^N M$ in $M(F)$ and the corresponding symplectic space $R^\perp/R = {}_{N,N}M$. Set $U_R := {}_{N,N}U \in \mathcal{L}({}_{N,N}M)$. We have the line bundle \mathcal{A}_N on $\mathcal{L}({}_{N,N}M)$ whose fibre at L is $\det({}_{0,N}M) \otimes \det L$. As above, $\tilde{\mathcal{L}}({}_{N,N}M)$ is the gerb of square roots of \mathcal{A}_N . Let

$$U_R^0 = (U_R, \det({}_{0,N}U)) \in \tilde{\mathcal{L}}({}_{N,N}M)$$

equipped with a canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism $\det({}_{0,N}U)^2 \xrightarrow{\sim} \det U_R \otimes \det({}_{0,N}M)$.

Let $H_R = {}_{N,N}M \times \mathbb{A}^1$ denote the corresponding Heisenberg group, it has the subgroup $\bar{U}_R = U_R \times \mathbb{A}^1$ equipped with the character $\chi_{U,R} : \bar{U}_R \rightarrow \bar{\mathbb{Q}}_\ell^*$ given by $\chi_{U,R}(u, a) = \psi(a)$, $a \in \mathbb{A}^1$. In the classical setting, \mathcal{H}_{U_R} is the space of functions on H_R , which are $(\bar{U}_R, \chi_{U,R})$ -equivariant under the left multiplication. Set $\text{Shr}_U^R = \{f \in \text{Shr}_U \mid f(h(r, 0)) = f(h), r \in R, h \in H\}$.

Lemma 9. *In the classical setting there is an isomorphism*

$$\text{Shr}_U^R \xrightarrow{\sim} \mathcal{H}_{U_R} \quad (25)$$

Proof Write $\mathcal{H}'_{U_R} = \{\phi' : R^\perp/R \rightarrow \bar{\mathbb{Q}}_\ell \mid \phi'(m+u) = \psi(\frac{1}{2}\langle m, u \rangle)\phi'(m), u \in U_R\}$. We identify $\mathcal{H}_{U_R} \xrightarrow{\sim} \mathcal{H}'_{U_R}$ via the map $\phi \mapsto \phi'$, where $\phi'(m) = \phi(m, 0)$. Given $f \in \text{Shr}_U^R$ for $m \in t^{-N}M$ the value $f(m, 0)$ depends only on the image \bar{m} of m under $t^{-N}M \rightarrow {}_{N,N}M$. The isomorphism (25) sends f to $\phi' \in \mathcal{H}'_{U_R}$ given by $\phi'(\bar{m}) = f(m, 0)$. \square

In the geometric setting \mathcal{H}_{U_R} is the category of $(\bar{U}_R, \chi_{U,R})$ -equivariant perverse sheaves on H_R . We identify it with $\text{P}({}_{N,N}(U^* \otimes \Omega))$ as follows. Let $m_U : \bar{U}_R \times {}_{N,N}(U^* \otimes \Omega) \rightarrow H_R$ be the isomorphism sending (\bar{u}, h) to their product $\bar{u}h$ in H_R . The functor $\text{D}({}_{N,N}(U^* \otimes \Omega)) \rightarrow \text{D}\mathcal{H}_{U_R}$ sending K to

$$(m_U)_!(\chi_{U,R} \boxtimes K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \bar{U}_R}$$

is an equivalence (exact for the perverse t-structures).

Let $N' \geq N$ and $S = t^{N'}M$. The corresponding transition functor (23) now yields a functor denoted $T_{S,R}^U : \text{D}\mathcal{H}_{U_R} \rightarrow \text{D}\mathcal{H}_{U_S}$.

Let $L^0 \in \tilde{\mathcal{L}}_d(M(F))$ be a k -point over $L \in \mathcal{L}_d(M(F))$. Assume that N is large enough so that $L \cap R = 0$. Let L_R^0 denote the image of L^0 under (18). Define $U_S^0, L_S^0 \in \tilde{\mathcal{L}}(S^\perp/S)$ similarly.

Lemma 10. *The diagram is canonically 2-commutative*

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_R} & \xrightarrow{T_{S,R}^U} & \text{D}\mathcal{H}_{U_S} \\ \downarrow \mathcal{F}_{L_R^0, U_R^0} & & \downarrow \mathcal{F}_{L_S^0, U_S^0} \\ \text{D}\mathcal{H}_{L_R} & \xrightarrow{T_{S,R}^L} & \text{D}\mathcal{H}_{L_S} \end{array}$$

Proof Set $W = t^{N'}U \oplus t^N(U^* \otimes \Omega)$. The subspace $W/S \subset S^\perp/S$ is isotropic, and $U_S \cap (W/S) = L_S \cap (W/S) = 0$. Write $H_W = (W^\perp/W) \times \mathbb{A}^1$ for the corresponding Heisenberg group. Set $U_W = U_S \cap (W^\perp/S)$, $L_W = L_S \cap (W^\perp/S)$. Applying Proposition 2, we get a 2-commutative diagram

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_W} & \xrightarrow{T_{S,W}^U} & \text{D}\mathcal{H}_{U_S} \\ \downarrow \mathcal{F}_{L_W^0, U_W^0} & & \downarrow \mathcal{F}_{L_S^0, U_S^0} \\ \text{D}\mathcal{H}_{L_W} & \xrightarrow{T_{S,W}^L} & \text{D}\mathcal{H}_{L_S} \end{array}$$

Now $R/W \subset W^\perp/W$ is an isotropic subspace, and $R/W \subset U_W$, $R/W \cap L_W = 0$. Note that $U_R = U_W/(R/W)$. Applying Proposition 3, we get a 2-commutative diagram

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_R} & \xrightarrow{T_{W,R}^U} & \text{D}\mathcal{H}_{U_W} \\ \downarrow \mathcal{F}_{L_R^0, U_R^0} & & \downarrow \mathcal{F}_{L_W^0, U_W^0} \\ \text{D}\mathcal{H}_{L_R} & \xrightarrow{T_{W,R}^L} & \text{D}\mathcal{H}_{L_W} \end{array}$$

Our assertion easily follows. \square

Proof of Proposition 5

Passing to the limit as N goes to infinity, the functors $\mathcal{F}_{L_R^0, U_R^0} : D\mathcal{H}_{U_R} \rightarrow D\mathcal{H}_{L_R}$ from Lemma 10 yield the desired functor (24). The second assertion follows by construction. \square

Definition 5. Let $\mathcal{F}_{U(F)} : D(U^* \otimes \Omega(F)) \rightarrow D(\widetilde{\mathcal{L}}_d(M(F)))$ denote the composition

$$D(U^* \otimes \Omega(F)) \xrightarrow{\mathcal{F}_{U(F), L^0}} D\mathcal{H}_L \xrightarrow{\mathcal{F}_{L^0}} D(\widetilde{\mathcal{L}}_d(M(F)))$$

By Theorem 2 and Proposition 5, it does not depend on the choice of a k -point $L^0 \in \widetilde{\mathcal{L}}_d(M(F))$. By construction, $\mathcal{F}_{U(F)}$ is exact for the perverse t-structures.

We have a morphism of group stacks $\mathrm{GL}(U)(F) \rightarrow \widetilde{G}(F)$ sending $g \in \mathrm{GL}(U)(F)$ to $(g, \mathcal{B} = \det(U : gU))$ equipped with a canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det(M : gM) \xrightarrow{\sim} \det(U : gU) \otimes \det(U^* \otimes \Omega : g(U^* \otimes \Omega)) \xrightarrow{\sim} \det(U : gU)^{\otimes 2}$$

Let $\mathrm{GL}(U)(F)$ act on $\widetilde{\mathcal{L}}_d(M(F))$ via this homomorphism, let it also act naturally on $U^* \otimes \Omega(F)$. Then one may show that $\mathcal{F}_{U(F)}$ commutes with the action of $\mathrm{GL}(U)(F)$.

Note also that over $\mathrm{GL}(U)(\mathcal{O})$ the sections $\mathrm{GL}(U)(F) \rightarrow \widetilde{G}(F)$ and $G(\mathcal{O}) \rightarrow \widetilde{G}(F)$ are compatible.

7. GLOBAL APPLICATION

7.1 Assume k algebraically closed. Let X be a smooth connected projective curve. Let Ω be the canonical invertible sheaf on X . Let G be the group scheme over X of automorphisms of $\mathcal{O}_X^d \oplus \Omega^d$ preserving the symplectic form $\wedge^2(\mathcal{O}_X^d \oplus \Omega^d) \rightarrow \Omega$.

Write Bun_G for the stack of G -torsors on X , it classifies a rank $2d$ -vector bundle \mathcal{M} on X together with a symplectic form $\wedge^2 \mathcal{M} \rightarrow \Omega$. Let \mathcal{A} be the $(\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) line bundle on Bun_G whose fibre at \mathcal{M} is $\det \mathrm{R}\Gamma(X, \mathcal{M})$. Write $\widetilde{\mathrm{Bun}}_G$ for the gerb of square roots of \mathcal{A} over Bun_G .

Remind the definition of the theta-sheaf Aut on $\widetilde{\mathrm{Bun}}_G$ ([10], Definition 1). Let ${}_i \mathrm{Bun}_G \hookrightarrow \mathrm{Bun}_G$ be the locally closed substack given by $\dim H^0(X, \mathcal{M}) = i$ for $\mathcal{M} \in \mathrm{Bun}_G$. Write ${}_i \widetilde{\mathrm{Bun}}_G$ for the restriction of $\widetilde{\mathrm{Bun}}_G$ to ${}_i \mathrm{Bun}_G$.

Let ${}_i \mathcal{B}$ be the line bundle on ${}_i \mathrm{Bun}_G$ whose fibre at $\mathcal{M} \in {}_i \mathrm{Bun}_G$ is $\det H^0(X, \mathcal{M})$, we view it as $\mathbb{Z}/2\mathbb{Z}$ -graded of degree $i \bmod 2$. For each i we have a canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism ${}_i \mathcal{B}^2 \xrightarrow{\sim} \mathcal{A}$, it yields a trivialization ${}_i \widetilde{\mathrm{Bun}}_G \xrightarrow{\sim} {}_i \mathrm{Bun}_G \times B(\mu_2)$.

Define $\mathrm{Aut}_g \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$ (resp., $\mathrm{Aut}_s \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$) as the intermediate extension of

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathrm{Bun}_G}$$

(resp., of $(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathrm{Bun}_G - 1}$) under ${}_i \widetilde{\mathrm{Bun}}_G \hookrightarrow \widetilde{\mathrm{Bun}}_G$. Set $\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s$.

7.2 Fix a closed point $x \in X$. Write \mathcal{O}_x for the completed local ring of X at x , F_x for its fraction field. Fix a G -torsor over $\text{Spec } \mathcal{O}_x$, we think of it as a free \mathcal{O}_x -module M of rank $2d$ with symplectic form $\wedge^2 M \rightarrow \Omega(\mathcal{O}_x)$ and an action of $G(\mathcal{O}_x)$. We have a map

$$\xi_x : \text{Bun}_G \rightarrow \mathcal{L}_d(M(F_x))/G(\mathcal{O}_x),$$

where $\mathcal{L}_d(M(F_x))/G(\mathcal{O}_x)$ is the stack quotient. It sends $\mathcal{M} \in \text{Bun}_G$ to the Tate space $\mathcal{M}(F_x)$ with lagrangian c-lattice $\mathcal{M}(\mathcal{O}_x)$ and lagrangian d-lattice $H^0(X-x, \mathcal{M})$.

The line bundle \mathcal{A}_d on $\mathcal{L}_d(M(F_x))/G(\mathcal{O}_x)$ is that of Section 5.3. Write $\tilde{\mathcal{L}}_d(M(F_x))/G(\mathcal{O}_x)$ for the gerb of square roots of \mathcal{A}_d .

We have canonically $\xi_x^* \mathcal{A}_d \xrightarrow{\sim} \mathcal{A}$, so ξ lifts naturally to a map of gerbs

$$\tilde{\xi}_x : \widetilde{\text{Bun}}_G \rightarrow \tilde{\mathcal{L}}_d(M(F_x))/G(\mathcal{O}_x)$$

For $r \geq 0$ let ${}_{rx} \text{Bun}_G \subset \text{Bun}_G$ be the open substack given by $H^0(X, \mathcal{M}(-rx)) = 0$. Write ${}_{rx} \widetilde{\text{Bun}}_G$ for the restriction of the gerb $\widetilde{\text{Bun}}_G$ to ${}_{rx} \text{Bun}_G$. If $r' \geq r$ then ${}_{rx} \widetilde{\text{Bun}}_G \subset {}_{r'x} \widetilde{\text{Bun}}_G$ is an open substack, so we consider the projective 2-limit

$$2\text{-}\lim_{r \rightarrow \infty} \text{D}({}_{rx} \widetilde{\text{Bun}}_G)$$

Note that $2\text{-}\lim_{r \rightarrow \infty} \text{P}({}_{rx} \widetilde{\text{Bun}}_G) \xrightarrow{\sim} \text{P}(\widetilde{\text{Bun}}_G)$ is a full subcategory in the above limit. Let us define the restriction functor

$$\tilde{\xi}_x^* : \text{D}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F))) \rightarrow 2\text{-}\lim_{r \rightarrow \infty} \text{D}({}_{rx} \widetilde{\text{Bun}}_G) \quad (26)$$

To do so, for $N \geq r \geq 0$ and $r_1 \geq 2N$ let

$$\xi_N : {}_{rx} \text{Bun}_G \rightarrow {}_r \mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}) \quad (27)$$

be the map sending \mathcal{M} to the lagragian subspace $H^0(X, \mathcal{M}(Nx)) \subset {}_{N,N} \mathcal{M}$. If $N_1 \geq N \geq r$ and $r_1 \geq 2N_1$ then the diagram commutes

$$\begin{array}{ccc} {}_{rx} \text{Bun}_G & \xrightarrow{\xi_N} & {}_r \mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}) \\ & \searrow \xi_{N_1} & \uparrow p \\ & & {}_r \mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}) \end{array}$$

It induces a similar diagram between the gerbs (cf. Section 5.3 for their definition)

$$\begin{array}{ccc} {}_{rx} \widetilde{\text{Bun}}_G & \xrightarrow{\tilde{\xi}_N} & ({}_r \mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim \\ & \searrow \tilde{\xi}_{N_1} & \uparrow \\ & & ({}_r \mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}))^\sim \end{array}$$

The functors $K \mapsto \tilde{\xi}_N^* K \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\xi_N)}$ from $\text{D}_{G(\mathcal{O})}({}_r \tilde{\mathcal{L}}_{(N,N)M})$ to $\text{D}({}_{rx} \widetilde{\text{Bun}}_G)$ are compatible with the transition functors, so yield a functor

$${}_r \tilde{\xi}_x^* : \text{D}_{G(\mathcal{O})}({}_r \tilde{\mathcal{L}}_d(M(F))) \rightarrow \text{D}({}_{rx} \widetilde{\text{Bun}}_G)$$

Passing to the limit by r , one gets the desired functor (26).

Theorem 3. *The object $\tilde{\xi}_x^* S_{M(F_x)}$ lies in $\mathrm{P}(\widetilde{\mathrm{Bun}}_G)$, and there is an isomorphism of perverse sheaves*

$$\tilde{\xi}_x^* S_{M(F_x)} \xrightarrow{\sim} \mathrm{Aut}$$

Proof For $r \geq 0$ consider the map

$$\tilde{\xi}_r : {}_{rx} \widetilde{\mathrm{Bun}}_G \rightarrow (\mathcal{L}_{(r,r}M)/G(\mathcal{O}/t^{2r}))^\sim$$

Set $Y = \mathcal{L}_{(r,r}M) \times \mathcal{L}_{(r,r}M)$. Write \mathcal{Y} for the stack quotient of Y by the diagonal action of $\mathrm{Sp}_{(r,r}M)$. Let $\mathcal{A}_{\mathcal{Y}}$ be the $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on \mathcal{Y} with fibre $\det L_1 \otimes \det L_2$ at (L_1, L_2) . Write $\tilde{\mathcal{Y}}$ for the gerb of square roots of $\mathcal{A}_{\mathcal{Y}}$ over \mathcal{Y} . The map $\mathcal{L}_{(r,r}M) \rightarrow Y$ sending L_1 to $({}_{0,r}M, L_1) \in Y$ yields a morphism of stacks

$$\rho : (\mathcal{L}_{(r,r}M)/G(\mathcal{O}/t^{2r}))^\sim \rightarrow \tilde{\mathcal{Y}}$$

Write $S_{r,r}M$ for the perverse sheaf on $\tilde{\mathcal{Y}}$ introduced in (Section 3.2, Definition 1). Set $\tau = \rho \circ \tilde{\xi}_r$. It suffices to establish for any $r \geq 0$ a canonical isomorphism

$$\tau^* S_{r,r}M \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\mathrm{rel}(\tau)} \xrightarrow{\sim} \mathrm{Aut} \quad (28)$$

over ${}_{rx} \widetilde{\mathrm{Bun}}_G$.

Remind that $Y_i \subset Y$ is the locally closed subscheme given by $\dim(L_1 \cap L_2) = i$ for $(L_1, L_2) \in Y$. Let \mathcal{Y}_i be the stack quotient of Y_i by the diagonal action of $\mathrm{Sp}_{(r,r}M)$, set $\tilde{\mathcal{Y}}_i = \mathcal{Y}_i \times_{\mathcal{Y}} \tilde{\mathcal{Y}}$. Set

$${}_{rx,i} \widetilde{\mathrm{Bun}}_G = {}_{rx} \widetilde{\mathrm{Bun}}_G \cap_i \widetilde{\mathrm{Bun}}_G \quad \text{and} \quad {}_{rx,i} \mathrm{Bun}_G = {}_{rx} \mathrm{Bun}_G \cap_i \mathrm{Bun}_G$$

For each i the map τ fits into a cartesian square

$$\begin{array}{ccc} {}_{rx,i} \widetilde{\mathrm{Bun}}_G & \xrightarrow{\tau_i} & \tilde{\mathcal{Y}}_i \\ \downarrow & & \downarrow \\ {}_{rx} \widetilde{\mathrm{Bun}}_G & \xrightarrow{\tau} & \tilde{\mathcal{Y}} \end{array}$$

Indeed, for $\mathcal{M} \in {}_{rx} \mathrm{Bun}_G$ the space $\mathrm{H}^0(X, \mathcal{M})$ equals the intersection of $\mathcal{M}/\mathcal{M}(-rx)$ and $\mathrm{H}^0(X, \mathcal{M}(rx))$ inside $\mathcal{M}(rx)/\mathcal{M}(-rx)$. By ([10], Theorem 1), the $*$ -restriction of Aut to ${}_i \widetilde{\mathrm{Bun}}_G \xrightarrow{\sim} {}_i \mathrm{Bun}_G \times B(\mu_2)$ identifies with

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathrm{Bun}_G - i}$$

Similarly, by ([10], Proposition 1 and 5), the $*$ -restriction of S_M to $\tilde{\mathcal{Y}}_i \xrightarrow{\sim} \mathcal{Y}_i \times B(\mu_2)$ identifies with

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{Y} - i}$$

Since the map τ_i is compatible with our trivializations of the corresponding gerbs, we get the isomorphism (28) over ${}_{rx,i} \widetilde{\mathrm{Bun}}_G$ for each i . Since Aut is perverse, this also shows that the LHS

of (28) is placed in perverse degrees ≤ 0 , and its $*$ -restriction to ${}_{\leq 2}\widetilde{\text{Bun}}_G$ is placed in perverse degrees < 0 .

The map τ is not smooth, we overcome this difficulty as follows. Let us show that the LHS of (28) is placed in perverse degrees ≥ 0 . Consider the stack \mathcal{X} classifying $(\mathcal{M}, \mathcal{B}) \in {}_{rx}\widetilde{\text{Bun}}_G$ and a trivialization

$$\mathcal{M} |_{\text{Spec } \mathcal{O}_x/t_x^{2r}} \xrightarrow{\sim} M |_{\text{Spec } \mathcal{O}_x/t_x^{2r}}$$

of the corresponding G -torsor. Let $\nu : \mathcal{X} \rightarrow \tilde{Y}$ be the map sending a point of \mathcal{X} to the triple $(\mathcal{M}/\mathcal{M}(-rx), H^0(X, \mathcal{M}(rx)), \mathcal{B})$. Define \mathcal{X}_1 and \mathcal{X}_3 by the cartesian squares

$$\begin{array}{ccc} \mathcal{X}_3 & \rightarrow & C_3 \\ \downarrow \pi_{\mathcal{X}_3} & & \downarrow \pi_C \\ \mathcal{X}_1 & \rightarrow & U \times_{\mathcal{L}(r,rM)} U \\ \downarrow & & \downarrow \pi_Y \\ \mathcal{X} & \xrightarrow{\nu} & \tilde{Y}, \end{array}$$

Using (7), we get an isomorphism

$$\mu^* \tau^* S_{r,rM} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\mu)+\dim.\text{rel}(\tau)} \xrightarrow{\sim} (\pi_{\mathcal{X}_3})_! \mathcal{E} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{X}_3}$$

for some rank one local system \mathcal{E} on \mathcal{X}_3 . Here $\mu : \mathcal{X}_1 \rightarrow {}_{rx}\widetilde{\text{Bun}}_G$ is the projection, it is smooth. Since $\pi_{\mathcal{X}_3}$ is affine and \mathcal{X}_3 is smooth, the LHS of (28) is placed in perverse degrees ≥ 0 .

Thus, there exists an exact sequence of perverse sheaves $0 \rightarrow K \rightarrow K_1 \rightarrow \text{Aut} \rightarrow 0$ on ${}_{rx}\widetilde{\text{Bun}}_G$, where $K_1 = \tau^* S_{r,rM} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\tau)}$, and K is the extension by zero from ${}_{\leq 2}\widetilde{\text{Bun}}_G$. But we know already that K_1 and Aut are isomorphic in the Grothendieck group of ${}_{rx}\widetilde{\text{Bun}}_G$. So, K vanishes in this Grothendieck group, hence $K = 0$. We are done. \square

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