

# Geometric Weil representation: local field case

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**ABSTRACT** Let  $k$  be an algebraically closed field of characteristic  $> 2$ ,  $F = k((t))$  and  $G = \mathrm{Sp}_{2d}$ . In this paper we propose a geometric analog of the Weil representation of the metaplectic group  $\tilde{G}(F)$ . This is a category of certain perverse sheaves on some stack, on which  $\tilde{G}(F)$  acts by functors. This construction will be used in [11] (and subsequent publications) for the proof of the geometric Langlands functoriality for some dual reductive pairs.

## 1. INTRODUCTION

1.1 This paper followed by [11] form a series, where we prove the geometric Langlands functoriality for the dual reductive pair  $\mathrm{Sp}_{2n}, \mathrm{SO}_{2m}$  (in the everywhere nonramified case).

Let  $k = \mathbb{F}_q$  with  $q$  odd, set  $\mathcal{O} = k[[t]] \subset F = k((t))$ . Write  $\Omega$  for the completed module of relative differentials of  $\mathcal{O}$  over  $k$ . Let  $M$  be a free  $\mathcal{O}$ -module of rank  $2d$  with symplectic form  $\wedge^2 M \rightarrow \Omega$ , set  $G = \mathrm{Sp}(M)$ . The group  $G(F)$  admits a nontrivial metaplectic extension

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{G}(F) \rightarrow G(F) \rightarrow 1$$

(defined up to a unique isomorphism). Let  $\psi : k \rightarrow \bar{\mathbb{Q}}_\ell^*$  be a nontrivial additive character, let  $\chi : \Omega(F) \rightarrow \bar{\mathbb{Q}}_\ell^*$  be given by  $\chi(\omega) = \psi(\mathrm{Res} \omega)$ . Write  $H = M \oplus \Omega$  for the Heisenberg group of  $M$  with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\omega\langle m_1, m_2 \rangle) \quad m_i \in M, a_i \in \Omega$$

Denote by  $\mathcal{S}_\psi$  the Weil representation of  $H(M)(F)$  with central character  $\chi$ . As a representation of  $\tilde{G}(F)$ , it decomposes  $\mathcal{S}_\psi \xrightarrow{\sim} \mathcal{S}_{\psi, \text{odd}} \oplus \mathcal{S}_{\psi, \text{even}}$  into a direct sum of two irreducible smooth representations, where the even (resp., the odd) part is unramified (resp., ramified).

The discovery of this representation by A. Weil in [14] had a major influence on the theory of automorphic forms (among numerous developpements and applications are Howe duality for reductive dual pairs, particular cases of classical Langlands functoriality, Siegel-Weil formulas, relation with L-functions, representation-theoretic approach to the theory of theta-series. We refer the reader to [3], [9], [7], [12], [13] for history and details).

In this paper we introduce a geometric analog of the Weil representation  $\mathcal{S}_\psi$ . The pioneering work in this direction is due to P. Deligne [2], where a geometric approach to the Weil

representation of a symplectic group over a finite field was set up. It was further extended by Gurevich-Hadani in [4, 5]. The point of this paper is to develop the geometric theory in the case when a finite field is replaced by a local non-archimedean field.

First, we introduce a  $k$ -scheme  $\mathcal{L}_d(M(F))$  of discrete lagrangian lattices in  $M(F)$  and a certain  $\mu_2$ -gerb  $\tilde{\mathcal{L}}_d(M(F))$  over it. We view the metaplectic group  $\tilde{G}(F)$  as a group stack over  $k$ . We construct a category

$$W(\tilde{\mathcal{L}}_d(M(F)))$$

of certain perverse sheaves on  $\tilde{\mathcal{L}}_d(M(F))$ , which provides a geometric analog of  $\mathcal{S}_{\psi, \text{even}}$ . The metaplectic group  $\tilde{G}(F)$  acts on the category  $W(\tilde{\mathcal{L}}_d(M(F)))$  by functors. This action is *geometric* in the sense that it comes from a natural action of  $\tilde{G}(F)$  on  $\tilde{\mathcal{L}}_d(M(F))$  (cf. Theorem 2).

The category  $W(\tilde{\mathcal{L}}_d(M(F)))$  has a distinguished object  $S_{M(F)}$  corresponding to the unique non-ramified vector of  $\mathcal{S}_{\psi, \text{even}}$ .

Our category  $W(\tilde{\mathcal{L}}_d(M(F)))$  is obtained from Weil representations of symplectic groups  $\text{Sp}_{2r}(k)$  by some limit procedure. This uses a construction of geometric canonical intertwining operators for such representations. A similar result has been announced by Gurevich and Hadani in [4] and proved for  $d = 1$  in [5]. We give a proof for any  $d$  (cf. Theorem 1). When this paper has already been written we learned about a new preprint [6], where a result similar to our Theorem 1 is claimed to be proved for all  $d$ . However, the sheaves of canonical intertwining operators constructed in *loc.cit.* and in this paper live on different bases.

Finally, in Section 7 we give a global application. Let  $X$  be a smooth projective curve. Write  $\Omega_X$  for the canonical line bundle on  $X$ . Let  $G$  denote the sheaf of automorphisms of  $\mathcal{O}_X^d \oplus \Omega_X^d$  preserving the natural symplectic form  $\wedge^2(\mathcal{O}_X^d \oplus \Omega_X^d) \rightarrow \Omega_X$ .

Our Theorem 3 relates  $S_{M(F)}$  with the theta-sheaf  $\text{Aut}$  on the moduli stack  $\widetilde{\text{Bun}}_G$  of metaplectic bundles on  $X$  introduced in [10]. This result will play an important role in [11].

**1.2 NOTATION** In Section 2 we let  $k = \mathbb{F}_q$  of characteristic  $p > 2$ . Starting from Section 3 we assume  $k$  either finite as above or algebraically closed with a fixed inclusion  $\mathbb{F}_q \hookrightarrow k$ . All the schemes (or stacks) we consider are defined over  $k$ .

Fix a prime  $\ell \neq p$ . For a scheme (or stack)  $S$  write  $\text{D}(S)$  for the bounded derived category of  $\ell$ -adic étale sheaves on  $S$ , and  $\text{P}(S) \subset \text{D}(S)$  for the category of perverse sheaves.

Fix a nontrivial character  $\psi : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell^*$ , write  $\mathcal{L}_\psi$  for the corresponding Artin-Schreier sheaf on  $\mathbb{A}^1$ . Fix a square root  $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$  of the sheaf  $\bar{\mathbb{Q}}_\ell(1)$  on  $\text{Spec } \mathbb{F}_q$ . Isomorphism classes of such correspond to square roots of  $q$  in  $\bar{\mathbb{Q}}_\ell$ .

If  $V \rightarrow S$  and  $V^* \rightarrow S$  are dual rank  $n$  vector bundles over a stack  $S$ , we normalize the Fourier transform  $\text{Four}_\psi : \text{D}(V) \rightarrow \text{D}(V^*)$  by  $\text{Four}_\psi(K) = (p_{V^*})_!(\xi^* \mathcal{L}_\psi \otimes p_V^* K)[n](\frac{n}{2})$ , where  $p_V, p_{V^*}$  are the projections, and  $\xi : V \times_S V^* \rightarrow \mathbb{A}^1$  is the pairing.

Our conventions about  $\mathbb{Z}/2\mathbb{Z}$ -gradings are those of [10].

## 2. CANONICAL INTERWINING OPERATORS: FINITE FIELD CASE

2.1 Let  $M$  be a symplectic  $k$ -vector space of dimension  $2d$ . The symplectic form on  $M$  is denoted  $\omega\langle \cdot, \cdot \rangle$ . The Heisenberg group  $H = M \times \mathbb{A}^1$  with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\omega\langle m_1, m_2 \rangle) \quad m_i \in M, a_i \in \mathbb{A}^1$$

is algebraic over  $k$ . Set  $G = \mathrm{Sp}(M)$ . Write  $\mathcal{L}(M)$  for the variety of lagrangian subspaces in  $M$ . Fix a one-dimensional  $k$ -vector space  $\mathcal{J}$  (purely of degree  $d \bmod 2$  as  $\mathbb{Z}/2\mathbb{Z}$ -graded). Let  $\mathcal{A}$  be the (purely of degree zero as  $\mathbb{Z}/2\mathbb{Z}$ -graded) line bundle over  $\mathcal{L}(M)$  with fibre  $\mathcal{J} \otimes \det L$  at  $L \in \mathcal{L}(M)$ . Write  $\tilde{\mathcal{L}}(M)$  for the gerb of square roots of  $\mathcal{A}$ . The line bundle  $\mathcal{A}$  is  $G$ -equivariant, so  $G$  acts naturally on  $\tilde{\mathcal{L}}(M)$ .

For a  $k$ -point  $L \in \mathcal{L}(M)$  write  $L^0$  for a  $k$ -point of  $\tilde{\mathcal{L}}(M)$  over  $L$ . Write

$$\bar{L} = L \oplus k,$$

this is a subgroup of  $H(k)$  equipped with the character  $\chi_L : \bar{L} \rightarrow \bar{\mathbb{Q}}_\ell^*$  given by  $\chi_L(l, a) = \psi(a)$ ,  $l \in L, a \in k$ . Write

$$\mathcal{H}_L = \{f : H(k) \rightarrow \bar{\mathbb{Q}}_\ell \mid f(\bar{l}h) = \chi_L(\bar{l})f(h), \text{ for } \bar{l} \in \bar{L}, h \in H\}$$

This is a representation of  $H(k)$  by right translations. Write  $\mathcal{S}(H)$  for the space of all  $\bar{\mathbb{Q}}_\ell$ -valued functions on  $H(k)$ . The group  $G$  acts naturally in  $\mathcal{S}(H)$ . For  $L \in \mathcal{L}(M), g \in G$  we have an isomorphism  $\mathcal{H}_L \rightarrow \mathcal{H}_{gL}$  sending  $f$  to  $gf$ .

The purpose of Sections 2 and 3 is to study the canonical interwining operators (and their geometric analogs) between various models  $\mathcal{H}_L$  of the Weil representation. The corresponding results for a finite field were formulated by Gurevich and Hadani [4] without a proof (we give all proofs for the sake of completeness). Besides, our setting is a bit different from *loc.cit*, we work with gerbs instead of the total space of the corresponding line bundles.

2.2 For  $k$ -points  $L^0, N^0 \in \tilde{\mathcal{L}}(M)$  we will define a canonical interwining operator

$$F_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$$

They will satisfy the properties

- $F_{L^0, L^0} = \mathrm{id}$
- $F_{R^0, N^0} \circ F_{N^0, L^0} = F_{R^0, L^0}$  for any  $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)$
- for any  $g \in G$  we have  $g \circ F_{N^0, L^0} \circ g^{-1} = F_{gN^0, gL^0}$ .
- under the natural action of  $\mu_2$  on the set  $\tilde{\mathcal{L}}(M)(k)$  of (isomorphism classes of)  $k$ -points,  $F_{N^0, L^0}$  is odd as a function of  $N^0$  and of  $L^0$ .

In (Remark 2, Section 3.1) we will define a function  $F^{cl}$  on the set of  $k$ -points of  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ , which we denote  $F_{N^0, L^0}(h)$  for  $h \in H$ . It will realize the operator  $F_{N^0, L^0}$  by

$$(F_{N^0, L^0} f)(h_1) = \int_{h_2 \in H} F_{N^0, L^0}(h_1 h_2^{-1}) f(h_2) dh_2$$

All our measures on finite sets are normalized by requiring the volume of a point to be one. Given two functions  $f_1, f_2 : H \rightarrow \bar{\mathbb{Q}}_\ell$  their convolution  $f_1 * f_2 : H \rightarrow \bar{\mathbb{Q}}_\ell$  is defined by

$$(f_1 * f_2)(h) = \int_{v \in H} f_1(hv^{-1}) f_2(v) dv \quad h \in H$$

The function  $F_{N^0, L^0}$  will satisfy the following:

- $F_{N^0, L^0}(\bar{n}h\bar{l}) = \chi_N(\bar{n})\chi_L(\bar{l})F_{N^0, L^0}(h)$  for  $\bar{l} \in \bar{L}, \bar{n} \in \bar{N}, h \in H$ .
- $F_{gN^0, gL^0}(gh) = F_{N^0, L^0}(h)$  for  $g \in G, h \in H$ .
- Convolution property:  $F_{R^0, L^0} = F_{R^0, N^0} * F_{N^0, L^0}$  for any  $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)$ .

2.3 First, we define the non-normalized function  $\tilde{F}_{N, L} : H \rightarrow \bar{\mathbb{Q}}_\ell$ , it will depend only on  $N, L \in \mathcal{L}(M)$ , not of their inanced structure.

Given  $N, L \in \mathcal{L}(M)$  let  $\chi_{NL} : \bar{N}\bar{L} \rightarrow \bar{\mathbb{Q}}_\ell$  be the function given by

$$\chi_{NL}(\bar{n}\bar{l}) = \chi_N(\bar{n})\chi_L(\bar{l}),$$

it is correctly defined. Note that  $\bar{N}\bar{L} = \bar{L}\bar{N}$  but  $\chi_{NL} \neq \chi_{LN}$  in general. Set

$$\tilde{F}_{N, L}(h) = \begin{cases} \chi_{NL}(h), & \text{if } h \in \bar{N}\bar{L} \\ 0, & \text{otherwise} \end{cases}$$

Note that  $\chi_{LL} = \chi_L$ .

Given  $L, R, N \in \mathcal{L}(M)$  with  $N \cap L = N \cap R = 0$ , define  $\theta(R, N, L) \in \bar{\mathbb{Q}}_\ell$  as follows. There is a unique map  $b : L \rightarrow N$  such that  $R = \{l + b(l) \in L \oplus N \mid l \in L\}$ . Set

$$\theta(R, N, L) = \int_{l \in L} \psi\left(\frac{1}{2}\omega\langle l, b(l) \rangle\right) dl$$

This expression has been considered in ([10], Appendix B).

**Lemma 1.** 1) Let  $L, N \in \mathcal{L}(M)$ . If  $L \cap N = 0$  then  $\tilde{F}_{L, N} * \tilde{F}_{N, L} = q^{2d+1}\tilde{F}_{L, L}$ .

2) Let  $L, R, N \in \mathcal{L}(M)$  with  $N \cap L = N \cap R = 0$ . Then  $\tilde{F}_{R, N} * \tilde{F}_{N, L} = q^{d+1}\theta(R, N, L)\tilde{F}_{R, L}$

*Proof* 2) Using  $L \oplus N = N \oplus R = M$ , for  $h \in H$  we get

$$(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) = q^{d+1} \int_{v \in \tilde{N} \setminus H} \chi_{RN}(hv^{-1}) \chi_{NL}(v) dv = q^{d+1} \int_{r \in R} \chi_{RN}(h(-r, 0)) \chi_{NL}(r, 0) dr$$

Because of the equivariance property of  $\tilde{F}_{R,N} * \tilde{F}_{N,L}$ , we may assume  $h = (n, 0), n \in N$ . We get

$$\begin{aligned} (\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) &= q^{d+1} \int_{r \in R} \chi_{RN}((n, 0)(-r, 0)) \chi_{NL}(r, 0) dr \\ &= q^{d+1} \int_{r \in R} \psi(\omega\langle r, n \rangle) \chi_{NL}(r, 0) dr \quad (1) \end{aligned}$$

The latter formula essentially says that the resulting function on  $N$  is the Fourier transform of some local system on  $R$  (the symplectic form on  $M$  induces an isomorphism  $R \xrightarrow{\sim} N^*$ ). This will be used for geometrization in Lemma 2.

There is a unique map  $b : L \rightarrow N$  such that  $R = \{l + b(l) \in L \oplus N \mid l \in L\}$ . So, the above integral rewrites

$$\begin{aligned} (\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) &= q^{d+1} \int_{l \in L} \psi(\omega\langle l, n \rangle) \chi_{NL}((l + b(l), 0)) dl = \\ &= q^{d+1} \int_{l \in L} \psi(\omega\langle l, n \rangle) \chi_{NL}((b(l), \frac{1}{2}\omega\langle l, b(l) \rangle)(l, 0)) dl = q^{d+1} \int_{l \in L} \psi(\omega\langle l, n \rangle + \frac{1}{2}\omega\langle l, b(l) \rangle) dl \quad (2) \end{aligned}$$

Note that if  $R = L$  then  $b = 0$  and the latter formula yields 1).

Let us identify  $N \xrightarrow{\sim} L^*$  via the map sending  $n \in N$  to the linear functional  $l \mapsto \omega\langle l, n \rangle$ . Denote by  $\langle \cdot, \cdot \rangle$  the symmetric pairing between  $L$  and  $L^*$ . By Sublemma 1 below, the value (2) vanishes unless  $n \in (R + L) \cap N = \text{Im } b$ . In the latter case pick  $l_1 \in L$  with  $b(l_1) = n$ . Then

$$\chi_{RL}(n, 0) = \psi(-\frac{1}{2}\omega\langle l_1, b(l_1) \rangle)$$

So, we get for  $L' = \text{Ker } b$

$$(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) = q^{d+1+\dim L'} \chi_{RL}(h) \int_{l \in L/L'} \psi(\frac{1}{2}\omega\langle l, b(l) \rangle) dl$$

We are done.  $\square$

**Sublemma 1.** *Let  $L$  be a  $d$ -dimensional  $k$ -vector space,  $b \in \text{Sym}^2 L^*$  and  $u \in L^*$ . View  $b$  as a map  $b : L \rightarrow L^*$ , let  $L'$  be the kernel of  $b$ . Then*

$$\int_{l \in L} \psi(\langle l, u \rangle + \frac{1}{2}\langle l, b(l) \rangle) dl \quad (3)$$

*is supported at  $u \in (L/L')^*$  and there equals*

$$q^{\dim L'} \psi(-\frac{1}{2}\langle b^{-1}u, u \rangle) \int_{L/L'} \psi(\frac{1}{2}\langle l, b(l) \rangle) dl,$$

where  $b : L/L' \xrightarrow{\sim} (L/L')^*$ , so that  $b^{-1}u \in L/L'$ . (Here the scalar product is between  $L$  and  $L^*$ , so is symmetric).

*Proof* Let  $L' \subset L$  denote the kernel of  $b : L \rightarrow L^*$ . Integrating first along the fibres of the projection  $L \rightarrow L/L'$  we will get zero unless  $u \in (L/L')^*$ . For any  $l_0 \in L$  the integral (3) equals

$$\int_{l \in L} \psi(\langle l+l_0, u \rangle + \frac{1}{2} \langle l+l_0, b(l)+b(l_0) \rangle) dl = \psi(\langle l_0, u \rangle + \frac{1}{2} \langle l_0, b(l_0) \rangle) \int_{l \in L} \psi(\langle l, u+b(l_0) \rangle + \frac{1}{2} \langle l, b(l) \rangle) dl$$

Assuming  $u \in (L/L')^*$  take  $l_0$  such that  $u = -b(l_0)$ . Then (3) becomes

$$\psi(\frac{1}{2} \langle l_0, u \rangle) \int_{l \in L} \psi(\frac{1}{2} \langle l, b(l) \rangle) dl$$

We are done.  $\square$

*Remark 1.* The expression (3) is the Fourier transform from  $L$  to  $L^*$ . In the geometric setting we will use 2) of Lemma 1 only under the additional assumption  $R \cap L = 0$ .

### 3. GEOMETRIZATION

3.1 Let  $M, H, \mathcal{L}(M)$  and  $\tilde{\mathcal{L}}(M)$  be as in Section 2.1. Remind that  $G = \mathrm{Sp}(M)$ . For each  $L \in \mathcal{L}(M)$  we have a rank one local system  $\chi_L$  on  $\bar{L} = L \times \mathbb{A}^1$  defined by  $\chi_L = \mathrm{pr}^* \mathcal{L}_\psi$ , where  $\mathrm{pr} : L \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$  is the projection. Let  $\mathcal{H}_L$  denote the category of perverse sheaves on  $H$  which are  $(\bar{L}, \chi_L)$ -equivariant under the left multiplication, this is a full subcategory in  $\mathrm{P}(H)$ . Write  $\mathrm{D}\mathcal{H}_L \subset \mathrm{D}(H)$  for the full subcategory of objects whose all perverse cohomologies lie in  $\mathcal{H}_L$ .

Denote by  $C \rightarrow \mathcal{L}(M)$  (resp.,  $\bar{C} \rightarrow \mathcal{L}(M)$ ) the vector bundle whose fibre over  $L \in \mathcal{L}(M)$  is  $L$  (resp.,  $\bar{L} = L \times \mathbb{A}^1$ ). Its inverse image to  $\tilde{\mathcal{L}}(M)$  is denoted by the same symbol.

Write  $\chi_{\bar{C}}$  for the local system  $p^* \mathcal{L}_\psi$  on  $\bar{C}$ , where  $p : \bar{C} \rightarrow \mathbb{A}^1$  is the projection on the center sending  $(L \in \mathcal{L}(M), (l, a) \in \bar{L})$  to  $a$ . Consider the maps

$$\mathrm{pr}, \mathrm{act}_{lr} : \bar{C} \times \bar{C} \times H \rightarrow \mathcal{L}(M) \times \mathcal{L}(M) \times H \times H$$

where  $\mathrm{act}_{lr}$  sends  $(\bar{n} \in \bar{N}, \bar{l} \in \bar{L}, h)$  to  $(N, L, \bar{n}h\bar{l})$ , and  $\mathrm{pr}$  sends the above point to  $(N, L, h)$ . We say that a perverse sheaf  $K$  on  $\mathcal{L}(M) \times \mathcal{L}(M) \times H$  is  $\mathrm{act}_{lr}$ -equivariant if it admits an isomorphism

$$\mathrm{act}_{lr}^* K \xrightarrow{\sim} \mathrm{pr}^* K \otimes \mathrm{pr}_1^* \chi_{\bar{C}} \otimes \mathrm{pr}_2^* \chi_{\bar{C}}$$

satisfying the usual associativity condition and whose restriction to the unit section is the identity (such isomorphism is unique if it exists). One has a similar definition for  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ .

Let

$$\mathrm{act}_G : G \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$$

be the action map sending  $(g, N^0, L^0, h)$  to

$$(gN^0, gL^0, gh)$$

For this map we have a usual notion of a  $G$ -equivariant perverse sheaf on  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ . As  $G$  is connected, a perverse sheaf on  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$  admits at most one  $G$ -equivariant structure.

If  $S$  is a stack then for  $K, F \in \mathbf{D}(S \times H)$  define their convolution  $K * F \in \mathbf{D}(S \times H)$  by

$$K * F = \text{mult}_!(\text{pr}_1^* K \otimes \text{pr}_2^* F) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{d+1-2 \dim \mathcal{L}(M)},$$

here  $\text{pr}_i : S \times H \times H \rightarrow S \times H$  is the projection to the  $i$ -th component in the pair  $H \times H$  (and the identity on  $S$ ). The multiplication map  $\text{mult} : H \times H \rightarrow H$  sends  $(h_1, h_2)$  to  $h_1 h_2$ .

Let

$$(\mathcal{L}(M) \times H)_\Delta \hookrightarrow \mathcal{L}(M) \times H \tag{4}$$

be the closed subscheme of those  $(L \in \mathcal{L}(M), h \in H)$  for which  $h \in \bar{L}$ . Let

$$\alpha_\Delta : (\mathcal{L}(M) \times H)_\Delta \rightarrow \mathbb{A}^1$$

be the map sending  $(L, h)$  to  $a$ , where  $h = (l, a)$ ,  $l \in L, a \in \mathbb{A}^1$ . Define a perverse sheaf

$$\tilde{F}_\Delta = \alpha_\Delta^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{d+1+\dim \mathcal{L}(M)},$$

which we extend by zero under (4).

Since  $\tilde{\mathcal{L}}(M) \rightarrow \mathcal{L}(M)$  is a  $\mu_2$ -gerb,  $\mu_2$  acts on each  $K \in \mathbf{D}(\tilde{\mathcal{L}}(M))$ , and we say that  $K$  is *genuine* if  $-1 \in \mu_2$  acts on  $K$  as  $-1$ .

**Theorem 1.** *There exists an irreducible perverse sheaf  $F$  on  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$  (pure of weight zero) with the following properties:*

- for the diagonal map  $i : \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$  the complex  $i^* F$  identifies canonically with the inverse image of

$$\tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

under the projection  $\tilde{\mathcal{L}}(M) \times H \rightarrow \mathcal{L}(M) \times H$ .

- $F$  is  $\text{act}_{l_r}$ -equivariant;
- $F$  is  $G$ -equivariant;
- $F$  is genuine in the first and the second variable;
- convolution property for  $F$  holds, namely for the  $ij$ -th projections

$$q_{ij} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$$

inside the triple  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M)$  we have  $(q_{12}^* F) * (q_{23}^* F) \xrightarrow{\sim} q_{13}^* F$  canonically.

The proof of Theorem 1 is given in Sections 3.2-3.4.

*Remark 2.* In the case  $k = \mathbb{F}_q$  define  $F^{cl}$  as the trace of the geometric Frobenius on  $F$ .

3.2 Let  $U \subset \mathcal{L}(M) \times \mathcal{L}(M)$  be the open subset of pairs  $(N, L) \in \mathcal{L}(M) \times \mathcal{L}(M)$  such that  $N \cap L = 0$ . Define a perverse sheaf  $\tilde{F}_U$  on  $U \times H$  as follows. Let

$$\alpha_U : U \times H \rightarrow \mathbb{A}^1$$

be the map sending  $(N, L, h)$  to  $a + \frac{1}{2}\omega\langle l, n \rangle$ , where  $l \in L, n \in N, a \in \mathbb{A}^1$  are uniquely defined by  $h = (n + l, a)$ . Set

$$\tilde{F}_U = \alpha_U^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim H + 2 \dim \mathcal{L}(M)} \quad (5)$$

Write  $U \times_{\mathcal{L}(M)} U \subset \mathcal{L}(M) \times \mathcal{L}(M) \times \mathcal{L}(M)$  for the open subscheme classifying  $(R, N, L)$  with  $N \cap L = N \cap R = 0$ . Let

$$q_i : U \times_{\mathcal{L}(M)} U \rightarrow U$$

be the projection on the  $i$ -th factor, so  $q_1$  (resp.,  $q_2$ ) sends  $(R, N, L)$  to  $(R, N)$  (resp., to  $(N, L)$ ). Let  $q : U \times_{\mathcal{L}(M)} U \rightarrow \mathcal{L}(M) \times \mathcal{L}(M)$  be the map sending  $(R, N, L)$  to  $(R, L)$ . Write

$$(U \times_{\mathcal{L}(M)} U)_0 = q^{-1}(U)$$

The geometric analog of  $\theta(R, N, L)$  is the following (shifted) perverse sheaf  $\Theta$  on  $U \times_{\mathcal{L}(M)} U$ . Let  $\pi_C : C_3 \rightarrow U \times_{\mathcal{L}(M)} U$  be the vector bundle whose fibre over  $(R, N, L)$  is  $L$ . We have a map  $\beta : C_3 \rightarrow \mathbb{A}^1$  defined as follows. Given a point  $(R, N, L) \in U \times_{\mathcal{L}(M)} U$ , there is a unique map  $b : L \rightarrow N$  such that  $R = \{l + b(l) \in L \oplus N = M \mid l \in L\}$ . Set  $\beta(R, N, L, l) = \frac{1}{2}\omega\langle l, b(l) \rangle$ . Set

$$\Theta = (\pi_C)_! \beta^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^d$$

Write  $Y = \mathcal{L}(M) \times \mathcal{L}(M)$ , let  $\mathcal{A}_Y$  be the ( $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) line bundle on  $Y$  whose fibre at  $(R, L)$  is  $\det R \otimes \det L$ . Write  $\tilde{Y}$  for the gerb of square roots of  $\mathcal{A}_Y$ . Note that  $\mathcal{A}_Y$  is  $G$ -equivariant, so  $G$  acts on  $\tilde{Y}$  naturally.

The following perverse sheaf  $S_M$  on  $\tilde{Y}$  was introduced in ([10], Definition 2). Let  $Y_i \subset Y$  be the locally closed subscheme given by  $\dim(R \cap L) = i$  for  $(R, L) \in Y_i$ . The restriction of  $\mathcal{A}_Y$  to each  $Y_i$  admits the following  $G$ -equivariant square root. For a point  $(R, L) \in Y_i$  we have an isomorphism  $L/(R \cap L) \xrightarrow{\sim} (R/(R \cap L))^*$  sending  $l$  to the functional  $r \mapsto \omega\langle r, l \rangle$ . It induces a  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  $\det R \otimes \det L \xrightarrow{\sim} \det(R \cap L)^2$ .

So, for the restriction  $\tilde{Y}_i$  of the gerb  $\tilde{Y} \rightarrow Y$  to  $Y_i$  we get a trivialization

$$\tilde{Y}_i \xrightarrow{\sim} Y_i \times B(\mu_2) \quad (6)$$

Write  $W$  for the nontrivial local system of rank one on  $B(\mu_2)$  corresponding to the covering  $\text{Spec } k \rightarrow B(\mu_2)$ .

**Definition 1.** Let  $S_{M,g}$  (resp.,  $S_{M,s}$ ) denote the intermediate extension of

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim Y}$$

from  $\tilde{Y}_0$  to  $\tilde{Y}$  (resp., of  $(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim Y-1}$  from  $\tilde{Y}_1$  to  $\tilde{Y}$ ). Set  $S_M = S_{M,g} \oplus S_{M,s}$ .

Let

$$\pi_Y : U \times_{\mathcal{L}(M)} U \rightarrow \tilde{Y}$$

be the map sending  $(R, N, L)$  to

$$(R, L, \mathcal{B}, \epsilon : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L),$$

where  $\mathcal{B} = \det L$  and  $\epsilon$  is the isomorphism induced by  $\epsilon_0$ . Here  $\epsilon_0 : L \xrightarrow{\sim} R$  is the isomorphism sending  $l \in L$  to  $l + b(l) \in R$ . In other words,  $\epsilon_0$  sends  $l$  to the unique  $r \in R$  such that  $r = l \bmod N \in M/N$ . Write also  $\tilde{U} = \tilde{Y}_0$ .

Define  $\mathcal{E} \in \mathbf{D}(\mathrm{Spec} k)$  by

$$\mathcal{E} = \mathrm{R}\Gamma_c(\mathbb{A}^1, \beta_0^* \mathcal{L}_\psi) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2}),$$

where  $\beta_0 : \mathbb{A}^1 \rightarrow \mathbb{A}^1$  sends  $x$  to  $x^2$ . Then  $\mathcal{E}$  is a 1-dimensional vector space placed in cohomological degree zero. The geometric Frobenius  $\mathrm{Fr}_{\mathbb{F}_q}$  acts on  $\mathcal{E}^2$  by 1 if  $-1 \in (\mathbb{F}_q^*)^2$  and by  $-1$  otherwise. A choice of  $\sqrt{-1} \in k$  yields an isomorphism  $\mathcal{E}^2 \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$ , so  $\mathcal{E}^4 \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$  canonically.

As in ([10], Proposition 5), one gets a canonical isomorphism

$$\pi_Y^*(S_{M,g} \otimes \mathcal{E}^d \oplus S_{M,s} \otimes \mathcal{E}^{d-1}) \xrightarrow{\sim} \Theta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M)} \quad (7)$$

Since  $d \geq 1$ , the restriction  $\pi_Y : (U \times_{\mathcal{L}(M)} U)_0 \rightarrow \tilde{U}$  is smooth of relative dimension  $\dim \mathcal{L}(M)$ , with geometrically connected fibres. It is convenient to introduce a rank one local system  $\Theta_U$  on  $\tilde{U}$  equipped with a canonical isomorphism

$$\Theta \xrightarrow{\sim} \pi_Y^* \Theta_U \quad (8)$$

over  $(U \times_{\mathcal{L}(M)} U)_0$ . The local system  $\Theta_U$  is defined up to a unique isomorphism.

Let  $i_U : U \rightarrow U \times_{\mathcal{L}(M)} U$  be the map sending  $(L, N)$  to  $(L, N, L)$ . Let  $p_1 : U \rightarrow \mathcal{L}(M)$  be the projection sending  $(L, N)$  to  $L$ .

**Lemma 2.** 1) *The complex*

$$(q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

*is an irreducible perverse sheaf on  $U \times_{\mathcal{L}(M)} U \times H$  pure of weight zero. We have canonically*

$$i_U^*((q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U)) \xrightarrow{\sim} p_1^* \tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

over  $U \times H$ .

2) There is a canonical isomorphism

$$(q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \xrightarrow{\sim} q^* \tilde{F}_U \otimes \Theta$$

over  $(U \times_{\mathcal{L}(M)} U)_0 \times H$ .

*Proof* 1) Follows from the properties of the Fourier transform as in Lemma 1, formula (1).

2) The proof of Lemma 1 goes through in the geometric setting. Our additional assumption that  $(R, N, L) \in (U \times_{\mathcal{L}(M)} U)_0$  means that  $b : L \rightarrow N$  is an isomorphism (it simplifies the argument a little).  $\square$

*Remark 3.* Let  $i_\Delta : \mathcal{L}(M) \rightarrow \tilde{Y}$  be the map sending  $L$  to  $(L, L, \mathcal{B} = \det L)$  equipped with the isomorphism  $\text{id} : \mathcal{B}^2 \xrightarrow{\sim} \det L \otimes \det L$ . The commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{i_U} & U \times_{\mathcal{L}(M)} U \\ \downarrow p_1 & & \downarrow \pi_Y \\ \mathcal{L}(M) & \xrightarrow{i_\Delta} & \tilde{Y} \end{array} \quad (9)$$

together with (7) yield a canonical isomorphism

$$i_\Delta^* S_M \xrightarrow{\sim} \begin{cases} \mathcal{E}^{-d} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M) - d}, & d \text{ is even} \\ \mathcal{E}^{1-d} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M) - d}, & d \text{ is odd} \end{cases}$$

3.3 Consider the following diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{q}_1} & (U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_2} & \tilde{U} \\ & & \downarrow \tilde{q} & & \\ & & \tilde{U} & & \end{array}$$

Here  $\tilde{q}$  is the restriction of  $\pi_Y$ , and the map  $\tilde{q}_i$  is the lifting of  $q_i$  defined as follows. We set  $\tilde{q}_1(R, N, L) = \tilde{q}(R, L, N)$  and  $\tilde{q}_2(R, N, L) = \tilde{q}(N, R, L)$ .

The following property is a geometric counterpart of the way the Maslov index of  $(R, N, L)$  changes under permutations of three lagrangian subspaces.

**Lemma 3.** 1) For  $i = 1, 2$  we have canonically over  $(U \times_{\mathcal{L}(M)} U)_0$

$$\tilde{q}_i^* \Theta_U \otimes \tilde{q}^* \Theta_U \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$$

2) We have  $\Theta_U^2 \xrightarrow{\sim} \mathcal{E}^{2d}$  canonically, so  $\Theta_U^4 \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell$  canonically.

*Proof* 1) The two isomorphisms are obtained similarly, we consider only the case  $i = 2$ . For a point  $(R, N, L) \in (U \times_{\mathcal{L}(M)} U)_0$  we have isomorphisms  $b : L \xrightarrow{\sim} N$  and  $b_0 : L \xrightarrow{\sim} R$  such that

$R = \{l + b(l) \mid l \in L\}$  and  $N = \{l + b_0(l) \mid l \in L\}$ . Clearly,  $b_0(-l) = l + b(l)$  for  $l \in L$ . Let  $\beta_2 : L \times L \rightarrow \mathbb{A}^1$  be the map sending  $(l, l_0)$  to  $\frac{1}{2}\omega\langle l, b(l)\rangle + \frac{1}{2}\omega\langle l, b_0(l)\rangle$ . We must show that

$$\mathrm{R}\Gamma_c(L \times L, \beta_2^* \mathcal{L}_\psi) \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell[2d](d)$$

The quadratic form  $(l, l_0) \mapsto \omega\langle l, b(l)\rangle - \omega\langle l_0, b(l_0)\rangle$  is hyperbolic on  $L \oplus L$ . Consider the isotopic subspace  $Q = \{(l, l) \in L \times L \mid l \in L\}$ . Integrating first along the fibres of the projection  $L \times L \rightarrow (L \times L)/Q$  and then over  $(L \times L)/Q$ , one gets the desired isomorphism.

2) This follows from (7).  $\square$

Define a perverse sheaf  $F_U$  on  $\tilde{U} \times H$  by

$$F_U = \mathrm{pr}_1^* \Theta_U \otimes \tilde{F}_U,$$

it is understood that we take the inverse image of  $\tilde{F}_U$  under the projection  $\tilde{U} \times H \rightarrow U \times H$  is the above formula. Let  $F$  be the intermediate extension of  $F_U$  under the open immersion  $\tilde{U} \times H \subset \tilde{Y} \times H$ .

*Remark 4.* In the case  $d = 0$  we have  $H = \mathbb{A}^1$  and  $\tilde{Y} = B(\mu_2)$ . In this case by definition  $F = W \boxtimes \mathcal{L}_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})$  over  $\tilde{Y} \times H = B(\mu_2) \times \mathbb{A}^1$ .

Combining Lemma 3 and 2) of Lemma 2, we get the following.

**Lemma 4.** *We have canonically  $(\tilde{q}_1^* F_U) * (\tilde{q}_2^* F_U) \xrightarrow{\sim} \tilde{q}^* F_U \otimes \mathcal{E}^{2d}$  over  $(U \times_{\mathcal{L}(M)} U)_0 \times H$ .*

We have a map  $\xi : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \rightarrow \tilde{Y}$  sending  $(\mathcal{B}_1, N, \mathcal{B}_1^2 \xrightarrow{\sim} \mathcal{J} \otimes \det N; \mathcal{B}_2, L, \mathcal{B}_2^2 \xrightarrow{\sim} \mathcal{J} \otimes \det L)$  to  $(N, L, \mathcal{B})$ , where  $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{J}^{-1}$  is equipped with the natural isomorphism  $\mathcal{B}^2 \xrightarrow{\sim} \det N \otimes \det L$ . The restriction of  $F$  under

$$\xi \times \mathrm{id} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{Y} \times H$$

is also denoted by  $F$ . Clearly,  $F$  is an irreducible perverse sheaf of weight zero.

Consider the cartesian square

$$\begin{array}{ccc} (U \times_{\mathcal{L}(M)} U)_0 \times H & \hookrightarrow & (U \times_{\mathcal{L}(M)} U) \times H \\ \downarrow \pi_Y \times \mathrm{id} & & \downarrow \pi_Y \times \mathrm{id} \\ \tilde{U} \times H & \hookrightarrow & \tilde{Y} \times H \end{array}$$

This diagram together with Lemma 2 yield a canonical isomorphism over  $(U \times_{\mathcal{L}(M)} U) \times H$

$$(\pi_Y \times \mathrm{id})^* F \xrightarrow{\sim} (q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \tag{10}$$

by intermediate extension from  $(U \times_{\mathcal{L}(M)} U)_0 \times H$ . This gives an explicit formula for  $F$ .

Consider the diagram

$$\begin{array}{ccc} U \times H & \xrightarrow{i_U \times \mathrm{id}} & U \times_{\mathcal{L}(M)} U \times H \\ \downarrow p_1 \times \mathrm{id} & & \downarrow \pi_Y \times \mathrm{id} \\ \mathcal{L}(M) \times H & \xrightarrow{i_\Delta \times \mathrm{id}} & \tilde{Y} \times H \end{array}$$

obtained from (9) by multiplication with  $H$ . By Lemma 2 and (10), we get canonically

$$(p_1 \times \text{id})^*(i_\Delta \times \text{id})^*F \xrightarrow{\sim} (p_1 \times \text{id})^*\tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

Since  $\tilde{F}_\Delta$  is perverse and  $p_1$  has connected fibres, this isomorphism descends to a uniquely defined isomorphism

$$(i_\Delta \times \text{id})^*F \xrightarrow{\sim} \tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

By construction,  $F$  is  $\text{act}_{l_r}$ -equivariant and  $G$ -equivariant (this holds for  $F_U$  and this property is preserved by the intermediate extension).

3.4 To finish the proof of Theorem 1, it remains to establish the convolution property of  $F$ . We actually prove it in the following form.

Write  $\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$  for the stack classifying  $R, N, L \in \mathcal{L}(M)$ , one dimensional  $k$ -vector spaces  $\mathcal{B}_1, \mathcal{B}_2$  equipped with isomorphisms  $\mathcal{B}_1^2 \xrightarrow{\sim} \det R \otimes \det N$  and  $\mathcal{B}_2^2 \xrightarrow{\sim} \det N \otimes \det L$ . We have a diagram

$$\begin{array}{ccccc} \tilde{Y} & \xleftarrow{\tau_1} & \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y} & \xrightarrow{\tau_2} & \tilde{Y} \\ & & \downarrow \tau & & \\ & & \tilde{Y} & & \end{array}$$

where  $\tau_1$  (resp.,  $\tau_2$ ) sends the above collection to  $(R, N, \mathcal{B}_1) \in \tilde{Y}$  (resp.,  $(N, L, \mathcal{B}_2) \in \tilde{Y}$ ). The map  $\tau$  sends the above collection to  $(R, L, \mathcal{B})$ , where  $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes (\det N)^{-1}$  is equipped with  $\mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L$ .

**Proposition 1.** *There is a canonical isomorphism over  $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$*

$$(\tau_1^*F) * (\tau_2^*F) \xrightarrow{\sim} \tau^*F \tag{11}$$

*Proof*

**Step 1.** Consider the diagram

$$\begin{array}{ccc} (U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_1 \times \tilde{q}_2} & (\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \\ & \searrow \tilde{q} & \downarrow \tau \\ & & \tilde{U} \end{array}$$

It becomes 2-commutative over  $\text{Spec } \mathbb{F}_q(\sqrt{-1})$ . More precisely, for  $K \in \text{D}(\tilde{U})$  we have a canonical isomorphism functorial in  $K$

$$\tilde{q}^*K \otimes \mathcal{E}^{2d} \xrightarrow{\sim} (\tilde{q}_1 \times \tilde{q}_2)^* \tau^*K$$

Indeed, let  $(R, N, L)$  be a  $k$ -point of  $(U \times_{\mathcal{L}(M)} U)_0$ , let  $(R, N, L, \mathcal{B}_1, \mathcal{B}_2)$  be its image under  $\tilde{q}_1 \times \tilde{q}_2$ . So,  $\mathcal{B}_1 = \det N$  and  $\pi_Y(R, L, N) = (R, N, \mathcal{B}_1)$ ,  $\mathcal{B}_2 = \det L$  and  $\pi_Y(N, R, L) = (N, L, \mathcal{B}_2)$ . Write

$$\tau(R, N, L, \mathcal{B}_1, \mathcal{B}_2) = (R, L, \mathcal{B}, \delta : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L)$$

Write  $\tilde{q}(R, N, L) = (R, L, \mathcal{B}, \delta_0 : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L)$ . It suffices to show that  $\delta_0 = (-1)^d \delta$ .

Let  $\epsilon_1 : N \xrightarrow{\sim} R$  be the isomorphism sending  $n \in N$  to  $r \in R$  such that  $r = n \pmod L$ . Write  $\epsilon_2 : L \xrightarrow{\sim} N$  for the isomorphism sending  $l \in L$  to  $n \in N$  such that  $l = n \pmod R$ . Let  $\epsilon_0 : L \xrightarrow{\sim} R$  be the isomorphism sending  $l \in L$  to  $r \in R$  such that  $r = l \pmod N$ . We get two isomorphisms

$$\text{id} \otimes \det \epsilon_0, \det \epsilon_1 \otimes \det \epsilon_2 : \det N \otimes \det L \xrightarrow{\sim} \det R \otimes \det N$$

We must show that  $\text{id} \otimes \det \epsilon_0 = (-1)^d \det \epsilon_1 \otimes \det \epsilon_2$ . Pick a base  $\{n_1, \dots, n_d\}$  in  $N$ . Define  $r_i \in R, l_i \in L$  by  $n_i = r_i + l_i$ . Then

$$\epsilon_1(n_i) = r_i, \quad \epsilon_2(l_i) = n_i, \quad \epsilon_0(l_i) = -r_i$$

So,  $\epsilon_0(l_1 \wedge \dots \wedge l_d) = (-1)^d r_1 \wedge \dots \wedge r_d$ . On the other hand,  $\det \epsilon_1 \otimes \det \epsilon_2$  sends

$$(n_1 \wedge \dots \wedge n_d) \otimes (l_1 \wedge \dots \wedge l_d)$$

to  $(r_1 \wedge \dots \wedge r_d) \otimes (n_1 \wedge \dots \wedge n_d)$ .

**Step 2.** The isomorphism (6) for  $i = 0$  yields  $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \xrightarrow{\sim} (U \times_{\mathcal{L}(M)} U)_0 \times B(\mu_2) \times B(\mu_2)$ . The corresponding 2-automorphisms  $\mu_2 \times \mu_2$  of  $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y})$  act in the same way on both sides of (11). Now from Step 1 it follows that the isomorphism of Lemma 4 descends under  $\tilde{q}_1 \times \tilde{q}_2$  to the desired isomorphism (11) over  $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H$ .

**Step 3.** To finish the proof it suffices to show that  $(\tau_1^* F) * (\tau_2^* F)$  is perverse, the intermediate extension under the open immersion

$$(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H \subset (\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$$

Let us first explain the idea informally, at the level of functions. In this step for  $(N, R, \mathcal{B}) \in \tilde{Y}$  we denote by  $F_{N,R,\mathcal{B}} : H \rightarrow \bar{\mathbb{Q}}_\ell$  the function trace of Frobenius of the sheaf  $F$ .

Given  $(R, N, \mathcal{B}_1) \in \tilde{Y}$  and  $(N, L, \mathcal{B}_2) \in \tilde{Y}$  pick any  $S, T \in \mathcal{L}(M)$  such that  $(R, S, N) \in U \times_{\mathcal{L}(M)} U$ ,  $(N, T, L) \in U \times_{\mathcal{L}(M)} U$  and  $S \cap T = S \cap L = 0$ . Assuming

$$(R, N, \mathcal{B}_1) = \pi_Y(R, S, N) \quad \text{and} \quad (N, L, \mathcal{B}_2) = \pi_Y(N, T, L),$$

by (10) we get

$$\begin{aligned} F_{R,N,\mathcal{B}_1} * F_{N,L,\mathcal{B}_2} &= (\tilde{F}_{R,S} * \tilde{F}_{S,N}) * (\tilde{F}_{N,T} * \tilde{F}_{T,L}) = q^{d+1} \theta(S, N, T) \tilde{F}_{R,S} * \tilde{F}_{S,T} * \tilde{F}_{T,L} \\ &= q^{2d+2} \theta(S, N, T) \theta(S, T, L) \tilde{F}_{R,S} * \tilde{F}_{S,L} = q^{2d+2} \theta(S, N, T) \theta(S, T, L) F_{R,L,\mathcal{B}}, \end{aligned}$$

where  $(R, L, \mathcal{B}) = \pi_Y(R, S, L)$ . Now we turn back to the geometric setting.

**Step 4.** Consider the scheme  $\mathcal{W}$  classifying  $(R, S, N) \in U \times_{\mathcal{L}(M)} U$  and  $(N, T, L) \in U \times_{\mathcal{L}(M)} U$  such that  $S \cap T = S \cap L = 0$ . Let

$$\kappa : \mathcal{W} \rightarrow \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$$

be the map sending the above point to  $(R, N, L, \mathcal{B}_1, \mathcal{B}_2)$ , where  $(R, N, \mathcal{B}_1) = \pi_Y(R, S, N)$  and  $(N, L, \mathcal{B}_2) = \pi_Y(N, T, L)$ . The map  $\kappa$  is smooth and surjective. It suffices to show that

$$\kappa^*((\tau_1^*F) * (\tau_2^*F))$$

is a shifted perverse sheaf, the intermediate extension from  $\kappa^{-1}(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0$ .

Let  $\mu : \mathcal{W} \rightarrow U \times_{\mathcal{L}(M)} U$  be the map sending a point of  $\mathcal{W}$  to  $(R, S, L)$ . Applying (10) several times as in Step 3, we learn that there is a local system of rank one and order two, say  $\mathcal{I}$  on  $\mathcal{W}$  such that

$$\kappa^*((\tau_1^*F) * (\tau_2^*F)) \xrightarrow{\sim} \mathcal{I} \otimes \mu^* \pi_Y^* F$$

Since  $F$  is an irreducible perverse sheaf, our assertion follows.  $\square$

Thus, Theorem 1 is proved.

3.5 Now given  $k$ -points  $N^0, L^0 \in \tilde{\mathcal{L}}(M)$ , let  $F_{N^0, L^0} \in \mathbf{D}(H)$  be the  $*$ -restriction of  $F$  under  $(N^0, L^0) \times \text{id} : H \hookrightarrow \tilde{Y} \times H$ . Define the functor  $\mathcal{F}_{N^0, L^0} : \mathbf{D}\mathcal{H}_L \rightarrow \mathbf{D}\mathcal{H}_N$  by

$$\mathcal{F}_{N^0, L^0}(K) = F_{N^0, L^0} * K$$

To see that it preserves perversity we can pick  $S^0 \in \tilde{\mathcal{L}}(M)$  with  $N \cap S = L \cap S = 0$  and use  $\mathcal{F}_{N^0, L^0} = \mathcal{F}_{N^0, S^0} \circ \mathcal{F}_{S^0, L^0}$ . This reduces the question to the case  $N \cap L = 0$ , in the latter case  $\mathcal{F}_{N^0, L^0}$  is nothing but the Fourier transform.

By Theorem 1, for  $N^0, L^0, R^0 \in \tilde{\mathcal{L}}(M)$  the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathbf{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{R^0, L^0}} & \mathbf{D}\mathcal{H}_R \\ & \searrow \mathcal{F}_{N^0, L^0} & \downarrow \mathcal{F}_{N^0, R^0} \\ & & \mathbf{D}\mathcal{H}_N \end{array}$$

### 3.6 NONRAMIFIED WEIL CATEGORY

For a  $k$ -point  $L^0 \in \tilde{\mathcal{L}}(M)$  let  $i_{L^0} : \tilde{\mathcal{L}}(M) \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$  be the map sending  $N^0$  to  $(N^0, L^0, 0)$ . We get a functor  $\mathcal{F}_{L^0} : \mathbf{D}\mathcal{H}_L \rightarrow \mathbf{D}(\tilde{\mathcal{L}}(M))$  sending  $K$  to the complex

$$i_{L^0}^*(F * \text{pr}_3^* K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M) - 2d - 1}$$

For any  $k$ -points  $L^0, N^0 \in \tilde{\mathcal{L}}(M)$  the diagram commutes

$$\begin{array}{ccc} \mathbf{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathbf{D}(\tilde{\mathcal{L}}(M)) \\ & \searrow \mathcal{F}_{L^0, N^0} & \uparrow \mathcal{F}_{N^0} \\ & & \mathbf{D}\mathcal{H}_N \end{array} \tag{12}$$

One checks that  $\mathcal{F}_{L^0}$  is exact for the perverse t-structure.

**Definition 2.** *The non-ramified Weil category  $W(\tilde{\mathcal{L}}(M))$  is the essential image of  $\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow \mathbf{P}(\tilde{\mathcal{L}}(M))$ . This is a full subcategory in  $\mathbf{P}(\tilde{\mathcal{L}}(M))$  independent of  $L^0$ , because (12) is commutative.*

The group  $G$  acts naturally on  $\tilde{\mathcal{L}}(M)$ , hence also on  $\mathbf{P}(\tilde{\mathcal{L}}(M))$ . This action preserves the full subcategory  $W(\tilde{\mathcal{L}}(M))$ .

At the classical level, for  $L \in \mathcal{L}(M)$  the  $G$ -representation  $\mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{L,odd} \oplus \mathcal{H}_{L,even}$  is a direct sum of two irreducible ones consisting of odd and even functions respectively. The category  $W(\tilde{\mathcal{L}}(M))$  is a geometric analog of the space  $\mathcal{H}_{L,even}$ . The geometric analog of the whole Weil representation  $\mathcal{H}_L$  is as follows.

**Definition 3.** Let  $\text{act}_l : \bar{C} \times H \rightarrow \tilde{\mathcal{L}}(M) \times H$  be the map sending  $(L^0, h, \bar{l} \in \bar{L})$  to  $(L^0, \bar{l}h)$ . A perverse sheaf  $K \in \mathbf{P}(\tilde{\mathcal{L}}(M) \times H)$  is  $(\bar{C}, \chi_{\bar{C}})$ -equivariant if it is equipped with an isomorphism

$$\text{act}_l^* K \xrightarrow{\sim} \text{pr}_1^* K \otimes \text{pr}_1^* \chi_{\bar{C}}$$

satisfying the usual associativity property, and whose restriction to the unit section is the identity.

The complete Weil category  $W(M)$  is the category of pairs  $(K, \sigma)$ , where  $K \in \mathbf{P}(\tilde{\mathcal{L}}(M) \times H)$  is a  $(\bar{C}, \chi_{\bar{C}})$ -equivariant perverse sheaf, and

$$\sigma : F * \text{pr}_{23}^* K \xrightarrow{\sim} \text{pr}_{13}^* K$$

is an isomorphism for the projections  $\text{pr}_{13}, \text{pr}_{23} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times H$ . The map  $\sigma$  must be compatible with the associativity constraint and the unit section constraint of  $F$ .

The group  $G$  acts on  $\tilde{\mathcal{L}}(M) \times H$  sending  $(g \in G, L^0, h)$  to  $(gL^0, gh)$ . This action extends to an action of  $G$  on the category  $W(M)$ .

#### 4. COMPATIBILITY PROPERTY

4.1 In this section we establish the following additional property of the canonical intertwining operators. Let  $V \subset M$  be an isotropic subspace,  $V^\perp \subset M$  its orthogonal complement. Let  $\mathcal{L}(M)_V \subset \mathcal{L}(M)$  be the open subscheme of  $L \in \mathcal{L}(M)$  such that  $L \cap V = 0$ . Set  $M_0 = V^\perp/V$ . We have a map  $p_V : \mathcal{L}(M)_V \rightarrow \mathcal{L}(M_0)$  sending  $L$  to  $L_V := L \cap V^\perp$ .

Write  $Y = \mathcal{L}(M) \times \mathcal{L}(M)$  and  $Y_V = \mathcal{L}(M)_V \times \mathcal{L}(M)_V$ . The gerb  $\tilde{Y}$  is defined as in Section 3.2, write  $\tilde{Y}_V$  for its restriction to  $Y_V$ . Set  $Y_0 = \mathcal{L}(M_0) \times \mathcal{L}(M_0)$ , we have the corresponding gerb  $\tilde{Y}_0$  defined as in Section 3.2. We extend the map  $p_V \times p_V$  to a map

$$\pi_V : \tilde{Y}_V \rightarrow \tilde{Y}_0$$

sending  $(L_1, L_2, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det L_1 \otimes \det L_2)$  to

$$(L_{1,V}, L_{2,V}, \mathcal{B}_0, \mathcal{B}_0^2 \xrightarrow{\sim} \det L_{1,V} \otimes \det L_{2,V})$$

Here  $L_{i,V} = L_i \cap V^\perp$  and  $\mathcal{B}_0 = \mathcal{B} \otimes \det V$ . We used the exact sequences

$$0 \rightarrow L_{i,V} \rightarrow L_i \rightarrow M/V^\perp \rightarrow 0$$

yielding canonical ( $\mathbb{Z}/2\mathbb{Z}$ -graded) isomorphisms  $\det L_{i,V} \otimes \det V^* \xrightarrow{\sim} \det L_i$ .

Write  $H_0 = M_0 \oplus k$  for the Heisenberg group of  $M_0$ . For  $L \in \mathcal{L}(M)_V$  we have the categories  $\mathcal{H}_L$  and  $\mathcal{H}_{L_V}$  of certain perverse sheaves on  $H$  and  $H_0$  respectively. To such  $L$  we associate a transition functor  $T^L : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_L$  which will be fully faithful and exact for the perverse t-structures.

Write for brevity  $H^V = V^\perp \times \mathbb{A}^1$ . First, at the level of functions, given  $f \in \mathcal{H}_{L_V}$  consider it as a function on  $H^V$  via the composition  $H^V \xrightarrow{\alpha_V} H_0 \xrightarrow{f} \bar{\mathbb{Q}}_\ell$ , where  $\alpha_V$  sends  $(v, a)$  to  $(v \bmod V, a)$ . Then there is a unique  $f_1 \in \mathcal{H}_L$  such that  $f_1(m) = q^{\dim V} f(m)$  for all  $m \in H^V$ . We use the property  $V^\perp + L = M$ . We set

$$(T^L)(f) = f_1 \tag{13}$$

The image of  $T^L$  is

$$\{f_1 \in \mathcal{H}_L \mid f(h(v, 0)) = f(h), \quad h \in H, v \in V\}$$

Note that  $H^V \subset H$  is a subgroup, and  $V = \{(v, 0) \in H^V \mid v \in V\} \subset H^V$  is a normal subgroup lying in the center of  $H^V$ . The operator  $T^L : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_L$  commutes with the action of  $H^V$ . It is understood that on  $\mathcal{H}_{L_V}$  this group acts via its quotient  $H^V \xrightarrow{\alpha_V} H_0$ .

On the geometric level, consider the map  $s : L \times H^V \rightarrow H$  sending  $(l, (v, a))$  to the product in the Heisenberg group  $(l, 0)(v, a) \in H$ . Note that  $s$  is smooth and surjective, an affine fibration of rank  $\dim L_V$ . Given  $K \in \mathcal{H}_{L_V}$  there is a (defined up to a unique isomorphism) perverse sheaf  $T^L K \in \mathcal{H}_L$  equipped with

$$s^*(T^L K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim L_V} \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell \boxtimes \alpha_V^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim V + \dim L}$$

The *compatibility property* of the canonical intertwining operators is as follows.

**Proposition 2.** *Let  $(L, N, \mathcal{B}) \in \tilde{Y}_V$ , write  $(L_V, N_V, \mathcal{B}_0)$  for the image of  $(L, N, \mathcal{B})$  under  $\pi_V$ . Write  $\mathcal{F}_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$  and  $\mathcal{F}_{N_V^0, L_V^0} : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_{N_V}$  for the corresponding functors defined as in Section 3.5. Then the diagram of categories is canonically 2-commutative*

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^L} & \mathcal{H}_L \\ \downarrow \mathcal{F}_{N_V^0, L_V^0} & & \downarrow \mathcal{F}_{N^0, L^0} \\ \mathcal{H}_{N_V} & \xrightarrow{T^N} & \mathcal{H}_N \end{array}$$

One may also replace  $\mathcal{H}$  by  $D\mathcal{H}$  in the above diagram.

4.2 First, we realize the functors  $T^L$  by a universal kernel, namely, we define a perverse sheaf  $T$  on  $\mathcal{L}(M)_V \times H \times H_0$  as follows.

Remind the vector bundle  $\bar{C} \rightarrow \mathcal{L}(M)$ , its fibre over  $L$  is  $\bar{L} = L \times \mathbb{A}^1$ . Write  $\bar{C}_V$  for the restriction of  $\bar{C}$  to the open subscheme  $\mathcal{L}(M)_V$ . We have a closed immersion

$$i_0 : \bar{C}_V \times H^V \rightarrow \mathcal{L}(M)_V \times H \times H_0$$

sending  $(\bar{l} \in \bar{L}, u \in H^V)$  to  $(L, \bar{l}u, \alpha_V(u))$ , where the product  $\bar{l}u$  is taken in  $H$ . The perverse sheaf  $T$  is defined by

$$T = (i_0)_! \operatorname{pr}_1^* \chi_{\bar{C}} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \bar{C} + \dim V + \dim H_0},$$

here  $\operatorname{pr}_1 : \bar{C}_V \times H^V \rightarrow \bar{C}_V$  is the projection, and  $\chi_{\bar{C}}$  was defined in 3.1.

For  $L \in \mathcal{L}(M)_V$  let  $T_L$  be the  $*$ -restriction of  $T$  under  $(L, \operatorname{id}) : H \times H_0 \rightarrow \mathcal{L}(M)_V \times H \times H_0$ . Define  $T^L : D\mathcal{H}_{L_V} \rightarrow D\mathcal{H}_L$  by

$$T^L(K) \xrightarrow{\sim} \operatorname{pr}_{1!}(T_L \otimes \operatorname{pr}_2^* K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim V - d - \dim \mathcal{L}(M)} \quad (14)$$

for the diagram of projections  $H \xleftarrow{\operatorname{pr}_1} H \times H_0 \xrightarrow{\operatorname{pr}_2} H_0$ . It is exact for the perverse t-structures.

The sheaf  $T$  has the following properties. At the level of functions, the corresponding function  $T_L : H \times H_0 \rightarrow \bar{\mathbb{Q}}_\ell$  satisfies

$$T_L(\bar{l}h, \bar{l}_0 h_0) = \chi_L(\bar{l}) \chi_{L_V}(\bar{l}_0)^{-1} T_L(h, h_0), \quad \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V$$

The geometric analog is as follows. Let  ${}^0\bar{C} \rightarrow \mathcal{L}(M)_V$  be the vector bundle, whose fibre over  $L \in \mathcal{L}(M)_V$  is  $\bar{L} \times \bar{L}_V$ . Consider the diagram

$$\mathcal{L}(M)_V \times H \times H_0 \xleftarrow{\operatorname{pr}^V} {}^0\bar{C} \times H \times H_0 \xrightarrow{\operatorname{act}_{r'}^V} \mathcal{L}(M)_V \times H \times H_0,$$

where  $\operatorname{pr}^V$  is the projection, and  $\operatorname{act}_{r'}^V$  sends

$$(L \in \mathcal{L}(M)_V, \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V, h \in H, h_0 \in H_0)$$

to  $(L, \bar{l}h, \bar{l}_0 h_0)$ . Let  ${}^0p : {}^0\bar{C} \rightarrow \mathbb{A}^1$  be the map sending

$$(L \in \mathcal{L}(M)_V, \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V)$$

to  $p(\bar{l}) - p(\bar{l}_0)$ . Here  $p : \bar{L} \rightarrow \mathbb{A}^1$  and  $p : \bar{L}_V \rightarrow \mathbb{A}^1$  are the projections on the center. Set  ${}^0\chi = ({}^0p)^* \mathcal{L}_\psi$ . Then  $T$  is  $\operatorname{act}_{r'}^V$ -equivariant, that is, it admits an isomorphism

$$(\operatorname{act}_{r'}^V)^* T \xrightarrow{\sim} (\operatorname{pr}^V)^* T \otimes \operatorname{pr}_1^* ({}^0\chi),$$

satisfying the usual associativity property, and its restriction to the unit section is the identity.

4.3 We will prove a geometric version of the equality (up to an explicit power of  $q$ )

$$\int_{u \in H} F_{N^0, L^0}(hu^{-1}) T_L(u, h_0) du = \int_{v \in H_0} T_N(h, v) F_{N_V^0, L_V^0}(vh_0^{-1}) dv$$

for  $h \in H, h_0 \in H_0$ . Here  $(N^0, L^0) \in \tilde{Y}_V$  and

$$(N_V^0, L_V^0) = \pi_V(N^0, L^0)$$

Write  $\text{inv} : H \xrightarrow{\sim} H$  for the map sending  $h$  to  $h^{-1}$ , set  $\text{inv}^* F = (\text{id} \times \text{inv})^* F$  for  $\text{id} \times \text{inv} : \tilde{Y} \times H \rightarrow \tilde{Y} \times H$ . For  $i = 1, 2$  write  $p_i : \tilde{Y}_V \rightarrow \mathcal{L}(M)_V$  for the projection on the  $i$ -th factor. Let  $q_0$  denote the composition

$$\tilde{Y}_V \times H \times H_0 \xrightarrow{\text{pr}_{13}} \tilde{Y}_V \times H_0 \xrightarrow{\pi_V \times \text{id}} \tilde{Y}_0 \times H_0$$

Proposition 2 is an immediate consequence of the following.

**Lemma 5.** *There is a canonical isomorphism over  $\tilde{Y}_V \times H \times H_0$*

$$(\text{pr}_{12}^* F) *_H (p_2 \times \text{id})^* T \xrightarrow{\sim} (q_0^*(\text{inv}^* F)) *_H (p_1 \times \text{id})^* T$$

where  $\text{pr}_{12} : \tilde{Y}_V \times H \times H_0 \rightarrow \tilde{Y}_V \times H$  and  $p_1 \times \text{id}, p_2 \times \text{id} : \tilde{Y}_V \times H \times H_0 \rightarrow \mathcal{L}(M)_V \times H \times H_0$ .

Let  $i_V : H^V \hookrightarrow H$  be the natural closed immersion. It is elementary to check that Lemma 5 is equivalent to the following.

**Lemma 6.** *There is a canonical isomorphism of (shifted) perverse sheaves*

$$(\text{id} \times \alpha_V)_! i_V^* F \xrightarrow{\sim} (\pi_V \times \text{id})^* F \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\pi_V) + \dim V} \quad (15)$$

for the diagram

$$\begin{array}{ccc} \tilde{Y}_V \times H^V & \xrightarrow{i_V} & \tilde{Y}_V \times H \\ \downarrow \text{id} \times \alpha_V & & \\ \tilde{Y}_0 \times H_0 & \xleftarrow{\pi_V \times \text{id}} & \tilde{Y}_V \times H_0 \end{array}$$

*Proof* Write  $U(M_0)$  for the scheme  $U$  constructed out of the symplectic space  $M_0$ , it classifies pairs of lagrangian subspaces in  $M_0$  that do not intersect. We have a 2-commutative diagram

$$\begin{array}{ccccc} U(M_0) \times_{\mathcal{L}(M_0)} U(M_0) & \xleftarrow{\pi_W} & W_V & \xrightarrow{i_W} & U \times_{\mathcal{L}(M)} U \\ \downarrow \pi_{Y_0} & & \downarrow \pi_{Y,V} & \swarrow \pi_Y & \\ \tilde{Y}_0 & \xleftarrow{\pi_V} & \tilde{Y}_V & & \end{array}$$

where the square is cartesian thus defining  $W_V, \pi_W$ , and  $\pi_{Y,V}$ . The map  $i_W$  is a locally closed immersion. Write a point of  $W_V$  as a triple  $(N, R, L) \in \mathcal{L}(M)$  such that  $N, L \in \mathcal{L}(M)_V$ ,  $V \subset R \subset V^\perp$ , and  $N \cap R = R \cap L = 0$ . The map  $\pi_W$  sends  $(N, R, L)$  to  $(N_V, R_V, L_V)$  with  $R_V = R/V$ .

Let us establish the isomorphism (15) after restriction under  $\pi_{Y,V} \times \alpha_V : W_V \times H^V \rightarrow \tilde{Y}_V \times H_0$ . We first give the argument at the level of functions and then check that it holds through in the geometric setting.

Consider a point of  $W_V$  given by a triple  $(N, R, L) \in \mathcal{L}(M)$ , so  $N, L \in \mathcal{L}(M)_V$ ,  $V \subset R \subset V^\perp$ , and  $N \cap R = R \cap L = 0$ . We have  $V^\perp = R \oplus L_V$ . Let  $h \in H^V$ , write  $h = (r, a)(l_1, 0)$  for uniquely

defined  $r \in R, l_1 \in L_V, a \in k$ . Write  $(N^0, L^0) \in \tilde{Y}_V$  for the image of  $(N, R, L)$  under  $\pi_{Y,V}$ . Using (10), we get

$$\begin{aligned} \int_{v \in V} F_{N^0, L^0}(h(v, 0)) dv &= q^{\dim \mathcal{L}(M) - \frac{d+1}{2}} \int_{v \in V, u \in H} \tilde{F}_{N,R}(u) \tilde{F}_{R,L}(u^{-1}h(v, 0)) dv du = \\ &= q^{\dim \mathcal{L}(M) + \frac{d+1}{2}} \int_{v \in V, u \in H/\tilde{R}} \tilde{F}_{N,R}(u) \tilde{F}_{R,L}(u^{-1}(r, a)(v, 0)) dv du = \\ &= q^{\dim \mathcal{L}(M) + \frac{d+1}{2}} \int_{v \in V, l \in L} \tilde{F}_{N,R}(l, 0) \tilde{F}_{R,L}((-l, 0)(r, a)(v, 0)) dv dl \end{aligned}$$

Since  $(-l, 0)(r + v, a) = (r + v, a + \omega\langle r + v, l \rangle)(-l, 0)$ , the latter expression equals

$$q^{-\frac{d}{2}} \int_{v \in V, l \in L} \tilde{F}_{N,R}(l, 0) \psi(a + \omega\langle r + v, l \rangle) dv dl = q^{\dim V - \frac{d}{2}} \int_{l \in L_V} \tilde{F}_{N,R}(l, 0) \psi(a + \omega\langle r, l \rangle) dl$$

For  $l \in L_V$  we get  $\tilde{F}_{N,R}(l, 0) = q^{\dim \mathcal{L}(M_0) - \dim \mathcal{L}(M) - \dim V} \tilde{F}_{N_V, R_V}(l, 0)$ . Indeed, since  $V^\perp = R \oplus N_V$ , there are unique  $r_1 \in R, n_1 \in N_V$  such that  $l = n_1 + r_1$ . For  $\bar{r}_1 = r_1 \bmod V \in M_0$  we get

$$\begin{aligned} \tilde{F}_{N,R}(l, 0) &= q^{-\dim \mathcal{L}(M) - \frac{2d+1}{2}} \chi_{NR}(l, 0) = q^{-\dim \mathcal{L}(M) - \frac{2d+1}{2}} \psi\left(\frac{1}{2}\omega\langle r_1, n_1 \rangle\right) = \\ &= q^{-\dim \mathcal{L}(M) - \frac{2d+1}{2}} \chi_{N_V R_V}(\bar{r}_1 + n_1, 0) = q^{\dim \mathcal{L}(M_0) - \dim \mathcal{L}(M) - \dim V} \tilde{F}_{N_V, R_V}(l, 0) \end{aligned}$$

Further, we claim that

$$\tilde{F}_{R_V, L_V}((-l, 0)\alpha_V(h)) = q^{-\dim \mathcal{L}(M_0) - \frac{\dim H_0}{2}} \psi(a + \omega\langle r, l \rangle)$$

This follows from definition (5) of  $\tilde{F}_U$  and the formula  $(-l, 0)(r, a) = (r, a + \omega\langle r, l \rangle)(-l, 0)$ .

Combing the above we get

$$\begin{aligned} \int_{v \in V} F_{N^0, L^0}(h(v, 0)) dv &= q^c \int_{l \in L_V} \tilde{F}_{N_V, R_V}(l, 0) \tilde{F}_{R_V, L_V}((-l, 0)\alpha_V(h)) dl = \\ &= q^{c + \dim V - d - 1} \int_{u \in H_0} \tilde{F}_{N_V, R_V}(u) \tilde{F}_{R_V, L_V}(u^{-1}\alpha_V(h)) du \end{aligned}$$

with  $c = \frac{\dim H_0 - d}{2} + 2 \dim \mathcal{L}(M_0) - \dim \mathcal{L}(M)$ . By (10), the latter expression identifies with  $F_{N_V^0, L_V^0}(h)$  up to an explicit power of  $q$ .

The argument holds through in the geometric setting yielding the desired isomorphism  $\gamma$  over  $W_V \times H^V$ . For any point  $(N_V, L_V \mathcal{B}_0) \in \tilde{Y}_0$  such that  $N_V \neq L_V$  the fibre of  $\pi_{Y_0}$  over this point is geometrically connected. So, for  $\dim V < d$  the isomorphism  $\gamma$  descends to a uniquely defined isomorphism (15). The case  $\dim V = d$  is easier and is left to the reader.  $\square$

*Remark 5.* Let  $i_H : \text{Spec } k \hookrightarrow H$  denote the zero section. Arguing as in Lemma 6, for the map  $\text{id} \times i_H : \tilde{Y} \rightarrow \tilde{Y} \times H$  one gets a canonical isomorphism

$$(\text{id} \times i_H)^* F \xrightarrow{\sim} (S_{M,g} \otimes \mathcal{E}^d \oplus S_{M,s} \otimes \mathcal{E}^{d-1}) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim H},$$

it will not be used in this paper.

4.4 The functors  $T^L$  satisfy the following transitivity property. Assume that  $V_1 \subset V$  is another isotropic subspace in  $M$ . Let  $M_1 = V_1^\perp/V_1$  and  $H_1 = M_1 \times \mathbb{A}^1$  be the corresponding Heisenberg group. Then for  $L \in \mathcal{L}(M)_V$  we also have  $L_{V_1} := L \cap V_1^\perp$  and the category  $\mathcal{H}_{L_{V_1}}$  of certain perverse sheaves on  $H_1$ . Then the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^{L_{V_1}}} & \mathcal{H}_{L_{V_1}} \\ & \searrow T^L & \downarrow T^L \\ & & \mathcal{H}_L \end{array}$$

4.5 We will need also one more compatibility property of the canonical interwining operators. Let  $V \subset V^\perp \subset M$  be as in 4.1. Write  $i_{0,V} : \mathcal{L}(M_0) \rightarrow \mathcal{L}(M)$  for the closed immersion sending  $L_0$  to the preimage of  $L_0$  under  $V^\perp \rightarrow V^\perp/V$ .

For  $L \in \mathcal{L}(M)$  with  $V \subset L$  set  $L_V = L/V \in \mathcal{L}(M_0)$ . Let  $(\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\sim$  denote the restriction of the gerb  $\tilde{Y}$  under

$$\mathcal{L}(M_0) \times \mathcal{L}(M)_V \xrightarrow{i_{0,V} \times \text{id}} \mathcal{L}(M) \times \mathcal{L}(M)_V \subset Y$$

Define  $\pi_{0,V} : (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\sim \rightarrow \tilde{Y}_0$  as the map sending  $(L, N, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det L \otimes \det N)$  to

$$(L_V, N_V, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det L_V \otimes \det N_V)$$

Here  $L \in \mathcal{L}(M)$  with  $V \subset L$ . We have used the canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  $\det L \otimes \det N \xrightarrow{\sim} \det L_V \otimes \det N_V$ .

Remind the closed immersion  $i_V : H^V \hookrightarrow H$ . For  $L \in \mathcal{L}(M)$  with  $V \subset L$  define the transition functor  $T^L : \mathcal{H}_{L_V} \rightarrow \mathcal{H}_L$  by

$$T^L(K) = i_{V!} \alpha_V^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim V}$$

The proof of the following is similar to that of Proposition 2 and is left to the reader.

**Proposition 3.** *Let  $(L, N, \mathcal{B}) \in (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\sim$ , let  $(L_V, N_V, \mathcal{B})$  denote its image under  $\pi_{0,V}$ . Write  $\mathcal{F}_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$  and  $\mathcal{F}_{N_V^0, L_V^0}$  for the corresponding functors defined as in Section 3.5. Then the diagram of categories is canonically 2-commutative*

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^L} & \mathcal{H}_L \\ \downarrow \mathcal{F}_{N_V^0, L_V^0} & & \downarrow \mathcal{F}_{N^0, L^0} \\ \mathcal{H}_{N_V} & \xrightarrow{T^N} & \mathcal{H}_N \end{array}$$

One may also replace  $\mathcal{H}$  by  $D\mathcal{H}$  in the above diagram.  $\square$

## 5. DISCRETE LAGRANGIAN LATTICES AND THE METAPLECTIC GROUP

5.1 Set  $\mathcal{O} = k[[t]] \subset F = k((t))$ . Denote by  $\Omega$  the completed module of relative differentials of  $\mathcal{O}$  over  $k$ . Let  $M$  be a free  $\mathcal{O}$ -module of rank  $2d$  with symplectic form  $\wedge^2 M \rightarrow \Omega$ . Write  $G$  for the group scheme over  $\text{Spec } \mathcal{O}$  of automorphisms of  $M$  preserving the symplectic form. Consider the Tate space  $M(F)$  (cf. [1], 4.2.13 for the definition), it is equipped with the symplectic form  $(m_1, m_2) \mapsto \text{Res } \omega \langle m_1, m_2 \rangle$ .

For a  $k$ -subspace  $L \subset M(F)$  write

$$L^\perp = \{m \in M(F) \mid \text{Res } \omega \langle m, l \rangle = 0 \text{ for all } l \in L\}$$

For two  $k$ -subspaces  $L_1, L_2 \subset M$  we get  $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$ . For a finite-dimensional symplectic  $k$ -vector space  $U$  write  $\mathcal{L}(U)$  for the variety of lagrangian subspaces in  $U$ .

As in *loc.cit*, we say that an  $\mathcal{O}$ -submodule  $R \subset M(F)$  is a *c-lattice* if  $M(-N) \subset R \subset M(N)$  for some integer  $N$ . A *lagrangian d-lattice* in  $M(F)$  is a  $k$ -vector subspace  $L \subset M(F)$  such that  $L^\perp = L$  and there exists a c-lattice  $R$  with  $R \cap L = 0$ . Note that the condition  $R \cap L = 0$  implies  $R^\perp + L = M(F)$ . Let  $\mathcal{L}_d(M(F))$  denote the set of lagrangian d-lattices in  $M(F)$ .

For a given c-lattice  $R \subset M(F)$  write

$$\mathcal{L}_d(M(F))_R = \{L \in \mathcal{L}_d(M(F)) \mid L \cap R = 0\}$$

If  $R$  is a c-lattice in  $M(F)$  with  $R \subset R^\perp$  then  $\mathcal{L}_d(M(F))_R$  is a naturally a  $k$ -scheme (not of finite type over  $k$ ). Indeed, for each c-lattice  $R_1 \subset R$  we have the variety

$$\mathcal{L}(R_1^\perp/R_1)_R := \{L_1 \in \mathcal{L}(R_1^\perp/R_1) \mid L_1 \cap R/R_1 = 0\}$$

For  $R_2 \subset R_1 \subset R$  we get a map  $p_{R_2, R_1} : \mathcal{L}(R_2^\perp/R_2)_R \rightarrow \mathcal{L}(R_1^\perp/R_1)_R$  sending  $L_2$  to

$$L_1 = (L_2 \cap (R_1^\perp/R_2)) + R_1$$

The map  $p_{R_2, R_1}$  is a composition of two affine fibrations of constant rank. Then  $\mathcal{L}_d(M(F))_R$  is the inverse limit of  $\mathcal{L}(R_1^\perp/R_1)_R$  over the partially ordered set of c-lattices  $R_1 \subset R$ .

If  $R' \subset R$  is another c-lattice then  $\mathcal{L}_d(M(F))_R \subset \mathcal{L}_d(M(F))_{R'}$  is an open immersion (as it is an open immersion on each term of the projective system). So,  $\mathcal{L}_d(M(F))$  is a  $k$ -scheme that can be seen as the inductive limit of  $\mathcal{L}_d(M(F))_R$ .

Let us define the categories  $\mathbf{P}(\mathcal{L}_d(M(F)))$  and  $\mathbf{P}_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$  of perverse sheaves and  $G(\mathcal{O})$ -equivariant perverse sheaves on  $\mathcal{L}_d(M(F))$ .

For  $r \geq 0$  set

$${}_r\mathcal{L}_d(M(F)) = \mathcal{L}_d(M(F))_{M(-r)},$$

the group  $G(\mathcal{O})$  acts on  ${}_r\mathcal{L}_d(M(F))$  naturally. First, define the category  $\mathbf{D}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$  as follows.

For  $N + r \geq 0$  set  ${}_{N,r}M = t^{-N}M/t^rM$ . For  $N \geq r \geq 0$  the action of  $G(\mathcal{O})$  on  ${}_r\mathcal{L}_{(N,N)M} := \mathcal{L}_{(N,N)M}_{M(-r)}$  factors through  $G(\mathcal{O}/t^{2N})$ . For  $r_1 \geq 2N$  the kernel

$$\text{Ker}(G(\mathcal{O}/t^{r_1})) \rightarrow G(\mathcal{O}/t^{2N})$$

is unipotent, so that we have an equivalence (exact for the perverse t-structures)

$$D_{G(\mathcal{O}/t^{2N})}({}_r\mathcal{L}_{(N,N)}M) \xrightarrow{\sim} D_{G(\mathcal{O}/t^{r_1})}({}_r\mathcal{L}_{(N,N)}M)$$

Define  $D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N,N)}M)$  as  $D_{G(\mathcal{O}/t^{r_1})}({}_r\mathcal{L}_{(N,N)}M)$  for any  $r_1 \geq 2N$ . It is equipped with the perverse t-structure.

For  $N_1 \geq N \geq r \geq 0$  the fibres of the above projection

$$p : {}_r\mathcal{L}_{(N_1, N_1)}M \rightarrow {}_r\mathcal{L}_{(N, N)}M$$

are isomorphic to affine spaces of fixed dimension, and  $p$  is smooth and surjective. Hence, this map yields transition functors (exact for the perverse t-structures and fully faithful embeddings)

$$D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N, N)}M) \rightarrow D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N_1, N_1)}M)$$

and

$$D({}_r\mathcal{L}_{(N, N)}M) \rightarrow D({}_r\mathcal{L}_{(N_1, N_1)}M)$$

We define  $D_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$  as the inductive 2-limit of  $D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N, N)}M)$  as  $N$  goes to plus infinity. The category  $D({}_r\mathcal{L}_d(M(F)))$  is defined similarly. Both they are equipped with perverse t-structures.

If  $N_1 \geq N \geq r_1 \geq r \geq 0$  we have a diagram

$$\begin{array}{ccc} {}_r\mathcal{L}_{(N_1, N_1)}M & \xrightarrow{p} & {}_r\mathcal{L}_{(N, N)}M \\ \downarrow j & & \downarrow j \\ {}_{r_1}\mathcal{L}_{(N_1, N_1)}M & \xrightarrow{p} & {}_{r_1}\mathcal{L}_{(N, N)}M, \end{array}$$

where  $j$  are natural open immersions. The restriction functors  $j^* : D_{G(\mathcal{O})}({}_{r_1}\mathcal{L}_{(N, N)}M) \rightarrow D_{G(\mathcal{O})}({}_r\mathcal{L}_{(N, N)}M)$  yield (in the limit as  $N$  goes to plus infinity) the functors

$$j_{r_1, r}^* : D_{G(\mathcal{O})}({}_{r_1}\mathcal{L}_d(M(F))) \rightarrow D_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$$

of restriction with respect to the open immersion  $j_{r_1, r} : {}_r\mathcal{L}_d(M(F)) \hookrightarrow {}_{r_1}\mathcal{L}_d(M(F))$ . Define  $D_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$  as the projective 2-limit of

$$D_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$$

as  $r$  goes to plus infinity. Similarly,  $P_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$  is defined as the projective 2-limit of  $P_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$ . Along the same lines, one defines the categories  $P(\mathcal{L}_d(M(F)))$  and  $D(\mathcal{L}_d(M(F)))$ .

**5.2 RELATIVE DETERMINANT** For a pair of c-lattices  $M_1, M_2$  in  $M(F)$  define the relative determinant  $\det(M_1 : M_2)$  as the following  $\mathbb{Z}/2\mathbb{Z}$ -graded 1-dimensional  $k$ -vector space. If  $R$  is a c-lattice in  $M(F)$  such that  $R \subset M_1 \cap M_2$  then

$$\det(M_1 : M_2) \xrightarrow{\sim} \det(M_1/R) \otimes \det(M_2/R)^{-1},$$

it is defined up to a unique isomorphism.

Write  $\text{Gr}_G$  for the affine grassmanian  $G(F)/G(\mathcal{O})$  of  $G$  (cf. [1], Section 4.5). For  $R \in \text{Gr}_G, L \in \mathcal{L}_d(M(F))$  define the relative determinant  $\det(R : L)$  as the following ( $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) 1-dimensional vector space. Pick a c-lattice  $R_1 \subset R$  such that  $R_1 \cap L = 0$ . Then in  $R_1^\perp/R_1$  one gets two lagrangian subspaces  $R/R_1$  and  $L_{R_1} := L \cap R_1^\perp$ . Set

$$\det(R : L) = \det(R/R_1) \otimes \det(L_{R_1})$$

If  $R_2 \subset R_1$  is another c-lattice then the exact sequence

$$0 \rightarrow L_{R_1} \rightarrow L \cap R_2^\perp \rightarrow R_2^\perp/R_1^\perp \rightarrow 0$$

yields a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det(R/R_2) \otimes \det(L_{R_2}) \xrightarrow{\sim} \det(R_1/R_2) \otimes \det(R/R_1) \otimes \det(L_{R_1}) \otimes \det(R_2^\perp/R_1^\perp) \xrightarrow{\sim} \det(R/R_1) \otimes \det(L_{R_1})$$

So,  $\det(R : L)$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded line defined up to a unique isomorphism. Another way to say is as follows. Consider the complex  $R \oplus L \xrightarrow{s} M(F)$  placed in cohomological degrees 0 and 1, where  $s(r, l) = r + l$ . It has finite-dimensional cohomologies and

$$\det(R : L) = \det(R \oplus L \xrightarrow{s} M(F))$$

For  $g \in G(F)$  we have canonically

$$\det(gR : gL) \xrightarrow{\sim} \det(R : L)$$

For  $R_1, R_2 \in \text{Gr}_G, L \in \mathcal{L}_d(M(F))$  we have canonically

$$\det(R_1 : L) \xrightarrow{\sim} \det(R_1 : R_2) \otimes \det(R_2 : L)$$

5.3 Write  $\mathcal{A}_d$  for the line bundle on  $\mathcal{L}_d(M(F))$  with fibre  $\det(M : L)$  at  $L \in \mathcal{L}_d(M(F))$ . Clearly,  $\mathcal{A}_d$  is  $G(\mathcal{O})$ -equivariant, so we may see  $\mathcal{A}_d$  as the line bundle on the stack quotient  $\mathcal{L}_d(M(F))/G(\mathcal{O})$ . Let  $\tilde{\mathcal{L}}_d(M(F))$  denote the  $\mu_2$ -gerb of square roots of  $\mathcal{A}_d$ .

The categories of the corresponding perverse sheaves  $\text{P}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$  and  $\text{P}(\tilde{\mathcal{L}}_d(M(F)))$  are defined as above. Namely, first for  $r \geq 0$  define

$$\text{D}_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$$

as follows. For  $N \geq r$  take  $r_1 \geq 2N$  and consider the stack quotient  ${}_r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1})$ . We have the line bundle, say  $\mathcal{A}_N$  on this stack whose fibre at  $L$  is  $\det(M/M(-N)) \otimes \det L$ . Here  $L \subset {}_N, {}_N M$  is a Lagrangian subspace such that  $L \cap (M(-r)/M(-N)) = 0$ . Write

$$({}_r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\vee$$

for the gerb of square roots of this line bundle. Let  $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M})$  denote the category

$$D((r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim)$$

for any  $r_1 \geq 2N$  (we have canonical equivalences exact for the perverse t-structures between such categories for various  $r_1$ ).

Assume  $N_1 \geq N \geq r$  and  $r_1 \geq 2N_1$ . For the projection

$$p : r\mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}) \rightarrow r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1})$$

we have a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  $p^* \mathcal{A}_N \xrightarrow{\sim} \mathcal{A}_{N_1}$ . This yields a transition map

$$(r\mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}))^\sim \rightarrow (r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim$$

The corresponding inverse image yields a transition functor

$$D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M}) \rightarrow D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N_1,N_1)M}) \quad (16)$$

exact for the perverse t-structures (and a fully faithful embedding). We define  $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$  as the inductive 2-limit of  $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M})$  as  $N$  goes to plus infinity.

For  $N \geq r' \geq r$  and  $r_1 \geq 2N$  we have an open immersion

$$\tilde{j} : (r\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim \subset (r'\mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim$$

hence the \*-restriction functors

$$\tilde{j}^* : D_{G(\mathcal{O})}(r'\tilde{\mathcal{L}}_{(N,N)M}) \rightarrow D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_{(N,N)M})$$

compatible with the transition functors (16). Passing to the limit as  $N$  goes to plus infinity, we get the functors

$$\tilde{j}_{r',r}^* : D_{G(\mathcal{O})}(r'\tilde{\mathcal{L}}_d(M(F))) \rightarrow D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$$

Define  $D_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$  as the projective 2-limit of  $D_{G(\mathcal{O})}(r\tilde{\mathcal{L}}_d(M(F)))$  as  $r$  goes to plus infinity, and similarly for  $P_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$ .

Along the same lines one defines the categories  $P(\tilde{\mathcal{L}}_d(M(F)))$  and  $D(\tilde{\mathcal{L}}_d(M(F)))$ .

**5.4 METAPLECTIC GROUP** Let  $\mathcal{A}_G$  be the line bundle on the ind-scheme  $G(F)$  whose fibre at  $g$  is  $\det(M : gM)$ . Write  $\tilde{G}(F) \rightarrow G(F)$  for the gerb of square roots of  $\mathcal{A}_G$ . The stack  $\tilde{G}(F)$  has a structure of a group stack. The product map  $m : \tilde{G}(F) \times \tilde{G}(F) \rightarrow \tilde{G}(F)$  sends

$$(g_1, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \xrightarrow{\sim} \det(M : g_1M)), (g_2, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \xrightarrow{\sim} \det(M : g_2M))$$

to the collection  $(g_1g_2, \mathcal{B}, \sigma : \mathcal{B}^2 \xrightarrow{\sim} \det(M : g_1g_2M))$ , where  $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$  and  $\sigma$  is the composition

$$\begin{aligned} (\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \xrightarrow{\sigma_1 \otimes \sigma_2} \det(M : g_1M) \otimes \det(M : g_2M) &\xrightarrow{\text{id} \otimes g_1} \det(M : g_1M) \otimes \det(g_1M : g_1g_2M) \\ &\xrightarrow{\sim} \det(M : g_1g_2M) \end{aligned}$$

Informally speaking, one may think of the exact sequence of group stacks

$$1 \rightarrow B(\mu_2) \rightarrow \tilde{G}(F) \rightarrow G(F) \rightarrow 1$$

We also have a canonical section  $G(\mathcal{O}) \rightarrow \tilde{G}(F)$  sending  $g$  to

$$(g, \mathcal{B} = k, \text{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(M : M))$$

The group stack  $\tilde{G}(F)$  acts naturally on  $\tilde{\mathcal{L}}_d(M(F))$ , the action map  $\tilde{G}(F) \times \tilde{\mathcal{L}}_d(M(F)) \rightarrow \tilde{\mathcal{L}}_d(M(F))$  sends

$$(g, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \xrightarrow{\sim} \det(M : gM)), (L, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \xrightarrow{\sim} \det(M : L))$$

to the collection  $(gL, \mathcal{B})$ , where  $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$  is equipped with the isomorphism

$$(\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \xrightarrow{\sigma_1 \otimes \sigma_2} \det(M : gM) \otimes \det(M : L) \xrightarrow{\text{id} \otimes g} \det(M : gM) \otimes \det(gM : gL) \xrightarrow{\sim} \det(M : gL)$$

5.5 For  $g \in G(F)$  and a c-lattice  $R \subset R^\perp$  in  $M(F)$  we have an isomorphism of symplectic spaces  $g : R^\perp/R \xrightarrow{\sim} (gR)^\perp/gR$ . For each c-lattice  $R_1 \subset R$  we have a diagram

$$\begin{array}{ccc} \mathcal{L}(R_1^\perp/R_1)_R & \xrightarrow{g} & \mathcal{L}(gR_1^\perp/gR_1)_{gR} \\ \downarrow p & & \downarrow p \\ \mathcal{L}(R^\perp/R) & \xrightarrow{g} & \mathcal{L}(gR^\perp/gR) \end{array}$$

Let  $\mathcal{A}_{R_1}$  be the ( $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) line bundle on  $\mathcal{L}(R_1^\perp/R_1)_R$  whose fibre at  $L$  is  $\det L \otimes \det(M : R_1)$ . Assume that  $\tilde{g} = (g, \mathcal{B}, \mathcal{B}^2 \xrightarrow{\sim} \det(M : gM))$  is a  $k$ -point of  $\tilde{G}(F)$  over  $g$ . It yields a diagram

$$\begin{array}{ccc} \tilde{\mathcal{L}}(R_1^\perp/R_1)_R & \xrightarrow{\tilde{g}} & \tilde{\mathcal{L}}(gR_1^\perp/gR_1)_{gR} \\ \downarrow p & & \downarrow p \\ \tilde{\mathcal{L}}(R^\perp/R) & \xrightarrow{\tilde{g}} & \tilde{\mathcal{L}}(gR^\perp/gR) \end{array}$$

Here the top horizontal arrow sends  $(L, \mathcal{B}_1, \mathcal{B}_1^2 \xrightarrow{\sim} \det L \otimes \det(M : R_1))$  to

$$(gL, \mathcal{B}_2, \sigma : \mathcal{B}_2^2 \xrightarrow{\sim} \det(gL) \otimes \det(M : gR_1)),$$

where  $\mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B}$  and  $\sigma$  is the composition

$$\begin{aligned} (\mathcal{B}_1 \otimes \mathcal{B})^2 &\xrightarrow{\sim} \det L \otimes \det(M : R_1) \otimes \det(M : gM) \xrightarrow{g \otimes \text{id}} \\ &\det(gL) \otimes \det(gM : gR_1) \otimes \det(M : gM) \xrightarrow{\sim} \det(gL) \otimes \det(M : gR_1) \end{aligned}$$

In the limit by  $R_1$  the corresponding functors  $\tilde{g}^* : \mathbb{P}(\tilde{\mathcal{L}}(gR_1^\perp/gR_1)_{gR}) \xrightarrow{\sim} \mathbb{P}(\tilde{\mathcal{L}}(R_1^\perp/R_1)_R)$  yield an equivalence

$$\tilde{g}^* : \mathbb{P}(\tilde{\mathcal{L}}_d(M(F))_{gR}) \xrightarrow{\sim} \mathbb{P}(\tilde{\mathcal{L}}_d(M(F))_R)$$

Taking one more limit by the partially ordered set of c-lattices  $R$ , one gets an equivalence

$$\tilde{g}^* : \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))) \xrightarrow{\sim} \mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$$

In this sense  $\tilde{G}(F)$  acts on  $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$ .

## 6. CANONICAL INTERWINING OPERATORS: LOCAL FIELD CASE

6.1 Keep notations of Section 5. Write  $H = M \oplus \Omega$  for the Heisenberg group defined as in Section 2.1, this is a group scheme over  $\mathrm{Spec} \mathcal{O}$ .

For  $L \in \mathcal{L}_d(M(F))$  we have the subgroup  $\bar{L} = L \oplus \Omega(F) \subset H(F)$  and the character  $\chi_L : \bar{L} \rightarrow \bar{\mathbb{Q}}_\ell^*$  given by  $\chi_L(l, a) = \chi(a)$ . Here  $\chi : \Omega(F) \rightarrow \bar{\mathbb{Q}}_\ell^*$  sends  $a$  to  $\psi(\mathrm{Res} a)$ . In the classical setting we let  $\mathcal{H}_L$  denote the space of functions  $f : H(F) \rightarrow \bar{\mathbb{Q}}_\ell$  satisfying

C1)  $f(\bar{l}h) = \chi_L(\bar{l})f(h)$ , for  $h \in H, \bar{l} \in \bar{L}$ ;

C2) there exists a c-lattice  $R \subset M(F)$  such that  $f(h(r, 0)) = f(h)$  for  $r \in R, h \in H$ .

Note that such  $f$  has automatically compact support modulo  $\bar{L}$ . The group  $H(F)$  acts on  $\mathcal{H}_L$  by right translations, this is a model of the Weil representation. Let us introduce a geometric analog of  $\mathcal{H}_L$ .

Given a c-lattice  $R \subset M(F)$  such that  $R \subset R^\perp$  write  $H_R = (R^\perp/R) \oplus k$  for the Heisenberg group corresponding to the symplectic space  $R^\perp/R$ . If  $L \in \mathcal{L}_d(M(F))_R$  then  $L_R := L \cap R^\perp \subset R^\perp/R$  is lagrangian. Set  $\bar{L}_R = L_R \oplus k \subset H_R$ . Let  $\chi_{L,R} : \bar{L}_R \rightarrow \bar{\mathbb{Q}}_\ell^*$  be the character sending  $(l, a)$  to  $\psi(a)$ . Set

$$\mathcal{H}_{L_R} = \{f : H_R \rightarrow \bar{\mathbb{Q}}_\ell \mid f(\bar{l}h) = \chi_{L,R}(\bar{l})f(h), h \in H_R, \bar{l} \in \bar{L}_R\}$$

**Lemma 7.** *There is a canonical embedding  $T_R^L : \mathcal{H}_{L_R} \hookrightarrow \mathcal{H}_L$  whose image is the subspace of those  $f \in \mathcal{H}_L$  which satisfy*

$$f(h(r, 0)) = f(h) \text{ for } r \in R, h \in H \tag{17}$$

*Proof* Set

$$' \mathcal{H}_{L_R} = \{\phi : R^\perp/R \rightarrow \bar{\mathbb{Q}}_\ell \mid \phi(r+l) = \chi(\frac{1}{2}\omega(r, l))\phi(r), r \in R^\perp/R, l \in L_R\}$$

We have an isomorphism  $\mathcal{H}_{L_R} \xrightarrow{\sim} ' \mathcal{H}_{L_R}$  sending  $f$  to  $\phi$  given by  $\phi(r) = f(r, 0)$ . Given  $f \in \mathcal{H}_L$  satisfying (17), we associate to  $f$  a function  $\phi \in ' \mathcal{H}_{L_R}$  given by

$$\phi(r) = q^{\frac{1}{2} \dim R^\perp/R} f(r, 0)$$

for  $r \in R^\perp$ . This defines the map  $T_R^L$ .  $\square$

Assume that  $S \subset R \subset M(F)$  are c-lattices and  $R \cap L = 0$ . Remind the operator  $\mathcal{H}_{L_R} \xrightarrow{T^{L_S}}$   $\mathcal{H}_{L_S}$  given by (13), it corresponds to the isotropic subspace  $R/S \subset S^\perp/S$ . The composition  $\mathcal{H}_{L_R} \xrightarrow{T^{L_S}} \mathcal{H}_{L_S} \xrightarrow{T_S^L} \mathcal{H}_L$  equals  $T_R^L$ .

The geometric analog of  $\mathcal{H}_L$  is as follows. For a c-lattice  $R$  such that  $R \cap L = 0$  and  $R \subset R^\perp$  we have the category  $\mathcal{H}_{L_R}$  of perverse sheaves on  $H_R$  which are  $(\bar{L}_R, \chi_{L,R})$ -equivariant, and the corresponding category  $D\mathcal{H}_{L_R}$ . For  $S \subset R$  as above we have an (exact for the perverse structure and fully faithful) transition functor (14), which we now denote by

$$T_{S,R}^L : D\mathcal{H}_{L_R} \rightarrow D\mathcal{H}_{L_S}$$

Define  $\mathcal{H}_L$  (resp.,  $D\mathcal{H}_L$ ) as the inductive 2-limit of  $\mathcal{H}_{L_R}$  (resp., of  $D\mathcal{H}_{L_R}$ ) over the partially ordered set of c-lattices  $R$  such that  $R \cap L = 0$  and  $R \subset R^\perp$ . So,  $\mathcal{H}_L$  is abelian and  $D\mathcal{H}_L$  is a triangulated category.

6.2 Let  $R \subset R^\perp$  be a c-lattice in  $M(F)$ . We have a projection

$$\mathcal{L}_d(M(F))_R \rightarrow \mathcal{L}(R^\perp/R)$$

sending  $L$  to  $L_R$ . Let  $\mathcal{A}_R$  be the  $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on  $\mathcal{L}(R^\perp/R)$  whose fibre at  $L_1$  is  $\det L_1 \otimes \det(M : R)$ . Write  $\tilde{\mathcal{L}}(R^\perp/R)$  for the gerb of square roots of  $\mathcal{A}_R$ . The restriction of  $\mathcal{A}_R$  to  $\mathcal{L}_d(M(F))_R$  identifies canonically with  $\mathcal{A}_d$ . The above projection lifts naturally to a morphism of gerbs

$$\tilde{\mathcal{L}}_d(M(F))_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R) \quad (18)$$

Given  $k$ -points  $N^0, L^0 \in \tilde{\mathcal{L}}_d(M(F))$  we are going to associate to them in a canonical way a functor

$$\mathcal{F}_{N^0, L^0} : D\mathcal{H}_L \rightarrow D\mathcal{H}_N \quad (19)$$

sending  $\mathcal{H}_L$  to  $\mathcal{H}_N$ . To do so, consider a c-lattice  $R \subset R^\perp$  in  $M(F)$  such that  $L, N \in \mathcal{L}_d(M(F))_R$ . Write  $N_R^0, L_R^0 \in \tilde{\mathcal{L}}(R^\perp/R)$  for the images of  $N^0$  and  $L^0$  under (18). By definition, the enhanced structure on  $L_R$  and  $N_R$  is given by one-dimensional vector spaces  $\mathcal{B}_L, \mathcal{B}_N$  equipped with

$$\mathcal{B}_L^2 \xrightarrow{\sim} \det L_R \otimes \det(M : R), \quad \mathcal{B}_N^2 \xrightarrow{\sim} \det N_R \otimes \det(M : R),$$

hence an isomorphism  $\mathcal{B}^2 \xrightarrow{\sim} \det L_R \otimes \det N_R$  for  $\mathcal{B} := \mathcal{B}_L \otimes \mathcal{B}_N \otimes \det(M : R)^{-1}$ . We denote by

$$\mathcal{F}_{N_R^0, L_R^0} : D\mathcal{H}_{L_R} \rightarrow D\mathcal{H}_{N_R}$$

the canonical intertwining functor defined in Section 3.5 corresponding to  $(N_R, L_R, \mathcal{B}) \in \tilde{Y}$ , here  $Y = \mathcal{L}(R^\perp/R) \times \mathcal{L}(R^\perp/R)$ . The following is an immediate consequence of Proposition 2.

**Proposition 4.** *Let  $S \subset R \subset R^\perp \subset S^\perp$  be c-lattices such that  $L^0, N^0 \in \tilde{\mathcal{L}}_d(M(F))_R$ . Then the following diagram of categories is canonically 2-commutative*

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{T_{S,R}^L} & D\mathcal{H}_{L_S} \\ \downarrow \mathcal{F}_{N_R^0, L_R^0} & & \downarrow \mathcal{F}_{N_S^0, L_S^0} \\ D\mathcal{H}_{N_R} & \xrightarrow{T_{S,R}^N} & D\mathcal{H}_{N_S} \end{array}$$

Define (19) as the limit of functors  $\mathcal{F}_{N_R^0, L_R^0}$  over the partially ordered set of c-lattices  $R \subset R^\perp$  such that  $L, N \in \mathcal{L}_d(M(F))_R$ . As in Section 3.5, one shows that for  $L^0, N^0, R^0 \in \tilde{\mathcal{L}}_d(M(F))$  the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathrm{DH}_L & \xrightarrow{\mathcal{F}_{R^0, L^0}} & \mathrm{DH}_R \\ & \searrow \mathcal{F}_{N^0, L^0} & \downarrow \mathcal{F}_{N^0, R^0} \\ & & \mathrm{DH}_N \end{array}$$

Our main result in the local field case is as follows.

**Theorem 2.** *For each  $k$ -point  $L^0 \in \tilde{\mathcal{L}}_d(M(F))$  there is a canonical functor*

$$\mathcal{F}_{L^0} : \mathrm{DH}_L \rightarrow \mathrm{D}(\tilde{\mathcal{L}}_d(M(F))) \quad (20)$$

*sending  $\mathcal{H}_L$  to  $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$ . For a pair of  $k$ -points  $(L^0, N^0)$  in  $\tilde{\mathcal{L}}_d(M(F))$  the diagram*

$$\begin{array}{ccc} \mathrm{DH}_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathrm{D}(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow \mathcal{F}_{N^0, L^0} & \nearrow \mathcal{F}_{N^0} & \\ \mathrm{DH}_N & & \end{array} \quad (21)$$

*is canonically 2-commutative. Let  $W(\tilde{\mathcal{L}}_d(M(F)))$  be the essential image of*

$$\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))),$$

*this is a full subcategory independent of  $L^0$ . Besides,  $W(\tilde{\mathcal{L}}_d(M(F)))$  is preserved under the natural action of  $\tilde{G}(F)$  on  $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$ .*

We will refer to  $W(\tilde{\mathcal{L}}_d(M(F)))$  as *the non-ramified Weil category on  $\tilde{\mathcal{L}}_d(M(F))$* . Remind that in the classical setting

$$\mathcal{H}_L = \mathcal{H}_{L, \text{odd}} \oplus \mathcal{H}_{L, \text{even}}$$

is a direct sum of two irreducible representations of the metaplectic group (consisting of odd and even functions respectively). The representation  $\mathcal{H}_{L, \text{odd}}$  is ramified, whence  $\mathcal{H}_{L, \text{even}}$  is not. The category  $W(\tilde{\mathcal{L}}_d(M(F)))$  together with the action of  $\tilde{G}(F)$  is a geometric counterpart of the representation  $\mathcal{H}_{L, \text{even}}$ . The proof of Theorem 2 is given in Sections 6.3-6.4.

6.3 Let  $L^0$  be a  $k$ -point of  $\tilde{\mathcal{L}}_d(M(F))$ . Let  $R \subset R^\perp$  be a c-lattice with  $L \cap R = 0$ . Write  $L_R^0$  for the image of  $L^0$  under (18). Applying the construction of Section 3.6 to the symplectic space  $R^\perp/R$  with  $L_R^0 \in \tilde{\mathcal{L}}(R^\perp/R)$ , one gets the functor

$$\mathcal{F}_{L_R^0} : \mathrm{DH}_{L_R} \rightarrow \mathrm{D}(\tilde{\mathcal{L}}(R^\perp/R))$$

If  $N^0$  is another  $k$ -point of  $\tilde{\mathcal{L}}_d(M(F))_R$  then writing  $N_R^0$  for the image of  $N^0$  in  $\tilde{\mathcal{L}}(R^\perp/R)$  we also get that the diagram

$$\begin{array}{ccc} \mathrm{DH}_{L_R} & \xrightarrow{\mathcal{F}_{L_R^0}} & \mathrm{D}(\tilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow \mathcal{F}_{N_R^0, L_R^0} & \nearrow \mathcal{F}_{N_R^0} & \\ \mathrm{DH}_{N_R} & & \end{array} \quad (22)$$

is canonically 2-commutative.

Let now

$${}^R\mathcal{F}_{L^0} : D\mathcal{H}_{L_R} \rightarrow D(\tilde{\mathcal{L}}_d(M(F)))_R$$

denote the composition of  $\mathcal{F}_{L^0}$  with the (exact for the perverse t-structures) restriction functor  $D(\tilde{\mathcal{L}}(R^\perp/R)) \rightarrow D(\tilde{\mathcal{L}}_d(M(F)))_R$  for the projection (18).

Let  $S \subset R$  be another c-lattice. As in Section 5.3, for the open immersion  $j_{S,R} : \tilde{\mathcal{L}}_d(M(F))_R \hookrightarrow \tilde{\mathcal{L}}_d(M(F))_S$  we have the restriction functors  $j_{S,R}^* : D(\tilde{\mathcal{L}}_d(M(F)))_S \rightarrow D(\tilde{\mathcal{L}}_d(M(F)))_R$ .

**Lemma 8.** *The diagram of functors is canonically 2-commutative*

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{{}^R\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}_d(M(F)))_R \\ \downarrow T_{S,R}^L & & \uparrow j_{S,R}^* \\ D\mathcal{H}_{L_S} & \xrightarrow{{}^S\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}_d(M(F)))_S \end{array}$$

*Proof* We have an open immersion  $j : \tilde{\mathcal{L}}(S^\perp/S)_R \hookrightarrow \tilde{\mathcal{L}}(S^\perp/S)$  and a projection  $p_{R/S} : \tilde{\mathcal{L}}(S^\perp/S)_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R)$ . Set  $P_{R/S} = p_{R/S}^* \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(p_{R/S})}$ . It suffices to show that the diagram is canonically 2-commutative

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0,R}} & D(\tilde{\mathcal{L}}(R^\perp/R)) & \xrightarrow{P_{R/S}} & D(\tilde{\mathcal{L}}(S^\perp/S)_R) \\ \downarrow T_{S,R}^L & & & \nearrow j^* & \\ D\mathcal{H}_{L_S} & \xrightarrow{\mathcal{F}_{L^0,S}} & D(\tilde{\mathcal{L}}(S^\perp/S)) & & \end{array}$$

This follows from Lemma 5.  $\square$

Define  $\mathcal{F}_{L^0,R} : D\mathcal{H}_{L_R} \rightarrow D(\tilde{\mathcal{L}}_d(M(F)))$  as the functor sending  $K_1$  to the following object  $K_2$ . For a c-lattice  $S \subset R$  we declare the restriction of  $K_2$  to  $\tilde{\mathcal{L}}_d(M(F))_S$  to be

$$({}^S\mathcal{F}_{L^0} \circ T_{S,R}^L)(K_1)$$

By Lemma 8, the corresponding projective system defines an object  $K_2$  of  $D(\tilde{\mathcal{L}}_d(M(F)))$ .

Finally, for  $S \subset R$  with  $R \cap L = 0$  the diagram

$$\begin{array}{ccc} D\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0,R}} & D(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow T_{S,R}^L & \nearrow \mathcal{F}_{L^0,S} & \\ D\mathcal{H}_{L_S} & & \end{array}$$

is canonically 2-commutative. We define (20) as the limit of the functors  $\mathcal{F}_{L^0,R}$  over the partially ordered set of c-lattices  $R \subset R^\perp$  such that  $L \cap R = 0$ . The commutativity of (21) follows from the commutativity of (22).

**Definition 4.** The non-ramified Weil category  $W(\tilde{\mathcal{L}}_d(M(F)))$  is the essential image of the functor  $\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow P(\tilde{\mathcal{L}}_d(M(F)))$ . It does not depend on a choice of a  $k$ -point  $L^0$  of  $\tilde{\mathcal{L}}_d(M(F))$ .

6.4 Let  $R \subset R^\perp$  be a c-lattice in  $M(F)$ , let  $\tilde{g} \in \tilde{G}(F)$  be a  $k$ -point, write  $g$  for its image in  $G(F)$ . As in Section 5.5, we have an isomorphism  $g : H_R \xrightarrow{\sim} H_{gR}$  of algebraic groups over  $k$  sending  $(x, a) \in (R^\perp/R) \times \mathbb{A}^1$  to  $(gx, a) \in (gR^\perp/gR) \times \mathbb{A}^1$ . For  $L \in \mathcal{L}_d(M(F))_R$  it induces an equivalence

$$g : \mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}$$

If  $L^0 \in \tilde{\mathcal{L}}_d(M(F))_R$  is a  $k$ -point then the  $G$ -equivariance of  $F$  implies that the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0}} & \mathrm{P}(\tilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0}} & \mathrm{P}(\tilde{\mathcal{L}}(gR^\perp/gR)) \end{array}$$

This, in turn, implies that the diagram is 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0, R}} & \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0, gR}} & \mathrm{P}(\tilde{\mathcal{L}}_d(M(F))) \end{array}$$

Thus, Theorem 2 is proved.

6.5 THETA-SHEAF Let  $L \in \mathcal{L}_d(M(F))_M$ , this is equivalent to saying that  $L \subset M(F)$  is a lagrangian d-lattice such that  $L \oplus M = M(F)$ . Then the category  $\mathcal{H}_{L_M}$  has a distinguished object  $\mathcal{L}_\psi$  on  $\mathbb{A}^1 = \mathrm{H}_M$ . Write  $S_L$  for its image under  $\mathcal{H}_{L_M} \rightarrow \mathcal{H}_L$ . The line bundle  $\mathcal{A}_d$  over  $\mathcal{L}_d(M(F))_M$  is canonically trivialized, so  $L$  has a distinguished enhanced structure

$$(L, \mathcal{B}) = L^0 \in \tilde{\mathcal{L}}_d(M(F))_M,$$

where  $\mathcal{B} = k$  is equipped with  $\mathrm{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(M : L)$ . The *theta-sheaf*  $S_{M(F)}$  over  $\tilde{\mathcal{L}}_d(M(F))$  is defined as  $\mathcal{F}_{L^0}(S_L)$ . It does not depend on  $L \in \mathcal{L}_d(M(F))_M$  in the sense that for another  $N \in \mathcal{L}_d(M(F))_M$  the diagram (21) yields a canonical isomorphism  $\mathcal{F}_{L^0}(S_L) \xrightarrow{\sim} \mathcal{F}_{N^0}(S_N)$ . The perverse sheaf  $S_{M(F)}$  has a natural  $G(\mathcal{O})$ -equivariant structure.

## 6.6 RELATION WITH THE SCHRÖDINGER MODEL

Assume in addition that  $M$  is decomposed as  $M \xrightarrow{\sim} U \oplus U^* \otimes \Omega$ , where  $U$  is a free  $\mathcal{O}$ -module of rank  $d$ , both  $U$  and  $U^* \otimes \Omega$  are isotropic, and the form  $\omega : \wedge^2 M \rightarrow \Omega$  is given by  $\omega\langle u, u^* \rangle = \langle u, u^* \rangle$  for  $u \in U, u^* \in U^* \otimes \Omega$ , where  $\langle \cdot, \cdot \rangle$  is the natural pairing between  $U$  and  $U^*$ . Let  $\bar{U} = U(F) \oplus \Omega(F)$  viewed as a subgroup of  $H(F)$ , it is equipped with the character  $\chi_U : \bar{U} \rightarrow \bar{\mathbb{Q}}_\ell^*$  given by  $\chi_U(u, a) = \psi(\mathrm{Res} a), a \in \Omega(F), u \in U(F)$ . Write

$$\mathrm{Shr}_U = \{f : H(F) \rightarrow \bar{\mathbb{Q}}_\ell \mid f(\bar{u}h) = \chi_U(\bar{u})f(h), \bar{u} \in \bar{U}, h \in H(F), f \text{ is smooth, of compact support modulo } \bar{U}\},$$

$H(F)$  acts on it by right translations. This is the Schrödinger model of the Weil representation, it identifies naturally with the Schwarz space  $\mathcal{S}(U^* \otimes \Omega(F))$ .

Remind the definition of the derived category  $D(U^* \otimes \Omega)$  and its subcategory of perverse sheaves  $P(U^* \otimes \Omega)$  given in ([11], Section 4). For  $N, r \in \mathbb{Z}$  with  $N + r \geq 0$  we write  ${}_{N,r}U = t^{-N}U/t^rU$ .

For  $N_1 \geq N_2, r_1 \geq r_2$  we have a diagram

$${}_{N_2, r_2}(U^* \otimes \Omega) \xleftarrow{p} {}_{N_2, r_1}(U^* \otimes \Omega) \xrightarrow{i} {}_{N_1, r_1}(U^* \otimes \Omega),$$

where  $p$  is the smooth projection and  $i$  is a closed immersion. We have a transition functor

$$D({}_{N_2, r_2}(U^* \otimes \Omega)) \rightarrow D({}_{N_1, r_1}(U^* \otimes \Omega)) \quad (23)$$

sending  $K$  to  $i_! p^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(p)}$ , it is fully faithful and exact for the perverse t-structures. Then  $D(U^* \otimes \Omega(F))$  (resp.,  $P(U^* \otimes \Omega(F))$ ) is defined as the inductive 2-limit of  $D({}_{N,r}(U^* \otimes \Omega))$  (resp., of  $P({}_{N,r}(U^* \otimes \Omega))$ ) as  $r, N$  go to infinity. The category  $P(U^* \otimes \Omega(F))$  is the geometric analog of the space  $\text{Shr}_U$ .

In this section we prove the following.

**Proposition 5.** *For each  $k$ -point  $L^0 \in \tilde{\mathcal{L}}_d(M(F))$  there is a canonical equivalence*

$$\mathcal{F}_{U(F), L^0} : D(U^* \otimes \Omega(F)) \rightarrow D\mathcal{H}_L \quad (24)$$

which identifies  $P(U^* \otimes \Omega(F))$  with the category  $\mathcal{H}_L$ . For  $L^0, N^0 \in \tilde{\mathcal{L}}_d(M(F))$  the diagram is canonically 2-commutative

$$\begin{array}{ccc} D(U^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U(F), L^0}} & D\mathcal{H}_L \\ \downarrow \mathcal{F}_{U(F), N^0} & \nearrow \mathcal{F}_{L^0, N^0} & \\ D\mathcal{H}_N & & \end{array}$$

For  $N \geq 0$  consider the c-lattice  $R = t^N M$  in  $M(F)$  and the corresponding symplectic space  $R^\perp/R = {}_{N,N}M$ . Set  $U_R := {}_{N,N}U \in \mathcal{L}({}_{N,N}M)$ . We have the line bundle  $\mathcal{A}_N$  on  $\mathcal{L}({}_{N,N}M)$  whose fibre at  $L$  is  $\det({}_{0,N}M) \otimes \det L$ . As above,  $\tilde{\mathcal{L}}({}_{N,N}M)$  is the gerb of square roots of  $\mathcal{A}_N$ . Let

$$U_R^0 = (U_R, \det({}_{0,N}U)) \in \tilde{\mathcal{L}}({}_{N,N}M)$$

equipped with a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  $\det({}_{0,N}U)^2 \xrightarrow{\sim} \det U_R \otimes \det({}_{0,N}M)$ .

Let  $H_R = {}_{N,N}M \times \mathbb{A}^1$  denote the corresponding Heisenberg group, it has the subgroup  $\bar{U}_R = U_R \times \mathbb{A}^1$  equipped with the character  $\chi_{U,R} : \bar{U}_R \rightarrow \bar{\mathbb{Q}}_\ell^*$  given by  $\chi_{U,R}(u, a) = \psi(a)$ ,  $a \in \mathbb{A}^1$ . In the classical setting,  $\mathcal{H}_{U_R}$  is the space of functions on  $H_R$ , which are  $(\bar{U}_R, \chi_{U,R})$ -equivariant under the left multiplication. Set  $\text{Shr}_U^R = \{f \in \text{Shr}_U \mid f(h(r, 0)) = f(h), r \in R, h \in H\}$ .

**Lemma 9.** *In the classical setting there is an isomorphism*

$$\text{Shr}_U^R \xrightarrow{\sim} \mathcal{H}_{U_R} \quad (25)$$

*Proof* Write  $\mathcal{H}'_{U_R} = \{\phi' : R^\perp/R \rightarrow \bar{\mathbb{Q}}_\ell \mid \phi'(m+u) = \psi(\frac{1}{2}\langle m, u \rangle)\phi'(m), u \in U_R\}$ . We identify  $\mathcal{H}_{U_R} \xrightarrow{\sim} \mathcal{H}'_{U_R}$  via the map  $\phi \mapsto \phi'$ , where  $\phi'(m) = \phi(m, 0)$ . Given  $f \in \text{Shr}_U^R$  for  $m \in t^{-N}M$  the value  $f(m, 0)$  depends only on the image  $\bar{m}$  of  $m$  under  $t^{-N}M \rightarrow {}_{N,N}M$ . The isomorphism (25) sends  $f$  to  $\phi' \in \mathcal{H}'_{U_R}$  given by  $\phi'(\bar{m}) = f(m, 0)$ .  $\square$

In the geometric setting  $\mathcal{H}_{U_R}$  is the category of  $(\bar{U}_R, \chi_{U,R})$ -equivariant perverse sheaves on  $H_R$ . We identify it with  $\text{P}({}_{N,N}(U^* \otimes \Omega))$  as follows. Let  $m_U : \bar{U}_R \times {}_{N,N}(U^* \otimes \Omega) \rightarrow H_R$  be the isomorphism sending  $(\bar{u}, h)$  to their product  $\bar{u}h$  in  $H_R$ . The functor  $\text{D}({}_{N,N}(U^* \otimes \Omega)) \rightarrow \text{D}\mathcal{H}_{U_R}$  sending  $K$  to

$$(m_U)_!(\chi_{U,R} \boxtimes K) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \bar{U}_R}$$

is an equivalence (exact for the perverse t-structures).

Let  $N' \geq N$  and  $S = t^{N'}M$ . The corresponding transition functor (23) now yields a functor denoted  $T_{S,R}^U : \text{D}\mathcal{H}_{U_R} \rightarrow \text{D}\mathcal{H}_{U_S}$ .

Let  $L^0 \in \tilde{\mathcal{L}}_d(M(F))$  be a  $k$ -point over  $L \in \mathcal{L}_d(M(F))$ . Assume that  $N$  is large enough so that  $L \cap R = 0$ . Let  $L_R^0$  denote the image of  $L^0$  under (18). Define  $U_S^0, L_S^0 \in \tilde{\mathcal{L}}(S^\perp/S)$  similarly.

**Lemma 10.** *The diagram is canonically 2-commutative*

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_R} & \xrightarrow{T_{S,R}^U} & \text{D}\mathcal{H}_{U_S} \\ \downarrow \mathcal{F}_{L_R^0, U_R^0} & & \downarrow \mathcal{F}_{L_S^0, U_S^0} \\ \text{D}\mathcal{H}_{L_R} & \xrightarrow{T_{S,R}^L} & \text{D}\mathcal{H}_{L_S} \end{array}$$

*Proof* Set  $W = t^{N'}U \oplus t^N(U^* \otimes \Omega)$ . The subspace  $W/S \subset S^\perp/S$  is isotropic, and  $U_S \cap (W/S) = L_S \cap (W/S) = 0$ . Write  $H_W = (W^\perp/W) \times \mathbb{A}^1$  for the corresponding Heisenberg group. Set  $U_W = U_S \cap (W^\perp/S)$ ,  $L_W = L_S \cap (W^\perp/S)$ . Applying Proposition 2, we get a 2-commutative diagram

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_W} & \xrightarrow{T_{S,W}^U} & \text{D}\mathcal{H}_{U_S} \\ \downarrow \mathcal{F}_{L_W^0, U_W^0} & & \downarrow \mathcal{F}_{L_S^0, U_S^0} \\ \text{D}\mathcal{H}_{L_W} & \xrightarrow{T_{S,W}^L} & \text{D}\mathcal{H}_{L_S} \end{array}$$

Now  $R/W \subset W^\perp/W$  is an isotropic subspace, and  $R/W \subset U_W$ ,  $R/W \cap L_W = 0$ . Note that  $U_R = U_W/(R/W)$ . Applying Proposition 3, we get a 2-commutative diagram

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_R} & \xrightarrow{T_{W,R}^U} & \text{D}\mathcal{H}_{U_W} \\ \downarrow \mathcal{F}_{L_R^0, U_R^0} & & \downarrow \mathcal{F}_{L_W^0, U_W^0} \\ \text{D}\mathcal{H}_{L_R} & \xrightarrow{T_{W,R}^L} & \text{D}\mathcal{H}_{L_W} \end{array}$$

Our assertion easily follows.  $\square$

*Proof of Proposition 5*

Passing to the limit as  $N$  goes to infinity, the functors  $\mathcal{F}_{L_R^0, U_R^0} : D\mathcal{H}_{U_R} \rightarrow D\mathcal{H}_{L_R}$  from Lemma 10 yield the desired functor (24). The second assertion follows by construction.  $\square$

**Definition 5.** Let  $\mathcal{F}_{U(F)} : D(U^* \otimes \Omega(F)) \rightarrow D(\widetilde{\mathcal{L}}_d(M(F)))$  denote the composition

$$D(U^* \otimes \Omega(F)) \xrightarrow{\mathcal{F}_{U(F), L^0}} D\mathcal{H}_L \xrightarrow{\mathcal{F}_{L^0}} D(\widetilde{\mathcal{L}}_d(M(F)))$$

By Theorem 2 and Proposition 5, it does not depend on the choice of a  $k$ -point  $L^0 \in \widetilde{\mathcal{L}}_d(M(F))$ . By construction,  $\mathcal{F}_{U(F)}$  is exact for the perverse t-structures.

We have a morphism of group stacks  $\mathrm{GL}(U)(F) \rightarrow \widetilde{G}(F)$  sending  $g \in \mathrm{GL}(U)(F)$  to  $(g, \mathcal{B} = \det(U : gU))$  equipped with a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det(M : gM) \xrightarrow{\sim} \det(U : gU) \otimes \det(U^* \otimes \Omega : g(U^* \otimes \Omega)) \xrightarrow{\sim} \det(U : gU)^{\otimes 2}$$

Let  $\mathrm{GL}(U)(F)$  act on  $\widetilde{\mathcal{L}}_d(M(F))$  via this homomorphism, let it also act naturally on  $U^* \otimes \Omega(F)$ . Then one may show that  $\mathcal{F}_{U(F)}$  commutes with the action of  $\mathrm{GL}(U)(F)$ .

Note also that over  $\mathrm{GL}(U)(\mathcal{O})$  the sections  $\mathrm{GL}(U)(F) \rightarrow \widetilde{G}(F)$  and  $G(\mathcal{O}) \rightarrow \widetilde{G}(F)$  are compatible.

## 7. GLOBAL APPLICATION

7.1 Assume  $k$  algebraically closed. Let  $X$  be a smooth connected projective curve. Let  $\Omega$  be the canonical invertible sheaf on  $X$ . Let  $G$  be the group scheme over  $X$  of automorphisms of  $\mathcal{O}_X^d \oplus \Omega^d$  preserving the symplectic form  $\wedge^2(\mathcal{O}_X^d \oplus \Omega^d) \rightarrow \Omega$ .

Write  $\mathrm{Bun}_G$  for the stack of  $G$ -torsors on  $X$ , it classifies a rank  $2d$ -vector bundle  $\mathcal{M}$  on  $X$  together with a symplectic form  $\wedge^2 \mathcal{M} \rightarrow \Omega$ . Let  $\mathcal{A}$  be the  $(\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero) line bundle on  $\mathrm{Bun}_G$  whose fibre at  $\mathcal{M}$  is  $\det \mathrm{R}\Gamma(X, \mathcal{M})$ . Write  $\widetilde{\mathrm{Bun}}_G$  for the gerb of square roots of  $\mathcal{A}$  over  $\mathrm{Bun}_G$ .

Remind the definition of the theta-sheaf  $\mathrm{Aut}$  on  $\widetilde{\mathrm{Bun}}_G$  ([10], Definition 1). Let  ${}_i \mathrm{Bun}_G \hookrightarrow \mathrm{Bun}_G$  be the locally closed substack given by  $\dim H^0(X, \mathcal{M}) = i$  for  $\mathcal{M} \in \mathrm{Bun}_G$ . Write  ${}_i \widetilde{\mathrm{Bun}}_G$  for the restriction of  $\widetilde{\mathrm{Bun}}_G$  to  ${}_i \mathrm{Bun}_G$ .

Let  ${}_i \mathcal{B}$  be the line bundle on  ${}_i \mathrm{Bun}_G$  whose fibre at  $\mathcal{M} \in {}_i \mathrm{Bun}_G$  is  $\det H^0(X, \mathcal{M})$ , we view it as  $\mathbb{Z}/2\mathbb{Z}$ -graded of degree  $i \bmod 2$ . For each  $i$  we have a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  ${}_i \mathcal{B}^2 \xrightarrow{\sim} \mathcal{A}$ , it yields a trivialization  ${}_i \widetilde{\mathrm{Bun}}_G \xrightarrow{\sim} {}_i \mathrm{Bun}_G \times B(\mu_2)$ .

Define  $\mathrm{Aut}_g \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$  (resp.,  $\mathrm{Aut}_s \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$ ) as the intermediate extension of

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathrm{Bun}_G}$$

(resp., of  $(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathrm{Bun}_G - 1}$ ) under  ${}_i \widetilde{\mathrm{Bun}}_G \hookrightarrow \widetilde{\mathrm{Bun}}_G$ . Set  $\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s$ .

7.2 Fix a closed point  $x \in X$ . Write  $\mathcal{O}_x$  for the completed local ring of  $X$  at  $x$ ,  $F_x$  for its fraction field. Fix a  $G$ -torsor over  $\text{Spec } \mathcal{O}_x$ , we think of it as a free  $\mathcal{O}_x$ -module  $M$  of rank  $2d$  with symplectic form  $\wedge^2 M \rightarrow \Omega(\mathcal{O}_x)$  and an action of  $G(\mathcal{O}_x)$ . We have a map

$$\xi_x : \text{Bun}_G \rightarrow \mathcal{L}_d(M(F_x))/G(\mathcal{O}_x),$$

where  $\mathcal{L}_d(M(F_x))/G(\mathcal{O}_x)$  is the stack quotient. It sends  $\mathcal{M} \in \text{Bun}_G$  to the Tate space  $\mathcal{M}(F_x)$  with lagrangian c-lattice  $\mathcal{M}(\mathcal{O}_x)$  and lagrangian d-lattice  $H^0(X-x, \mathcal{M})$ .

The line bundle  $\mathcal{A}_d$  on  $\mathcal{L}_d(M(F_x))/G(\mathcal{O}_x)$  is that of Section 5.3. Write  $\tilde{\mathcal{L}}_d(M(F_x))/G(\mathcal{O}_x)$  for the gerb of square roots of  $\mathcal{A}_d$ .

We have canonically  $\xi_x^* \mathcal{A}_d \xrightarrow{\sim} \mathcal{A}$ , so  $\xi$  lifts naturally to a map of gerbs

$$\tilde{\xi}_x : \widetilde{\text{Bun}}_G \rightarrow \tilde{\mathcal{L}}_d(M(F_x))/G(\mathcal{O}_x)$$

For  $r \geq 0$  let  ${}_{rx} \text{Bun}_G \subset \text{Bun}_G$  be the open substack given by  $H^0(X, \mathcal{M}(-rx)) = 0$ . Write  ${}_{rx} \widetilde{\text{Bun}}_G$  for the restriction of the gerb  $\widetilde{\text{Bun}}_G$  to  ${}_{rx} \text{Bun}_G$ . If  $r' \geq r$  then  ${}_{rx} \widetilde{\text{Bun}}_G \subset {}_{r'x} \widetilde{\text{Bun}}_G$  is an open substack, so we consider the projective 2-limit

$$2\text{-}\lim_{r \rightarrow \infty} \text{D}({}_{rx} \widetilde{\text{Bun}}_G)$$

Note that  $2\text{-}\lim_{r \rightarrow \infty} \text{P}({}_{rx} \widetilde{\text{Bun}}_G) \xrightarrow{\sim} \text{P}(\widetilde{\text{Bun}}_G)$  is a full subcategory in the above limit. Let us define the restriction functor

$$\tilde{\xi}_x^* : \text{D}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F))) \rightarrow 2\text{-}\lim_{r \rightarrow \infty} \text{D}({}_{rx} \widetilde{\text{Bun}}_G) \quad (26)$$

To do so, for  $N \geq r \geq 0$  and  $r_1 \geq 2N$  let

$$\xi_N : {}_{rx} \text{Bun}_G \rightarrow {}_r \mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}) \quad (27)$$

be the map sending  $\mathcal{M}$  to the lagragian subspace  $H^0(X, \mathcal{M}(Nx)) \subset {}_{N,N} \mathcal{M}$ . If  $N_1 \geq N \geq r$  and  $r_1 \geq 2N_1$  then the diagram commutes

$$\begin{array}{ccc} {}_{rx} \text{Bun}_G & \xrightarrow{\xi_N} & {}_r \mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}) \\ & \searrow \xi_{N_1} & \uparrow p \\ & & {}_r \mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}) \end{array}$$

It induces a similar diagram between the gerbs (cf. Section 5.3 for their definition)

$$\begin{array}{ccc} {}_{rx} \widetilde{\text{Bun}}_G & \xrightarrow{\tilde{\xi}_N} & ({}_r \mathcal{L}_{(N,N)M}/G(\mathcal{O}/t^{r_1}))^\sim \\ & \searrow \tilde{\xi}_{N_1} & \uparrow \\ & & ({}_r \mathcal{L}_{(N_1,N_1)M}/G(\mathcal{O}/t^{r_1}))^\sim \end{array}$$

The functors  $K \mapsto \tilde{\xi}_N^* K \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\xi_N)}$  from  $\text{D}_{G(\mathcal{O})}({}_r \tilde{\mathcal{L}}_{(N,N)M})$  to  $\text{D}({}_{rx} \widetilde{\text{Bun}}_G)$  are compatible with the transition functors, so yield a functor

$${}_r \tilde{\xi}_x^* : \text{D}_{G(\mathcal{O})}({}_r \tilde{\mathcal{L}}_d(M(F))) \rightarrow \text{D}({}_{rx} \widetilde{\text{Bun}}_G)$$

Passing to the limit by  $r$ , one gets the desired functor (26).

**Theorem 3.** *The object  $\tilde{\xi}_x^* S_{M(F_x)}$  lies in  $\mathrm{P}(\widetilde{\mathrm{Bun}}_G)$ , and there is an isomorphism of perverse sheaves*

$$\tilde{\xi}_x^* S_{M(F_x)} \xrightarrow{\sim} \mathrm{Aut}$$

*Proof* For  $r \geq 0$  consider the map

$$\tilde{\xi}_r : {}_{rx} \widetilde{\mathrm{Bun}}_G \rightarrow (\mathcal{L}_{(r,r}M)/G(\mathcal{O}/t^{2r}))^\sim$$

Set  $Y = \mathcal{L}_{(r,r}M) \times \mathcal{L}_{(r,r}M)$ . Write  $\mathcal{Y}$  for the stack quotient of  $Y$  by the diagonal action of  $\mathrm{Sp}_{(r,r}M)$ . Let  $\mathcal{A}_{\mathcal{Y}}$  be the  $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on  $\mathcal{Y}$  with fibre  $\det L_1 \otimes \det L_2$  at  $(L_1, L_2)$ . Write  $\tilde{\mathcal{Y}}$  for the gerb of square roots of  $\mathcal{A}_{\mathcal{Y}}$  over  $\mathcal{Y}$ . The map  $\mathcal{L}_{(r,r}M) \rightarrow Y$  sending  $L_1$  to  $({}_{0,r}M, L_1) \in Y$  yields a morphism of stacks

$$\rho : (\mathcal{L}_{(r,r}M)/G(\mathcal{O}/t^{2r}))^\sim \rightarrow \tilde{\mathcal{Y}}$$

Write  $S_{r,r}M$  for the perverse sheaf on  $\tilde{\mathcal{Y}}$  introduced in (Section 3.2, Definition 1). Set  $\tau = \rho \circ \tilde{\xi}_r$ . It suffices to establish for any  $r \geq 0$  a canonical isomorphism

$$\tau^* S_{r,r}M \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\mathrm{rel}(\tau)} \xrightarrow{\sim} \mathrm{Aut} \quad (28)$$

over  ${}_{rx} \widetilde{\mathrm{Bun}}_G$ .

Remind that  $Y_i \subset Y$  is the locally closed subscheme given by  $\dim(L_1 \cap L_2) = i$  for  $(L_1, L_2) \in Y$ . Let  $\mathcal{Y}_i$  be the stack quotient of  $Y_i$  by the diagonal action of  $\mathrm{Sp}_{(r,r}M)$ , set  $\tilde{\mathcal{Y}}_i = \mathcal{Y}_i \times_{\mathcal{Y}} \tilde{\mathcal{Y}}$ . Set

$${}_{rx,i} \widetilde{\mathrm{Bun}}_G = {}_{rx} \widetilde{\mathrm{Bun}}_G \cap_i \widetilde{\mathrm{Bun}}_G \quad \text{and} \quad {}_{rx,i} \mathrm{Bun}_G = {}_{rx} \mathrm{Bun}_G \cap_i \mathrm{Bun}_G$$

For each  $i$  the map  $\tau$  fits into a cartesian square

$$\begin{array}{ccc} {}_{rx,i} \widetilde{\mathrm{Bun}}_G & \xrightarrow{\tau_i} & \tilde{\mathcal{Y}}_i \\ \downarrow & & \downarrow \\ {}_{rx} \widetilde{\mathrm{Bun}}_G & \xrightarrow{\tau} & \tilde{\mathcal{Y}} \end{array}$$

Indeed, for  $\mathcal{M} \in {}_{rx} \mathrm{Bun}_G$  the space  $\mathrm{H}^0(X, \mathcal{M})$  equals the intersection of  $\mathcal{M}/\mathcal{M}(-rx)$  and  $\mathrm{H}^0(X, \mathcal{M}(rx))$  inside  $\mathcal{M}(rx)/\mathcal{M}(-rx)$ . By ([10], Theorem 1), the  $*$ -restriction of  $\mathrm{Aut}$  to  ${}_i \widetilde{\mathrm{Bun}}_G \xrightarrow{\sim} {}_i \mathrm{Bun}_G \times B(\mu_2)$  identifies with

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathrm{Bun}_G - i}$$

Similarly, by ([10], Proposition 1 and 5), the  $*$ -restriction of  $S_M$  to  $\tilde{\mathcal{Y}}_i \xrightarrow{\sim} \mathcal{Y}_i \times B(\mu_2)$  identifies with

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{Y} - i}$$

Since the map  $\tau_i$  is compatible with our trivializations of the corresponding gerbs, we get the isomorphism (28) over  ${}_{rx,i} \widetilde{\mathrm{Bun}}_G$  for each  $i$ . Since  $\mathrm{Aut}$  is perverse, this also shows that the LHS

of (28) is placed in perverse degrees  $\leq 0$ , and its  $*$ -restriction to  ${}_{\leq 2}\widetilde{\text{Bun}}_G$  is placed in perverse degrees  $< 0$ .

The map  $\tau$  is not smooth, we overcome this difficulty as follows. Let us show that the LHS of (28) is placed in perverse degrees  $\geq 0$ . Consider the stack  $\mathcal{X}$  classifying  $(\mathcal{M}, \mathcal{B}) \in {}_{rx}\widetilde{\text{Bun}}_G$  and a trivialization

$$\mathcal{M} |_{\text{Spec } \mathcal{O}_x/t_x^{2r}} \xrightarrow{\sim} M |_{\text{Spec } \mathcal{O}_x/t_x^{2r}}$$

of the corresponding  $G$ -torsor. Let  $\nu : \mathcal{X} \rightarrow \tilde{Y}$  be the map sending a point of  $\mathcal{X}$  to the triple  $(\mathcal{M}/\mathcal{M}(-rx), H^0(X, \mathcal{M}(rx)), \mathcal{B})$ . Define  $\mathcal{X}_1$  and  $\mathcal{X}_3$  by the cartesian squares

$$\begin{array}{ccc} \mathcal{X}_3 & \rightarrow & C_3 \\ \downarrow \pi_{\mathcal{X}_3} & & \downarrow \pi_C \\ \mathcal{X}_1 & \rightarrow & U \times_{\mathcal{L}(r,rM)} U \\ \downarrow & & \downarrow \pi_Y \\ \mathcal{X} & \xrightarrow{\nu} & \tilde{Y}, \end{array}$$

Using (7), we get an isomorphism

$$\mu^* \tau^* S_{r,rM} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\mu)+\dim.\text{rel}(\tau)} \xrightarrow{\sim} (\pi_{\mathcal{X}_3})_! \mathcal{E} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim \mathcal{X}_3}$$

for some rank one local system  $\mathcal{E}$  on  $\mathcal{X}_3$ . Here  $\mu : \mathcal{X}_1 \rightarrow {}_{rx}\widetilde{\text{Bun}}_G$  is the projection, it is smooth. Since  $\pi_{\mathcal{X}_3}$  is affine and  $\mathcal{X}_3$  is smooth, the LHS of (28) is placed in perverse degrees  $\geq 0$ .

Thus, there exists an exact sequence of perverse sheaves  $0 \rightarrow K \rightarrow K_1 \rightarrow \text{Aut} \rightarrow 0$  on  ${}_{rx}\widetilde{\text{Bun}}_G$ , where  $K_1 = \tau^* S_{r,rM} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\dim.\text{rel}(\tau)}$ , and  $K$  is the extension by zero from  ${}_{\leq 2}\widetilde{\text{Bun}}_G$ . But we know already that  $K_1$  and  $\text{Aut}$  are isomorphic in the Grothendieck group of  ${}_{rx}\widetilde{\text{Bun}}_G$ . So,  $K$  vanishes in this Grothendieck group, hence  $K = 0$ . We are done.  $\square$

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